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SHARP ASYMPTOTICS FOR THE EIGENVALUE FUNCTION OF AHARONOV-BOHM OPERATORS WITH A MOVING POLE

Abstract. In this breif note we present several results obtained in collaboration with V. Felli. They concern the behavior of eigenvalues for a magnetic Aharonov-Bohm operator with half-integer circulation and Dirichlet boundary conditions in a planar domain. In particular, they contain sharp asymptotics for eigenvalues as the pole is moving in the interior of the domain, approaching a zero of an eigenfunction of the limiting problem along a general direction.

1. Introduction

In this brief note we present several results obtained in collaboration with V. Felli which are essentially proved in the papers [1] and [2]. They concern the behavior of eigenvalues for a magnetic Aharonov-Bohm operator with half-integer circulation and Dirichlet boundary conditions in a planar domain. As it will appear in the sequel, these operators are special as they present a strong singularity at a point (pole), for which they cannot be considered small perturbations of the standard Laplacian. In particular, the two aforementioned papers address the challenging question about the possible determination of the first term of the Taylor expansion of the function $a \mapsto \lambda_a$, where a is the operator's pole and λ_a is one of its simple eigenvalues.

Indeed, if Ω is a Lipschitz open bounded and simply connected set in $\mathbb{R}^2$ and if λ_0 is a simple eigenvalue for the Aharonov-Bohm operator with the pole located at 0, the function $a \mapsto \lambda_a$ can be shown to be analytic in a neighborhood of 0 ([21]).

We devote the second section to show the functional setting and to state the results in a rigorous way. This part will take some technical definitions and several references.

In the third section we briefly present how the general problem and the particular results presented here deal with the so-called *spectral minimal partitions*. This is a wider research topic, it was initiated by some seminal papers by B. Helffer, T. Hoffmann-Ostenhof, S. Terracini and others. In this section we would like to stress our contribution in this direction.

The last section contains the main ideas for the proofs of the results. The interested reader can find them in full details in the papers [1] and [2].

2. Statement of the results

For $a = (a_1, a_2) \in \mathbb{R}^2$ and $\gamma \in \mathbb{R}\setminus\mathbb{Z}$, we consider the vector potential

$$A_a^\gamma(x) = \gamma\left(\frac{-(x_2-a_2)}{(x_1-a_1)^2+(x_2-a_2)^2}, \frac{x_1-a_1}{(x_1-a_1)^2+(x_2-a_2)^2}\right),$$
$$x = (x_1,x_2) \in \mathbb{R}^2\setminus\{a\},$$

which generates the Aharonov-Bohm magnetic field in $\mathbb{R}^2$ with pole a and circulation γ; such a field is produced by an infinitely long thin solenoid intersecting perpendicularly the plane (x_1,x_2) at the point a, as the radius of the solenoid goes to zero and the magnetic flux remains constantly equal to γ.

We will focus on the case of half-integer circulation, so we will assume $\gamma = 1/2$ and denote

$$A_a(x) = A_a^{1/2}(x) = A_0(x-a), \quad \text{where} \quad A_0(x_1,x_2) = \frac{1}{2}\left(-\frac{x_2}{x_1^2+x_2^2}, \frac{x_1}{x_1^2+x_2^2}\right).$$

In the spirit [7], [22] and [23], we are interested in studying the dependence on the pole a of the spectrum of Schrödinger operators with Aharonov-Bohm vector potentials, i.e. of operators $(i\nabla + A_a)^2$ acting on functions $u : \mathbb{R}^2 \to \mathbb{C}$ as

$$(i\nabla + A_a)^2 u = -\Delta u + 2iA_a \cdot \nabla u + |A_a|^2 u.$$

Let $\Omega \subset \mathbb{R}^2$ be a bounded, open and simply connected domain. For every $a \in \Omega$, we introduce the space $H^{1,a}(\Omega,\mathbb{C})$ as the completion of $\{u \in H^1(\Omega,\mathbb{C}) \cap C^\infty(\Omega,\mathbb{C}) :$ u vanishes in a neighborhood of $a\}$ with respect to the norm

$$\|u\|_{H^{1,a}(\Omega,\mathbb{C})} = \left(\|\nabla u\|^2_{L^2(\Omega,\mathbb{C}^2)} + \|u\|^2_{L^2(\Omega,\mathbb{C})} + \left\|\frac{u}{|x-a|}\right\|^2_{L^2(\Omega,\mathbb{C})}\right)^{1/2}.$$

