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BRUXELLES-TORINO TALKS IN PDE'S –TURIN, MAY 2–5, 2016

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PREFACE

This issue of "Rendiconti" contains most of the talks given at the conference "Bruxelles-Turin talks in PDE's" held in Turin, May 2–5, 2016 and with scientific committee composed of Veronica Felli, Angela Pistoia, Susanna Terracini, Denis Bonheure, Enrico Serra, and the second editor of the present issue. This conference was a joint initiative of the research groups in Turin and Brussels to give the floor to young researchers in nonlinear analysis and partial differential equations. During four days, the invited speakers have given an overview of the new trends in several topics of nonlinear analysis and PDE. We thank the above-named colleagues, Annalisa Piccolo, member of the organizing committee and, of course, all the participants.

The Editors
Marino Badiale, Paolo Caldiroli

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C. Léna

EXAMPLES OF SPECTRAL MINIMAL PARTITIONS

Abstract. We study a minimal partition problem on the flat rectangular torus. We give a partial review of the existing literature, and present some numerical and theoretical work recently published elsewhere by V. Bonnaillie-Noël and the author, with some improvements.

1. Introduction

1.1. Minimal partitions

The topic of spectral minimal partitions has been actively investigated by the shape optimization community during recent years. In addition to its intrinsic interest, it has many applications, for instance in condensed matter physics, mathematical ecology or data sorting. In this review, we focus on one specific problem, for which the quantity to be optimized depends on the Dirichlet Laplacian eigenvalues. This problem is intimately connected with the nodal patterns of Laplacian eigenfunctions. Although we begin by recalling quite general results on minimal partitions in two dimensions, the paper then focus on the model problem of the flat rectangular torus. For the most part, we review the numerical and theoretical results obtained by the author in collaboration with V. Bonnaillie-Noël in [2]. We also present a new lower bound on transition values which improves existing estimates (Proposition 2). We point out that the authors previously studied circular sectors in a similar way [3].

Let Ω be a bounded open set in $\mathbb{R}^2$ or in a 2-dimensional Riemannian manifold. For any open subset D of Ω, let $(\lambda_k(D))_{k\geqslant 1}$ be the eigenvalues of the Dirichlet Laplacian in D, arranged in non-decreasing order and counted with multiplicities. A *k-partition* of Ω is a family $\mathcal{D} = (D_1, \dots, D_k)$ of open, connected and mutually disjoint subsets. We define its *energy* as $\Lambda_k(\mathcal{D}) = \max_{1\leqslant i\leqslant k} \lambda_1(D_i)$. A k-partition $\mathcal{D}^*$ is called *minimal* if it has minimal energy, which we denote by $\mathfrak{L}_k(\Omega)$.

Let us introduce some additional notions, which enable us to describe the regularity of minimal partitions. We say that the k-partition $\mathcal{D} = (D_1, \dots, D_k)$ is *strong* if it fills the set Ω, that is to say if

$$\Omega = \mathrm{Int}\left(\cup_{i=1}^k \overline{D}_i\right) \setminus \partial\Omega.$$

In that case, we define the *boundary* of $\mathcal{D}$ as $N(\mathcal{D}) := \overline{\cup_{i=1}^k \partial D_i \setminus \partial\Omega}$. We say that $\mathcal{D}$ is *regular* if it is strong and if $N(\mathcal{D})$ satisfies the following properties.

i. It is a union of regular arcs connecting a finite number of singular points (inside Ω or possibly on $\partial\Omega$).

ii. At the singular points, the arcs meet with equal angles (taking into account $\partial\Omega$ if necessary).

Point ii is called the *equal angle meeting property*. Let us note that these properties of $N(\mathcal{D})$ are also satisfied by the nodal set of a Dirichlet Laplacian eigenfunction. However, in this latter case, the singular points inside Ω are crossing points, so the number of arcs meeting there must be even. This number can be odd in the case of a minimal partition.

Existence and regularity of minimal partitions follow from the work of several authors: D. Bucur, G. Buttazzo, and A. Henrot [5]; L. Caffarelli and F.-H. Lin [6]; M. Conti, S. Terracini, and G. Verzini [7]; B. Helffer, T. Hoffmann-Ostenhof, and S. Terracini [11]. In the rest of the paper, we refer to the results by Helffer, Hoffmann-Ostenhof, and Terracini.

THEOREM 1. *Let Ω be a bounded open set in $\mathbb{R}^2$ with a piecewise-$C^{1,+}$ boundary and satisfying the interior cone property. Then, for any positive integer k,*

i. there exists a minimal k-partition of Ω;

ii. any minimal k-partition of Ω is regular up to 0-capacity sets.

Reference [11] also establishes the *subpartition property*, which we use at the end of the present paper.

THEOREM 2. *Let $\mathcal{D} = (D_i)_{1\leqslant i\leqslant k}$ be a minimal k-partition of Ω. Let I be a subset of $\{1,\dots,k\}$ with $k' := \sharp I$, $k' < k$, such that*

$$\Omega_I := Int\left(\bigcup_{i\in I}\overline{D}_i\right)$$

is a connected open set. Then the sub-partition $\mathcal{D}_I = (D_i)_{i\in I}$ is the unique minimal k'-partition of Ω_I (up to 0-capacity sets).

COROLLARY 1 (pair compatibility condition). *Let $\mathcal{D} = (D_i)_{1\leqslant i\leqslant k}$ ($k \geqslant 3$) be a minimal k-partition of Ω. For any two neighbors D_i and D_j, the second eigenvalue of the Dirichlet Laplacian on $D_{i,j} := Int\left(\overline{D}_i \cup \overline{D}_j\right)$ is simple, and D_i and D_j are the nodal domains of an eigenfunction associated with $\lambda_2(D_{i,j})$.*

1.2. Nodal partitions

If u is an eigenfunction of the Dirichlet Laplacian in Ω, the connected components of the complement of its zero set are called its nodal domains. Let us denote by $\nu(u)$ the number of nodal domain of u. The family $\mathcal{D}_u = (D_i)_{1\leqslant i\leqslant \nu(u)}$ of all the nodal domains of u is the nodal partition associated with u. Given a regular k-partition $\mathcal{D} = (D_i)_{1\leqslant i\leqslant k}$, we say that two domains D_i and D_j are neighbors if they have a common boundary not reduced to points, that is to say if the set $D_{i,j} := \mathrm{Int}\left(\overline{D}_i \cup \overline{D}_j\right)$ is connected.

THEOREM 3. *A minimal k-partition of Ω is nodal if, and only if, it is bipartite, that is to say if we can color its domains with only two colors such that two neighbors have a different color.*

THEOREM 4 (Courant, 1923). *If u is an eigenfunction associated with $\lambda_k(\Omega)$, $\nu(u) \leqslant k$.*

THEOREM 5 (Courant-sharp characterization). *The nodal partition associated with the eigenfunction u is minimal if, and only if, u is Courant-sharp, that is to say associated with $\lambda_k(\Omega)$, where $k = \nu(u)$.*

In particular, a minimal 2-partition is always the nodal partition associated with a second eigenfunction. Theorem 5 allows one to give explicit examples of minimal partitions, in domains Ω for which the eigenvalues and eigenfunctions of the Laplacian are explicitly known: see for instance [11, 1]. Combined with topogical arguments and covering surfaces, it can also be used to produce examples of non-nodal minimal partitions [12, 9, 15]. Let us add that while minimal partitions are in general not nodal for the Dirichlet Laplacian [11, Corollary 7.8], they are always nodal for a magnetic Laplacian, with a suitable magnetic potential of Aharonov-Bohm type, as was proved by B. Helffer and T. Hoffmann-Ostenhof [8] (see also [13, 1, 10]).

2. Transitions for the flat torus

2.1. Statement of the problem

Let us now describe our model problem. We consider the flat rectangular torus of length a and width b: $\mathsf{T}(a,b) = (\mathbb{R}/a\mathbb{Z}) \times (\mathbb{R}/b\mathbb{Z})$. The set of its eigenvalues is

$$\{\lambda_{m,n}(a,b)\,;\,(m,n) \in \mathbb{N}_0^2\},$$

with

$$\lambda_{m,n}(a,b) = 4\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right),$$

and a corresponding basis of eigenfunctions is given by

$$u_{m,n}^{a,b}(x,y) = \varphi\left(\frac{2m\pi x}{a}\right)\psi\left(\frac{2n\pi y}{b}\right),$$

where $\varphi, \psi \in \{\cos, \sin\}$.

We first consider the partition of $\mathsf{T}(a,b)$ into k equal vertical strips: $\mathcal{D}_k(a,b) = (D_i)_{1\leqslant i\leqslant k}$, with

$$D_i = \left(\frac{i-1}{k}a, \frac{i}{k}a\right) \times (0,b)\,.$$

Its energy is $\Lambda_k(\mathcal{D}_k(a,b)) = k^2\pi^2/a^2$. We investigate the following question: for which values of $b \in (0,1]$ is $\mathcal{D}_k(1,b)$ a minimal partition of $\mathsf{T}(1,b)$. More specifically, let us define the transition value

$$b_k = \sup\{b \in (0,1]\,;\, \mathcal{D}_k(1,b) \text{ is a minimal } k\text{-partition of } \mathsf{T}(1,b)\}.$$

The following result justifies the term *transition value* (see [2, Proposition 2.1]).

PROPOSITION 1. *The partition $\mathcal{D}_k(1,b)$ is minimal for all $b \in (0, b_k]$.*

We want to localize as precisely as possible this transition value. Let us first recall a result of Helffer and Hoffmann-Ostenhof [9].

THEOREM 6. *If k is even, $b_k = 2/k$. If k is odd, $b_k \geqslant 1/k$.*

We want to improve the lower bound when k is odd. This can be done by considering the following auxiliary optimization problem. For $b \in (0,1]$, we consider the infinite strip $\mathsf{S}_b = \mathbb{R} \times (0,b)$ and we define

$$b_k^{\mathsf{S}} = \sup\left\{b \in (0,1]\,;\, j(b) > k^2\pi^2\right\},\ \text{with } j(b) = \inf_{\Omega \subset \mathsf{S}_b, |\Omega| \leqslant b} \lambda_1(\Omega).$$

As seen in [2, Theorem 1.9], $b_k \geqslant b_k^{\mathsf{S}}$ if k is odd. The following estimate gives a quantitative improvement of Theorem 6 and of [2, Theorem 1.9]

PROPOSITION 2. *For any integer $k \geqslant 2$, $1/\sqrt{k^2 - 1/8} \leqslant b_k^{\mathsf{S}} < 1/\sqrt{k^2-1}$.*

As was pointed out to us by Bernard Helffer, the method of covering surfaces in [9] leads quite naturally to the following conjecture.

CONJECTURE 1. For any odd integer $k \geqslant 3$, $b_k = 2/\sqrt{k^2-1}$.

It can actually be proved that $b_k \leqslant 2/\sqrt{k^2-1}$ (see [2, Proposition 2.8]). The conjecture is supported by the numerical study. Proposition 2 shows that, for any odd integer $k \geqslant 3$, $b_k^{\mathsf{S}} < 2/\sqrt{k^2-1}$. New ideas would therefore be needed to prove Conjecture 1.

2.2. Proof of Proposition 2

Let us sketch the proof of Proposition 2. It is a direct consequence of the following proposition, after rescaling.

PROPOSITION 3. *For $V \geqslant 1/2$,*

$$\pi^2\left(1 + \frac{1}{8V^2}\right) \leqslant J(V) < \pi^2\left(1+\frac{1}{V^2}\right),\ \textit{where } J(V) := \inf_{\Omega \subset \mathsf{S}_1, |\Omega| \leqslant V} \lambda_1(\Omega).$$

Let us note that in Proposition 3, and in the rest of this section, we define $\lambda_1(\Omega)$ for any open set in $\mathbb{R}^2$, possibly unbounded and of infinite volume, as the infimum of a Rayleigh quotient:

$$\lambda_1(\Omega) := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\, dx}{\int_\Omega u^2\, dx}.$$

The upper bound of $J(V)$ is obtained immediately by considering the rectangle $(0,V) \times (0,1)$, which cannot be minimal, since the normal derivative of the first eigenfunction on its free boundary is not constant. The lower bound is harder to prove. The first part of the proof relies on a symmetrization argument. For all $V > 0$, we define an open subset C_V of S by

$$\mathsf{C}_V := \left\{ (x_1, x_2) \in \mathbb{R}^2 \, : \, \left| x_2 - \frac{1}{2} \right| < g(x_1) \right\} \text{ with } g(x_1) := \min \left(\frac{1}{2}, \frac{V}{4x_1} \right).$$

LEMMA 1. *For all $V > 0$, $J(V) \geqslant \lambda_1(\mathsf{C}_V)$.*

Proof. Let Ω be an open subset of S, of volume V. We perform two successive Steiner symmetrizations, with respect to the lines $x_1 = 0$ and $x_2 = \frac{1}{2}$, and denote by Ω^* the resulting set. We have, according to the definition of Steiner symmetrization,

$$\Omega^* = \left\{ (x_1, x_2) \in \mathbb{R}^2 \, : \, \left| x_2 - \frac{1}{2} \right| < f(x_1) \right\},$$

where $f : \mathbb{R} \to \left[0, \frac{1}{2}\right]$ is an even function, non-increasing in $[0, +\infty)$. Since f is non-increasing, we have, for all $x_1 \in (0, +\infty)$,

$$x_1 f(x_1) \leqslant \int_0^{x_1} f(t)\, dt \leqslant \int_0^{+\infty} f(t)\, dt = \frac{V}{4},$$

and therefore $f(x_1) \leqslant \frac{V}{4x_1}$. This implies that $f(x_1) \leqslant g(x_1)$, and therefore $\Omega^* \subset \mathsf{C}_V$. Since the first Dirichlet Laplacian eigenvalue is non-increasing with respect to Steiner symmetrization and the inclusion of domains, we obtain

$$\lambda_1(\mathsf{C}_V) \leqslant \lambda_1(\Omega^*) \leqslant \lambda_1(\Omega).$$

Passing to the infimum, we get the desired result. □

To conclude the proof of Proposition 3, we obtain an explicit lower bound of $\lambda_1(\mathsf{C}_V)$. For $h > 0$, let us define the ordinary differential operator P_h by

$$P_h := -h^2 \frac{d^2}{dt^2} + \pi^2 (t^2 - 1)_+,$$

with $(t^2 - 1)_+ := \max\left(0, t^2 - 1\right)$. This operator is positive and self-adjoint, with compact resolvent. It therefore has discrete spectrum, and we denote by $\mu_1(h)$ its first eigenvalue.

LEMMA 2. *For all $V > 0$, $\lambda_1(\mathsf{C}_V) \geqslant \pi^2 + \mu_1\left(\frac{2}{V}\right)$.*

Proof. Let u be a smooth function compactly supported in C_V. We have

$$\int_{\mathsf{C}_V} |\nabla u|^2 \, dx = \int_{-\infty}^{+\infty} dx_1 \int_{\frac{1}{2} - g(x_1)}^{\frac{1}{2} + g(x_1)} dx_2 \left(|\partial_{x_1} u|^2 + |\partial_{x_2} u|^2 \right).$$

For a given x_1, the one-dimensional Poincaré inequality on the segment

$$\left(\frac{1}{2} - g(x_1), \frac{1}{2} + g(x_1) \right)$$

gives us

$$\int_{\frac{1}{2}-g(x_1)}^{\frac{1}{2}+g(x_1)} |\partial_{x_2} u|^2 \, dx_2 \geqslant \frac{\pi^2}{4g(x_1)^2} \int_{\frac{1}{2}-g(x_1)}^{\frac{1}{2}+g(x_1)} u^2 \, dx_2.$$

We obtain therefore

$$\int_{\mathsf{C}_V} |\nabla u|^2 \, dx \geqslant \int_{\frac{1}{2}-g(x_1)}^{\frac{1}{2}+g(x_1)} dx_2 \int_{-\infty}^{+\infty} dx_1 \left(|\partial_{x_1} u|^2 + \frac{\pi^2}{4g(x_1)^2} u^2 \right).$$

We now denote by $\nu_1(V)$ the first eigenvalue of the ordinary differential operator

$$Q_V := -\frac{d^2}{dx_1^2} + \frac{\pi^2}{4g(x_1)^2}.$$

According to the variational characterization of $\nu_1(V)$, we get

$$\int_{-\infty}^{+\infty} dx_1 \left(|\partial_{x_1} u|^2 + \frac{\pi^2}{4g(x_1)^2} u^2 \right) \geqslant \nu_1(V) \int_{-\infty}^{+\infty} u^2 \, dx_1$$

for all $x_2 \in (0,1)$, and therefore

$$\int_{\mathsf{C}_V} |\nabla u|^2 \, dx \geqslant \nu_1(V) \int_{\mathsf{C}_V} u^2 \, dx.$$

By density, the inequality holds for any $u \in H_0^1(\mathsf{C}_V)$, and therefore $\lambda_1(\mathsf{C}_V) \geqslant \nu_1(V)$. The change of variable $x_1 = (V/2)t$ shows that Q_V is unitarily equivalent to $P_h + \pi^2$ with $h = \frac{2}{V}$, which establishes the desired result. □

LEMMA 3. *If $h \leqslant 4$, $\mu_1(h) \geqslant \frac{\pi^2 h^2}{32}$.*

Proof. For any $h > 0$, $R_h \leqslant P_h$, where R_h is the differential operator

$$R_h := -h^2 \frac{d^2}{dt^2} + W(t), \text{ with } W(t) := \begin{cases} 0 & \text{if } |t| < \sqrt{2}; \\ \pi^2 & \text{if } |t| \geqslant \sqrt{2}. \end{cases}$$

We therefore have $\mu_1(h) \geqslant \xi_1(h)$, with $\xi_1(h)$ the first eigenvalue of R_h.

The spectrum of R_h is known explicitly (it is a Schrödinger operator with a square well potential, studied in most textbooks on quantum mechanics, see for instance [14, Chapter 2, Section 9]). We find $\xi_1(h) = \frac{h^2}{2}\rho_1^2(h)$, where $\rho_1(h)$ is the smallest positive solution of the equation $\rho \tan(\rho) = \sqrt{2\pi^2/h^2 - \rho^2}$. It is easily seen that the assumption $h \leqslant 4$ implies $\rho_1(h) \geqslant \pi/4$, and thus $\mu_1(h) \geqslant \xi_1(h) \geqslant \pi^2 h^2/32$. □

Gathering all the previous estimates, we obtain, when $V \geqslant 1/2$,

$$J(V) \geqslant \lambda_1(\mathsf{C}_V) \geqslant \nu_1(V) \geqslant \pi^2 + \mu_1\left(\frac{2}{V}\right) \geqslant \pi^2 + \frac{\pi^2}{8V^2}.$$

3. Numerical study of the flat torus

3.1. Algorithm and results

We performed in [2] a numerical study of our model problem, using the method introduced by B. Bourdin, D. Bucur and É. Oudet [4], with some modifications. In their work, they looked for partitions which are optimal with respect to the sum of the eigenvalues. They passed to a relaxed formulation, looking for indicator functions instead of domains, and penalizing overlapping supports. They then discretized the resulting optimization problem, through a five points finite difference method for the Laplacian, and performed the optimization iteratively, with the projected gradient algorithm. We made the following changes to their algorithm. First, we considered general ℓ^p-norms for the energy, rather than just the ℓ^1-norm, in order to approach the maximum by taking a larger p. We also added a last step, in which we built a strong partition from the result of the optimization algorithm, and evaluated its energy without relaxation. As pointed out in [4], the algorithm proves to be quite sensitive to the initial condition, due to the non-convexity of the problem. For each value of k and b, we therefore ran the algorithm several times with different initial conditions, and chose the results giving the lowest energy.

(a) $b = 0.7$ (b) $b = 0.72$ (c) $b = 1$

Figure 1: 3-partitions of $T(1,b)$ for some values of b.

Figure 1 presents some results of the numerical optimization. Comparing the partitions in Figures 1(a) and 1(b), we see that b_3 seems close to the value $1/\sqrt{2} \simeq 0.7071$ given by Conjecture 1. It appears in fact slightly higher in our numerical computations, possibly because of the approximations introduced in the algorithm. These results also suggest a transition mechanism from Figure 1(a) to Figure 1(b). Indeed, we can construct a 3-partition of $\mathsf{T}(1,1/\sqrt{2})$, with the same energy as $\mathcal{D}_3(1,1/\sqrt{2})$ but of a different topological type. It is represented on Figure 2, and is obtained by projecting on $\mathsf{T}(1,1/\sqrt{2})$ a nodal 6-partition of the double covering $\mathsf{T}(2,1/\sqrt{2})$ (see [2, Section 2.3]). The partition on Figure 1(b) could then be obtained by a deformation which splits each singular point of order 4 into two singular points of order 3.

Finally, Figure 1(c) strongly suggests that for b quite larger than $1/\sqrt{2}$, min-

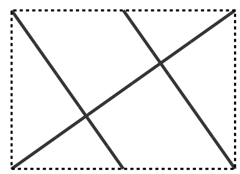

Figure 2: 3-partition of $\mathsf{T}(1,1/\sqrt{2})$

imal partitions of $\mathsf{T}(1,b)$ are close to hexagonal tilings. These tilings can be explicitly constructed, and their energy is an upper bound of $\mathfrak{L}_3(\mathsf{T}(1,b))$, smaller that $\Lambda_3(\mathcal{D}_3(1,b))$ for some values of b. Figure 3 summarizes the information thus obtained on $\mathfrak{L}_3(\mathsf{T}(1,b))$. The solid line represents λ_1 for a tiling hexagon, which is the energy of the hexagonal tiling, the dashed line the energy of $\mathcal{D}_3(1,b)$, and the crosses the results of the numerical optimization. The transition around $b = 1/\sqrt{2}$ clearly appears. We obtained similar results for $k \in \{4,5\}$.

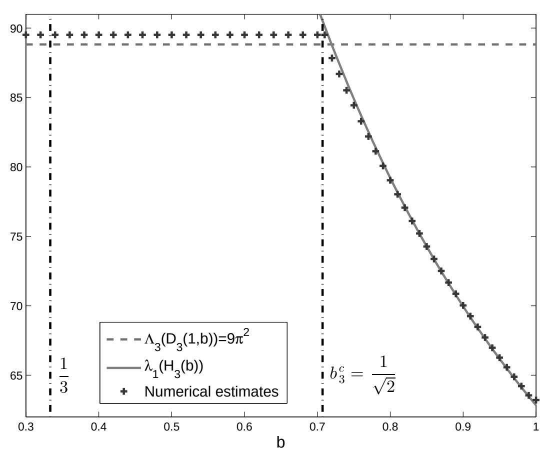

Figure 3: Upper bounds of $\mathfrak{L}_3(\mathsf{T}(1,b))$

We constructed explicitly hexagonal tilings of the same topological type as the numerical results and satisfying the equal angle meeting property [2, Section 4]. The results are summarized in the following theorem, from [2, Section 1.3].

THEOREM 7. *For $k \in \{3,4,5\}$, there exists $b_k^{\mathsf{H}} \in (0,1)$ such that, for any $b \in (b_k^{\mathsf{H}},1]$, there exists a tiling of $\mathsf{T}(1,b)$ by k hexagons that satisfies the equal angle meeting property. We denote by $\mathsf{H}_k(b)$ the corresponding tiling domain, and we have*

$$\mathfrak{L}_k(\mathsf{T}(1,b)) \leqslant \min\left(k^2\pi^2, \lambda_1(\mathsf{H}_k(b))\right), \quad \forall b \in (b_k^{\mathsf{H}},1].$$

More explicitly, we can choose

$$b_3^{\mathsf{H}} = \frac{\sqrt{11}-\sqrt{3}}{4} \simeq 0.396\,, \quad b_4^{\mathsf{H}} = \frac{1}{2\sqrt{3}} \simeq 0.289 < b_4 = \frac{1}{2}\,,$$

and

$$b_5^{\mathsf{H}} = \frac{\sqrt{291}-5\sqrt{3}}{36} \simeq 0.233\,.$$

In order to test the minimality of these tilings, we used the pair compatibility condition (see [2, Section 4.5]). Indeed, if one of these tilings is a minimal k-partition of $\mathsf{T}(1,b)$, Corollary 1 implies that $\lambda_1(\mathsf{H}_k(b)) = \lambda_2(2\mathsf{H}_k(b))$, with $2\mathsf{H}_k(b)$ any one of the polygonal domains obtained by gluing two copies of $\mathsf{H}_k(b)$ along corresponding sides. Numerically, this condition does not seem to be met for b close to $1/\sqrt{2}$ when $k=3$, to $1/2$ when $k=4$, and to $1/\sqrt{6}$ and 1 when $k=5$. Hexagonal tilings therefore appear not to be minimal under these conditions. This idea is supported by the numerical values of the energy, and by the slight curvature visible in the boundary of the numerically obtained partitions. Let us finally point out that when $k=5$ and $b=1$, the numerical result is very close to the partition into 5 squares represented on Figure 4 (see [2, Section 4.4]).

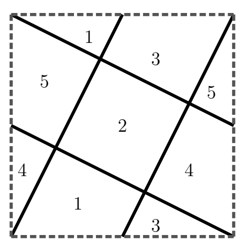

Figure 4: 5-partition of $\mathsf{T}(1,1)$

These numerical findings reveal a rich structure for minimal partitions of the flat rectangular torus. A better understanding would however require faster numerical algorithms and new theoretical methods.