It is easy to verify that $H^{1,a}(\Omega,\mathbb{C}) = \left\{u \in H^1(\Omega,\mathbb{C}) : \frac{u}{|x-a|} \in L^2(\Omega,\mathbb{C})\right\}$. We also observe that, in view of the Hardy type inequality proved in [20], an equivalent norm in $H^{1,a}(\Omega,\mathbb{C})$ is given by

$$\left(\|(i\nabla + A_a)u\|^2_{L^2(\Omega,\mathbb{C}^2)} + \|u\|^2_{L^2(\Omega,\mathbb{C})}\right)^{1/2}.$$

We also consider the space $H_0^{1,a}(\Omega,\mathbb{C})$ as the completion of $C_c^\infty(\Omega\setminus\{a\},\mathbb{C})$ with respect to the norm $\|\cdot\|_{H_a^1(\Omega,\mathbb{C})}$, so that $H_0^{1,a}(\Omega,\mathbb{C}) = \left\{u \in H_0^1(\Omega,\mathbb{C}) : \frac{u}{|x-a|} \in L^2(\Omega,\mathbb{C})\right\}$.

For every $a \in \Omega$, we consider the eigenvalue problem

$$(E_a) \qquad \begin{cases} (i\nabla + A_a)^2 u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

in a weak sense, i.e. we say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (E_a) if there exists $u \in H_0^{1,a}(\Omega,\mathbb{C})\setminus\{0\}$ (called eigenfunction) such that

$$\int_\Omega (i\nabla u + A_a u)\cdot \overline{(i\nabla v + A_a v)}\,dx = \lambda \int_\Omega u\bar{v}\,dx \quad \text{for all } v \in H_0^{1,a}(\Omega,\mathbb{C}).$$

From classical spectral theory, the eigenvalue problem (E_a) admits a sequence of real diverging eigenvalues $\{\lambda_k^a\}_{k\geqslant 1}$ with finite multiplicity; in the enumeration $\lambda_1^a \leqslant \lambda_2^a \leqslant \cdots \leqslant \lambda_j^a \leqslant \ldots$, we repeat each eigenvalue as many times as its multiplicity. We are interested in the behavior of the function $a \mapsto \lambda_j^a$ in a neighborhood of a fixed point $b \in \Omega$. Up to a translation, it is not restrictive to consider $b = 0$. Thus, we assume that $0 \in \Omega$.

In [7, Theorem 1.1] and [21, Theorem 1.2] it is proved that, for all $j \geqslant 1$,

(1) $$\text{the function } a \mapsto \lambda_j^a \text{ is continuous in } \Omega.$$

A strong improvement of the regularity (1) holds under simplicity of the eigenvalue. Indeed in [7, Theorem 1.3] it is proved that, if there exists $n_0 \geqslant 1$ such that

(2) $$\lambda_{n_0}^0 \quad \text{is simple},$$

then the function $a \mapsto \lambda_{n_0}^a$ is of class C^∞ in a neighborhood of 0; this regularity result is improved in [21, Theorem 1.3], where, in the more general setting of Aharonov-Bohm operators with many singularities, it is shown that, under assumption (2) the function $a \mapsto \lambda_{n_0}^a$ is analytic in a neighborhood of 0. Then the question of what is the leading term in the asymptotic expansion of such a function (at least on a single straight path around the limit point 0) naturally arises. This may also shed some light on the nature of 0 as a critical point for the map $a \mapsto \lambda_a$ when the limit eigenfunction has in 0 a zero of order $k/2$ with $k \geqslant 3$ odd.

Let us assume that there exists $n_0 \geqslant 1$ such that (2) holds and denote $\lambda_0 = \lambda_{n_0}^0$ and, for any $a \in \Omega$, $\lambda_a = \lambda_{n_0}^a$. From (1) it follows that, if $a \to 0$, then $\lambda_a \to \lambda_0$. Let $\varphi_0 \in H_0^{1,0}(\Omega,\mathbb{C})\setminus\{0\}$ be an eigenfunction of problem (E_0) associated to the eigenvalue $\lambda_0 = \lambda_{n_0}^0$, i.e. solving

$$\begin{cases} (i\nabla + A_0)^2\varphi_0 = \lambda_0\varphi_0, & \text{in } \Omega, \\ \varphi_0 = 0, & \text{on } \partial\Omega, \end{cases}$$

such that

$$\int_\Omega |\varphi_0(x)|^2\,dx = 1.$$

In view of [11, Theorem 1.3] we have that

(3) $$\varphi_0 \text{ has at 0 a zero of order } \frac{k}{2} \text{ for some odd } k \in \mathbb{N},$$

see [7, Definition 1.4]. We recall from [11, Theorem 1.3] and [23, Theorem 1.5] that (3) implies that the eigenfunction φ_0 has got exactly k nodal lines meeting at 0 and dividing the whole angle into k equal parts.

A first result relating the rate of convergence of λ_a to λ_0 with the order of vanishing of φ_0 at 0 can be found in [7], where the following estimate is proved.

THEOREM 1 ([7], Theorem 1.7). *If assumptions* (2) *and* (3) *with* $k \geqslant 3$ *are satisfied, then*

$$|\lambda_a - \lambda_0| \leqslant C|a|^{\frac{k+1}{2}} \qquad \text{as } a \to 0$$

for a constant $C > 0$ *independent of* a*.*

As already mentioned, the latter theorem pursue the idea that the asymptotic expansion of the function $a \mapsto \lambda_a$ has to do with the nodal properties of the related limit eigenfunction.

The first result we present is essentially proved in the paper [1] and establishes the exact order of the asymptotic expansion of $\lambda_a - \lambda_0$ along a suitable direction as $|a|^k$, where k is the number of nodal lines of φ_0 at 0 which coincides with twice the order of vanishing of φ_0 in assumption (3). In addition, we detected the sharp coefficient of the asymptotics, which can be characterized in terms of the limit profile of a blow-up sequence obtained by a suitable scaling of approximating eigenfunctions.

In order to state properly the first result, we need to recall some known facts and to introduce some notation. By [11, Theorem 1.3], if φ_0 is an eigenfunction of $(i\nabla + A_0)^2$ on Ω satisfying assumption (3), there exist $\beta_1, \beta_2 \in \mathbb{C}$ such that $(\beta_1, \beta_2) \neq (0,0)$ and

$$r^{-k/2}\varphi_0(r(\cos t, \sin t)) \to \beta_1 e^{i\frac{t}{2}} \cos\left(\frac{k}{2}t\right) + \beta_2 e^{i\frac{t}{2}} \sin\left(\frac{k}{2}t\right) \quad \text{in } C^{1,\tau}([0,2\pi],\mathbb{C}) \tag{4}$$

as $r \to 0^+$ for any $\tau \in (0,1)$.

Let s_0 be the positive half-axis $s_0 = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 = 0 \text{ and } x_1 \geqslant 0\}$. We observe that, for every odd natural number k, there exists a unique (up to a multiplicative constant) function ψ_k which is harmonic on $\mathbb{R}^2 \backslash s_0$, homogeneous of degree $k/2$ and vanishing on s_0. Such a function is given by

$$\psi_k(r\cos t, r\sin t) = r^{k/2} \sin\left(\frac{k}{2}t\right), \quad r \geqslant 0, \quad t \in [0,2\pi]. \tag{5}$$