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SHARP ASYMPTOTICS FOR THE EIGENVALUE FUNCTION OF AHARONOV-BOHM OPERATORS WITH A MOVING POLE

Abstract. In this breif note we present several results obtained in collaboration with V. Felli. They concern the behavior of eigenvalues for a magnetic Aharonov-Bohm operator with half-integer circulation and Dirichlet boundary conditions in a planar domain. In particular, they contain sharp asymptotics for eigenvalues as the pole is moving in the interior of the domain, approaching a zero of an eigenfunction of the limiting problem along a general direction.

1. Introduction

In this brief note we present several results obtained in collaboration with V. Felli which are essentially proved in the papers [1] and [2]. They concern the behavior of eigenvalues for a magnetic Aharonov-Bohm operator with half-integer circulation and Dirichlet boundary conditions in a planar domain. As it will appear in the sequel, these operators are special as they present a strong singularity at a point (pole), for which they cannot be considered small perturbations of the standard Laplacian. In particular, the two aforementioned papers address the challenging question about the possible determination of the first term of the Taylor expansion of the function $a \mapsto \lambda_a$, where a is the operator's pole and λ_a is one of its simple eigenvalues.

Indeed, if Ω is a Lipschitz open bounded and simply connected set in $\mathbb{R}^2$ and if λ_0 is a simple eigenvalue for the Aharonov-Bohm operator with the pole located at 0, the function $a \mapsto \lambda_a$ can be shown to be analytic in a neighborhood of 0 ([21]).

We devote the second section to show the functional setting and to state the results in a rigorous way. This part will take some technical definitions and several references.

In the third section we briefly present how the general problem and the particular results presented here deal with the so-called *spectral minimal partitions*. This is a wider research topic, it was initiated by some seminal papers by B. Helffer, T. Hoffmann-Ostenhof, S. Terracini and others. In this section we would like to stress our contribution in this direction.

The last section contains the main ideas for the proofs of the results. The interested reader can find them in full details in the papers [1] and [2].

2. Statement of the results

For $a=(a_1,a_2)\in\mathbb{R}^2$ and $\gamma\in\mathbb{R}\setminus\mathbb{Z}$, we consider the vector potential

$$A_a^\gamma(x)=\gamma\left(\frac{-(x_2-a_2)}{(x_1-a_1)^2+(x_2-a_2)^2},\frac{x_1-a_1}{(x_1-a_1)^2+(x_2-a_2)^2}\right),$$
$$x=(x_1,x_2)\in\mathbb{R}^2\setminus\{a\},$$

which generates the Aharonov-Bohm magnetic field in $\mathbb{R}^2$ with pole a and circulation γ; such a field is produced by an infinitely long thin solenoid intersecting perpendicularly the plane (x_1,x_2) at the point a, as the radius of the solenoid goes to zero and the magnetic flux remains constantly equal to γ.

We will focus on the case of half-integer circulation, so we will assume $\gamma=1/2$ and denote

$$A_a(x)=A_a^{1/2}(x)=A_0(x-a),\quad \text{where}\quad A_0(x_1,x_2)=\frac{1}{2}\left(-\frac{x_2}{x_1^2+x_2^2},\frac{x_1}{x_1^2+x_2^2}\right).$$

In the spirit [7], [22] and [23], we are interested in studying the dependence on the pole a of the spectrum of Schrödinger operators with Aharonov-Bohm vector potentials, i.e. of operators $(i\nabla+A_a)^2$ acting on functions $u:\mathbb{R}^2\to\mathbb{C}$ as

$$(i\nabla+A_a)^2u=-\Delta u+2iA_a\cdot\nabla u+|A_a|^2u.$$

Let $\Omega\subset\mathbb{R}^2$ be a bounded, open and simply connected domain. For every $a\in\Omega$, we introduce the space $H^{1,a}(\Omega,\mathbb{C})$ as the completion of $\{u\in H^1(\Omega,\mathbb{C})\cap C^\infty(\Omega,\mathbb{C}):$ u vanishes in a neighborhood of $a\}$ with respect to the norm

$$\|u\|_{H^{1,a}(\Omega,\mathbb{C})}=\left(\|\nabla u\|^2_{L^2(\Omega,\mathbb{C}^2)}+\|u\|^2_{L^2(\Omega,\mathbb{C})}+\left\|\frac{u}{|x-a|}\right\|^2_{L^2(\Omega,\mathbb{C})}\right)^{1/2}.$$

It is easy to verify that $H^{1,a}(\Omega,\mathbb{C})=\left\{u\in H^1(\Omega,\mathbb{C}):\frac{u}{|x-a|}\in L^2(\Omega,\mathbb{C})\right\}$. We also observe that, in view of the Hardy type inequality proved in [20], an equivalent norm in $H^{1,a}(\Omega,\mathbb{C})$ is given by

$$\left(\|(i\nabla+A_a)u\|^2_{L^2(\Omega,\mathbb{C}^2)}+\|u\|^2_{L^2(\Omega,\mathbb{C})}\right)^{1/2}.$$

We also consider the space $H_0^{1,a}(\Omega,\mathbb{C})$ as the completion of $C_c^\infty(\Omega\setminus\{a\},\mathbb{C})$ with respect to the norm $\|\cdot\|_{H_a^1(\Omega,\mathbb{C})}$, so that $H_0^{1,a}(\Omega,\mathbb{C})=\left\{u\in H_0^1(\Omega,\mathbb{C}):\frac{u}{|x-a|}\in L^2(\Omega,\mathbb{C})\right\}$.

For every $a\in\Omega$, we consider the eigenvalue problem

$$(E_a)\qquad \begin{cases}(i\nabla+A_a)^2u=\lambda u, & \text{in }\Omega,\\ u=0, & \text{on }\partial\Omega,\end{cases}$$

in a weak sense, i.e. we say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (E_a) if there exists $u \in H^{1,a}_0(\Omega,\mathbb{C})\setminus\{0\}$ (called eigenfunction) such that

$$\int_\Omega (i\nabla u + A_a u)\cdot \overline{(i\nabla v + A_a v)}\,dx = \lambda \int_\Omega u\bar{v}\,dx \quad \text{for all } v \in H^{1,a}_0(\Omega,\mathbb{C}).$$

From classical spectral theory, the eigenvalue problem (E_a) admits a sequence of real diverging eigenvalues $\{\lambda^a_k\}_{k\geqslant 1}$ with finite multiplicity; in the enumeration $\lambda^a_1 \leqslant \lambda^a_2 \leqslant \dots \leqslant \lambda^a_j \leqslant \dots$, we repeat each eigenvalue as many times as its multiplicity. We are interested in the behavior of the function $a \mapsto \lambda^a_j$ in a neighborhood of a fixed point $b \in \Omega$. Up to a translation, it is not restrictive to consider $b = 0$. Thus, we assume that $0 \in \Omega$.

In [7, Theorem 1.1] and [21, Theorem 1.2] it is proved that, for all $j \geqslant 1$,

$$\text{the function } a \mapsto \lambda^a_j \text{ is continuous in } \Omega. \tag{1}$$

A strong improvement of the regularity (1) holds under simplicity of the eigenvalue. Indeed in [7, Theorem 1.3] it is proved that, if there exists $n_0 \geqslant 1$ such that

$$\lambda^0_{n_0} \quad \text{is simple,} \tag{2}$$

then the function $a \mapsto \lambda^a_{n_0}$ is of class C^∞ in a neighborhood of 0; this regularity result is improved in [21, Theorem 1.3], where, in the more general setting of Aharonov-Bohm operators with many singularities, it is shown that, under assumption (2) the function $a \mapsto \lambda^a_{n_0}$ is analytic in a neighborhood of 0. Then the question of what is the leading term in the asymptotic expansion of such a function (at least on a single straight path around the limit point 0) naturally arises. This may also shed some light on the nature of 0 as a critical point for the map $a \mapsto \lambda_a$ when the limit eigenfunction has in 0 a zero of order $k/2$ with $k \geqslant 3$ odd.

Let us assume that there exists $n_0 \geqslant 1$ such that (2) holds and denote $\lambda_0 = \lambda^0_{n_0}$ and, for any $a \in \Omega$, $\lambda_a = \lambda^a_{n_0}$. From (1) it follows that, if $a \to 0$, then $\lambda_a \to \lambda_0$. Let $\varphi_0 \in H^{1,0}_0(\Omega,\mathbb{C})\setminus\{0\}$ be an eigenfunction of problem (E_0) associated to the eigenvalue $\lambda_0 = \lambda^0_{n_0}$, i.e. solving

$$\begin{cases} (i\nabla + A_0)^2\varphi_0 = \lambda_0\varphi_0, & \text{in } \Omega,\\ \varphi_0 = 0, & \text{on } \partial\Omega, \end{cases}$$

such that

$$\int_\Omega |\varphi_0(x)|^2\,dx = 1.$$

In view of [11, Theorem 1.3] we have that

$$\varphi_0 \text{ has at 0 a zero of order } \frac{k}{2} \text{ for some odd } k \in \mathbb{N}, \tag{3}$$

see [7, Definition 1.4]. We recall from [11, Theorem 1.3] and [23, Theorem 1.5] that (3) implies that the eigenfunction φ_0 has got exactly k nodal lines meeting at 0 and dividing the whole angle into k equal parts.

A first result relating the rate of convergence of λ_a to λ_0 with the order of vanishing of φ_0 at 0 can be found in [7], where the following estimate is proved.

THEOREM 1 ([7], Theorem 1.7). *If assumptions* (2) *and* (3) *with* $k \geqslant 3$ *are satisfied, then*

$$|\lambda_a - \lambda_0| \leqslant C|a|^{\frac{k+1}{2}} \qquad \text{as } a \to 0$$

for a constant $C > 0$ *independent of* a.

As already mentioned, the latter theorem pursue the idea that the asymptotic expansion of the function $a \mapsto \lambda_a$ has to do with the nodal properties of the related limit eigenfunction.

The first result we present is essentially proved in the paper [1] and establishes the exact order of the asymptotic expansion of $\lambda_a - \lambda_0$ along a suitable direction as $|a|^k$, where k is the number of nodal lines of φ_0 at 0 which coincides with twice the order of vanishing of φ_0 in assumption (3). In addition, we detected the sharp coefficient of the asymptotics, which can be characterized in terms of the limit profile of a blow-up sequence obtained by a suitable scaling of approximating eigenfunctions.

In order to state properly the first result, we need to recall some known facts and to introduce some notation. By [11, Theorem 1.3], if φ_0 is an eigenfunction of $(i\nabla + A_0)^2$ on Ω satisfying assumption (3), there exist $\beta_1, \beta_2 \in \mathbb{C}$ such that $(\beta_1, \beta_2) \neq (0,0)$ and

$$r^{-k/2}\varphi_0(r(\cos t, \sin t)) \to \beta_1 e^{i\frac{t}{2}} \cos\left(\frac{k}{2}t\right) + \beta_2 e^{i\frac{t}{2}} \sin\left(\frac{k}{2}t\right) \quad \text{in } C^{1,\tau}([0,2\pi],\mathbb{C}) \tag{4}$$

as $r \to 0^+$ for any $\tau \in (0,1)$.

Let s_0 be the positive half-axis $s_0 = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 = 0 \text{ and } x_1 \geqslant 0\}$. We observe that, for every odd natural number k, there exists a unique (up to a multiplicative constant) function ψ_k which is harmonic on $\mathbb{R}^2 \backslash s_0$, homogeneous of degree $k/2$ and vanishing on s_0. Such a function is given by

$$\psi_k(r\cos t, r\sin t) = r^{k/2} \sin\left(\frac{k}{2}t\right), \quad r \geqslant 0, \quad t \in [0,2\pi]. \tag{5}$$

Let $s := \{(x_1,x_2) \in \mathbb{R}^2 : x_2 = 0 \text{ and } x_1 \geqslant 1\}$ and $\mathbb{R}^2_+ = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 > 0)\}$. We denote as $\mathcal{D}^{1,2}_s(\mathbb{R}^2_+)$ the completion of $C^\infty_c(\overline{\mathbb{R}^2_+} \backslash s)$ under the norm $(\int_{\mathbb{R}^2_+} |\nabla u|^2\, dx)^{1/2}$. From the Hardy type inequality proved in [20] and a change of gauge, it follows that functions in $\mathcal{D}^{1,2}_s(\mathbb{R}^2_+)$ satisfy the following Hardy type inequality:

$$\int_{\mathbb{R}^2} |\nabla\varphi(x)|^2\, dx \geqslant \frac{1}{4}\int_{\mathbb{R}^2} \frac{|\varphi(x)|^2}{|x-\mathbf{e}|^2}\, dx, \quad \text{for all } \varphi \in \mathcal{D}^{1,2}_s(\mathbb{R}^2_+),$$

where $\mathbf{e} = (1,0)$. Then

$$\mathcal{D}^{1,2}_s(\mathbb{R}^2_+) = \Big\{u \in L^1_{loc}(\overline{\mathbb{R}^2_+} \backslash s) : \ \nabla u \in L^2(\mathbb{R}^2_+), \ \tfrac{u}{|x-\mathbf{e}|} \in L^2(\mathbb{R}^2_+), \text{ and } u = 0 \text{ on } s\Big\}.$$

The functional

$$J_k : \mathcal{D}^{1,2}_s(\mathbb{R}^2_+) \to \mathbb{R}, \quad J_k(u) = \frac{1}{2}\int_{\mathbb{R}^2_+} |\nabla u(x)|^2\, dx - \int_{\partial\mathbb{R}^2_+ \backslash s} u(x_1,0)\frac{\partial\psi_k}{\partial x_2}(x_1,0)\, dx_1,$$

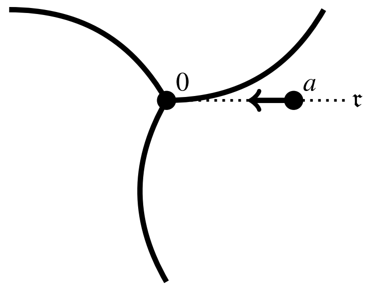

Figure 1: a approaches 0 along the tangent $\mathfrak{r}$ to a nodal line of φ_0.

is well-defined on the space $\mathcal{D}^{1,2}_s(\mathbb{R}^2_+)$; we notice that $\frac{\partial\psi_k}{\partial x_2}$ vanishes on $\partial\mathbb{R}^2_+\setminus s_0$, so that

$$\int_{\partial\mathbb{R}^2_+\setminus s} u(x_1,0)\frac{\partial\psi_k}{\partial x_2}(x_1,0)\,dx_1 = \int_0^1 u(x_1,0)\frac{\partial\psi_k}{\partial x_2}(x_1,0)\,dx_1.$$

By standard minimization methods, J_k achieves its minimum over the whole space $\mathcal{D}^{1,2}_s(\mathbb{R}^2_+)$ at some function $w_k\in\mathcal{D}^{1,2}_s(\mathbb{R}^2_+)$, i.e. there exists $w_k\in\mathcal{D}^{1,2}_s(\mathbb{R}^2_+)$ such that

$$\mathfrak{m}_k = \min_{u\in\mathcal{D}^{1,2}_s(\mathbb{R}^2_+)} J_k(u) = J_k(w_k). \tag{6}$$

We note that

$$\mathfrak{m}_k = J_k(w_k) = -\frac{1}{2}\int_{\mathbb{R}^2_+}|\nabla w_k(x)|^2\,dx = -\frac{1}{2}\int_0^1\frac{\partial_+\psi_k}{\partial x_2}(x_1,0)\,w_k(x_1,0)\,dx_1<0, \tag{7}$$

where, for all $x_1>0$, $\frac{\partial_+\psi_k}{\partial x_2}(x_1,0)=\lim_{t\to 0^+}\frac{\psi_k(x_1,t)-\psi_k(x_1,0)}{t}=\frac{k}{2}x_1^{\frac{k}{2}-1}$.

We are now in position to state the first result.

THEOREM 2 ([1]). *Let $\Omega\subset\mathbb{R}^2$ be a bounded, open and simply connected domain such that $0\in\Omega$ and let $n_0\geqslant 1$ be such that the n_0-th eigenvalue $\lambda_0=\lambda^0_{n_0}$ of $(i\nabla+A_0)^2$ on Ω is simple with associated eigenfunctions having in 0 a zero of order $k/2$ with $k\in\mathbb{N}$ odd. For $a\in\Omega$ let $\lambda_a=\lambda^a_{n_0}$ be the n_0-th eigenvalue of $(i\nabla+A_a)^2$ on Ω. Let $\mathfrak{r}$ be the half-line tangent to a nodal line of eigenfunctions associated to λ_0 ending at 0. Then, as $a\to 0$ with $a\in\mathfrak{r}$,*

$$\frac{\lambda_0-\lambda_a}{|a|^k}\to -4\left(|\beta_1|^2+|\beta_2|^2\right)\mathfrak{m}_k$$

with $(\beta_1,\beta_2)\neq(0,0)$ being as in (4) and $\mathfrak{m}_k$ being as in (6)–(7).

Once stated the first main result, we would like to recall the following result, established in the paper [7]:

PROPOSITION 1. *([7, Corollary 1.8]) Fix any $j \in \mathbb{N}$. If 0 is an extremal point of $a \mapsto \lambda_j^a$, then either λ_j^0 is not simple, or the eigenfunction of $(i\nabla + A_0)^2$ associated to λ_j^0 has at 0 a zero of order $k/2$ with $k \geqslant 3$ odd.*

This gives us the opportunity to list several of remarkable consequences of Theorem 2.

1. Due to the analyticity of $a \mapsto \lambda_a$, $\frac{\lambda_0 - \lambda_a}{|a|^k} \to 4\left(|\beta_1|^2 + |\beta_2|^2\right) \mathfrak{m}_k$ as $a \to 0$ along the opposite half-line. Hence, if λ_0 is simple, then 0 cannot be an extremal point of the map $a \mapsto \lambda_a$.
2. In view of Theorem 2 and the first consequence, we can exclude the second alternative in Proposition 1, producing the following claim: fix any $j \in \mathbb{N}$, if 0 is an extremal point of $a \mapsto \lambda_j^a$, then λ_j^0 is not simple.
3. If λ_0 is simple and $k \geqslant 3$, 0 is a saddle point for the map $a \mapsto \lambda_a$. In particular, 0 is a stationary and not extremal point.
4. If λ_0 is simple and $k = 1$, the gradient of $a \mapsto \lambda_a$ in 0 is different from zero, then 0 is not a stationary point, a fortiori not even an extremal point (see [23]).

From Theorem 2 we can easily deduce that, under assumption (2) and being k as in (3), the Taylor polynomials of the function $a \mapsto \lambda_0 - \lambda_a$ with center 0 and degree strictly smaller than k vanish.

LEMMA 1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, open and simply connected domain such that $0 \in \Omega$ and let $n_0 \geqslant 1$ be such that the n_0-th eigenvalue $\lambda_0 = \lambda_{n_0}^0$ of $(i\nabla + A_0)^2$ on Ω is simple with associated eigenfunctions having in 0 a zero of order $k/2$ with $k \in \mathbb{N}$ odd. For $a \in \Omega$ let $\lambda_a = \lambda_{n_0}^a$ be the n_0-th eigenvalue of $(i\nabla + A_a)^2$ on Ω. Then*

$$\lambda_0 - \lambda_a = P(a) + o(|a|^k), \quad \text{as } |a| \to 0^+, \tag{8}$$

for some homogeneous polynomial $P \not\equiv 0$ of degree k

$$P(a) = P(a_1, a_2) = \sum_{j=0}^{k} c_j a_1^{k-j} a_2^j. \tag{9}$$

The second result which we are going to show is contained in the paper [2] and here it is stated in Theorem 3. It provides the exact determination of all coefficients of the polynomial P (and hence the sharp asymptotic behavior of $\lambda_a - \lambda_0$ as $a \to 0$ along any direction, see Figure 2).

According to Equation (4), we define

$$\alpha_0 = \begin{cases} \frac{2}{k} \operatorname{arccot}\left(-\frac{\beta_2}{\beta_1}\right), & \text{if } \beta_1 \neq 0, \\ 0, & \text{if } \beta_1 = 0. \end{cases} \tag{10}$$

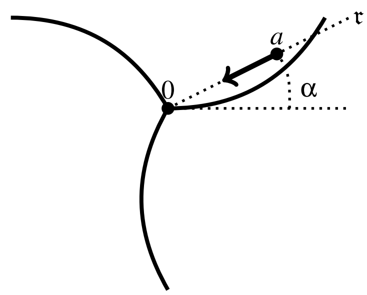

Figure 2: $a = |a|(\cos\alpha, \sin\alpha)$ approaches 0 along the direction determined by the angle α.

THEOREM 3 ([2]). *Under the same assumptions of Lemma 1, let* $\alpha \in [0, 2\pi)$. *Then*

$$\frac{\lambda_0 - \lambda_a}{|a|^k} \to C_0 \cos\big(k(\alpha - \alpha_0)\big) \qquad \textit{as } a \to 0 \textit{ with } a = |a|(\cos\alpha, \sin\alpha),$$

where α_0 *is defined in* (10) *and* $C_0 = -4(|\beta_1|^2 + |\beta_2|^2)\,\mathfrak{m}_k$.

REMARK 1. By Theorem 3 it follows that the polynomial (9) of Lemma 1 is given by

$$P(|a|(\cos\alpha, \sin\alpha)) = C_0 |a|^k \cos(k(\alpha - \alpha_0)).$$

Hence

$$P(a_1, a_2) = C_0\, \mathfrak{Re}\left(e^{-ik\alpha_0}(a_1 + i a_2)^k\right),$$

thus yielding $\Delta P = 0$, i.e. the polynomial P in (8)-(9) is *harmonic*.

3. Spectral minimal partitions

We would like to spend a few words about the choice of dimension equal to 2 and Aharonov-Bohm operators with half-integer circulation $\gamma = 1/2$. On the contrary with the other cases, these two choices together show particular features which have to do with a more general spectral analysis. We mean the fact that nodal domains of eigenfunctions of such operators are strongly related to spectral minimal partitions of the Dirichlet Laplacian with points of odd multiplicity, see [5, 23].

In the last few years much attention has been given to the so-called *spectral minimal partitions*. This research topic deals with the Dirichlet Laplacian in a bounded domain in $\mathbb{R}^2$ (in general with manifolds of dimension 2) and aims to analyze the relations between the nodal domains of the eigenfunctions and the so-called *minimal* partitions of Ω by k open sets D_i. Let us denote $\mathfrak{D}_k$ the set of all the possible k-partitions of Ω and $\mathcal{D} \in \mathfrak{D}_k$ a fixed k-partition formed by $\{D_i\}_{i=1,\ldots,k}$. In this context, a partition is said to be minimal in the sense that the maximum over the D_is of the ground state energy of the Dirichlet realization of the Laplacian is minimal. This can be visualized as

$$\mathfrak{L}_k := \inf_{\mathcal{D}\in\mathfrak{D}_k} \max_{1\leqslant i\leqslant k} \lambda(D_i) :$$
$$\lambda(D_i) \text{ is the ground state energy of the Dirichlet realization of } -\Delta.$$

Moreover, this problem can be seen as a strong competition limit of segregating species in population dynamics (see [8, 9] and references therein).

As already mentioned, the topic excited much interest and we refer the interested reader to the references [4, 6, 12, 13, 15, 16, 17, 18, 19] for details on the deep relation between behavior of eigenfunctions, their nodal domains and spectral minimal partitions. In this section, we aim at giving just an idea of the topic.

First of all, we find useful to recall the following result: from [8, 9, 10, 14, 17], the boundary of the optimal partition is the union of a finite number of regular arcs; moreover, if the minimal partition is bipartite, then it is nodal. This means that if the number of half lines meeting at each intersection point is even, then the partition components are nodal domains of a suitable ("Courant–sharp") eigenfunction of the Dirichlet Laplacian.

Nevertheless, in the aforementioned references we can find examples for k-partitions with k odd which can not be nodal. In all these cases the partition features a "point of odd multiplicity", that is an ending point of an odd number of half-lines. The researchers' attention was thus turned to spectral analysis of Aharonov-Bohm operator with half-integer circulation. Indeed, it was proved in [13, 23] (see also [11]) that in this case the eigenfunctions have an odd number of nodal lines ending at the pole a (see also (3)) and an even number of nodal lines meeting at zeros different from a. Of course, the zero set of an eigenfunction produces a regular partition only if j half-lines end at the pole with $j \geqslant 3$. Furthermore, numerical experiments (see e.g. [4]) show that the number of the half-lines ending at the pole depends even on the position of the pole in the domain. This suggests that a partition featuring a point of odd multiplicity could be generated by the zero set of a suitable ("Courant–sharp") eigenfunction even accordingly to suitable positions of the pole.

Related to this, the investigation carried out in [7, 21, 22, 23] highlighted a strong connection between nodal properties of eigenfunctions and the critical points of the map which associates eigenvalues of the operator A_a to the position of pole a, as well as our results presented here. We refer in particular to Consequence (3) stated above.

REMARK 2. Consequence (3) suggests that Aharonov-Bohm approach works for spectral minimal partitions only when the pole is located at a point with special features: it must be a point of k-multiplicity for a suitable eigenfunction with $k \geqslant 3$ odd. This can occur essentially in two cases: either the pole is an inflexion point for the map $a \mapsto \lambda_a$ or the eigenvalue is not simple. Thus, one could be led to think that stationary and extremal (regular or not) points for the map $a \mapsto \lambda_a$ are good candidates for positions of the pole in order to produce spectral minimal partitions as nodal partitions.

Finally, we stress again that the results obtained in [1, 2] and summarized here are significant not only from a pure "analytic" point of view (detecting of sharp asymptotics), but also from a quite theoretical point of view, which involves even spectral analysis for the classical Laplacian.

4. Idea of the proofs

For seek of completeness, in this last section we mention the main ideas which underlie the proofs of Theorem 2 and Theorem 3. As already mentioned, the interested reader can find the details in [1] and [2].

4.1. Sketched proof of Theorem 2

The proof of Theorem 2 is based on the Courant-Fisher minimax characterization of eigenvalues. The asymptotics for eigenvalues is derived by combining estimates from above and below of the Rayleigh quotient. To obtain sharp estimates, we construct proper test functions for the Rayleigh quotient by suitable manipulation of eigenfunctions. In this way, we obtain upper and lower bounds whose limit as $a \to 0$ can be explicitly computed taking advantage of a fine blow-up analysis for scaled eigenfunctions. More precisely, it can be proved that the blow-up sequence

$$\frac{\varphi_a(|a|x)}{|a|^{k/2}} \tag{11}$$

converges as $|a| \to 0^+$, $a \in \mathfrak{r}$, to a limit profile, which can be identified, up to a phase and a change of coordinates, with $w_k + \psi_k$, being w_k and ψ_k as in (6) and (5) respectively. The proof of the energy estimates for the blow-up sequence uses a monotonicity argument inspired by [3], based on the study of an Almgren-type frequency function given by the ratio of the local magnetic energy over mass near the origin; see [11, 22] for Almgren-type monotonicity formulae for elliptic operators with magnetic potentials. The main difficulty of the argument relies in the identification of the limit profile of the blow-up sequence (11). This difficulty was overcome by fine energy estimates of the difference between approximating and limit eigenfunctions, performed exploiting the invertibility of an operator associated to the limit eigenvalue problem.

4.2. Sketched proof of Theorem 3

The proof of Theorem 3 is based on a combination of estimates from above and below of the Rayleigh quotient associated to the eigenvalue problem with a fine blow-up analysis for scaled eigenfunctions (11), which gives a sharp characterization of upper and lower bounds for eigenvalues. Differently from the blow-up analysis performed in the previous case when poles are moving tangentially to nodal lines, in this general case when poles are moving along any direction we cannot explicitly construct the limit profile of the family (11). Such a difficulty is overcome by studying the dependence of the limit profile on the position of the pole and the symmetry/periodicity properties of its Fourier coefficient with respect to a basis of eigenvectors of an associated angular problem: such symmetry and periodicity turn into certain symmetry and periodicity invariances of the polynomial P. A complete classification of homogeneous k-degree polynomials with these particular periodicity/symmetry invariances allows us to conclude the proof.

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ON DIRICHLET DATA FOR THE SPECTRAL LAPLACIAN

Abstract. We present a construction for nontrivial harmonic functions associated to the *spectral fractional Laplacian* operator, that is a fractional power of the Dirichlet Laplacian giving rise to a nonlocal operator of fractional order. These harmonic functions present a divergent profile at the boundary of the prescribed domain, and they can be classified in terms of a singular boundary trace.

We introduce a notion of L^1-weak solution, in the spirit of Stampacchia, and we produce solutions of linear and nonlinear problems (possibly with measure data) where one prescribes such a singular boundary trace, therefore providing with a *nonhomogeneous* boundary value problem for this operator. We also present some results entailing the existence of *large* solutions in this context.

This is a summary of a joint work with Louis Dupaigne, see [3].

1. Introduction

Given a bounded domain Ω of the $\mathbb{R}^N$, the *spectral (or Navier) fractional Laplacian* operator $(-\triangle|_\Omega)^s,\ s\in(0,1)$, is defined as a fractional power of the Laplacian with homogeneous Dirichlet boundary conditions, see (2) below. This provides a nonlocal operator of elliptic type with *homogeneous* boundary conditions. Recent bibliography on this operator can be found *e.g.* in [8, 4].

One aspect of the theory is however left unanswered: the formulation of natural *nonhomogeneous* boundary conditions. We provide a well-posed weak formulation for linear problems of the form

$$(-\triangle|_\Omega)^s u \;=\; \mu \quad \text{in } \Omega, \qquad \frac{u}{h_1} \;=\; \zeta \quad \text{on } \partial\Omega \tag{1}$$

where h_1 is a reference function, see (7) below, with prescribed singular behaviour at the boundary. Namely, h_1 is bounded above and below by constant multiples of $\delta^{-(2-2s)}$, where $\delta(x) := \mathrm{dist}(x,\partial\Omega)$ is the distance to the boundary. Unlike the classical Dirichlet problem for the Laplace operator, nonhomogeneous boundary conditions must be singular. In fact, for the special case of positive s-harmonic functions, the singular boundary condition was already identified in previous works emphasizing the probabilistic and potential theoretic aspects of the problem: see *e.g.* [11, 6, 10].

Turning to nonlinear problems, even more singular boundary conditions arise: in the above system, if $\mu=-u^p$ for suitable values of p, one may choose $\zeta=+\infty$, in the sense that the solution u will blow up at a higher rate than $\delta^{-(2-2s)}$.

DEFINITION 1. *Let $\Omega\subset\mathbb{R}^N$ a bounded domain and let $\{(\lambda_j,\varphi_j)\}_{j\in\mathbb{N}}$, $\varphi_j\in$*

$H^1_0(\Omega)\cap C^\infty(\Omega)$ and $-\triangle\varphi_j=\lambda_j\varphi_j$ in Ω. Given $s\in(0,1)$, consider the Hilbert space

$$H(2s):=\left\{v=\sum_{j=1}^\infty \widehat{v}_j\varphi_j\in L^2(\Omega):\|v\|^2_{H(2s)}=\sum_{j=0}^\infty \lambda_j^{2s}|\widehat{v}_j|^2<\infty\right\}.$$

The spectral fractional Laplacian of $u\in H(2s)$ is the function

$$(-\triangle|_\Omega)^s u \;=\; \sum_{j=1}^\infty \lambda_j^s\widehat{u}_j\,\varphi_j. \tag{2}$$

Alternatively, for almost every $x\in\Omega$,

$$(-\triangle|_\Omega)^s u(x) \;=\; p.v.\int_\Omega [u(x)-u(y)]J(x,y)\,dy+\kappa(x)u(x), \tag{3}$$

where, letting $p_\Omega(t,x,y)$ denote the heat kernel of $-\triangle|_\Omega$,

$$J(x,y)=\frac{s}{\Gamma(1-s)}\int_0^\infty \frac{p_\Omega(t,x,y)}{t^{1+s}}\,dt,\ \kappa(x)=\frac{s}{\Gamma(1-s)}\int_\Omega\left(1-\int_\Omega p_\Omega(t,x,y)\,dy\right)\frac{dt}{t^{1+s}} \tag{4}$$

are respectively the *jumping kernel* and the *killing measure**.

We assume from now on that Ω is of class $C^{1,1}$. In this case, sharp bounds are known for p_Ω, see (17), and provide in turn sharp estimates for $J(x,y)$, see (19), so that the right-hand side of (3) remains well-defined for every $x\in\Omega$ under the assumption that $u\in C^{2s+\varepsilon}_{loc}(\Omega)\cap L^1(\Omega,\delta(x)dx)$ for some $\varepsilon>0$. This allows us to *define* the spectral fractional Laplacian of functions which *do not* vanish on the boundary of Ω.

DEFINITION 2. *The Green function and the Poisson kernel of the spectral fractional Laplacian are defined respectively by*

$$G^s_\Omega(x,y) \;=\; \frac{1}{\Gamma(s)}\int_0^\infty p_\Omega(t,x,y)\,t^{s-1}\,dt, \qquad x,y\in\Omega, x\neq y,\ s\in(0,1], \tag{5}$$

and by

$$P^s_\Omega(x,y) \;:=\; -\frac{\partial}{\partial\nu_y}\,G^s_\Omega(x,y), \qquad x\in\Omega, y\in\partial\Omega. \tag{6}$$

where ν is the outward unit normal to $\partial\Omega$.

DEFINITION 3. *Consider the function space $\mathcal{T}(\Omega) \;:=\; (-\triangle|_\Omega)^{-s}C^\infty_c(\Omega)$ and*

$$h_1(x)=\int_{\partial\Omega} P^s_\Omega(x,y)\,d\sigma(y), \qquad x\in\Omega. \tag{7}$$

*In the language of potential theory of killed stochastic processes. Note that the integral in (3) must be understood in the sense of principal values. To see this, look at (19).

Given two Radon measures $\mu \in \mathcal{M}(\Omega)$ and $\zeta \in \mathcal{M}(\partial\Omega)$ with

$$\int_\Omega \delta(x)\, d|\mu|(x) \;<\; \infty, \qquad |\zeta|(\partial\Omega) \;<\; \infty, \tag{8}$$

a function $u \in L^1_{loc}(\Omega)$ is a weak solution to

$$(-\triangle|_\Omega)^s u = \;\mu \quad \text{in } \Omega, \qquad \frac{u}{h_1} \;=\; \zeta \quad \text{on } \partial\Omega \tag{9}$$

if, for any $\psi \in \mathcal{T}(\Omega)$,

$$\int_\Omega u\,(-\triangle|_\Omega)^s\psi \;=\; \int_\Omega \psi\, d\mu - \int_{\partial\Omega} \frac{\partial\psi}{\partial\nu}\, d\zeta. \tag{10}$$

We present some facts about harmonic functions in Section 2.1 with an eye kept on their singular boundary trace. We prove the well-posedness of (9) in Section 2.1. In Section 3 we solve nonlinear Dirichlet problems.

THEOREM 1. *Given two Radon measures $\mu \in \mathcal{M}(\Omega)$ and $\zeta \in \mathcal{M}(\partial\Omega)$ such that* (8) *holds, there exists a weak solution $u \in L^1_{loc}(\Omega)$ to* (9)*. Moreover, for a.e. $x \in \Omega$,*

$$u(x) \;=\; \int_\Omega G^s_\Omega(x,y)\, d\mu(y) + \int_{\partial\Omega} P^s_\Omega(x,y)\, d\zeta(y). \tag{11}$$

THEOREM 2. *Let $g(x,t) : \Omega \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a Carathéodory function such that $g(x,0) = 0$, and $h : \mathbb{R}^+ \to \mathbb{R}^+$ a nondecreasing function with*

$$0 \leqslant g(x,t) \leqslant h(t) \quad \text{for a.e. } x \in \Omega \text{ and all } t > 0, \qquad h(\delta^{-(2-2s)})\delta \in L^1(\Omega).$$

Then, problem

$$(-\triangle|_\Omega)^s u = -g(x,u) \quad \text{in } \Omega, \qquad \frac{u}{h_1} = \zeta \quad \text{on } \partial\Omega \tag{12}$$

has a solution $u \in L^1(\Omega, \delta(x)dx)$ for any $\zeta \in C(\partial\Omega), \zeta \geqslant 0$. In addition, if $t \mapsto g(x,t)$ is nondecreasing then the solution is unique.

THEOREM 3. *Let $p \in \left(1+s, \frac{1}{1-s}\right)$. There exists a function $u \in L^1(\Omega, \delta(x)dx) \cap C^\infty(\Omega)$ solving*

$$(-\triangle|_\Omega)^s u = -u^p \quad \text{in } \Omega, \qquad \frac{u}{h_1} = +\infty \quad \text{on } \partial\Omega \tag{13}$$

in the following sense: the first equality holds pointwise and in the sense of distributions, the boundary condition is understood as a pointwise limit. In addition, there exists a constant $C = C(\Omega, N, s, p)$ such that $0 \leqslant u \leqslant C\delta^{-\frac{2s}{p-1}}$.

2. Green function and Poisson kernel

In the following three lemmas, we review some useful identities for the Green function defined by (5). Compare them also with [7, formulas (17) and (8) respectively].

LEMMA 1. *Let $f \in L^2(\Omega)$. For almost every $x \in \Omega$, $G^s_\Omega(x,\cdot)f \in L^1(\Omega)$ and*

$$(-\triangle|_\Omega)^{-s} f(x) = \int_\Omega G^s_\Omega(x,y) f(y)\, dy \qquad \text{for a.e. } x \in \Omega.$$

LEMMA 2. *For a.e. $x,y \in \Omega$, $\int_\Omega G^{1-s}_\Omega(x,\xi) G^s_\Omega(\xi,y)\, d\xi = G^1_\Omega(x,y)$.*

Proof. Clearly $(-\triangle|_\Omega)^{-s}(-\triangle|_\Omega)^{s-1}\varphi_j = \lambda_j^{-s}\lambda_j^{s-1}\varphi_j = (-\triangle|_\Omega)^{-1}\varphi_j$ for any eigenfunction φ_j, so $(-\triangle|_\Omega)^{-s} \circ (-\triangle|_\Omega)^{s-1} = (-\triangle|_\Omega)^{-1}$ in $L^2(\Omega)$. By the previous lemma and Fubini's theorem, we deduce that for $\varphi \in L^2(\Omega)$ and a.e. $x \in \Omega$,

$$\int_\Omega \int_\Omega G^{1-s}_\Omega(x,\xi) G^s_\Omega(\xi,y) \varphi(y)\, d\xi\, dy = \int_\Omega G^1_\Omega(x,y)\varphi(y)\, dy$$

and so (2) holds almost everywhere. □

LEMMA 3. *For any $\psi \in C^\infty_c(\Omega)$,*

$$(-\triangle|_\Omega)^s \psi \;=\; (-\triangle) \circ (-\triangle|_\Omega)^{s-1} \psi \;=\; (-\triangle|_\Omega)^{s-1} \circ (-\triangle)\psi \tag{14}$$

Proof. The identity clearly holds if ψ is an eigenfunction. If $\psi \in C^\infty_c(\Omega)$, its spectral coefficients have fast (more than algebraic) decay and the result follows by writing the spectral decomposition of ψ. □

LEMMA 4. *The function $P^s_\Omega(x,y) := -\frac{\partial}{\partial \nu_y} G^s_\Omega(x,y)$ is well-defined for $x \in \Omega, y \in \partial\Omega$ and $P^s_\Omega(x,\cdot) \in C(\partial\Omega)$ for any $x \in \Omega$. Furthermore, there exists a constant $C > 0$ depending on N,s,Ω only such that*

$$\frac{1}{C}\frac{\delta(x)}{|x-y|^{N+2-2s}} \leqslant P^s_\Omega(x,y) \leqslant C\frac{\delta(x)}{|x-y|^{N+2-2s}}. \tag{15}$$

and

$$\int_\Omega G^{1-s}_\Omega(x,\xi) P^s_\Omega(\xi,y)\, d\xi \;=\; P^1_\Omega(x,y). \tag{16}$$

Proof. The proof is technical. The interested reader can find it in [3]. A main ingredient is the sharp double-sided estimate for the heat kernel (cf. [5]):

$$\left[\frac{\delta(x)\delta(y)}{t} \wedge 1\right] \frac{1}{c_1 t^{N/2}} e^{-|x-y|^2/(c_2 t)} \leqslant p_\Omega(t,x,y) \leqslant \left[\frac{\delta(x)\delta(y)}{t} \wedge 1\right] \frac{c_1}{t^{N/2}} e^{-c_2|x-y|^2/t}. \tag{17}$$

Recall that the Poisson kernel of the Dirichlet Laplacian can be defined via the very same formula, just by considering $s = 1$. □

REMARK 1. Thanks to bound (17), the following estimates also hold:

$$\frac{1}{C|x-y|^{N-2s}}\left(1\wedge\frac{\delta(x)\,\delta(y)}{|x-y|^2}\right)\leqslant G^s_\Omega(x,y)\leqslant\frac{C}{|x-y|^{N-2s}}\left(1\wedge\frac{\delta(x)\,\delta(y)}{|x-y|^2}\right)\tag{18}$$

for some constant $C=C(\Omega,N,s)$, and

$$\frac{1}{C|x-y|^{N+2s}}\left(1\wedge\frac{\delta(x)\delta(y)}{|x-y|^2}\right)\leqslant J(x,y)\leqslant\frac{C}{|x-y|^{N+2s}}\left(1\wedge\frac{\delta(x)\delta(y)}{|x-y|^2}\right).\tag{19}$$

2.1. Harmonic functions

DEFINITION 4. *A function $h\in L^1(\Omega,\delta(x)dx)$ is s-harmonic in Ω if*

$$\int_\Omega h\,(-\triangle|_\Omega)^s\psi\;=\;0\qquad \textit{for any } \psi\in C^\infty_c(\Omega).$$

LEMMA 5. *For any $\psi\in C^\infty_c(\Omega)$, $(-\triangle|_\Omega)^s\psi\in C^1_0(\overline{\Omega})$ and there exists a constant $C=C(s,N,\Omega,\psi)>0$ such that $|(-\triangle|_\Omega)^s\psi|\;\leqslant\;C\delta$ in Ω.*

Proof. One has

$$\left|\frac{(-\triangle|_\Omega)^s\psi}{\delta}\right|\leqslant\sum_{j=1}^\infty\lambda_j^s|\hat{\psi}_j|\left\|\frac{\varphi_j}{\delta}\right\|_{L^\infty(\Omega)}<\infty.$$

□

LEMMA 6. *The function $P^s_\Omega(\cdot,z)\in L^1(\Omega,\delta(x)dx)$ is s-harmonic in Ω.*

Proof. Thanks to (15), $P^s_\Omega(\cdot,z)\in L^1(\Omega,\delta(x)dx)$. Pick $\psi\in C^\infty_c(\Omega)$ and exploit (14). □

LEMMA 7. *For any finite Radon measure $\zeta\in\mathcal{M}(\partial\Omega)$, let*

$$h(x)\;=\;\int_{\partial\Omega}P^s_\Omega(x,z)\,d\zeta(z),\qquad x\in\Omega.\tag{20}$$

Then, h is s-harmonic in Ω.

Proof. Since $P^s_\Omega(x,\cdot)$ is continuous, h is well-defined. Pick $\psi\in C^\infty_c(\Omega)$:

$$\int_\Omega h(x)\,(-\triangle|_\Omega)^s\psi(x)\,dx=\int_{\partial\Omega}\left(\int_\Omega P^s_\Omega(x,z)\,(-\triangle|_\Omega)^s\psi(x)\,dx\right)d\zeta(z)=0$$

in view of Lemma 6. □

As a matter of fact, a computation shows how the reference function h_1 possess a precise boundary behaviour, which is

$$\frac{1}{C}\,\delta^{-(2-2s)}\leqslant h_1\leqslant C\delta^{-(2-2s)}.\tag{21}$$

for some constant $C = C(N,\Omega,s) > 0$. In the following we will use the notation $\mathbb{P}^s_\Omega g := \int_{\partial\Omega} P^s_\Omega(\cdot,\theta)\, g(\theta)\, d\sigma(\theta)$ where σ denotes the Hausdorff measure on $\partial\Omega$, whenever $g \in L^1(\Omega)$.

PROPOSITION 1. *Let $\zeta \in C(\partial\Omega)$. Then, for any $z \in \partial\Omega$,*

$$\frac{\mathbb{P}^s_\Omega \zeta(x)}{h_1(x)} \xrightarrow[x\to z]{x\in\Omega} \zeta(z) \qquad \textit{uniformly on } \partial\Omega. \tag{22}$$

Proof. Let us write

$$\begin{aligned}
&\left|\frac{\mathbb{P}^s_\Omega \zeta(x)}{h_1(x)} - \zeta(z)\right| = \left|\frac{1}{h_1(x)}\int_{\partial\Omega} P^s_\Omega(x,\theta)\,\zeta(\theta)\, d\sigma(\theta) - \frac{h_1(x)\,\zeta(z)}{h_1(x)}\right| \leqslant \\
&\leqslant \frac{1}{h_1(x)}\int_{\partial\Omega} P^s_\Omega(x,\theta)|\zeta(\theta)-\zeta(z)|\, d\sigma(\theta) \leqslant C\delta(x)^{3-2s}\int_{\partial\Omega}\frac{|\zeta(\theta)-\zeta(z)|}{|x-\theta|^{N+2-2s}}\, d\sigma(\theta) \leqslant \\
&\qquad \leqslant C\delta(x)\int_{\partial\Omega}\frac{|\zeta(\theta)-\zeta(z)|}{|x-\theta|^{N}}\, d\sigma(\theta).
\end{aligned}$$

It suffices now to repeat the computations in [1, Lemma 3.1.5] to show that the obtained quantity converges to 0 as $x \to z$. □

LEMMA 8. $\mathcal{T}(\Omega) \subseteq C^1_0(\overline{\Omega}) \cap C^\infty(\Omega)$. *Moreover, for any $\psi \in \mathcal{T}(\Omega)$ and $z \in \partial\Omega$,*

$$-\frac{\partial\psi}{\partial\nu}(z) = \int_\Omega P^s_\Omega(y,z)\,(-\triangle|_\Omega)^s\psi(y)\, dy. \tag{23}$$

Proof. Take $\psi \in \mathcal{T}(\Omega)$ and let $f = (-\triangle|_\Omega)^s\psi$. Since $f \in C^\infty_c(\Omega)$, the spectral coefficients of f have fast (more than algebraic) decay and so the same holds true for ψ. It follows that $\psi \in C^1_0(\overline{\Omega})$ and $\mathcal{T}(\Omega) \subseteq C^1_0(\overline{\Omega})$. By Lemma 1, for all $x \in \overline{\Omega}$, $\psi(x) = \int_\Omega G^s_\Omega(x,y)\, f(y)\, dy$. Using Lemma 4 and the dominated convergence theorem, (23) follows. Since $(-\triangle|_\Omega)^s$ is self-adjoint in $H(2s)$, we know that the equality $(-\triangle|_\Omega)^s\psi = f$ holds in $\mathcal{D}'(\Omega)$. □

LEMMA 9 (Maximum principle for classical solutions). *Let $u \in C^{2s+\varepsilon}_{loc}(\Omega) \cap L^1(\Omega,\delta(x)\,dx)$ such that*

$$(-\triangle|_\Omega)^s u \geqslant 0 \ \text{ in } \Omega, \qquad \liminf_{x\to\partial\Omega} u(x) \geqslant 0.$$

Then $u \geqslant 0$ in Ω. In particular this holds when $u \in \mathcal{T}(\Omega)$.

Proof. Suppose $x^* \in \Omega$ such that $u(x^*) = \min_\Omega u < 0$. Then

$$(-\triangle|_\Omega)^s u(x^*) = \int_\Omega [u(x^*) - u(y)]\, J(x,y)\, dy + \kappa(x^*)u(x^*) < 0,$$

a contradiction. □

LEMMA 10. *Let $\mu \in \mathcal{M}(\Omega)$, $\zeta \in \mathcal{M}(\partial\Omega)$ be two Radon measures satisfying* (8) *with $\mu \geqslant 0$ and $\zeta \geqslant 0$. Consider $u \in L^1_{loc}(\Omega)$ a weak solution to the Dirichlet problem* (9). *Then $u \geqslant 0$ a.e. in Ω.*

Proof. Take $f \in C^\infty_c(\Omega)$, $f \geqslant 0$ and $\psi = (-\triangle|_\Omega)^{-s} f \in \mathcal{T}(\Omega)$. By Lemma 9, $\psi \geqslant 0$ in Ω and by Lemma 8 $-\frac{\partial\psi}{\partial\nu} \geqslant 0$ on $\partial\Omega$. Thus, by (10), $\int_\Omega uf \geqslant 0$. Since this is true for every $f \in C^\infty_c(\Omega)$, the result follows. □

Proof of Theorem 1. Uniqueness is a direct consequence of the comparison principle, Lemma 10. Let us prove that formula (11) defines the desired weak solution. Observe that if u is given by (11), then $u \in L^1(\Omega, \delta(x)dx)$. Indeed,

$$\int_\Omega \left|\varphi_1(x) \int_\Omega G^s_\Omega(x,y) d\mu(y)\right| dx \leqslant \int_\Omega \int_\Omega G^s_\Omega(x,y)\varphi_1(x) dx\, d|\mu|(y) \tag{24}$$
$$= \frac{1}{\lambda_1^s} \int_\Omega \varphi_1(y)\, d|\mu|(y) \leqslant C\|\delta\mu\|_{\mathcal{M}(\Omega)}$$

This, along with Lemma 7, proves that $u \in L^1(\Omega, \delta(x)dx)$. Now, pick $\psi \in \mathcal{T}(\Omega)$ and compute, via the Fubini's Theorem, Lemma 1 and Lemma 8,

$$\begin{aligned}
&\int_\Omega u(x)(-\triangle|_\Omega)^s \psi(x)\, dx \;= \\
&= \int_\Omega \int_\Omega G^s_\Omega(x,y) d\mu(y)(-\triangle|_\Omega)^s \psi(x)\, dx + \int_\Omega \int_{\partial\Omega} P^s_\Omega(x,z)\, d\zeta(z)(-\triangle|_\Omega)^s \psi(x)\, dx \\
&= \int_\Omega \int_\Omega G^s_\Omega(x,y)(-\triangle|_\Omega)^s \psi(x)\, dx\, d\mu(y) + \int_{\partial\Omega} \int_\Omega P^s_\Omega(x,z)(-\triangle|_\Omega)^s \psi(x)\, dx\, d\zeta(z) \\
&= \int_\Omega \psi(y)\, d\mu(y) - \int_{\partial\Omega} \frac{\partial\psi}{\partial\nu}(z)\, d\zeta(z).
\end{aligned}$$

□

3. The nonlinear problem

THEOREM 4. *Let $f(x,t) : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function. Assume that there exists a subsolution and a supersolution $\underline{u}, \overline{u} \in L^1(\Omega, \delta(x)dx) \cap L^\infty_{loc}(\Omega)$ to*

$$(-\triangle|_\Omega)^s u = f(x,u) \quad \text{in } \Omega, \qquad \frac{u}{h_1} = 0 \quad \text{on } \partial\Omega \tag{25}$$

Assume in addition that $f(\cdot, v) \in L^1(\Omega, \delta(x)dx)$ for every $v \in L^1(\Omega, \delta(x)dx)$ such that $\underline{u} \leqslant v \leqslant \overline{u}$ a.e. Then, there exist weak solutions $u_1, u_2 \in L^1(\Omega, \delta(x)dx)$ in $[\underline{u}, \overline{u}]$ such that any solution in the interval $[\underline{u}, \overline{u}]$ satisfies

$$\underline{u} \leqslant u_1 \leqslant u \leqslant u_2 \leqslant \overline{u} \qquad a.e.$$

Moreover, if the nonlinearity f is decreasing in the second variable, then the solution is unique.