Let $s := \{(x_1,x_2) \in \mathbb{R}^2 : x_2 = 0 \text{ and } x_1 \geqslant 1\}$ and $\mathbb{R}^2_+ = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 > 0)\}$. We denote as $\mathcal{D}^{1,2}_s(\mathbb{R}^2_+)$ the completion of $C^\infty_c(\overline{\mathbb{R}^2_+} \backslash s)$ under the norm $(\int_{\mathbb{R}^2_+} |\nabla u|^2\, dx)^{1/2}$. From the Hardy type inequality proved in [20] and a change of gauge, it follows that functions in $\mathcal{D}^{1,2}_s(\mathbb{R}^2_+)$ satisfy the following Hardy type inequality:

$$\int_{\mathbb{R}^2} |\nabla\varphi(x)|^2\, dx \geqslant \frac{1}{4}\int_{\mathbb{R}^2} \frac{|\varphi(x)|^2}{|x - \mathbf{e}|^2}\, dx, \quad \text{for all } \varphi \in \mathcal{D}^{1,2}_s(\mathbb{R}^2_+),$$

where $\mathbf{e} = (1,0)$. Then

$$\mathcal{D}^{1,2}_s(\mathbb{R}^2_+) = \left\{ u \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}^2_+} \backslash s) : \ \nabla u \in L^2(\mathbb{R}^2_+),\ \tfrac{u}{|x-\mathbf{e}|} \in L^2(\mathbb{R}^2_+),\ \text{and } u = 0 \text{ on } s \right\}.$$

The functional

$$J_k : \mathcal{D}^{1,2}_s(\mathbb{R}^2_+) \to \mathbb{R}, \quad J_k(u) = \frac{1}{2}\int_{\mathbb{R}^2_+} |\nabla u(x)|^2\, dx - \int_{\partial\mathbb{R}^2_+ \backslash s} u(x_1,0)\frac{\partial\psi_k}{\partial x_2}(x_1,0)\, dx_1,$$

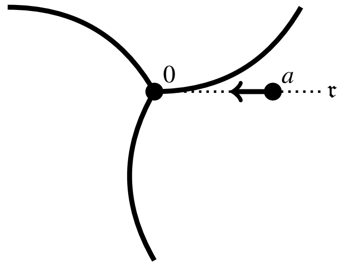

Figure 1: a approaches 0 along the tangent $\mathfrak{r}$ to a nodal line of φ_0.

is well-defined on the space $\mathcal{D}_s^{1,2}(\mathbb{R}^2_+)$; we notice that $\frac{\partial \psi_k}{\partial x_2}$ vanishes on $\partial\mathbb{R}^2_+ \setminus s_0$, so that

$$\int_{\partial\mathbb{R}^2_+ \setminus s} u(x_1,0)\frac{\partial \psi_k}{\partial x_2}(x_1,0)\,dx_1 = \int_0^1 u(x_1,0)\frac{\partial \psi_k}{\partial x_2}(x_1,0)\,dx_1.$$

By standard minimization methods, J_k achieves its minimum over the whole space $\mathcal{D}_s^{1,2}(\mathbb{R}^2_+)$ at some function $w_k \in \mathcal{D}_s^{1,2}(\mathbb{R}^2_+)$, i.e. there exists $w_k \in \mathcal{D}_s^{1,2}(\mathbb{R}^2_+)$ such that

$$\mathfrak{m}_k = \min_{u\in\mathcal{D}_s^{1,2}(\mathbb{R}^2_+)} J_k(u) = J_k(w_k). \tag{6}$$

We note that

$$\mathfrak{m}_k = J_k(w_k) = -\frac{1}{2}\int_{\mathbb{R}^2_+} |\nabla w_k(x)|^2\,dx = -\frac{1}{2}\int_0^1 \frac{\partial_+ \psi_k}{\partial x_2}(x_1,0)\,w_k(x_1,0)\,dx_1 < 0, \tag{7}$$

where, for all $x_1 > 0$, $\frac{\partial_+ \psi_k}{\partial x_2}(x_1,0) = \lim_{t\to 0^+} \frac{\psi_k(x_1,t)-\psi_k(x_1,0)}{t} = \frac{k}{2}x_1^{\frac{k}{2}-1}$.

We are now in position to state the first result.