Proof. The proof can be performed by adapting the one in [9] to the fractional case. More details can be found in [3]. □

Proof of Theorem 2. Problem (12) is equivalent to

$$(26) \qquad (-\triangle|_\Omega)^s v = g(x, \mathbb{P}^s_\Omega\zeta - v) \quad \text{in } \Omega, \qquad \frac{v}{h_1} = 0 \quad \text{on } \partial\Omega$$

that possesses $\overline{u} = \mathbb{P}^s_\Omega\zeta$ as a supersolution and $\underline{u} = 0$ as a subsolution. Indeed, by equation (21) we have $0 \leqslant \mathbb{P}^s_\Omega\zeta \leqslant \|\zeta\|_{L^\infty(\Omega)} h_1 \leqslant C\|\zeta\|_{L^\infty(\Omega)}\delta^{-(2-2s)}$. Thus any $v \in L^1(\Omega, \delta(x)dx)$ such that $0 \leqslant v \leqslant \mathbb{P}^s_\Omega\zeta$ satisfies

$$g(x,v) \leqslant h(v) \leqslant h(c\delta^{-(2-2s)}) \in L^1(\Omega, \delta(x)dx).$$

So, all hypotheses of Theorem 4 are satisfied and the result follows. □

4. Large solutions

Consider the sequence $\{u_j\}_{j\in\mathbb{N}}$ built by solving

$$(27) \qquad (-\triangle|_\Omega)^s u_j = -u_j^p \quad \text{in } \Omega, \qquad \frac{u_j}{h_1} = j \quad \text{on } \partial\Omega.$$

Theorem 2 guarantees the existence of such a sequence if $\delta^{-(2-2s)p} \in L^1(\Omega, \delta(x)dx)$, i.e. $p < 1/(1-s)$. Moreover, $\{u_j\}_{j\in\mathbb{N}}$ is increasing with j, thus it admits a pointwise limit, possibly infinite.

LEMMA 11. *There exist $\delta_0, C > 0$ such that $(-\triangle|_\Omega)^s\delta^{-\alpha} \geqslant -C\delta^{-\alpha p}$, for $\delta < \delta_0$ and $\alpha = \frac{2s}{p-1}$.*

Proof. We use the expression in equation (3). Obviously,

$$(-\triangle|_\Omega)^s\delta^{-\alpha}(x) = \int_\Omega [\delta(x)^{-\alpha} - \delta(y)^{-\alpha}] J(x,y)\, dy + \delta(x)^{-\alpha}\kappa(x) \geqslant$$
$$\geqslant \int_\Omega [\delta(x)^{-\alpha} - \delta(y)^{-\alpha}] J(x,y)\, dy.$$

□

LEMMA 12. *If a function $v \in L^1(\Omega, \delta(x)dx)$ satisfies*

$$(28) \qquad (-\triangle|_\Omega)^s v \in L^\infty_{loc}(\Omega), \qquad (-\triangle|_\Omega)^s v(x) \geqslant -C\, v(x)^p, \quad \text{when } \delta(x) < \delta_0,$$

for some $C, \delta_0 > 0$, then there exists $\overline{u} \in L^1(\Omega, \delta(x)dx)$ such that

$$(29) \qquad (-\triangle|_\Omega)^s \overline{u}(x) \geqslant -\overline{u}(x)^p, \qquad \text{throughout } \Omega.$$

Proof. Let $\lambda := C^{1/(p-1)} \vee 1$ and $\Omega_0 = \{x \in \Omega : \delta(x) < \delta_0\}$, then

$$(-\triangle|_\Omega)^s (\lambda v) \ \geqslant \ -(\lambda v)^p, \qquad \text{in } \Omega_0.$$

Let also $\mu := \lambda \|(-\triangle|_\Omega)^s v\|_{L^\infty(\Omega\setminus\Omega_0)}$ and define $\overline{u} = \mu \mathbb{G}^s_\Omega 1 + \lambda v$. On $\overline{u}$ we have $(-\triangle|_\Omega)^s \overline{u} = \mu + \lambda(-\triangle|_\Omega)^s v \geqslant \lambda|(-\triangle|_\Omega)^s v| + \lambda(-\triangle|_\Omega)^s v \geqslant -\overline{u}^p$ throughout Ω. □

COROLLARY 1. *There exists a function $\overline{u} \in L^1(\Omega, \delta(x)dx)$ such that*

$$(-\triangle|_\Omega)^s \overline{u} \ \geqslant \ -\overline{u}^p, \qquad \textit{in } \Omega,$$

holds in a pointwise sense. Moreover, $\overline{u} \asymp \delta^{-2s/(p-1)}$.

Proof. Apply Lemma 12 with $v = \delta^{-2s/(p-1)}$. □

LEMMA 13. *For any $j \in \mathbb{N}$, the solution u_j to problem (27) satisfies the upper bound $u_j \ \leqslant \ \overline{u}$, in Ω, where $\overline{u}$ is provided by Corollary 1.*

Proof. Write $u_j = jh_1 - v_j$ where

$$(-\triangle|_\Omega)^s v_j = (jh_1 - v_j)^p \quad \text{in } \Omega, \qquad \frac{v_j}{h_1} = 0 \quad \text{on } \partial\Omega.$$

and $0 \leqslant v_j \leqslant jh_1$. Since $(jh_1 - v_j)^p \in L^\infty_{loc}(\Omega)$, we deduce that $v_j \in C^\alpha_{loc}(\Omega)$ for any $\alpha \in (0, 2s)$. Now, we have that, by the boundary behaviour of $\overline{u}$ stated in Corollary 1, $u_j \leqslant \overline{u}$ close enough to $\partial\Omega$ (depending on the value of j) and

$$(-\triangle|_\Omega)^s (\overline{u} - u_j) \ \geqslant \ u_j^p - \overline{u}^p, \qquad \text{in } \Omega.$$

Since $u_j^p - \overline{u}^p \in C(\Omega)$ and $\lim_{x\to\partial\Omega} u_j^p - \overline{u}^p = -\infty$, then there exists $x_0 \in \Omega$ such that $u_j(x_0)^p - \overline{u}(x_0)^p = m =: \max_{x\in\Omega}(u_j(x)^p - \overline{u}(x)^p)$. If $m > 0$ then also $(-\triangle|_\Omega)^s(\overline{u} - u_j)(x_0) \geqslant m > 0$: this is a contradiction, as Definition 3 implies. Thus $m \leqslant 0$ and $u_j \leqslant \overline{u}$ throughout Ω. □

THEOREM 5. *For any $p \in \left(1+s, \frac{1}{1-s}\right)$ there exists $u \in L^1(\Omega, \delta(x)dx)$ solving*

$$(-\triangle|_\Omega)^s u = -u^p \quad \textit{in } \Omega, \qquad \delta^{2-2s} u = +\infty \quad \textit{on } \partial\Omega.$$

Proof. Consider the sequence $\{u_j\}_{j\in\mathbb{N}}$ provided by problem 27: it is increasing and locally bounded by Lemma 13, so it has a pointwise limit $u \leqslant \overline{u}$, where $\overline{u}$ is the function provided by Corollary 1. Since $p > 1+s$ and $\overline{u} \leqslant C\delta^{-2s/(p-1)}$, then $u \in L^1(\Omega, \delta(x)dx)$. Pick now $\psi \in C^\infty_c(\Omega)$, and recall that $\delta^{-1}(-\triangle|_\Omega)^s \psi \in L^\infty(\Omega)$: we have, by dominated convergence,

$$\int_\Omega u_j (-\triangle|_\Omega)^s \psi \xrightarrow[j\uparrow\infty]{} \int_\Omega u(-\triangle|_\Omega)^s \psi, \qquad \int_\Omega u_j^p \psi \xrightarrow[j\uparrow\infty]{} \int_\Omega u^p \psi$$

so we deduce

$$\int_\Omega u(-\triangle|_\Omega)^s \psi \ = \ -\int_\Omega u^p \psi.$$

Note now that for any compact $K \subset\subset \Omega$, applying some known elliptic regularity estimates we get for any $\alpha \in (0,2s)$

$$\|u_j\|_{C^\alpha(K)} \leqslant C\left(\|u_j\|^p_{L^\infty(K)} + \|u_j\|_{L^1(\Omega,\delta(x)dx)}\right) \leqslant C\left(\|\overline{u}\|^p_{L^\infty(K)} + \|\overline{u}\|_{L^1(\Omega,\delta(x)dx)}\right)$$

which means that $\{u_j\}_{j\in\mathbb{N}}$ is equibounded and equicontinuous in $C(K)$. By the Ascoli-Arzelà Theorem, its pointwise limit u will be in $C(K)$ too. Now, since

$$(-\triangle|_\Omega)^s u = -u^p \qquad \text{in } \mathcal{D}'(\Omega),$$

by bootstrapping the interior regularity we deduce $u \in C^\infty(\Omega)$. □

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SOME REMARKS ON CONVEX COMBINATIONS OF LOW EIGENVALUES

Abstract. In this survey we deal with shape optimization problems involving convex combinations of the first two eigenvalues of the Dirichlet Laplacian, mainly recalling and explaining some recent results. More precisely, we discuss some geometric properties of minimizers, in particular when they are no longer convex and the optimality of balls. This leads us to deal with the "attainable set" of the first two eigenvalues, which is a great source of open problems.

1. Introduction

The aim of this note is to introduce the reader to some shape optimization problems involving the first two eigenvalues of the Dirichlet Laplacian. In particular we focus on the minimization of convex combinations of these first two eigenvalues, among open subsets of the euclidean space with a measure constraint. Although this can seem a rather easy topic, as it often happens in shape optimization, there are many hidden difficulties and a lot of open conjectures. This work is mostly based on the papers [19] and [23], to which we refer for more details.

The topic of spectral optimization has received a lot of attention in the last years, see the books [8, 16, 18] for a broader introduction. The first issue for this kind of problems concerns the *existence* of an optimal shape: a result proved in the 1990s by Buttazzo and Dal Maso [13] is even now a cornerstone of the matter, and, for a large class of functionals, it ensures the existence of a solution in the class of quasi-open sets of fixed measure (*a priori* contained into a given box, which provides the necessary compactness to prove existence). Moreover, the *regularity* of an optimal shape is a highly difficult problem and a general regularity theory is nowadays not available: even a proof which guarantees that an optimal shape is open, and not merely quasi-open, is far from being trivial, see [12]. Another important point consists in proving some *geometric* properties of optimal shapes, such as connectedness, convexity, symmetry with respect to some axis, and this is the main topic of this note. In fact, only for few special functionals optimal shapes are explicitly known: classical examples are the lowest eigenvalues of the Dirichlet-Laplacian. We recall that, for a given integer $N \geqslant 2$ and an open set $\Omega \subset \mathbb{R}^N$ with finite measure, the *first* and *second eigenvalues* of the Dirichlet-Laplacian can be defined as

$$\lambda_1(\Omega) := \min_{u \in H^1_0(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u(x)|^2 dx}{\int_\Omega |u(x)|^2 dx}, \qquad \lambda_2(\Omega) := \min_{\substack{u \in H^1_0(\Omega)\setminus\{0\} \\ \int_\Omega u u_1 = 0}} \frac{\int_\Omega |\nabla u(x)|^2 dx}{\int_\Omega |u(x)|^2 dx},$$

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where these minima are attained, respectively, by the *first* and *second eigenfunctions* u_1 and u_2 (which are unique, up to a multiplicative constant).

The interest in the minimization of the first eigenvalue goes back to a conjecture due to Lord Rayleigh in 1877, then proved by Faber and Krahn in the 1920s. The *Faber-Krahn inequality* asserts that of all open sets of fixed measure, the ball has the minimum first eigenvalue: in formula, for every open set $\Omega \subset \mathbb{R}^N$ with unit measure

$$\lambda_1(\Omega) \geqslant \lambda_1(B) = \omega_N^{2/N} j_{N/2-1}^2, \tag{1}$$

where ω_N denotes the measure of the ball in $\mathbb{R}^N$ with unit radius, j_n the first positive zero of the Bessel function J_n, and B the open ball of unit measure in $\mathbb{R}^N$. Equality in (1) holds if and only if Ω is that ball (up to sets of capacity zero). The same issue for the second eigenvalue is known as the *Krahn-Szegö inequality*, which asserts that two disjoint open balls of half measure each are the unique (up to sets of capacity zero) minimizer, namely for every open set $\Omega \subset \mathbb{R}^N$ with unit measure

$$\lambda_2(\Omega) \geqslant \lambda_2(B_- \cup B_+) = 2^{2/N}\lambda_1(B) = (2\omega_N)^{2/N} j_{N/2-1}^2, \tag{2}$$

where $B_- \cup B_+$ is the union of two equal and disjoint open balls of half measure each, and equality in (2) holds if and only if $\Omega = B_- \cup B_+$.
Up to our knowledge, the only other functionals of eigenvalues for which an explicit minimizer is known are λ_1/λ_2 and λ_2/λ_3, see [3].

Starting with the important work of Keller and Wolf [21], there was a strong interest for *convex combinations* of the first two eigenvalues of the Dirichlet Laplacian, namely the functional F_t defined, for every $t \in (0,1)$, as

$$F_t(\Omega) := t\lambda_1(\Omega) + (1-t)\lambda_2(\Omega), \tag{3}$$

where $\Omega \subset \mathbb{R}^N$ is an open set of finite measure. Then, the corresponding spectral optimization problem writes as

$$\min\left\{F_t(\Omega)\ :\ \Omega \subset \mathbb{R}^N, \Omega \text{ open}, |\Omega| = 1\right\}. \tag{4}$$

The existence of a minimizer for this problem is now well understood and is guaranteed by a general theory recently developed in the works [7, 12, 22], all based on the above mentioned result [13], but with the new difficulty of working in the entire space $\mathbb{R}^N$. Notice that, all these results guarantee the existence of an optimal shape in the larger class of quasi-open sets, and only *a posteriori* one proves that a minimizer of problem (1.1) is in fact *open*, and so problem (1.1) is well-posed. Moreover, in [19] it was proved that, for every $t \in (0,1)$, minimizers of (1.1) are *connected* (more generally, this topological property was studied for minimizers of convex combinations of the first three eigenvalues). The main idea in order to prove connectedness of minimizers is to characterize the optimal *disconnected* configuration and then to find an explicit connected competitor, which is often either the ball or a perturbation of balls. In two dimensions ($N = 2$), some numerical computations on the shape of these minimizers appeared in [20]. We sum up all these results in the following theorem.

THEOREM 1. *For every $t \in (0,1)$ we consider the shape optimization problem:*

$$\min\left\{F_t(\Omega)\;:\;\Omega\subset\mathbb{R}^N,|\Omega|=1\right\}. \tag{5}$$

The following facts hold true for (5).

1. *There exists an optimal shape in the class of quasi-open sets (Buttazzo-Dal Maso [13], Bucur [7], Mazzoleni-Pratelli [22]),*
2. *Every optimal set is open and each of its first k eigenfunctions can be extended in $\mathbb{R}^N$ to a Lipschitz continuous function (Bucur-Mazzoleni-Pratelli-Velichkov [12]),*
3. *Every optimal set is bounded and has finite perimeter (Bucur [7]),*
4. *Every optimal set is connected (Iversen-Mazzoleni [19]).*

Notice that, if $t = 1$ the convex combination (3) is minimized by the ball with unit measure (because of the Faber-Krahn inequality (1)), while if $t = 0$, by two equal balls of half measure each (because of the Krahn-Szegö inequality (2)). Therefore, as t moves from 1 to 0, one expects the shape of a minimizer Ω_t deforming from a ball of unit measure to two balls of half measure each; in particular, it is natural to conjecture that at some value of t the convexity of all the minimizers in (1.1) is lost (as was numerically observed in [20], in two dimensions, the critical value for t is expected to be $1/2$). We give a first answer to this question, though non-optimal. All the results presented in this note, unless otherwise specified, will hold in every dimension $N \geqslant 2$.

THEOREM 2. *There exists a threshold $T > 0$ such that, for all $t \in (0,T)$, every minimizer in* (1.1) *is no longer convex.*

We provide a *quantitative* proof of this theorem, namely we explicitly construct the threshold T *via* the eigenvalues of the Dirichlet-Laplacian. Moreover, in two dimensions, it is possible to find a numerical lower bound on T using a quantitative Krahn-Szegö inequality involving the so-called Fraenkel 2-asymmetry, for this topic we refer the reader to [23], where it is studied an auxiliary purely geometrical problem: the minimization of the Fraenkel 2-asymmetry among *convex* sets of given area. It is possible to show that the *mobile*, i.e., the intersection of the convex hull of two tangent balls with a strip is the unique minimizer satisfying an isoperimetric inequality for the Fraenkel 2-asymmetry (16). An explicit value for the constant in the quantitative Krahn-Szegö inequality will be also needed. This opens a new area of application for quantitative inequalities, which can be found in [23, Appendix].

A second question to address is the optimality of a special convex set: the ball, and we present a generalization of a result from [21].

THEOREM 3. *For all $t \in (0,1)$ the ball B is never a minimizer in* (1.1).

The proof of this result follows from a more general proposition, i.e., that the second eigenvalue of a minimizer in (1.1) has to be simple and, as a consequence of the multiplicity of the second eigenvalue over balls, so we immediately get the result in Theorem 3. The proof of the simplicity of the second eigenvalue relies on some ideas developed in [16] and [10], with the help of a classical symmetry result due to Serrin [25] (see also [15]).

As an application of these results, we show how to get informations on the shape of the *attainable set*, namely the subset of the plane described by the range of the first two eigenvalues of the Dirichlet-Laplacian

$$\mathcal{E} := \left\{(x,y) \in \mathbb{R}^2 \,:\, x = \lambda_1(\Omega),\ y = \lambda_2(\Omega),\ \Omega \subset \mathbb{R}^N,\ \Omega \text{ open},\ |\Omega| = 1\right\}. \tag{6}$$

This set was introduced in [21], and then deeply studied in [9] (see also [1, 2, 5]), where several geometrical properties of $\mathcal{E}$ were discussed.

The link between problem (1.1) and the set $\mathcal{E}$ is the following: for a fixed $t \in (0,1)$ a minimizer Ω_t in (1.1) corresponds to the first point of $\mathcal{E}$ of coordinates $(\lambda_1(\Omega_t), \lambda_2(\Omega_t))$ that we reach with a line $tx + (1-t)y = a$ increasing the value a, that is $P_{\Omega_t} := (\lambda_1(\Omega_t), \lambda_2(\Omega_t))$ is one of the intersection points of the tangent line to $\mathcal{E}$ with slope $t/(t-1)$. In particular, if $t = 1$ the tangent line $x = \lambda_1(B)$ has a *unique* intersection point corresponding to the ball B (because of the Faber-Krahn inequality (1)), while if $t = 0$, the tangent line $y = \lambda_2(\Theta)$ has a *unique* intersection point corresponding to the two balls $B_- \cup B_+$ (because of the Krahn-Szegö inequality (2)).

Therefore, in Theorem 4, we will present a new strategy for studying the asymptotic behavior of the boundary of $\mathcal{E}$ near the points corresponding to B and $B_- \cup B_+$, extending to all dimensions a result proved in [21] only in two dimensions, and recovering the result proved in [5].

We suspect that to properly understand the boundary behavior of the attainable set, one has to carefully analyze problem (1.1). For this reason we restate the longstanding conjecture about the convexity of the attainable set in the language of the minimizers of convex combinations of the lowest Dirichlet eigenvalues.

The paper, which is mainly a review of a talk given by the author at the "BruTo PDE's Conference" held in Torino on May 2nd–5th, 2016, is organized as follows. In Section 2 we discuss Theorem 2, while in Section 3 we deal with Theorem 3 and with the attainable set (6).

2. Non-convexity of minimizers for problem (1.1)

In order to study the non-convexity of minimizers for problem (1.1) we first need to deal briefly with the study of optimal sets for λ_2 among convex bodies, which was the topic of an important paper by Henrot and Oudet [17]. Finding an explicit minimizer in this class seems a very difficult problem: a possible candidate to be the optimum is the stadium (i.e., the convex hull of two tangent balls), but this conjecture was refuted in [17]. Indeed any set which contains on the boundary some pieces of balls can not be a minimizer. Nevertheless, in [17] it was proved the existence of a convex minimizer Ω_{HO} so that, for every open and *convex* set $\Omega \subset \mathbb{R}^N$ with unit measure,

$$\lambda_2(\Omega) \geqslant \lambda_2(\Omega_{\text{HO}}), \tag{7}$$

(cf. (7) with the Krahn-Szegö inequality, where no-convexity constraint is required). Notice that, since Ω_{HO} has no pieces of balls on its boundary, in particular $\Omega_{\mathrm{HO}} \neq B$ and

$$\omega_N^{2/N} j_{N/2}^2 = \lambda_2(B) > \lambda_2(\Omega_{\mathrm{HO}}).$$

In two dimensions, Oudet in [24] and, more recently, Antunes and Henrot in [2], made some numerical computations, showing the shape of the optimal set Ω_{HO} and highlighting that Ω_{HO} is very close to the stadium, both from a geometrical and a numerical point of view; in particular

$$\text{(8)}\quad \lambda_2(B_- \cup B_+) = 2\pi j_0^2 \approx 36.336, \quad \lambda_2(\Omega_{\mathrm{HO}}) \approx 37.987, \quad \lambda_2(\Omega_{\mathrm{stadium}}) \approx 38.002,$$

where $\Omega_{\mathrm{stadium}}$ is the stadium with $|\Omega_{\mathrm{stadium}}| = 1$, i.e., a contracted version of the set hull(Θ). Note that we approximate all the numerically computed values only to the third decimal digit, for sake of simplicity.

Before proving Theorem 2 we need to set some notation. We say that Ω_t is a minimizer in (1.1) if for every admissible competitor Ω

$$\text{(9)}\qquad t\lambda_1(\Omega_t) + (1-t)\lambda_2(\Omega_t) \leqslant t\lambda_1(\Omega) + (1-t)\lambda_2(\Omega),$$

and equivalently, rearranging the terms

$$\text{(10)}\qquad \lambda_1(\Omega_t) - \lambda_1(\Omega) + \lambda_2(\Omega) - \lambda_2(\Omega_t) \leqslant \frac{1}{t}\Big(\lambda_2(\Omega) - \lambda_2(\Omega_t)\Big).$$

Proof of Theorem 2. From the Krahn-Szegö inequality (2) and the connectedness of Ω_t it follows that

$$\text{(11)}\qquad \lambda_2(B_- \cup B_+) < \lambda_2(\Omega_t),$$

which plugged into (9) with $\Omega = B_- \cup B_+$ yields

$$\text{(12)}\qquad \lambda_1(\Omega_t) < \lambda_1(B_- \cup B_+).$$

Taking $\Omega = B_- \cup B_+$ also in (10) and dividing therein by the negative quantity $\lambda_2(B_- \cup B_+) - \lambda_2(\Omega_t)$ (recall (11)) we get to

$$\text{(13)}\qquad \frac{\lambda_1(B_- \cup B_+) - \lambda_1(\Omega_t)}{\lambda_2(\Omega_t) - \lambda_2(B_- \cup B_+)} + 1 \geqslant \frac{1}{t}.$$

From (11) and (12), the ratio on the left-hand side of this inequality turns out to be a positive number; therefore, we can use the Faber-Krahn inequality (1) to estimate $\lambda_1(\Omega_t)$ at the numerator of this ratio. Moreover, if Ω_t would be a *convex* set, we could also use (7) to estimate $\lambda_2(\Omega_t)$ at the denominator of this ratio, obtaining the following uniform bound on t:

$$\text{(14)}\qquad t \geqslant \frac{1}{\frac{\lambda_1(B_- \cup B_+) - \lambda_1(B)}{\lambda_2(\Omega_{\mathrm{HO}}) - \lambda_2(B_- \cup B_+)} + 1}.$$

Calling T the quantity on the right-hand side of this inequality, the Krahn-Szegö inequality for convex sets gives $\lambda_2(\Omega_{\mathrm{HO}}) - \lambda_2(B_- \cup B_+) > 0$, thus $T > 0$. Therefore, if $t < T$, Ω_t can not be convex. $\square$

The proof of Theorem 2 is constructive and reveals an explicit expression for the threshold T in terms of the eigenvalues of the Dirichlet-Laplacian. In particular, the threshold T in Theorem 2 has the following expression:

$$T = 1 - \frac{(2^{2/N} - 1)\lambda_1(B)}{\lambda_2(\Omega_{\text{HO}}) - \lambda_1(B)}, \tag{15}$$

where Ω_{HO} is a minimizer in (7). As it is often the case, in the two dimensional case one can expect to be able to provide some explicit estimate for the constant T. This was done in [23] and we recall here only the result, with the needed explanations. First of all we have to define the Fraenkel 2-asymmetry of an open set $\Omega \subset \mathbb{R}^N$ with unit measure, that is,

(16)

$$\mathcal{A}_2(\Omega) := \min\{|\Omega \triangle (B_- \cup B_+)| \ : B_-, B_+ \text{ disjoint open balls}, |B_-| = |B_+| = 1/2\}.$$

Then we can state the quantitative Krahn–Szegö inequality (see, for example [4, 6]):

$$\frac{\lambda_2(\Omega)}{\lambda_2(B_- \cup B_+)} - 1 \geqslant \beta_{\text{KS}} \mathcal{A}_2(\Omega)^{\alpha}, \tag{17}$$

for some constant $\beta_{KS} > 0$ and an exponent $\alpha > 0$. At last, we consider the following shape optimization problem,

$$\inf\{\mathcal{A}_2(\Omega) : \ \Omega \subset \mathbb{R}^N, \Omega \text{ open and convex}, |\Omega| = 1\}. \tag{18}$$

In [23] it is showed that the set M (see [23, Definition 2.2]) defined as the intersection of the convex hull of two tangent balls with a strip is the unique minimizer satisfying an isoperimetric inequality for the Fraenkel 2-asymmetry (18).