THEOREM 2 ([1]). *Let $\Omega \subset \mathbb{R}^2$ be a bounded, open and simply connected domain such that $0 \in \Omega$ and let $n_0 \geqslant 1$ be such that the n_0-th eigenvalue $\lambda_0 = \lambda^0_{n_0}$ of $(i\nabla + A_0)^2$ on Ω is simple with associated eigenfunctions having in 0 a zero of order $k/2$ with $k \in \mathbb{N}$ odd. For $a \in \Omega$ let $\lambda_a = \lambda^a_{n_0}$ be the n_0-th eigenvalue of $(i\nabla + A_a)^2$ on Ω. Let $\mathfrak{r}$ be the half-line tangent to a nodal line of eigenfunctions associated to λ_0 ending at 0. Then, as $a \to 0$ with $a \in \mathfrak{r}$,*

$$\frac{\lambda_0 - \lambda_a}{|a|^k} \to -4\left(|\beta_1|^2 + |\beta_2|^2\right)\mathfrak{m}_k$$

with $(\beta_1,\beta_2) \neq (0,0)$ being as in (4) and $\mathfrak{m}_k$ being as in (6)–(7).

Once stated the first main result, we would like to recall the following result, established in the paper [7]:

PROPOSITION 1. *([7, Corollary 1.8]) Fix any $j \in \mathbb{N}$. If 0 is an extremal point of $a \mapsto \lambda_j^a$, then either λ_j^0 is not simple, or the eigenfunction of $(i\nabla + A_0)^2$ associated to λ_j^0 has at 0 a zero of order $k/2$ with $k \geqslant 3$ odd.*

This gives us the opportunity to list several of remarkable consequences of Theorem 2.

1. Due to the analyticity of $a \mapsto \lambda_a$, $\frac{\lambda_0 - \lambda_a}{|a|^k} \to 4\left(|\beta_1|^2 + |\beta_2|^2\right) \mathfrak{m}_k$ as $a \to 0$ along the opposite half-line. Hence, if λ_0 is simple, then 0 cannot be an extremal point of the map $a \mapsto \lambda_a$.
2. In view of Theorem 2 and the first consequence, we can exclude the second alternative in Proposition 1, producing the following claim: fix any $j \in \mathbb{N}$, if 0 is an extremal point of $a \mapsto \lambda_j^a$, then λ_j^0 is not simple.
3. If λ_0 is simple and $k \geqslant 3$, 0 is a saddle point for the map $a \mapsto \lambda_a$. In particular, 0 is a stationary and not extremal point.
4. If λ_0 is simple and $k = 1$, the gradient of $a \mapsto \lambda_a$ in 0 is different from zero, then 0 is not a stationary point, a fortiori not even an extremal point (see [23]).

From Theorem 2 we can easily deduce that, under assumption (2) and being k as in (3), the Taylor polynomials of the function $a \mapsto \lambda_0 - \lambda_a$ with center 0 and degree strictly smaller than k vanish.

LEMMA 1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, open and simply connected domain such that $0 \in \Omega$ and let $n_0 \geqslant 1$ be such that the n_0-th eigenvalue $\lambda_0 = \lambda_{n_0}^0$ of $(i\nabla + A_0)^2$ on Ω is simple with associated eigenfunctions having in 0 a zero of order $k/2$ with $k \in \mathbb{N}$ odd. For $a \in \Omega$ let $\lambda_a = \lambda_{n_0}^a$ be the n_0-th eigenvalue of $(i\nabla + A_a)^2$ on Ω. Then*

$$\lambda_0 - \lambda_a = P(a) + o(|a|^k), \quad \text{as } |a| \to 0^+, \tag{8}$$

for some homogeneous polynomial $P \not\equiv 0$ of degree k

$$P(a) = P(a_1, a_2) = \sum_{j=0}^{k} c_j a_1^{k-j} a_2^j. \tag{9}$$

The second result which we are going to show is contained in the paper [2] and here it is stated in Theorem 3. It provides the exact determination of all coefficients of the polynomial P (and hence the sharp asymptotic behavior of $\lambda_a - \lambda_0$ as $a \to 0$ along any direction, see Figure 2).

According to Equation (4), we define

$$\alpha_0 = \begin{cases} \frac{2}{k}\, \mathrm{arccot}\left(-\frac{\beta_2}{\beta_1}\right), & \text{if } \beta_1 \neq 0, \\ 0, & \text{if } \beta_1 = 0. \end{cases} \tag{10}$$

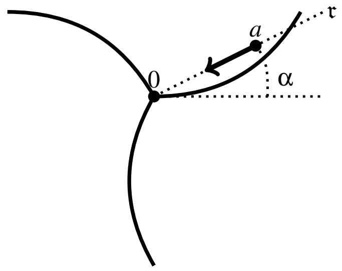

Figure 2: $a = |a|(\cos\alpha, \sin\alpha)$ approaches 0 along the direction determined by the angle α.

THEOREM 3 ([2]). *Under the same assumptions of Lemma 1, let* $\alpha \in [0, 2\pi)$. *Then*

$$\frac{\lambda_0 - \lambda_a}{|a|^k} \to C_0 \cos\big(k(\alpha - \alpha_0)\big) \qquad \text{as } a \to 0 \text{ with } a = |a|(\cos\alpha, \sin\alpha),$$

where α_0 *is defined in* (10) *and* $C_0 = -4(|\beta_1|^2 + |\beta_2|^2)\,\mathfrak{m}_k$.

REMARK 1. By Theorem 3 it follows that the polynomial (9) of Lemma 1 is given by

$$P(|a|(\cos\alpha, \sin\alpha)) = C_0 |a|^k \cos(k(\alpha - \alpha_0)).$$

Hence

$$P(a_1, a_2) = C_0\, \mathfrak{Re}\left(e^{-ik\alpha_0}(a_1 + i a_2)^k\right),$$

thus yielding $\Delta P = 0$, i.e. the polynomial P in (8)-(9) is *harmonic*.

3. Spectral minimal partitions

We would like to spend a few words about the choice of dimension equal to 2 and Aharonov-Bohm operators with half-integer circulation $\gamma = 1/2$. On the contrary with the other cases, these two choices together show particular features which have to do with a more general spectral analysis. We mean the fact that nodal domains of eigenfunctions of such operators are strongly related to spectral minimal partitions of the Dirichlet Laplacian with points of odd multiplicity, see [5, 23].

In the last few years much attention has been given to the so-called *spectral minimal partitions*. This research topic deals with the Dirichlet Laplacian in a bounded domain in $\mathbb{R}^2$ (in general with manifolds of dimension 2) and aims to analyze the relations between the nodal domains of the eigenfunctions and the so-called *minimal* partitions of Ω by k open sets D_i. Let us denote $\mathfrak{D}_k$ the set of all the possible k-partitions of Ω and $\mathcal{D} \in \mathfrak{D}_k$ a fixed k-partition formed by $\{D_i\}_{i=1,\dots,k}$. In this context, a partition is said to be minimal in the sense that the maximum over the D_is of the ground state energy of the Dirichlet realization of the Laplacian is minimal. This can be visualized as

$$\mathfrak{L}_k := \inf_{\mathcal{D}\in\mathfrak{D}_k} \max_{1\leqslant i\leqslant k} \lambda(D_i):$$
$$\lambda(D_i) \text{ is the ground state energy of the Dirichlet realization of } -\Delta.$$

Moreover, this problem can be seen as a strong competition limit of segregating species in population dynamics (see [8, 9] and references therein).

As already mentioned, the topic excited much interest and we refer the interested reader to the references [4, 6, 12, 13, 15, 16, 17, 18, 19] for details on the deep relation between behavior of eigenfunctions, their nodal domains and spectral minimal partitions. In this section, we aim at giving just an idea of the topic.

First of all, we find useful to recall the following result: from [8, 9, 10, 14, 17], the boundary of the optimal partition is the union of a finite number of regular arcs; moreover, if the minimal partition is bipartite, then it is nodal. This means that if the number of half lines meeting at each intersection point is even, then the partition components are nodal domains of a suitable ("Courant–sharp") eigenfunction of the Dirichlet Laplacian.