Then it is possible to see, in two dimensions, that

$$T \geqslant 1 - \frac{1}{1 + 2\beta_{\text{KS}} \mathcal{A}_2(M)^{9/2}} \approx 1.192 \cdot 10^{-14}, \tag{19}$$

where the constants β_{KS} and $\mathcal{A}_2(M)$ are as in (17) and (16) respectively. What is important to highlight of this bound is that it does not depend on the eigenvalues of the Dirichlet Laplacian.

REMARK 1. The explicit value for the lower bound to the threshold T is not very accurate, mostly due to the fact that the constant β_{KS} is not the optimal one, but we believe it is important to show that a numerical value can actually be provided. Moreover, if $N = 2$, plugging the numerical computation of $\lambda_2(\Omega_{\text{HO}})$ recalled in (8) into (15) and using $\lambda_1(B) = \pi j_0^2 \approx 18.168$, reveals a numerical approximation for the threshold defined by (15):

$$T \approx 0.083.$$

3. Optimality of the ball and attainable set

In order to deal with Theorem 3, the key point is to show the following, stronger Proposition. The main idea of the proof is to make careful perturbations of the boundary of an optimal set (provided it is regular enough) and then reduce the problem to an overdetermined PDE, which can be treated with techniques first developed by Serrin [25].

PROPOSITION 1. *For a fixed $t \in (0,1)$, let Ω_t be a minimizer of problem* (1.1). *If the boundary of Ω_t is of class C^2 and connected, then $\lambda_2(\Omega_t)$ is simple, namely $\lambda_1(\Omega_t) < \lambda_2(\Omega_t) < \lambda_3(\Omega_t)$. Moreover, on the boundary of Ω_t, the following optimality condition holds:*

$$t\,|\nabla u_1(x)|^2 + (1-t)\,|\nabla u_2(x)|^2 = \frac{2F_t(\Omega_t)}{N}, \qquad x \in \partial\Omega_t. \tag{20}$$

For a proof of the above Proposition we refer to [23], but then Theorem 3 follows easily.

Proof of Theorem 3. The proof is a straightforward consequence of Proposition 1: in every dimension, the second eigenvalue $\lambda_2(B)$ is not simple, therefore the ball B can not be a minimizer for any $t \in (0,1)$. □

REMARK 2. In two dimensions, the fact that balls are never minimizers was implicitly contained in the work [21]. For an arbitrary $\varepsilon > 0$ small enough, in [21] the authors constructed a nearly spherical competitor B_ε, with $|B_\varepsilon| = 1$, such that

$$\lambda_1(B_\varepsilon) \leqslant \lambda_1(B) + d_1\varepsilon^2, \quad \text{while} \quad \lambda_2(B_\varepsilon) \leqslant \lambda_2(B) - d_2\varepsilon,$$

for some positive constants d_1, d_2. Therefore, for every $t \in (0,1)$, it is possible to find $\varepsilon > 0$ so small so that

$$t\lambda_1(B_\varepsilon) + (1-t)\lambda_2(B_\varepsilon) < t\lambda_1(B) + (1-t)\lambda_2(B).$$

The last thing that we want to treat is the relation between problem (1.1) and Theorem 3 with the so called attainable set, defined in (6). We start listing the most important properties that are known on the attainable set $\mathcal{E}$ defined in (6) (for figures representing the set $\mathcal{E}$ we refer to [21, 9, 23]):

1. lies above the bisector $y = x$ (since by definition $\lambda_2(\Omega) \geqslant \lambda_1(\Omega)$ for every $\Omega \subset \mathbb{R}^N$).
2. lies on the right of the line $x = \lambda_1(B)$ (for the Faber-Krahn inequality (1)).
3. lies above the line $y = \lambda_2(B_- \cup B_+)$ (for the Krahn-Szegö inequality (2)).
4. lies below the line $y = \frac{\lambda_2(B)}{\lambda_1(B)}x$ (for the Ashbaugh-Benguria inequality [3]).
5. is conical with respect to the origin.

The numerical picture provided by Keller and Wolff suggests the following conjecture, which is still unsolved up to our knowledge.

Conjecture A. The attainable set $\mathcal{E}$ is convex.

The most important result in the direction of this conjecture was proposed by Bucur, Buttazzo and Figuereido in [9]. These authors proved that the attainable set (6), constructed through quasi-open sets instead of open sets, is convex in the vertical and in the horizontal direction and, as a consequence, that it is closed. Nevertheless the vertical and horizontal convexity *do not* imply convexity (think, for example to an L-shaped set).

From the properties of the set $\mathcal{E}$ listed above it is clear that the unique unknown part of the boundary of $\mathcal{E}$ is the curve $\mathcal{C}$ connecting the points $P_B = (\lambda_1(B), \lambda_2(B))$ and $P_{B_-\cup B_+} = (\lambda_1(B_- \cup B_+), \lambda_2(B_- \cup B_+))$. The convexity of $\mathcal{E}$ is then guaranteed as soon as $\mathcal{C}$ can be parametrized by a convex function. For this reason it is important to have more informations on the curve $\mathcal{C}$. In two dimensions, Keller and Wolf in [21] showed that the tangent of $\mathcal{C}$ at the point P_B corresponding to a ball B is vertical. They constructed a nearly spherical perturbation of B, as recalled in Remark 2, and then they computed the slope of the tangent to $\mathcal{C}$ as $\varepsilon \to 0$. Moreover, in all dimensions, Brasco, Nitsch and Pratelli showed that the tangent of $\mathcal{C}$ at the point $P_{B_-\cup B_+}$ corresponding to two balls $B_- \cup B_+$ is horizontal. In this case the limit as $\varepsilon \to 0$ was computed by overlapping the two balls B_- and B_+ of a quantity measured in terms of the parameter ε. In the following proposition we recover these limits relying on the minimality condition of the minimizers of convex combinations (9) without any explicit construction. Notice that the strategy that we adopt holds in all dimensions.

THEOREM 4. *For every dimension $N \geqslant 2$ and $t \in (0,1)$, let Ω_t be a minimizer of problem* (1.1)*. Then we have:*

i) the tangent of $\mathcal{C}$ at the point P_B corresponding to one ball is vertical*, namely*

$$\lim_{t\to 1} \frac{\lambda_2(\Omega_t) - \lambda_2(B)}{\lambda_1(\Omega_t) - \lambda_1(B)} = -\infty \tag{21}$$

ii) the tangent of $\mathcal{C}$ at the point $P_{B_-\cup B_+}$ corresponding to two identical balls is horizontal*, namely*

$$\lim_{t\to 0} \frac{\lambda_2(\Omega_t) - \lambda_2(B_-\cup B_+)}{\lambda_1(\Omega_t) - \lambda_1(B_-\cup B_+)} = 0. \tag{22}$$

Moreover, the following limits holds

$$\lim_{t\to 0} \lambda_2(\Omega_t) = \lambda_2(B_-\cup B_+) \qquad \textit{and} \qquad \lim_{t\to 1} \lambda_1(\Omega_t) = \lambda_1(B). \tag{23}$$

Proof. From the Faber-Krahn inequality (1) and Theorem 3 we find that

$$\lambda_1(B) < \lambda_1(\Omega_t), \tag{24}$$

which plugged into (9) with $\Omega = B$ yields

$$\lambda_2(\Omega_t) < \lambda_2(B). \tag{25}$$

Taking $\Omega = B$ in (10) and dividing therein by $\lambda_2(B) - \lambda_2(\Omega_t)$ (which from (25) is a strictly positive value) yields

$$\frac{\lambda_1(\Omega_t) - \lambda_1(B)}{\lambda_2(B) - \lambda_2(\Omega_t)} + 1 \leqslant \frac{1}{t}.$$

From (24) and (25) one can see that the ratio on the left-hand side of this inequality is a positive number, therefore, letting $t \uparrow 1$, necessarily, it holds the limit in (21). Moreover, repeating the computations made in the proof of Theorem 2 and letting $t \downarrow 0$ in (13), it follows the limit in (22).

Finally, the limits in (23) are a consequence of (21), (22) and of the boundedness of the denominator in (22) (because of (12)) and of the numerator in (21) (because of (25)). □

We finish this discussion formulating an *isospectral* conjecture on the minimizers of problem (1.1), which could be used to prove the convexity of the attainable set $\mathcal{E}$.

Conjecture B. Let $t \in (0,1)$ and assume $X, Y \subset \mathbb{R}^N$ to be minimizers of problem (1.1) with $F_t(X) = F_t(Y)$. Then, the lowest eigenvalues of X and Y coincide, namely

$$\lambda_1(X) = \lambda_1(Y) \qquad \text{and} \qquad \lambda_2(X) = \lambda_2(Y).$$

PROPOSITION 2. *The validity of Conjecture B implies that Conjecture A holds true.*

Proof. If $\mathcal{E}$ is not convex, then we can find two points $P_X, P_Y \in \mathcal{C}$, corresponding, respectively, to X, Y, and a straight line l passing through these points such that the curve $\mathcal{C}$ lies above l. Therefore, it is clear that l will be of the form $tx + (1-t)y = a$ for some fixed $t \in (0,1)$ and a real number a. Hence the sets X, Y are minimizers in (1.1) for such a t, but $\lambda_1(X) \neq \lambda_1(Y)$ and $\lambda_2(X) \neq \lambda_2(Y)$, a contradiction with Conjecture A. □

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GROUND STATE SOLUTIONS FOR A NONLINEAR CHOQUARD EQUATION

Abstract. We discuss the existence of ground state solutions for the Choquard equation

$$-\Delta u + u = (I_\alpha * F(u))F'(u) \qquad \text{in } \mathbb{R}^N.$$

We prove the existence of solutions under general hypotheses, investigating in particular the case of a homogeneous nonlinearity $F(u) = \frac{|u|^p}{p}$. The cases $N = 2$ and $N \geqslant 3$ are treated differently in some steps. The solutions are found through a variational mountain-pass strategy. The results presented are contained in the papers [8, 2].

1. Introduction

We investigate the existence of solutions for nonlinear Choquard equations of the form

$$-\Delta u + u = (I_\alpha * F(u))F'(u) \qquad \text{in } \mathbb{R}^N, \tag{1}$$

where Δ is the standard Euclidean laplacian, $*$ indicates the convolution, $F \in C^1(\mathbb{R}, \mathbb{R})$ is a smooth nonlinearity and $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is, for $\alpha \in (0, N)$, the Riesz potential:

$$I_\alpha(x) := \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\pi^{\frac{N}{2}}2^\alpha}\frac{1}{|x|^{N-\alpha}}. \tag{2}$$

Problem (1) can be seen as a non-local counterpart of the very well-known scalar field equation

$$-\Delta u + u = G'(u) \qquad \text{in } \mathbb{R}^N, \tag{3}$$

which can be formally recovered from (1) by letting α go to 0 and setting $G = \frac{F^2}{2}$.
Problem (3) has been widely studied since many years. General existence results were provided in [4] when $N \geqslant 3$ and [3] (when $N = 2$) under mild hypotheses on G.
Anyway, the argument from both [4] and [3] does not seem to be suitable to attack problem (3): roughly speaking, the authors use a constrained minimization technique and then a dilation to get rid of the Lagrangian multiplier, which does not work in our case because of the scaling properties of the Riesz potential (2).

We study the problem (1) variationally: its solutions are critical points of the following energy functional on $H^1\left(\mathbb{R}^N\right)$:

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2 + |u|^2\right) - \frac{1}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F'(u). \tag{4}$$

In particular, we look for solutions at a mountain-pass level b defined by

$$b := \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I(\gamma(t)), \tag{5}$$

with

$$\Gamma := \left\{\gamma \in C\left([0,1], H^1\left(\mathbb{R}^N\right)\right); \gamma(0) = 0,\ I(\gamma(1)) < 0\right\}.$$

In particular, we by-pass the issue of Palais-Smale sequences by a *scaling trick* introduced in [5], which basically allows us to consider Palais-Smale sequences also asymptotically satisfying the Pohožaev identity

$$\mathcal{P}(u) := \frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2 + \frac{N}{2}|u|^2 - \frac{N+\alpha}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u) = 0, \tag{6}$$

for which convergence is easier to be proved.
We can show existence of solutions under general hypotheses, in the same spirit of [4, 3]. In the particular yet very important case of a power-type nonlinearity $F(u) = \frac{|u|^p}{p}$ such hypotheses are equivalent to $1 + \frac{\alpha}{N} < p < \frac{N+\alpha}{N-2}$, which in [7] is shown to be also a necessary condition. This shows that the hypotheses we make are somehow natural.
We also show that the mountain-pass type solution is also a ground state, namely an energy-minimizing solution: it satisfies

$$I(u) = c := \inf\left\{I(v) : v \in H^1\left(\mathbb{R}^N\right)\setminus\{0\} \text{ solves } (1)\right\}. \tag{7}$$

We first show the existence of mountain-pass solutions in Section 2 and then in Section 3 we prove that they are actually ground states. Such results were originally presented in [8] for the dimension $N \geqslant 3$ and in [2] for the case $N = 2$.

2. Existence of mountain-pass solutions

We show here existence of a solution for (1) under general hypotheses on F.
First of all, we want to exclude the trivial case of an identically vanishing F:

(F_0) There exists $s_0 \in \mathbb{R}$ such that $F(s_0) \neq 0$.

Then, we also need some growth assumptions which give a well-posed variational formulation, namely a energy functional I being well-defined on $H^1\left(\mathbb{R}^N\right)$. Such assumptions are different depending whether the dimension is two or it is greater, since the limiting-case embeddings in Sobolev spaces are different: in the higher-dimensional case, we impose a power-type growth whereas in $\mathbb{R}^2$ we require one of exponential type:

$(N \geqslant 3)$ (F_1) There exists $C > 0$ such that $|F'(s)| \leqslant C\left(|s|^{\frac{\alpha}{N}} + |s|^{\frac{\alpha+2}{N-2}}\right)$ for any $s > 0$

$(N = 2)$ (F_1') For any $\theta > 0$ there exists $C_\theta > 0$ such that $|F'(s)| \leqslant C_\theta \min\left\{1, |s|^{\frac{\alpha}{2}}\right\} e^{\theta|s|^2}$ for any $s > 0$.

It is not hard to see that (F_1), combined with Sobolev and Hardy-Littlewood-Sobolev inequality, implies the finiteness of the term $\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)$, hence the well-posedness and smoothness of the functional I defined by (4). In dimension two we need, in place of Sobolev's inequality, a special form of the Moser-Trudinger inequality on the whole plane, which was given in [1]:

(8) $\forall\beta\in(0,4\pi)\exists C_\beta>0$ such that $\int_{\mathbb{R}^2}|\nabla u|^2\leqslant 1 \Rightarrow \int_{\mathbb{R}^2}\min\left\{1,u^2\right\}e^{\beta u^2}\leqslant C_\beta\int_{\mathbb{R}^2}|u|^2$

The last hypotheses we need is a sort of *sub-criticality* with respect to the critical power in Hardy-Littlewood-Sobolev inequality. Again, we state the condition differently depending on the dimension, since in dimension 2 there is no critical Sobolev exponent:

$(N\geqslant 3)$ (F_2) $\lim_{s\to 0}\frac{F(s)}{|s|^{1+\frac{\alpha}{N}}}=\lim_{s\to+\infty}\frac{F(s)}{|s|^{\frac{N+\alpha}{N-2}}}=0$

$(N=2)$ (F_2') $\lim_{s\to 0}\frac{F(s)}{|s|^{1+\frac{\alpha}{N}}}=0$

Precisely, the result we present is the following:

THEOREM 1. *Assume F satisfies $(F_0),(F_1),(F_2)$ if $N\geqslant 3$ and $(F_0),(F_1'),(F_2')$ if $N=2$. Then, the problem (1) has a non-trivial solution $u\in H^1\left(\mathbb{R}^N\right)\setminus\{0\}$.*

We start by showing the existence of a Pohožaev-Palais-Smale sequence. We argue as in [5] to get the asymptotical Pohožaev identity.

LEMMA 1. *Assume F satisfies $(F_0),(F_1)$ (or, in case $N=2$, $(F_0),(F_1')$). Then, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $H^1\left(\mathbb{R}^N\right)$ such that:*

$$I(u_n)\underset{n\to+\infty}{\to}b \qquad I'(u_n)\underset{n\to+\infty}{\to}0 \text{ in } H^1\left(\mathbb{R}^N\right)' \qquad \mathcal{P}(u_n)\underset{n\to+\infty}{\to}0$$

Proof. We divide the proof in three steps: first we show that the mountain-pass level (5) is not degenerate and then we apply a variant of the mountain-pass principle.

Step 1: $b>0$ We suffice to show that $\Gamma\neq\varnothing$, namely that there exists some $u_0\in H^1\left(\mathbb{R}^N\right)$ with $I(u_0)<0$.
By (F_0), we can choose s_0 such that $F(s_0)\neq 0$, therefore if we take a smooth v_0 approximating $s_0\mathbf{1}_{B_1}$ we easily get $\int_{\mathbb{R}^N}(I_\alpha * F(v_0))F(v_0)>0$. If now we consider $v_t=v_0\left(\frac{\cdot}{t}\right)$, we get

$$(9)\qquad I(v_t)=\frac{t^{N-2}}{2}\int_{\mathbb{R}^N}|\nabla v_0|^2+\frac{t^N}{2}\int_{\mathbb{R}^N}|v_0|^2-\frac{t^{N+\alpha}}{2}\int_{\mathbb{R}^N}(I_\alpha * F(v_0))F(v_0),$$

which is negative for large t, so we can take $u_0=v_t$ with $t\gg 1$.

Step 2: $b<+\infty$ We need to show that for any $\gamma\in\Gamma$ there exists t_γ such that $I(\gamma(t_\gamma))\geqslant\varepsilon>0$.
If $\int_{\mathbb{R}^N}\left(|\nabla u|^2+|u|^2\right)\leqslant\delta\ll 1$, then by assumption (F_2) and H-L-S

and Sobolev's inequality we get

$$\begin{aligned}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u) &\leqslant C\left(\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^{\frac{N+\alpha}{N-2}}+\left(\int_{\mathbb{R}^N}|u|^2\right)^{1+\frac{\alpha}{N}}\right)\\ &\leqslant \frac{1}{4}\int_{\mathbb{R}^N}\left(|\nabla u|^2+|u|^2\right),\end{aligned}$$

which means $I(u) \geqslant \frac{1}{4}\int_{\mathbb{R}^N}\left(|\nabla u|^2+|u|^2\right)$, and the same can be proved similarly when $N=2$.
Now, for any fixed $\gamma \in \Gamma$ we can take t_γ such that $\int_{\mathbb{R}^2}\left(|\nabla\gamma(t_\gamma)|^2+|\gamma(t_\gamma)|^2\right)=\delta$ and we get $I(\gamma(t_\gamma)) \geqslant \frac{\delta}{4} =: \varepsilon$.

Step 3: Conclusion Consider the functional $\widetilde{I}:\mathbb{R}\times H^1\left(\mathbb{R}^N\right)\to\mathbb{R}$ defined by

$$\begin{aligned}\widetilde{I}(\sigma,v) := I\left(v\left(e^{-\sigma}\cdot\right)\right) &= \frac{e^{(N-2)\sigma}}{2}\int_{\mathbb{R}^N}|\nabla v|^2+\frac{e^{N\sigma}}{2}\int_{\mathbb{R}^N}|v|^2\\ &\quad -\frac{e^{(N+\alpha)\sigma}}{2}\int_{\mathbb{R}^N}(I_\alpha * F(v))F(v).\end{aligned}$$

By applying to $\widetilde{I}$ the standard min-max principle (see [9] for instance) we get a sequence $(\sigma_n,v_n)_{n\in\mathbb{N}}$ with $\widetilde{I}(\sigma_n,v_n)\underset{n\to+\infty}{\to} b$ and $\widetilde{I}(\sigma_n,v_n)'\underset{n\to+\infty}{\to} 0$, which is equivalent to what the Lemma required. □

To prove Theorem 1 we need to show the convergence of the Pohožaev-Palais-Smale sequence we just found. Here we need the sub-criticality assumption $(F_2),(F_2')$

LEMMA 2. *Assume F satisfies $(F_1),(F_2)$ (or, in case $N=2$, $(F_1'),(F_2')$) and $(u_n)_{n\in\mathbb{N}}$ satisfies*

$I(u_n)$ *is bounded* $\qquad I'(u_n)\underset{n\to+\infty}{\to} 0$ *in* $H^1\left(\mathbb{R}^N\right)'$ $\qquad \mathcal{P}(u_n)\underset{n\to+\infty}{\to} 0.$

Then, up to subsequences,

- *either* $u_n \underset{n\to+\infty}{\to} 0$ *strongly in* $H^1\left(\mathbb{R}^N\right)$
- *or* $u_n(\cdot - x_n)\underset{n\to+\infty}{\rightharpoonup} u$ *weakly for some* $(x_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}^N$ *and* $u\in H^1\left(\mathbb{R}^N\right)\setminus\{0\}$.

Proof. Assume the first alternative does not occur. Then, we show it weakly converges to some $u\not\equiv 0$.

Step 1: $(u_n)_{n\in\mathbb{N}}$ is bounded It follows by just writing

$$\frac{\alpha+2}{2(N+\alpha)}\int_{\mathbb{R}^N}|\nabla u_n|^2+\frac{\alpha}{2(N+\alpha)}\int_{\mathbb{R}^N}|u_n|^2 = I(u_n)-\frac{\mathcal{P}(u_n)}{N+\alpha}\underset{n\to+\infty}{\to} b$$

Step 2: $\sup_{x\in\mathbb{R}^N}\int_{B_1(x)}|u_n|^p \geqslant \frac{1}{C}$ By using the asymptotic Pohožaev identity it is not hard to see that $\inf_n \int_{\mathbb{R}^N}(I_\alpha * F(u_n))F(u_n) > 0$. Moreover, (F_2) implies, for any $\varepsilon > 0, p \in \left(2, \frac{2N}{N-2}\right)$,

$$|F(s)|^{\frac{2N}{N+\alpha}} \leqslant \varepsilon\left(|s|^2 + |s|^{\frac{2N}{N-2}}\right) + C_\varepsilon |s|^p,$$

therefore, by the following inequality from [6]

$$\int_{\mathbb{R}^N}|u_n|^p \leqslant C\left(\int_{\mathbb{R}^N}\left(|\nabla u_n|^2 + |u_n|^2\right)\right)\left(\sup_{x\in\mathbb{R}^N}\int_{B_1(x)}|u_n|^p\right)^{1-\frac{2}{p}},$$

we get

$$\begin{aligned}\left(\sup_{x\in\mathbb{R}^N}\int_{B_1(x)}|u_n|^p\right)^{1-\frac{2}{p}} &\geqslant \frac{1}{C}\frac{\int_{\mathbb{R}^N}|u_n|^p}{\int_{\mathbb{R}^N}\left(|\nabla u_n|^2+|u_n|^2\right)}\\ &\geqslant \frac{1}{C_\varepsilon}\left(\int_{\mathbb{R}^N}|F(u_n)|^{\frac{2N}{N+\alpha}} - \varepsilon\int_{\mathbb{R}^N}\left(|u|^2 + |u|^{\frac{2N}{N-2}}\right)\right)\\ &\geqslant \frac{1}{C'_\varepsilon}\left(\left(\int_{\mathbb{R}^N}(I_\alpha * F(u_n))F(u_n)\right)^{\frac{N}{N+\alpha}} - C\varepsilon\int_{\mathbb{R}^N}\left(|\nabla u_n|^2 + |u_n|^2\right)\right) \geqslant \frac{1}{C}.\end{aligned}$$

and a similar estimate holds true in the case $N = 2$.

Step 3: $u_n(\cdot - x_n)$ converges We choose x_n such that $\liminf_{n\to+\infty}\int_{B_1}|u_n(\cdot - x_n)|^p > 0$, its weak limit (which exists because Step 1 ensures boundedness) must be some $u \not\equiv 0$.
By Sobolev embeddings, one can show that $(I_\alpha * F(u_n))F'(u_n) \underset{n\to+\infty}{\rightharpoonup} (I_\alpha * F(u))F'(u)$ in $L^p_{\mathrm{loc}}\left(\mathbb{R}^N\right)$.
This easily yields that u solves (1) □

Proof of Theorem 1. By Lemma 1, I admits a Pohožaev-Palais-Smale sequence $(u_n)_{n\in\mathbb{N}}$ at the energy level b. We apply Lemma 2 to the latter sequence: if the first alternative occurred, then we would have $I(u_n) \underset{n\to+\infty}{\to} I(0) = 0$, contradicting Lemma 2. Therefore, the second alternative must occur and in particular $u \not\equiv 0$ solves (1). □

We conclude this section by showing that Theorem 1 is actually sharp in the case of a power nonlinearity $F(u) = \frac{|u|^p}{p}$; in other words, we give a non-existence result for all the values p not matching the assumptions of Theorem 1. To show non-existence, we use a Pohožaev identity, which is a classical property of solutions of (1).

PROPOSITION 1. *Any solution u of* (1) *satisfies the Pohožaev identity* (6).

THEOREM 2. *If $F(u) = \frac{|u|^p}{p}$ then problem* (1) *admits a non-trivial solution if and only if $p \in \left(1 + \frac{\alpha}{N}, \frac{N+\alpha}{N-2}\right)$, with the latter condition to be read as $p > 1 + \frac{\alpha}{2}$ if $N = 2$.*

Proof. If $p \in \left(1+\frac{\alpha}{N},\frac{N+\alpha}{N-2}\right)$ then one can easily see that $F(u)=\frac{|u|^p}{p}$ satisfies $(F_0),(F_1),(F_2)$, hence the existence of non-trivial solutions follows from Theorem 1.
Conversely, assume p is outside that range and u solves (1), By testing both sides of against u we get

$$\int_{\mathbb{R}^N}\left(|\nabla u|^2+|u|^2\right)=\int_{\mathbb{R}^N}(I_\alpha*|u|^p)|u|^p.$$

Moreover, u satisfies the Pohožaev identity (6), which has the form

$$\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2+\frac{N}{2}\int_{\mathbb{R}^N}|u|^2-\frac{N+\alpha}{2p}\int_{\mathbb{R}^N}(I_\alpha*|u|^p)|u|^p=0.$$

A linear combination of the two formulas gives

$$\left(\frac{N-2}{2}-\frac{N+\alpha}{2p}\right)\int_{\mathbb{R}^N}|\nabla u|^2+\left(\frac{N}{2}-\frac{N+\alpha}{2p}\right)|u|^2,$$

which implies $u\equiv 0$ if $p\leqslant 1+\frac{\alpha}{N}$ or $p\geqslant\frac{N+\alpha}{N-2}$. □

3. From solutions to ground states

In the last part of this paper we show that the mountain pass solutions given by Theorem 1 are actually energy-minimizing, in the sense of (7).

THEOREM 3. *The mountain-pass solution found in Theorem 1 is actually a ground state, namely its energy level is given by* (7).

The previous Theorem can be easily proved by constructing, for any solution v of (1), a path $\gamma_v\in\Gamma$ which attains its maximum energy on v.

LEMMA 3. *Assume F satisfies (F_1) and $v\in H^1\left(\mathbb{R}^N\right)\setminus\{0\}$ solves* (1). *Then, there exists a path $\gamma_v\in\Gamma$ such that:*

$$\gamma(0)=0 \qquad \gamma_v\left(\frac{1}{2}\right)=v \qquad I(\gamma_v(t))<I(v)\ \textit{for any}\ t\neq\frac{1}{2} \qquad I(\gamma_v(t))<0$$

Proof. Fix a non-trivial solution v of (1) and consider the path $\overline{\gamma}_v =: [0,+\infty)\to H^1\left(\mathbb{R}^N\right)$ defined by $\overline{\gamma}_v(t)=\begin{cases} v\left(\frac{\cdot}{t}\right) & \text{if } t>0\\ 0 & \text{if } t=0\end{cases}$.
Along the path, the energy is given by (9), which is negative for $t\gg 1$. Moreover, due to the Pohožaev identity (6) we can also write

$$I(\overline{\gamma}_v(t))=\left(\frac{t^{N-2}}{2}-\frac{N-2}{2(N+\alpha)}t^{N+\alpha}\right)\int_{\mathbb{R}^N}|\nabla u|^2+\left(\frac{t^N}{2}-\frac{N}{2(N+\alpha)}t^{N+\alpha}\right)\int_{\mathbb{R}^N}|u|^2,$$

which has its maximum in $t=1$. Therefore, up to a rescaling of t, this path has all the required properties.

Anyway, being

$$\int_{\mathbb{R}^N}\left(|\nabla\overline{\gamma}_v(t)|^2+|\overline{\gamma}_v(t)|^2\right)=t^{N-2}\int_{\mathbb{R}^N}|\nabla v|^2+t^N\int_{\mathbb{R}^N}|v|^2,$$

$\overline{\gamma}_v$ is continuous at $t=0$ only if $N\geqslant 3$, so in the case $N=2$ we need a modification for t close to 0.

If $N=2$ we take $\overline{\gamma}_v(t)=\begin{cases} v\left(\frac{\cdot}{t}\right) & \text{if } t>t_0 \\ \frac{t}{t_0}v\left(\frac{\cdot}{t_0}\right) & \text{if } t\leqslant t_0\end{cases}$ for some suitable $t_0\ll 1$. We only need to verify that $I(\overline{\gamma}_v(t))\leqslant I(\overline{\gamma}_v(1))$ for $t\leqslant t_0$.

Using the assumption (F_1') and Moser-Trudinger's (8) and Hardy-Littlewood-Sobolev inequalities we get

$$\int_{\mathbb{R}^2}(I_\alpha * F(\overline{\gamma}_v(t)))F(\overline{\gamma}_v(t))\leqslant C\left(\frac{\int_{\mathbb{R}^2}|\overline{\gamma}_v(t)|^2}{\int_{\mathbb{R}^2}|\nabla\overline{\gamma}_v(t)|^2}\right)^{1+\frac{\alpha}{2}}=Ct_0^{2+\alpha}\left(\int_{\mathbb{R}^2}|v|^2\right)^{1+\frac{\alpha}{2}},$$

therefore using again Pohožaev identity we get, for t_0 small enough,

$$\begin{aligned} I(\overline{\gamma}_v(t)) &\leqslant \frac{1}{2}\int_{\mathbb{R}^2}|\nabla v|^2+\frac{t_0^2}{2}\int_{\mathbb{R}^2}|v|^2+Ct_0^{2+\alpha}\left(\int_{\mathbb{R}^2}|v|^2\right)^{1+\frac{\alpha}{2}} \\ &= I(v)+\left(\frac{t_0^2}{2}-\frac{\alpha}{2(2+\alpha)}\right)\int_{\mathbb{R}^2}|v|^2+Ct_0^{2+\alpha}\left(\int_{\mathbb{R}^2}|v|^2\right)^{1+\frac{\alpha}{2}}<I(v)\end{aligned}$$

and the proof is complete. □

Proof of Theorem 3. Let u be the mountain-pass solution found in Theorem 1. By the lower-semicontinuity of the norm we find $I(u)\leqslant b$, whereas the definition (7) of ground state yields $I(u)\geqslant c$.

Now, take another solution $v\in H^1\left(\mathbb{R}^N\right)\setminus\{0\}$ and apply Lemma 3: we get

$$I(v)=\sup_{t\in[0,1]}I(\gamma_v(t))\geqslant\inf_{\gamma\in\Gamma}\sup_{t\in[0,1]}I(\gamma(t))=b.$$

Being v arbitrary, we get $c\geqslant b$, hence $c\leqslant I(u)\leqslant b\leqslant c$, therefore $I(u)=b=c$. □

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PROPERTIES OF GROUND STATES OF NONLINEAR SCHRÖDINGER EQUATIONS UNDER A WEAK CONSTANT MAGNETIC FIELD

Abstract. We study the qualitative properties of ground states of the time-independent magnetic semilinear Schrödinger equation

$$-(\nabla + iA)^2 u + u = |u|^{p-2}u \qquad \text{in } \mathbb{R}^N$$

where the magnetic potential A induces a constant magnetic field. When the latter magnetic field is small enough, we show that the ground state solution is unique up to magnetic translations and rotations in the complex phase space and that ground state solutions share the rotational invariance of the magnetic field. This is based on an article in collaboration with D. Bonheure and J. VanSchaftingen [3].

1. Introduction

We are interested in the *time-independent magnetic semilinear Schrödinger equation*

$$-\Delta_A u + u = |u|^{p-2}u \qquad \text{in } \mathbb{R}^N, N \geqslant 2, \tag{1}$$

with a *linear magnetic potential* $A \in \mathrm{Lin}(\mathbb{R}^N, \bigwedge^1 \mathbb{R}^N)$ which allows to define the *magnetic Laplacian*

$$-\Delta_A := -\Delta - 2iA\cdot\nabla - i\mathrm{div}A + |A|^2,$$

and a subcritical power p in the nonlinearity, i.e. $2 < p < \frac{2N}{N-2}$.

In this work we are interested in the qualitative properties of the *ground states* of (1), which can be obtained and characterized as minimizers of the variational problem

$$\inf\big\{ I_A(u) \ : \ u \in H^1_A(\mathbb{R}^N,\mathbb{C})\backslash\{0\} \ \text{and} \ I'_A(u) = 0\big\}.$$

Here the *magnetic Sobolev space* $H^1_A(\mathbb{R}^N,\mathbb{C})$ is a *real* Hilbert space given by

$$H^1_A(\mathbb{R}^N,\mathbb{C}) := \big\{u \in L^2(\mathbb{R}^N,\mathbb{C}) \ : \ D_A u \in L^2(\mathbb{R}^N)\big\}$$

endowed with the norm

$$\|u\|^2_{H^1_A(\mathbb{R}^N,\mathbb{C})} = \int_{\mathbb{R}^N} |D_A u|^2 + |u|^2,$$

deriving from the *real scalar product*

$$(u|v)_{H^1_A(\mathbb{R}^N,\mathbb{C})} = \int_{\mathbb{R}^N} (D_A u|D_A v) + (u|v),$$

*

where $(\cdot|\cdot)$ denotes the canonical *real scalar product* of vectors in $\mathbb{C}$ and in $\mathrm{Lin}(\mathbb{R}^N, \mathbb{C})$. The *magnetic covariant derivative* is defined by

$$D_A u = Du + iAu,$$

and the functional $I_A : H^1_A(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}$ is defined for each function $u \in H^1_A(\mathbb{R}^N, \mathbb{C})$ by

$$I_A(u) := \frac{1}{2}\int_{\mathbb{R}^N} \big(|D_A u|^2 + |u|^2\big) - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p.$$

Critical points of the functional I_A correspond to weak solutions of the equation (1), that is solutions $u \in H^1_A(\mathbb{R}^N, \mathbb{C})$ such that for every $v \in H^1_A(\mathbb{R}^N, \mathbb{C})$,

$$\int_{\mathbb{R}^N} (D_A u | D_A v) + (u|v) = \int_{\mathbb{R}^N} |u|^{p-2}(u|v).$$

The aim of the present work is to understand the properties of the ground states of equation (1) and their dependence on the magnetic field $B = dA \in \bigwedge^2 \mathbb{R}^N$. In order to alleviate the statement of the results, we simply the problem by gauge fixing.

The *gauge invariance* means that if for some function $\psi \in C^1(\mathbb{R}^N)$, we set

$$\tilde{A} = A + d\psi \quad \text{and} \quad \tilde{u} = \mathrm{e}^{-i\psi} u, \tag{2}$$

then

$$D_{\tilde{A}} \tilde{u} = \mathrm{e}^{-i\psi} D_A u.$$

In particular, if $u \in H^1_A(\mathbb{R}^N, \mathbb{C})$, then $I_{\tilde{A}}(\tilde{u}) = I_A(u)$ and therefore solutions of equation (1) with A and $\tilde{A}$ can be related to each other using the relation (2). Since $d\tilde{A} = dA$ and $|\tilde{u}| = |u|$, the gauge invariance means that only the *magnetic field* dA plays a role in the physical behavior of the solutions of (1). When, as in the present work, the magnetic field dA is constant, one of the simplest gauge choice is to assume that A is linear and skew-symmetric. If A is linear, such a choice can be made by setting $\psi(x) = -A(x)[x]/2$ in (2). This choice is equivalent to the choice of the Coulomb gauge with a transversal boundary condition at infinity i.e.,

$$\begin{cases} \operatorname{div} A = 0 & \text{in } \mathbb{R}^N, \\ \dfrac{A(x)[x]}{|x|^2} \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{3}$$

In particular, if $B \in \bigwedge^2 \mathbb{R}^N$ is a constant skew-symmetric form, there exists a unique $A \in \mathrm{Lin}(\mathbb{R}^N, \bigwedge^1 \mathbb{R}^N)$ satisfying $dA = B$ and (3). As $I_{\tilde{A}}(\tilde{u}) = I_A(u)$ when (2) holds, the precise choice (3) allows to define the *ground-energy function* $\mathcal{E} : \bigwedge^2 \mathbb{R}^N \to \mathbb{R}$ by

$$\mathcal{E}(B) = \mathcal{E}(dA) := \inf\big\{ I_A(v) \;:\; v \in H^1_A(\mathbb{R}^N, \mathbb{C}) \setminus \{0\} \text{ and } I'_A(v) = 0 \big\}.$$

The function $u \in H^1_A(\mathbb{R}^N, \mathbb{C})$ is a *ground state* of (1) if u is a weak solution of (1) such that

$$I_A(u) = \mathcal{E}(dA).$$

Because of the presence of the magnetic potential, equation (1) is not invariant under translations in $\mathbb{R}^N$. However, it is still invariant under *magnetic translations with respect to the connection* D_A. For $a \in \mathbb{R}^N$ and $u \in H^1_A(\mathbb{R}^N, \mathbb{C})$, that magnetic translation is defined by

$$\tau^A_a u(x) = \mathrm{e}^{-iA(a)[x]} u(x-a).$$

This definition depends on the gauge fixing made above. This magnetic translation commutes with the covariant derivative D_A, i.e., $D_A \circ \tau^A_a = \tau^A_a \circ D_A$. We observe that in general, magnetic translations do not commute. Indeed, one has

$$\tau^A_b \circ \tau^A_a = \mathrm{e}^{iA(a)[b]} \tau^A_{a+b}. \tag{4}$$

Our first result establishes that the ground state of (1) is unique up to magnetic translations and multiplication by a complex phase for dA sufficiently small.

THEOREM 1. *For every $N \geqslant 2$ and $p \in (2, \frac{2N}{N-2})$, there exists $\varepsilon > 0$ such that if $A \in \mathrm{Lin}(\mathbb{R}^N, \bigwedge^1 \mathbb{R}^N)$ satisfies $|dA| \leqslant \varepsilon$, if u and v are solutions of (1) satisfying $I_A(u) \leqslant \mathcal{E}(0) + \varepsilon$ and if $I_A(v) \leqslant \mathcal{E}(0) + \varepsilon$, then $u = \mathrm{e}^{i\theta} \tau^A_a v$ for some $a \in \mathbb{R}^N$ and $\theta \in \mathbb{R}$.*

The main idea of the proof of Theorem 1 is to take advantage of the well-known uniqueness and non-degeneracy of the solutions of (1) under a vanishing magnetic field $A = 0$, see e.g. [5, 7], and to extend the uniqueness by an implicit function argument. The main difficulty in this proof consists in the fact that the natural function space $H^1_A(\mathbb{R}^N, \mathbb{C})$ for the functional I_A *depends* on the magnetic field: the norm and the elements of the space differ in general for different magnetic fields.

A direct consequence of Theorem 1 is that the solutions inherit the symmetries of the magnetic potential in a sense explained below.

THEOREM 2. *Let $N \geqslant 2$, $p \in (2, \frac{2N}{N-2})$ and $\varepsilon > 0$ be as in Theorem 1. If $A \in \mathrm{Lin}(\mathbb{R}^N, \bigwedge^1 \mathbb{R}^N)$ is skew-symmetric and satisfies $|dA| \leqslant \varepsilon$ and if u is a solution of (1) such that $I_A(u) \leqslant \mathcal{E}(0) + \varepsilon$, then there exists $a \in \mathbb{R}^N$ such that for every linear isometry R of $\mathbb{R}^N$ satisfying $|A \circ R|^2 = |A|^2$, one has*

$$u(R(x+a) - a) = \mathrm{e}^{iA(a)[R(x+a)-(x+a)]} u(x).$$

Moreover, the function u is nondecreasing along any ray starting from the point a.

Since equation (1) is invariant under magnetic translations Theorem 2 implies the existence of a unique ground state u such that its conclusion holds with $a = 0$, that is, for every linear isometry R of $\mathbb{R}^N$ such that $|A \circ R|^2 = |A|^2$, one has $u \circ R = u$. Alternatively, Theorem 2 states that a ground state can be translated in such a way that it only depends monotonically on the norms of the projections on the eigenspaces of the quadratic form $|A|^2$. Also, because of the antisymmetric structure of A, the group of isometries such that $|A \circ R|^2 = |A|^2$ can be written, up to an isometry of the Euclidean space, as a product of orthogonal groups $O(2n_1) \times O(2n_2) \times \cdots \times O(2n_k) \times$

$O(N-2n_1-2n_2-\cdots-2n_k)$, with $n_1,n_2,\dots,n_k\in\mathbb{N}$; when $N=3$ and $A\neq 0$, it is always of the form $O(2)\times O(1)$, corresponding to a decomposition in the transversal and longitudinal directions with respect to the magnetic field.

2. Preliminaries

We first introduce a preliminary lemma about the convergence in L^p-spaces.

LEMMA 1 (Continuous Sobolev embedding across magnetic Sobolev spaces). *Assume that, for every $n\in\mathbb{N}$, $(A_n)_{n\in\mathbb{N}}$ is a sequence in $L^2_{\mathrm{loc}}(\mathbb{R}^N,\bigwedge^1\mathbb{R}^N)$ and $u_n\in H^1_{A_n}(\mathbb{R}^N,\mathbb{C})$. If $A_n\to A$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, and $u_n\to u$ strongly in $L^2(\mathbb{R}^N)$, $D_{A_n}u_n\to D_Au$ strongly in $L^2(\mathbb{R}^N)$, then $u_n\to u$ strongly in $L^p(\mathbb{R}^N)$ for $2\leqslant p\leqslant\frac{2N}{N-2}$, as $n\to\infty$.*

2.1. Ground states

Here we recall the known properties of the ground states of the nonlinear Schrödinger equation (1), with or without magnetic potential. The first lemma comes from [4, Theorem 3.1].

LEMMA 2 (Existence and characterization of ground states). *For every magnetic potential $A\in\mathrm{Lin}(\mathbb{R}^N,\bigwedge^1\mathbb{R}^N)$, there exists $u\in H^1_A(\mathbb{R}^N,\mathbb{C})\setminus\{0\}$ such that*

$$I_A(u)=\mathcal{E}(dA)\ \text{and}\ \ I_A'(u)=0.$$

Moreover,

$$\mathcal{E}(dA)=\left(\tfrac{1}{2}-\tfrac{1}{p}\right)\inf_{v\in H^1_A(\mathbb{R}^N,\mathbb{C})\setminus\{0\}}\mathcal{Q}_A(v)^{\frac{p}{p-2}},$$

where the functional $\mathcal{Q}_A:H^1_A(\mathbb{R}^N,\mathbb{C})\setminus\{0\}\to\mathbb{R}$ is defined by

$$\mathcal{Q}_A(v):=\frac{\displaystyle\int_{\mathbb{R}^N}|D_Av|^2+|v|^2}{\left(\displaystyle\int_{\mathbb{R}^N}|v|^p\right)^{\frac{2}{p}}}.$$

We recall some well established results for the problem without a magnetic field

$$-\Delta u+u=|u|^{p-2}u,\qquad\text{in }\mathbb{R}^N. \tag{5}$$

This problem is a natural limit for (1) when $A\to 0$. The next result states that the ground state of (5) in $H^1(\mathbb{R}^N,\mathbb{C})$ is unique up to rotations in $\mathbb{C}$ and translations in $\mathbb{R}^N$.

PROPOSITION 1 (Uniqueness up to rotations in $\mathbb{C}$ and translations in $\mathbb{R}^N$). *If $u,v\in H^1(\mathbb{R}^N,\mathbb{C})$ satisfy $I_0(u)=I_0(v)=\mathcal{E}(0)$ and $I_0'(u)=I_0'(v)=0$, then there exist $\theta\in\mathbb{R}$ and $a\in\mathbb{R}^N$ such that $v=\mathrm{e}^{i\theta}\tau^0_a u$.*

It clearly follows that there exists a unique real, positive and radially symmetric ground state of (5), that we denote by u_0. The next proposition states the non-degeneracy property due to M. I. Weinstein [7] and Y.-G. Oh [6].

PROPOSITION 2 (Non-degeneracy of the ground states in absence of magnetic field). *Assume that* $u \in H^1(\mathbb{R}^N, \mathbb{C})$ *satisfies* $I_0(u) = \mathcal{E}(0)$ *and* $I_0'(u) = 0$. *If* $w \in H^1(\mathbb{R}^N, \mathbb{C})$ *satisfies*

$$-\Delta w + w = |u|^{p-2} w + (p-2)|u|^{p-4}(u|w)u, \tag{6}$$

then there exist $y \in \mathbb{R}^N$ *and* $\lambda \in \mathbb{R}$ *such that*

$$w = Du[y] + \lambda i u. \tag{7}$$

In particular if u is a solution of the equation (5) and if w is a solution of its linearised problem (6) given by (7), then u and w are orthogonal in the space $H^1(\mathbb{R}^N, \mathbb{C})$, i.e.

$$\int_{\mathbb{R}^N} (Du|Dw) + (u|w) = 0. \tag{8}$$

For every ground state $u \in H^1(\mathbb{R}^N, \mathbb{C})$ of (1), we can rewrite equation (6) as an eigenvalue equation in the following way

$$L_u w = \lambda w, \qquad w \in H^1(\mathbb{R}^N, \mathbb{C}),$$

where the operator $L_u : H^1(\mathbb{R}^N, \mathbb{C}) \to H^1(\mathbb{R}^N, \mathbb{C})$ is given by

$$L_u w := \big(-\Delta + 1\big)^{-1}\big(|u|^{p-2} w + (p-2)|u|^{p-4}(u \otimes u)[w]\big),$$

with

$$(u \otimes u)[w] := (u|w)u. \tag{9}$$

It is standard that the operator L_u is compact. Indeed, the ground states u of (5) decays as $|x|^{-(N-1)/2}\exp(-|x|)$, see e.g. [2, p332], so that they are in $L^q(\mathbb{R}^N)$ for every $q \geqslant 1$. For completeness Lemma 3 below gives a more general result including the one above. It is also standard, see e.g. [1, Remark 4.2], to check directly that the ground state u is the first eigenfunction of eigenvalue $\lambda_1(L_u) = (p-1) > 1$, while the functions w given in (7) are the following eigenfunctions corresponding to the eigenvalues $\lambda_i(L_u) = 1$, $i = 2, \dots, N+2$. Finally, $\lambda_i(L_u) < 1$ for $i > N+2$.

PROPOSITION 3. *The ground energy function* $\mathcal{E} : \bigwedge^2 \mathbb{R}^N \to \mathbb{R}$ *is continuous. Moreover, if* $(A_n)_{n\in\mathbb{N}}$ *is a sequence in* $L^2_{\mathrm{loc}}(\mathbb{R}^N, \bigwedge^1 \mathbb{R}^N)$ *such that* $A_n \to A$ *in* $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ *as* $n \to \infty$ *and if the sequence* $(u_n)_{n\in\mathbb{N}}$ *in* $H^1_{A_n}(\mathbb{R}^N, \mathbb{C})$ *satisfies* $I'_{A_n}(u_n) = 0$ *and* $I_{A_n}(u_n) = \mathcal{E}(dA_n)$, *then there exist* $u \in H^1_A(\mathbb{R}^N, \mathbb{C})$ *with* $I_A(u) = \mathcal{E}(dA)$ *and* $I'_A(u) = 0$, *a sequence* $(a_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}^N$ *and a subsequence such that* $\tau^{A_{n_\ell}}_{a_{n_\ell}} u_{n_\ell} \to u$ *and* $D_{A_{n_\ell}}(\tau^{A_{n_\ell}}_{a_{n_\ell}} u_{n_\ell}) \to D_A u$ *strongly in* $L^2(\mathbb{R}^N)$ *as* $\ell \to \infty$.

We note that the convergence $A_n \to A$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ is equivalent to the convergence $dA_n \to dA$ in the finite-dimensional space $\bigwedge^2 \mathbb{R}^N$.