Nevertheless, in the aforementioned references we can find examples for k-partitions with k odd which can not be nodal. In all these cases the partition features a "point of odd multiplicity", that is an ending point of an odd number of half-lines. The researchers' attention was thus turned to spectral analysis of Aharonov-Bohm operator with half-integer circulation. Indeed, it was proved in [13, 23] (see also [11]) that in this case the eigenfunctions have an odd number of nodal lines ending at the pole a (see also (3)) and an even number of nodal lines meeting at zeros different from a. Of course, the zero set of an eigenfunction produces a regular partition only if j half-lines end at the pole with $j \geqslant 3$. Furthermore, numerical experiments (see e.g. [4]) show that the number of the half-lines ending at the pole depends even on the position of the pole in the domain. This suggests that a partition featuring a point of odd multiplicity could be generated by the zero set of a suitable ("Courant–sharp") eigenfunction even accordingly to suitable positions of the pole.

Related to this, the investigation carried out in [7, 21, 22, 23] highlighted a strong connection between nodal properties of eigenfunctions and the critical points of the map which associates eigenvalues of the operator A_a to the position of pole a, as well as our results presented here. We refer in particular to Consequence (3) stated above.

REMARK 2. Consequence (3) suggests that Aharonov-Bohm approach works for spectral minimal partitions only when the pole is located at a point with special features: it must be a point of k-multiplicity for a suitable eigenfunction with $k \geqslant 3$ odd. This can occur essentially in two cases: either the pole is an inflexion point for the map $a \mapsto \lambda_a$ or the eigenvalue is not simple. Thus, one could be led to think that stationary and extremal (regular or not) points for the map $a \mapsto \lambda_a$ are good candidates for positions of the pole in order to produce spectral minimal partitions as nodal partitions.

Finally, we stress again that the results obtained in [1, 2] and summarized here are significant not only from a pure "analytic" point of view (detecting of sharp asymptotics), but also from a quite theoretical point of view, which involves even spectral analysis for the classical Laplacian.

4. Idea of the proofs

For seek of completeness, in this last section we mention the main ideas which underlie the proofs of Theorem 2 and Theorem 3. As already mentioned, the interested reader can find the details in [1] and [2].

4.1. Sketched proof of Theorem 2

The proof of Theorem 2 is based on the Courant-Fisher minimax characterization of eigenvalues. The asymptotics for eigenvalues is derived by combining estimates from above and below of the Rayleigh quotient. To obtain sharp estimates, we construct proper test functions for the Rayleigh quotient by suitable manipulation of eigenfunctions. In this way, we obtain upper and lower bounds whose limit as $a \to 0$ can be explicitly computed taking advantage of a fine blow-up analysis for scaled eigenfunctions. More precisely, it can be proved that the blow-up sequence

$$\frac{\varphi_a(|a|x)}{|a|^{k/2}} \tag{11}$$

converges as $|a| \to 0^+$, $a \in \mathfrak{r}$, to a limit profile, which can be identified, up to a phase and a change of coordinates, with $w_k + \psi_k$, being w_k and ψ_k as in (6) and (5) respectively. The proof of the energy estimates for the blow-up sequence uses a monotonicity argument inspired by [3], based on the study of an Almgren-type frequency function given by the ratio of the local magnetic energy over mass near the origin; see [11, 22] for Almgren-type monotonicity formulae for elliptic operators with magnetic potentials. The main difficulty of the argument relies in the identification of the limit profile of the blow-up sequence (11). This difficulty was overcome by fine energy estimates of the difference between approximating and limit eigenfunctions, performed exploiting the invertibility of an operator associated to the limit eigenvalue problem.

4.2. Sketched proof of Theorem 3

The proof of Theorem 3 is based on a combination of estimates from above and below of the Rayleigh quotient associated to the eigenvalue problem with a fine blow-up analysis for scaled eigenfunctions (11), which gives a sharp characterization of upper and lower bounds for eigenvalues. Differently from the blow-up analysis performed in the previous case when poles are moving tangentially to nodal lines, in this general case when poles are moving along any direction we cannot explicitly construct the limit profile of the family (11). Such a difficulty is overcome by studying the dependence of the limit profile on the position of the pole and the symmetry/periodicity properties of its Fourier coefficient with respect to a basis of eigenvectors of an associated angular problem: such symmetry and periodicity turn into certain symmetry and periodicity invariances of the polynomial P. A complete classification of homogeneous k-degree polynomials with these particular periodicity/symmetry invariances allows us to conclude the proof.

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