3. Uniqueness up to magnetic translations and rotations in $\mathbb{C}$ of the ground states

Our main tool to prove Theorem 1 is to prove the stability of the spectrum of a magnetic operator under perturbations of the potentials. We first study the spectrum of the sequence of linear operators

$$L_n : H^1_{A_n}(\mathbb{R}^N,\mathbb{C}) \to H^1_{A_n}(\mathbb{R}^N,\mathbb{C}) : v \mapsto L_n v := (-\Delta_{A_n}+1)^{-1} W_n[v],$$

where

(H_1) $(A_n)_{n\in\mathbb{N}}$ is a sequence in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$,

(H_2) $(W_n)_{n\in\mathbb{N}}$ is a sequence in $L^q(\mathbb{R}^N,\mathrm{Lin}(\mathbb{C},\mathbb{C}))$ with $q \geqslant \frac{N}{2}$ and $q > 1$,

(H_3) W_n is self-adjoint and $W_n \geqslant 0$ on $\mathbb{R}^N$, that is $(z|W_n[z]) \geqslant 0$ for every $z \in \mathbb{C}$.

LEMMA 3. *Under (H_1)–(H_3) the operator L_n is self-adjoint and compact.*

Lemma 3 and the positivity of W_n imply that L_n has a nonincreasing sequence of positive eigenvalues converging to 0 and by Fischer's min-max principle one has

$$\lambda_k(L_n) = \sup_{\substack{E\subset H^1_{A_n}(\mathbb{R}^N,\mathbb{C})\\ \dim E = k}} \inf_{v\in E} \frac{\displaystyle\int_{\mathbb{R}^N} (v|W_n[v])}{\displaystyle\int_{\mathbb{R}^N} |D_{A_n} v|^2 + |v|^2}.$$

PROPOSITION 4. *Assume that $(H_1)-(H_3)$ hold. If W_n converges strongly to W in $L^q(\mathbb{R}^N)$ and A_n converges strongly to A in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ as $n\to\infty$, then*

$$\lambda_k(L_n) \to \lambda_k(L),$$

where $\lambda_k(L_n)$, $\lambda_k(L)$ are respectively the k-th eigenvalues of L_n, L, and $L : H^1_A(\mathbb{R}^N,\mathbb{C}) \to H^1_A(\mathbb{R}^N,\mathbb{C})$ is defined as

$$Lv = (-\Delta_A + 1)^{-1} W[v].$$

Moreover, if $u_n \in H^1_{A_n}(\mathbb{R}^N,\mathbb{C})$ is an eigenfunction of L_n satisfying

$$L_n u_n = \lambda_k(L_n) u_n, \quad \text{and} \quad \int_{\mathbb{R}^N} |D_{A_n} u_n|^2 + |u_n|^2 = 1,$$

then there exist $u \in H^1_A(\mathbb{R}^N,\mathbb{C})$ and a subsequence $(n_\ell)_{\ell\in\mathbb{N}}$ such that $u_{n_\ell} \to u$ and $D_{A_{n_\ell}} u_{n_\ell} \to D_A u$ strongly in $L^2(\mathbb{R}^N)$.

With those ingredients we can now prove Theorem 1.

Proof of Theorem 1. We first assume that $\mathrm{d}A_n \to 0$ as $n \to +\infty$, that is $A_n \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ as $n \to +\infty$ since A_n is skew-symmetric, and that u_n and v_n are ground states solutions of (1) with A_n. Our aim is to show that there exist $\theta_n \in \mathbb{R}$ and $a_n \in \mathbb{R}^N$ such that $u_n = \mathrm{e}^{i\theta_n} \tau^{A_n}_{a_n} v_n$ for n large enough.

Let U be a solution of the limit problem (5). By Proposition 1, U is unique up to rotations in $\mathbb{C}$ and translations in $\mathbb{R}^N$.

CLAIM 1. There exist sequences $(\tilde{\theta}_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ and $(\tilde{a}_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ such that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|D_{A_n}(e^{i\tilde{\theta}_n}\tau_{\tilde{a}_n}^{A_n}u_n)-DU|^2+|e^{i\tilde{\theta}_n}\tau_{\tilde{a}_n}^{A_n}u_n-U|^2=0.$$

Proof of the claim. By Proposition 3, there exist a sequence $(b_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$, a subsequence $(n_\ell)_{\ell\in\mathbb{N}}$ in $\mathbb{N}$ and a function $V\in H^1(\mathbb{R}^N,\mathbb{C})$ such that $\tau_{b_{n_\ell}}^{A_{n_\ell}}u_{n_\ell}\to V$ and $D_{A_{n_\ell}}(\tau_{b_{n_\ell}}^{A_{n_\ell}}u_{n_\ell})\to DV$ strongly in $L^2(\mathbb{R}^N)$. Because of the uniqueness up to translations and rotations in $\mathbb{C}$ of the solution of (5), there exist $b\in\mathbb{R}^N$ and $\omega\in\mathbb{R}$ such that $V=e^{i\omega}\tau_b^0U$. We can therefore write that

$$\int_{\mathbb{R}^N}|D_{A_{n_\ell}}(\tau_{b_{n_\ell}}^{A_{n_\ell}}u_{n_\ell})-DV|^2=\int_{\mathbb{R}^N}|e^{-i\omega}\tau_{-b}^{A_{n_\ell}}D_{A_{n_\ell}}(\tau_{b_{n_\ell}}^{A_{n_\ell}}u_{n_\ell})-\tau_{-b}^{A_{n_\ell}}\tau_b^0DU|^2=$$
$$\int_{\mathbb{R}^N}|e^{-i\omega}e^{-iA_{n_\ell}(b_{n_\ell})[b]}D_{A_{n_\ell}}(\tau_{b_{n_\ell}-b}^{A_{n_\ell}}u_{n_\ell})-DU+DU-\tau_{-b}^{A_{n_\ell}}\tau_b^0U|^2\to 0,$$

as $\ell\to+\infty$. Here, we used the commutation between the translation and the connexion and (4). Moreover, by using Lebesgue dominated convergence, we have that

$$\int_{\mathbb{R}^N}|DU-\tau_{-b}^{A_{n_\ell}}\tau_b^0DU|^2\to 0,\qquad\text{as }\ell\to+\infty.$$

By the triangle inequality, we infer that

$$\int_{\mathbb{R}^N}|e^{-i\omega}e^{-iA_{n_\ell}(b_{n_\ell})[b]}D_{A_{n_\ell}}(\tau_{b_{n_\ell}-b}^{A_{n_\ell}}u_{n_\ell})-DU|^2\to 0,\qquad\text{as }\ell\to+\infty,$$

and proceeding exactly in the same way, we obtain

$$\int_{\mathbb{R}^N}|e^{-i\omega}e^{-iA_{n_\ell}(b_{n_\ell})[b]}\tau_{b_{n_\ell}-b}^{A_{n_\ell}}u_{n_\ell}-U|^2\to 0,\qquad\text{as }\ell\to+\infty.$$

Setting $\tilde{\theta}_{n_\ell}=-\omega-A_{n_\ell}(b_{n_\ell})[b]$ and $\tilde{a}_{n_\ell}=b_{n_\ell}-b$, the conclusion of the claim follows for this subsequence. The claim is true for the whole sequence. Indeed, if not we would find a subsequence n_ℓ for which the Claim does not hold, leading to a contradiction. $\diamond$

CLAIM 2. There exist sequences $(\theta_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ and $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ such that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|D_{A_n}(e^{i\theta_n}\tau_{a_n}^{A_n}u_n)-DU|^2+|e^{i\theta_n}\tau_{a_n}^{A_n}u_n-U|^2=0.$$

Moreover, when $n\in\mathbb{N}$ is large enough, for every $w\in\mathbb{R}^N$, we have the following orthogonality relations

$$\int_{\mathbb{R}^N}\big(D_{A_n}(e^{i\theta_n}\tau_{a_n}^{A_n}u_n)\big|D(DU[w])\big)+(e^{i\theta_n}\tau_{a_n}^{A_n}u_n|DU[w])=0,$$
$$\int_{\mathbb{R}^N}\big(D_{A_n}(e^{i\theta_n}\tau_{a_n}^{A_n}u_n)\big|DiU\big)+(e^{i\theta_n}\tau_{a_n}^{A_n}u_n|iU)=0.$$

Proof of the claim. We already proved Claim 1 with $\tilde{\theta}_n$ and $\tilde{a}_n$. Let us first prove the two orthogonality relations. For this, we define the map $\Phi_n \in C(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$ for each $(x,\tau) \in \mathbb{R}^{N+1}$ and $(w,s) \in \mathbb{R}^{N+1}$ by the following scalar product

$$\big((w,s)\big|\Phi_n(x,\tau)\big) = \int_{\mathbb{R}^N} \big(D_{A_n}(\mathrm{e}^{i(\tilde{\theta}_n+\tau)}\tau_x^{A_n}\tau_{\tilde{a}_n}^{A_n}u_n)\big|s(DiU)\big) + \big(\mathrm{e}^{i(\tilde{\theta}_n+\tau)}\tau_x^{A_n}\tau_{\tilde{a}_n}^{A_n}u_n\big|siU\big)$$
$$+ \int_{\mathbb{R}^N} \big(D_{A_n}(\mathrm{e}^{i(\tilde{\theta}_n+\tau)}\tau_x^{A_n}\tau_{\tilde{a}_n}^{A_n}u_n)\big|D(DU[w])\big) + \big(\mathrm{e}^{i(\tilde{\theta}_n+\tau)}\tau_x^{A_n}\tau_{\tilde{a}_n}^{A_n}u_n\big|DU[w]\big).$$

Since $D_{A_n} \circ \tau_x^{A_n}\tau_{\tilde{a}_n}^{A_n} = \tau_x^{A_n}\tau_{\tilde{a}_n}^{A_n} \circ D_{A_n}$, and thanks to the convergence proved in Claim 1, the sequence $(\Phi_n)_{n\in\mathbb{N}}$ converges to Φ uniformly over compact subsets, where the function $\Phi \in C(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$ is defined for every $(x,\tau) \in \mathbb{R}^{N+1}$ and $(w,s) \in \mathbb{R}^{N+1}$ by

$$\big((w,s)\big|\Phi(x,\tau)\big) = \int_{\mathbb{R}^N} \big(D(\mathrm{e}^{i\tau}\tau_x^0 U)\big|s(DiU)\big) + \big(\mathrm{e}^{i\tau}\tau_x^0 U\big|siU\big)$$
$$+ \int_{\mathbb{R}^N} \big(D(\mathrm{e}^{i\tau}\tau_x^0 U)\big|D(DU[w])\big) + \big(\mathrm{e}^{i\tau}\tau_x^0 U\big|DU[w]\big).$$

We first remark that $\Phi(0,0) = 0$. This is due to the fact that $DU[w] + siU$ belongs to the tangent space of U, see (8). Next, observe now that

$$\big((w,s)\big|D\Phi(0,0)[z,r]\big) = \int_{\mathbb{R}^N} \big(D(Du[z])\big|D(DU[w])\big) + \big(DU[z]\big|DU[w]\big)$$
$$+ \int_{\mathbb{R}^N} \big(r(DiU)\big|s(DiU)\big) + \big(riU\big|siU\big),$$

meaning that $D\Phi(0,0) \geqslant 0$. Therefore, for every small $\rho > 0$, the Brouwer topological degree $\deg(\Phi, B_\rho, 0)$ of Φ on B_ρ with respect to 0 is well-defined, and $\deg(\Phi, B_\rho, 0) = 1$. Hence, since we have the uniform convergence on compacts of the continuous functions Φ_n, for n large enough, we obtain that $\deg(\Phi_n, B_\rho, 0) = 1$. We conclude to the existence of a sequence (x_n, τ_n) such that $\Phi_n(x_n, \tau_n) = 0$ for every n large enough, and $(x_n, \tau_n) \to (0,0)$ as $n \to \infty$. Finally, setting $a_n = x_n + \tilde{a}_n$ and $\theta_n = \tilde{\theta}_n + \tau_n + iA(\tilde{a}_n)[x_n]$, we reach the conclusion in view of the composition formula for magnetic translations (4), and using again the Lebesgue dominated convergence. ◇

Applying the first two claims to the sequence $(v_n)_n$ and renaming $\tilde{u}_n = \mathrm{e}^{i\theta_n}\tau_{a_n}^{A_n}u_n$ and $\tilde{v}_n = \mathrm{e}^{i\varphi_n}\tau_{c_n}^{A_n}v_n$ (where the couple $(\varphi_n, c_n) \in \mathbb{R}^{N+1}$ is given by the claims), we can assume that $\tilde{u}_n$ satisfied

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |D_{A_n}\tilde{u}_n - DU|^2 + |\tilde{u}_n - U|^2 = 0,$$

and for every $w \in \mathbb{R}^N$,

$$\int_{\mathbb{R}^N} (D_{A_n}\tilde{u}_n|D(DU[w])) + (\tilde{u}_n|DU[w]) = 0, \quad \int_{\mathbb{R}^N} (D_{A_n}\tilde{u}_n|DiU) + (\tilde{u}_n|iU) = 0,$$

and the same for $\tilde{v}_n$.

CLAIM 3. There exists $W_n \in L^q(\mathbb{R}^N, \mathrm{Lin}(\mathbb{C},\mathbb{C}))$ such that

$$-\Delta_{A_n}(\tilde{u}_n - \tilde{v}_n) + (\tilde{u}_n - \tilde{v}_n) = W_n[\tilde{u}_n - \tilde{v}_n] \qquad \text{in } \mathbb{R}^N,$$

and

$$W_n \to |U|^{p-2} + (p-2)|U|^{p-4} U \otimes U$$

in $L^q(\mathbb{R}^N)$ for every $2 \leqslant q(p-2) \leqslant \frac{2N}{N-2}$, where $\otimes$ was defined in (9).

Proof of the claim. We define $W_n : \mathbb{R}^N \to \mathrm{Lin}(\mathbb{C},\mathbb{C})$ by

$$(w|W_n[z]) = \int_0^1 Df\big((1-t)\tilde{u}_n + t\tilde{v}_n\big)[w,z]\,\mathrm{d}t,$$

for $f(u) = |u|^{p-2}u$. Both claim 1 and Lemma 1 imply that $\tilde{u}_n \to U$ and $\tilde{v}_n \to U$ in $L^{q(p-2)}(\mathbb{R}^N)$, for $2 \leqslant q(p-2) \leqslant \frac{2N}{N-2}$. Then, it is clear that $W_n \in L^q(\mathbb{R}^N)$ and $W_n \to Df(U) = |U|^{p-2} + (p-2)|U|^{p-4} U \otimes U$ in $L^q(\mathbb{R}^N)$ as $n \to \infty$. $\diamond$

CONCLUSION The compact operator defined by $L_n = (-\Delta_{A_n} + 1)^{-1} W_n$ enters in the hypothesis of Proposition 4. We know that the spectrum converges, i.e., $\lambda_k(L_n) \to \lambda_k(L)$. Moreover, since the limit equation is (6), we also know that $\lambda_1(L) = p-1 > 1$, $\lambda_i(L) = 1$, for $i = 2, \dots, N+2$, and $\lambda_i(L) < 1$, for $i \geqslant N+3$.

We define the orthogonal projection operator P_n^+ on the first eigenvector, P_n^0 the projection on the eigenspace E_n^0 made by the $N+1$ following eigenvectors, and $P_n^- = I - P_n^+ - P_n^0$. We observe that L_n commutes with P_n^- and P_n^+ and $L_n(\tilde{u}_n - \tilde{v}_n) = \tilde{u}_n - \tilde{v}_n$. Moreover,

$$\begin{aligned}
\big\|P_+^n(\tilde{u}_n - \tilde{v}_n)\big\|^2_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})} &= \big(P_n^+(\tilde{u}_n - \tilde{v}_n)\big|P_n^+ L_n(\tilde{u}_n - \tilde{v}_n)\big)_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})} \\
&= \big(P_n^+(\tilde{u}_n - \tilde{v}_n)\big|L_n P_n^+(\tilde{u}_n - \tilde{v}_n)\big)_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})} \\
&= \lambda_1(L_n)\big\|P_n^+(\tilde{u}_n - \tilde{v}_n)\big\|^2_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})}.
\end{aligned}$$

Then, since $\lim_{n\to+\infty} \lambda_1(L_n) > 1$, $P_n^+(\tilde{u}_n - \tilde{v}_n) = 0$ for n large enough. Similarly,

$$\begin{aligned}
\big\|P_n^-(\tilde{u}_n - \tilde{v}_n)\big\|^2_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})} &= \big(P_n^-(\tilde{u}_n - \tilde{v}_n)\big|P_n^- L_n(\tilde{u}_n - \tilde{v}_n)\big)_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})} \\
&= \big(P_n^-(\tilde{u}_n - \tilde{v}_n)\big|L_n P_n^-(\tilde{u}_n - \tilde{v}_n)\big)_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})} \\
&\leqslant \lambda_{N+3}(L_n)\,\big\|P_n^-(\tilde{u}_n - \tilde{v}_n)\big\|^2_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})}.
\end{aligned}$$

Thus, since $\lim_{n\to\infty} \lambda_{N+3}(L_n) < 1$, $P_n^-(\tilde{u}_n - \tilde{v}_n) = 0$ for n large enough. Assume now by contradiction that, for every $n \in \mathbb{N}$, $\tilde{u}_n \neq \tilde{v}_n$. Then, the function

$$z_n = \frac{\tilde{u}_n - \tilde{v}_n}{\|\tilde{u}_n - \tilde{v}_n\|_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})}}$$

is in the eigenspace E_n^0 and is a linear combination of eigenvectors. By Proposition 4, there exists $z = DU[w] + \lambda iU \in E^0$, where E^0 is the eigenspace of L corresponding the eigenvalue 1 (see §2.1), such that, up to a subsequence still denoted by n,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |D_{A_n} z_n - Dz|^2 + |z_n - z|^2 = 0.$$

By Claim 2, we also know that

$$\int_{\mathbb{R}^N} (D_{A_n} z_n | Dz) + (z_n | z) = 0,$$

for n large enough. Finally

$$\begin{aligned} \|\tilde{u}_n - \tilde{v}_n\|^2_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})} &= \int_{\mathbb{R}^N} (D_{A_n}(\tilde{u}_n - \tilde{v}_n) | D_{A_n}(\tilde{u}_n - \tilde{v}_n)) + (\tilde{u}_n - \tilde{v}_n | \tilde{u}_n - \tilde{v}_n) \\ &= \|\tilde{u}_n - \tilde{v}_n\|^2_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})} \int_{\mathbb{R}^N} (D_{A_n} z_n | D_{A_n} z_n) + (z_n | z_n) \\ &= \|\tilde{u}_n - \tilde{v}_n\|^2_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})} \int_{\mathbb{R}^N} (D_{A_n} z_n - Dz | D_{A_n} z_n) + (z_n - z | z_n) \\ &\leqslant \|\tilde{u}_n - \tilde{v}_n\|^2_{H^1_{A_n}(\mathbb{R}^N,\mathbb{C})} \int_{\mathbb{R}^N} |D_{A_n} z_n - Dz|^2 + |z_n - z|^2. \end{aligned}$$

This is impossible. Then, $\tilde{u}_n = \tilde{v}_n$ for n large. □

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POSITIVE PERIODIC SOLUTIONS TO NONLINEAR ODES WITH INDEFINITE WEIGHT: AN OVERVIEW

Abstract.
We discuss the periodic problem associated with the second order differential equation

$$u'' + \big(\lambda a^+(t) - \mu a^-(t)\big) g(u) = 0, \tag{1}$$

where λ, μ are positive parameters, $a(t)$ is a sign-changing periodic function and $g(u)$ is a nonlinear function having superlinear growth at zero and sublinear growth at infinity. More precisely, we show how various tools from Nonlinear Analysis and Dynamical Systems can be used to provide results about existence, multiplicity and chaotic dynamics of positive solutions to (1). This survey paper is based on a talk given by the author at the *Bru-To PDE's Conference* (University of Torino, May 2-5, 2016).

1. Introduction

The aim of this brief note is to collect together some recent results, obtained in collaboration with Guglielmo Feltrin, Maurizio Garrione and Fabio Zanolin (see [12, 13, 14, 16]), on the existence of *positive periodic solutions* to nonlinear ODEs with *indefinite weight*.

More precisely, throughout the paper we deal with the second order scalar equation

$$u'' + \big(\lambda a^+(t) - \mu a^-(t)\big) g(u) = 0, \tag{2}$$

where λ, μ are positive parameters, $a : \mathbb{R} \to \mathbb{R}$ is a locally integrable and T-periodic sign-changing function and $g : \mathbb{R}^+ := [0, +\infty) \to \mathbb{R}$ is a $\mathcal{C}^1$-function satisfying the sign condition

$$(g_*) \qquad g(0) = 0 \quad \text{and} \quad g(u) > 0, \text{ for any } u > 0,$$

as well as the growth conditions (at zero and at infinity)

$$(g_{**}). \qquad \lim_{u \to 0^+} \frac{g(u)}{u} = \lim_{u \to +\infty} \frac{g(u)}{u} = 0.$$

Due to this assumption, the nonlinear term $g(u)$ will be referred to as a *super-sublinear* function*; in this setting, the study of the periodic problem associated with (2) seems to

*This terminology could be a bit misleading, since any function satisfying (g_{**}) is clearly below any line (passing through the origin and having positive slope) both for u small and for u large; however, it is quite common and useful when both the behavior at zero and at infinity of a nonlinear function have to be emphasized, since pure power nonlinearities $g(u) = u^p$ can be referred to as sublinear (i.e., sub-sublinear) when $p < 1$ and superlinear (i.e., super-superlinear) when $p > 1$.

be an interesting and delicate topic and, maybe unexpectedly, several tools from Nonlinear Analysis and Dynamical Systems (topological degree theory, variational methods, Poincaré-Birkhoff theorem, shooting arguments...) have to be used to try to understand as much as possible about it (existence/nonexistence of solutions, multiplicity, chaotic dynamics...).

Before describing our results, let us spend some words about boundary value problems with indefinite weight, trying to better motivate our investigation (incidentally, it seems that the terminology "indefinite" - simply meaning that the coefficient of the nonlinear term is sign-changing - was first used in [5] in the context of a linear eigenvalue problem, and it has then become very popular in nonlinear problems starting with [35]).

The periodic problem associated with an equation like

$$u'' + q(t)g(u) = 0, \tag{3}$$

with $q(t)$ sign-changing, was first investigated by Butler in its pioneering papers [23, 24], dealing with cases when $g(u)$ is defined on the whole real line and has superlinear growth at infinity or sublinear growth at zero, respectively. Later on, along this line of research, several contributions followed (especially in the superlinear case) and a quite complete picture concerning existence and multiplicity of *oscillatory* solutions to various boundary value problems associated with (3) is available since fifteen years ago (see, among others, [25, 36, 37, 39]).

On the other hand, starting with the nineties, the existence of *positive* solutions to boundary value problems associated with the nonlinear PDE

$$\Delta u - V(x)u + q(x)g(u) = 0, \qquad x \in \Omega \subset \mathbb{R}^N, \tag{4}$$

with $q(x)$ sign-changing, has been considered, as well (see, among others, [2, 4, 6, 8, 9]). It is worth mentioning that the elliptic equation (4) naturally arises when searching for steady states of the corresponding evolutionary parabolic problem (see [1] for a recent nice survey on the topic). Such kind of equations has a typical interpretation in the context of population dynamics, with the unknown u playing the role of density of a species inhabiting the spatially heterogeneous domain Ω; accordingly, the (indefinite) sign of the coefficient q expresses saturation or autocatalytic behavior of the species u, when $q \leqslant 0$ or $q \geqslant 0$ respectively.

Needless to say, equation (3) can be meant as the one-dimensional case of (4) when $V(x) \equiv 0$; moreover, it is not difficult to realize that periodic boundary conditions for (3) exhibit strong analogies with Neumann conditions for the elliptic equation (4). To explain why (besides recalling the well-known fact that both these boundary conditions share the same principal eigenvalue $\lambda_0 = 0$) we observe that a mean value condition on the weight function is often necessary for the existence of a positive solution both for T-periodic and Neumann boundary conditions. Indeed, assuming the existence of a positive T-periodic solution to (3), we easily obtain - dividing the equation by $g(u)$ and integrating by parts -

$$\int_0^T q(t)\,dt = -\int_0^T \left(\frac{u'(t)}{g(u(t))}\right)^2 g'(u(t))\,dt.$$

As a consequence, the condition

$$\int_0^T q(t)\,dt < 0 \tag{5}$$

is necessary for the existence of a positive T-periodic solution whenever $g'(u) > 0$ for any $u > 0$ (we stress that this is not an assumption in our basic setting; however, since there are of course many increasing nonlinearities in the class of the nonlinear functions $g(u)$ satisfying (g_*) and (g_{**}), condition (5) has to be considered unavoidable in general). Essentially the same computation is valid when dealing with positive Neumann solutions to (4) (when $V(x) \equiv 0$); this was indeed first observed by Bandle, Pozio and Tesei in [6], showing that $\int_\Omega q(x)\,dx < 0$ is actually necessary and sufficient for the existence of a positive solution to the Neumann problem associated with $\Delta u + q(x)u^p = 0$ in the sublinear case $0 < p < 1$. The same result was then proved in the superlinear case $p > 1$ in [2, 8, 9].

The above discussion should explain how results about the existence of positive periodic solutions to (2) have to be interpreted if compared with the existing literature. In particular, we stress that they seem to be quite new from different points of view: on one hand, indeed, they can be meant as lying somewhat in the middle between results giving oscillatory periodic solutions to ODEs and results giving positive Neumann solutions to elliptic PDEs (it is worth mentioning that the existence of positive periodic solutions to equations like (3) was explicitely raised by Butler as an open problem in [23, p. 477]); on the other hand, growth conditions of super-sublinear type like (g_{**}) seem to represent a novelty in the indefinite setting (in particular, we are not aware of results dealing with positive Neumann solutions to $\Delta u + q(x)g(u) = 0$ when $g(u)$ is super-sublinear).

The rest of this paper will be devoted to the description of several results in this direction (existence/nonexistence, multiplicity, subharmonic solutions, chaotic dynamics..). From now on, we impose the following technical condition on the weight function:

(a_*) *there exist $m \geqslant 1$ intervals $I_1^+, \ldots, I_m^+$, closed and pairwise disjoint in the quotient space $\mathbb{R}/T\mathbb{Z}$, such that*

$$a(t) \geqslant 0, a(t) \not\equiv 0, \text{ on } I_i^+, \quad \text{for } i = 1, \ldots, m;$$

$$a(t) \leqslant 0, a(t) \not\equiv 0, \text{ on each connected component of } (\mathbb{R}/T\mathbb{Z}) \setminus \bigcup_{i=1}^m I_i^+,$$

we also define the value

$$\mu^\#(\lambda) := \lambda \frac{\int_0^T a^+(t)\,dt}{\int_0^T a^-(t)\,dt} \tag{6}$$

which is going to play an important role in what follows.

2. The T-periodic problem

We first focus on the existence of positive T-periodic solutions to (2), stating the following result.

THEOREM 1. *Let $a : \mathbb{R} \to \mathbb{R}$ be a locally integrable T-periodic function satisfying (a_*) and let $g : \mathbb{R}^+ \to \mathbb{R}$ be a continuously differentiable function satisfying (g_*) and (g_{**}). Then, there exists $\lambda^* > 0$ such that, for any $\lambda > \lambda^*$ and for any $\mu > \mu^{\#}(\lambda)$, equation (2) has at least two positive T-periodic solutions.*

According to this statement, the solvability of (2) (together with periodic conditions) seems to be ensured only when imposing some restrictions on the range of the parameters λ, μ (see also the discussion in Section 5); on the other hand, it is remarkable that *two* solutions can be obtained. The idea underlying this can be traced back to a classical result by Rabinowitz [38], proving indeed, for $\lambda > 0$ large enough, the existence of a pair of positive solutions for the Dirichlet problem associated with an equation like

$$\Delta u + \lambda f(x,u) = 0, \qquad x \in \Omega \subset \mathbb{R}^N,$$

when $f(x,\cdot)$ is (roughly speaking) super-sublinear (see also [3] for previous results in this direction, from a more abstract point of view). In the indefinite periodic case, the situation is however more subtle, since the restriction $\mu > \mu^{\#}(\lambda)$ also needs to be imposed. This is actually unavoidable in general, since

$$\int_0^T \big(\lambda a^+(t) - \mu a^-(t)\big)\, dt < 0 \quad \Longleftrightarrow \quad \mu > \mu^{\#}(\lambda)$$

and, recalling the discussion leading to (5), the above condition is necessary for the existence of a positive T-periodic solution when $g'(u) > 0$ for any $u > 0$. It can be interesting to observe that, from a functional analytic point of view, this average condition plays the role of pushing the super-sublinear function $g(u)$ below the principal eigenvalue $\lambda_0 = 0$ of the periodic problem, both at the origin and at the infinity (notice that this is not needed when Dirichlet boundary conditions are taken into account, since the principal eigevanlue is strictly positive).

Theorem 1 is proved in [13] using a topological degree argument (see also [28, 29]); more precisely, it is shown therein that the coincidence degree (for a suitable operator associated with (2)) is equal to 1 both on small balls and on large balls centered at the origin, being instead equal to 0 on balls with intermediate radius. The existence of two positive T-periodic solutions then follows from the excision property of the degree and maximum principle arguments. It is worth mentioning that the same strategy works for the damped equation

$$u'' + cu' + \big(\lambda a^+(t) - \mu a^-(t)\big) g(u) = 0, \tag{7}$$

where $c \in \mathbb{R}$ is an arbitrary constant. In the conservative case $c = 0$, a slighlty less general version of Theorem 1 was previously obtained in [18] using variational arguments. We mention this here, since the proof can be easily understood: the condition

$\mu > \mu^{\#}(\lambda)$ is used, together with the super-sublinearity of $g(u)$, to show that the action functional is bounded from below and has a strict local minimum at the origin; on the other hand, the largeness of λ ensures that the functional attains negative values. Two T-periodic solutions are then obtained via global minimization and a mountain pass procedure, respectively (by standard arguments, they can be shown to be positive).

3. Subharmonic solutions

Having proved the existence of positive T-periodic (i.e., harmonic) solutions, a further question which naturally arises is the existence of positive periodic solutions with larger minimal period: say, kT-periodic solutions, with $k \geqslant 2$ an integer number (i.e., subharmonic solutions). This issue, which is peculiar of the periodic setting, is typically quite delicate, the most difficult point consisting of course in the proof of the minimality of the period for the periodic solutions found (we refer to [20] for several remarks about the topic, as well as for an extensive bibliography). As for equation (2), we propose the following result.

THEOREM 2. *In the setting of Theorem 1, let us suppose further that $g(u)$ if of class C^2 on $[0,\rho)$ for some $\rho > 0$, with $g''(u) > 0$ for any $u \in (0,\rho)$. Then, for $\lambda > \lambda^*$ and $\mu > \mu^{\#}(\lambda)$, equation (2) has two positive T-periodic solutions, as well as positive subharmonic solutions of order k for any sufficiently large integer k (moreover, the number of positive subharmonics of order k goes at infinity for $k \to +\infty$).*

Let us clarify that by a subharmonic solution of order k we mean a kT-periodic solution which is not lT-periodic for any integer $l = 1,\dots,k-1$ (this is the most general definition of subharmonic solution, and is the natural one when just the T-periodicity of $a(t)$ is assumed; whenever T is the *minimal* period of $a(t)$, it is easy to see that subharmonic solutions of order k actually have kT as the minimal period, see [12, Remark 3.1]).

The proof of Theorem 2 is given in [12] and it consists in a non-standard application of the Poincaré-Birkhoff fixed point theorem to a suitable Poincaré operator associated with (2). More precisely, the crucial steps are the following: first, the local convexity assumption on $g(u)$ is used to prove (via a clever algebraic trick first used, in a slightly different context, by Brown and Hess [22]) that one of the T-periodic solutions given by Theorem 1, say $u^*(t)$, has Morse index different from zero; then, following ideas developed in [17, 20], the Poincaré-Birkhoff theorem is applied to give kT-periodic solutions $u_k(t)$ oscillating around $u^*(t)$: the information on the number of zeros of $u_k(t) - u^*(t)$ (which is an intrisic feature of the periodic solutions constructed with this techique) is then the key point in showing that kT is the minimal period of $u_k(t)$.

It is worth recalling that the possibility of applying the Poincaré-Birkhoff theorem strongly relies on the Hamiltonian structure of the equation: accordingly, this technique does not work for the damped equation (7). The other typical way to search for subarmonic solutions is the use of variational methods, which also require $c = 0$.

Therefore, investigating the existence of positive subharmonic solutions to (7) seems to be a challenging open problem.

4. When $\mu \to +\infty$: high multiplicity and chaotic dynamics

In this section, we focus on a different aspect of the dynamics of (2): roughly speaking, we show how it becomes extremely rich when the parameter μ is very large. More precisely, we state the following result (with obvious notation, we name I_i^+ all the intervals of positivity of the weigth function $a(t)$ on the real line, by letting the index i vary on $\mathbb{Z}$).

THEOREM 3. *Let $g(u)$ and $a(t)$ be as in Theorem 1. Then, given an arbitrary constant $\rho > 0$ there exists $\lambda^* = \lambda^*(\rho) > 0$ such that for each $\lambda > \lambda^*$ there exist two constants r, R with $0 < r < \rho < R$ and $\mu^*(\lambda) = \mu^*(\lambda, r, R) > 0$ such that for any $\mu > \mu^*(\lambda)$ the following holds: given any two-sided sequence $\mathcal{S} = (\mathcal{S}_i)_{i\in\mathbb{Z}}$ in the alphabet $\mathscr{A} := \{0,1,2\}$ which is not identically zero, there exists at least one positive solution $u(t)$ of (2) such that*

- $\max_{t\in I_i^+} u(t) < r$, *if* $\mathcal{S}_i = 0$;
- $r < \max_{t\in I_i^+} u(t) < \rho$, *if* $\mathcal{S}_i = 1$;
- $\rho < \max_{t\in I_i^+} u(t) < R$, *if* $\mathcal{S}_i = 2$.

Moreover, whenever the two-sided sequence $\mathcal{S}$ is k-periodic for some integer k, the corresponding positive solution $u(t)$ of (2) can be chosen to be a kT-periodic function (hence, a positive kT-periodic solution to (2)).

This result describes a typical picture of *symbolic dynamics*: globally defined positive solutions to (2) are constructed, having a multibump chaotic-like behavior coded by a double sequence $\mathcal{S} = (\mathcal{S}_i)_{i\in\mathbb{Z}}$ in an alphabet of three symbols. A remarkable feature, moreover, is that periodic sequences of symbols can be realized through periodic solutions of the equation. As a consequence, multiple positive kT-periodic solutions to (2) can be obtained for any $k \geqslant 2$: simply by checking the minimality of the period for the corresponding sequence $\mathcal{S} \in \{0,1,2\}^{\mathbb{Z}}$, many of these positive kT-periodic solutions can be shown to be positive subharmonics of order k.

Also, assuming $m \geqslant 2$ in (a_*), we easily obtain from Theorem 3 that equation (2) has at least $3^m - 1$ positive T-periodic solutions for $\lambda > \lambda^*$ and μ very large (typically, much larger than the sharp value $\mu^\#(\lambda)$ given in Theorem 1). We can interpret this *high multiplicity* result in a singular perturbation spirit. Indeed, it is possible to show that the solutions constructed in Theorem 3 converge, for $\mu \to +\infty$, to solutions of the Dirichlet problem associated with $u'' + \lambda a^+(t)g(u) = 0$ on each I_i^+ (and to zero elsewhere); since three non-negative solutions for this boundary value problem are always available (the trivial one, and two positive solutions - a small one and a large one - given by Rabinowitz's theorem [38], compare with the discussion after Theorem

1), the above Theorem 3 shows that, on the converse, positive T-periodic solutions of (2) can be obtained, when μ is very large, being either"very small" on I_i^+ (if $\mathcal{S}_i = 0$), "small" (if $\mathcal{S}_i = 1$) or "large" (if $\mathcal{S}_i = 2$).

The proof of Theorem 3 is given in [14], using in a very delicate way coincidence degree theory (a previous result about chaotic dynamics - on two symbols only - for (2) was given in [19] using a completely different technique based on topological horseshoes theory). We are confident that the same technique works also for the damped equation (7), thought this is not formally proved yet.

We like to mention that the possibility of finding multiple positive solutions of indefinite nonlinear problems by playing with the nodal behavior of the weight function was initially suggested in a paper by Gómez-Reñasco and López-Gómez [34] (therein, an interesting analogy is proposed with the celebrated papers by Dancer [26, 27] providing multiplicity of solutions to elliptic BVPs by playing with the shape of the domain). The first complete result in this direction was then given by Gaudenzi, Habets and Zanolin [31, 32] for the Dirichlet boundary value problem associated with the superlinear indefinite equation

$$u'' + \big(a^+(t) - \mu a^-(t)\big)u^p = 0, \quad \text{with } \ p > 1;$$

later on, along this line of research, several contributions followed [7, 10, 11, 29, 30, 33]), dealing both with ODEs and PDEs, with various boundary conditions, always in the superlinear case. Theorem 3 thus extends these ideas to the super-sublinear setting, showing that the corresponding dynamics is even richer.

5. Some complementary results

To conclude, we observe that the solvability picture described in Theorem 1 naturally suggests a couple of (quite subtle) questions:

(Q1) is the existence of positive T-periodic solutions still possible when $\lambda > 0$ is small?

(Q2) is the existence of positive T-periodic solutions still possible - for a non-monotone $g(u)$ - when $0 < \mu \leqslant \mu^{\#}(\lambda)$, that is, when the average of the weight function is non-negative?

In this final section we try to give partial answers, by imposing some further conditions on $a(t)$ and $g(u)$.

More precisely, as for (Q1) we propose the following result.

THEOREM 4. *Let us assume that $a(t)$ is even-symmetric, with $m = 1$ in (a_*); moreover, suppose that $g'(u) \neq 0$ for $u \notin [\eta, \frac{1}{\eta}]$ (with $\eta > 0$) and that*

$$\lim_{u \to +\infty} g'(u) = 0.$$

Then, for any $\lambda > 0$ there exists $\mu^+(\lambda) > \mu^{\#}(\lambda)$ such that equation has at least two positive T-periodic solutions for any $\mu \in (\mu^{\#}(\lambda), \mu^+(\lambda))$.

The meaning of the above result is, roughly speaking, the following: the existence of positive T-periodic solutions to (2) is certainly ensured for *any* $\lambda > 0$ provided that μ is not too large (from this point of view, Theorem 1 thus ensures that $\mu^+(\lambda) = +\infty$ for $\lambda > \lambda^*$). We mention however that, for the one-parameter equation $u'' + \lambda p(t)g(u) = 0$, with $\int_0^T p(t)\,dt < 0$ and $g(u)$ super-sublinear, non-existence of positive T-periodic solutions can often be proved when $\lambda > 0$ is small, see [13].

We finally turn to the question (Q2).

THEOREM 5. *In the setting of Theorem 4, assume further that $g'(u) < 0$ for $u > \frac{1}{\eta}$ (implying that $g(u)$ is non-monotone). Then, for any $\lambda > 0$ there exist $\mu^-(\lambda) \in (0, \mu^\#(\lambda))$ such that equation (2) has at least one positive T-periodic solution for $\mu = \mu^\#(\lambda)$ and at least two positive T-periodic solutions for any $\mu \in (\mu^-(\lambda), \mu^\#(\lambda))$.*

That is: the existence of positive T-periodic solutions to (2) is still ensured (for any $\lambda > 0$) provided that μ is not too small (namely, when the average of the weight function is non-negative but not too large). Notice in particular that equation (2) with $\mu = \mu^\#(\lambda)$ (that is, when the weight function has zero average) is always solvable (under the assumptions of Theorem 5, of course).

Both Theorem 4 and Theorem 5 are proved in [16], taking advantage of an ingenious change of variable introduced in [21] (see also [15]), which transforms the sign-indefinite equation (2) into a forced perturbation of an autonomous equation, that is

$$x'' = h(x)(x')^2 + \big(\lambda a^+(t) - \mu a^-(t)\big), \tag{8}$$

where $h(x)$ is a suitable function (obtained from $g(u)$) defined on the whole real line. A shooting argument is then used to find positive solutions to (8) satifying Neumann boundary conditions on $[0, \frac{T}{2}]$ and T-periodic solutions are finally obtained using the symmetry assumption on $a(t)$. This is of course a serious restriction, but it seems quite hard to prove the above results using functional analytic techniques.

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A POINTWISE FINITE-DIMENSIONAL REDUCTION METHOD FOR EINSTEIN-LICHNEROWCIZ TYPE SYSTEMS

Abstract. We explain the construction of non-compactness examples for the fully coupled Einstein-Lichnerowicz system in the focusing case recently obtained in [15]. The construction follows from a combination of pointwise a priori asymptotic analysis techniques with a finite-dimensional reduction and a fixed-point argument on the remainder part of the expected blow-up decomposition.

1. Introduction

1.1. Statement of the results

Let (M,g) be a closed Riemannian manifold of dimension $n \geqslant 6$. We investigate non-compactness issues in strong spaces for the set of positive solutions of the Einstein-Lichnerowicz system in M:

$$\begin{cases} \triangle_g u + hu = fu^{2^*-1} + \dfrac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{u^{2^*+1}} \\ \overrightarrow{\triangle}_g T = u^{2^*} X + Y. \end{cases} \tag{1.1}$$

The unknowns of (1.1) are u, a smooth positive function in M, and T, a smooth field of 1-forms in M. In (1.1) $\mathcal{L}_g T$ is the conformal Killing derivative of T and we have let, for any 1-form T: $\overrightarrow{\triangle}_g T = -\mathrm{div}_g\big(\mathcal{L}_g T\big)$. Also, in (1.1), $\triangle_g = -\mathrm{div}_g(\nabla \cdot)$ is the Laplace-Beltrami operator, h, f, π are smooth functions in M, σ is a smooth field of 2-forms with $\mathrm{tr}_g \sigma = 0$ and $\mathrm{div}_g \sigma = 0$ and X and Y are smooth fields of 1-forms in M. The exponent $2^* = \frac{2n}{n-2}$ is critical for the embedding of the Sobolev space $H^1(M)$ into Lebesgue spaces. We also assume that

$$f > 0 \text{ in } M \quad \text{(focusing case)}, \tag{1.2}$$

and that $\triangle_g + h$ is coercive (which is necessary in view of (1.2)). System (1.1) arises in the initial-value problem in Mathematical General Relativity as a conformal formulation of the constraint equations (see [2]). Assumption (1.2) covers the case of non-trivial non-gravitational physics data. Existence and multiplicity results for (1.1) in the focusing case (1.2) are in [9, 13, 14].

We are interested here in the stability features of system (1.1). Following [3] (see also [8]), we say that system (1.1) is *stable* if, for any sequence $(h_k, f_k, \pi_k, \sigma_k, X_k, Y_k)_k$ of coefficients converging towards $(h, f, \pi, \sigma, X, Y)$ as $k \to +\infty$ in some strong topology (to be precised), and for any sequence $(u_k, T_k)_k$ of solutions of

$$\begin{cases} \triangle_g u_k + h_k u_k = f_k {u_k}^{2^*-1} + \dfrac{|\mathcal{L}_g T_k + \sigma_k|_g^2 + \pi_k^2}{u_k^{2^*+1}} \\ \overrightarrow{\triangle}_g T_k = u_k^{2^*} X_k + Y_k, \end{cases} \tag{1.3}$$

with $u_k > 0$, there holds, up to a subsequence and up to elements in the kernel of $\mathcal{L}_g$, that $(u_k, T_k)_k$ converges to some positive solution (u_0, T_0) of (1.1) in $C^{1,\eta}(M)$ for all $0 < \eta < 1$. The *compactness* of (1.1) is defined analogously, for constant sequences of coefficients $(h_k, f_k, \pi_k, \sigma_k, X_k, Y_k)_k$. In the focusing case (1.2) stability results were first obtained in [4, 10, 14] for the decoupled system (when $X \equiv 0$). For the fully coupled case $X \not\equiv 0$, the stability of (1.1) has been investigated in [6] and [16] on locally conformally flat manifolds. In particular, system (1.1) is always stable in dimensions $3 \leqslant n \leqslant 5$ provided $\pi \not\equiv 0$. For higher dimensions, the picture is more nuanced: instability results can occur. In [17] a first *in*stability result for the physical case of (1.1) with $X \equiv 0$ was obtained. The instability behavior for the fully coupled case $X \not\equiv 0$ of system (1.1) was later addressed in [15], where the following non-compactness result was obtained.

THEOREM 1.1 (P., [15]). Let (M, g) be a closed Riemannian manifold of dimension $n \geqslant 6$ of positive Yamabe type and possessing no non-trivial conformal Killing fields. There exist regular coefficients $(h, f, \pi, \sigma, X, Y)$, with $\triangle_g + h$ coercive, $f > 0$, $\pi \not\equiv 0$ and $X \not\equiv 0$ such that the associated system of equations (1.1)

$$\begin{cases} \triangle_g u + hu = f u^{2^*-1} + \dfrac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{u^{2^*+1}} \\ \overrightarrow{\triangle}_g T = u^{2^*} X + Y \end{cases}$$

possesses a blowing-up sequence of solutions $(u_k, T_k)_k$, that is $\|u_k\|_{L^\infty(M)} \to +\infty$ and $\|\mathcal{L}_g T_k\|_{L^\infty(M)} \to +\infty$ as $k \to +\infty$. Also, the u_k are positive, possess a single blow-up point and blow-up with a non-zero limit profile.

A manifold (M, g) is said to be of positive Yamabe type if the operator $\triangle_g + \frac{n-2}{4(n-1)} S_g$ is coercive, where S_g is the scalar curvature of g. The assumption that (M, g) possesses no non-trivial conformal Killing fields is generic and implies that $\overrightarrow{\triangle}_g$ has no kernel. A striking consequence of Theorem 1.1 is the existence of an infinite number of solutions of (1.1), see [15]. This article is devoted to a presentation of the ideas of the proof of Theorem 1.1.

1.2. Strategy of the proof of Theorem 1.1.

In the fully coupled case $X \not\equiv 0$ treated here, because of the strong nonlinear coupling via the $(|\mathcal{L}_g T + \sigma|_g^2 + \pi^2) u^{-2^*-1}$ term, (1.1) does not possess a variational structure in $H^1(M)$. The only known existence results for (1.1) are therefore based on fixed-point methods in strong spaces. This is a serious obstacle to the application of the usual Lyapounov-Schmidt construction scheme (see [1, 20, 21] and the references therein) which proved to be a valuable tool in constructing instability examples for critical elliptic equations on manifolds ([7, 19, 18]). To prove Theorem 1.1 we therefore work in strong topologies. We construct a blowing-up sequence of solutions $(u_k, T_k)_k$ of (1.1) whose scalar component writes as

$$u_k = W_{k,t,p} + u + \varphi_{k,t,p}, \tag{1.4}$$

where $W_{k,t,p}$ denotes a positive bubbling profile depending on $(n+1)$ parameters (t, p) and u is a positive strictly stable function. But this time $\varphi_{k,t,p}$ is a *globally pointwise* small remainder, precisely

$$|\varphi_{k,t,p}| \leqslant \varepsilon_k \big(W_{k,t,p} + u\big) \quad \text{pointwise in } M, \tag{1.5}$$

for some $(\varepsilon_k)_k$, $\varepsilon_k \to 0$ as $k \to \infty$. The motivation for the choice of (1.4) comes from the a priori blow-up analysis of (1.1) performed in [6, 16] which shows that (1.5) holds, at least at a local scale, for blowing-up solutions of (1.1). See also [5] where a global control as (1.5) was proven to hold for sequences of solutions of critical stationary Schrödinger equations.

Since (1.1) is not variational, there is no canonical choice of a remainder $\varphi_{k,t,p}$ anymore. We construct it through an involved Banach-Picard fixed-point argument which goes through several steps: a semi-decoupling of (1.1) followed by a finite-dimensional reduction (Section 3), an accurate a priori pointwise description in strong spaces of the remainder constructed (Section 4), a Banach-Picard fixed-point argument in strong spaces for the remainders' mapping (Section 5) and a uniform expansion of the kernel coefficients (Section 6).

2. Setting of the problem and notations

In this article we only sketch the $n \geqslant 7$ case and refer to [15] for the six-dimensional one. Let $(\tau_k)_k$ be a sequence of positive real numbers such that $\sum_k \tau_k < +\infty$. We define a sequence $(\mu_k)_k$ as follows:

$$\mu_k = \begin{cases} \tau_k^{\frac{2}{n-6}} & \text{if } (M,g) \text{ is locally conformally flat or if } 7 \leqslant n \leqslant 10, \\ \tau_k^{\frac{1}{2}} & \text{if } n \geqslant 11 \text{ and } (M,g) \text{ is not locally conformally flat.} \end{cases} \tag{2.1}$$

Let $(\xi_k)_k$ be a sequence of points of M converging towards a given $\xi_0 \in M$ and satisfying $d_g(\xi_k, \xi_{k+1}) << \frac{1}{k^2}$ as $k \to +\infty$. Let $(\beta_k)_k$ be a sequence of positive numbers converging to zero as $k \to +\infty$ and satisfying

$$\beta_k >> \mu_k. \tag{2.2}$$

Let f be a smooth positive function, let σ be a smooth traceless and divergence-free $(2,0)$-tensor in M and let π be a smooth function in M with $\pi \not\equiv 0$. Let Y be a smooth field of 1-forms and denote by $\tilde{Y}$ the only solution of $\overrightarrow{\triangle}_g \tilde{Y} = Y$ in M. We let also H be a smooth nonnegative function in $\mathbb{R}^n$, compactly supported in $B_0(1)$ with $H(0) = 1$, and for which 0 is a *non-degenerate critical point*. We define

$$h = \frac{n-2}{4(n-1)} S_g + \sum_{k \geqslant 0} \tau_k H\left(\frac{1}{\beta_k}\left(\exp_{\xi_k}^{g_{\xi_k}}\right)^{-1}(x)\right). \tag{2.3}$$

Here $g_\xi = \Lambda_\xi^{\frac{4}{n-2}}$ is a conformal modification of the original metric g. The factor Λ_ξ is chosen in light of the conformal normal coordinates result of [12] to achieve the highest precision in the expansion of the volume element of g_ξ around ξ, see [15]. Note, with (2.1) and (2.2), that for any $r \in \mathbb{N}^*$, one can always choose β_k as in (2.2) so that $h \in C^r(M)$. Let u_0 be a smooth, positive, strictly stable solution of the following Einstein-Lichnerowicz equation:

$$\triangle_g u_0 + h u_0 = f u_0^{2^*-1} + \frac{|\mathcal{L}_g \tilde{Y} + \sigma|_g^2 + \pi^2}{u_0{}^{2^*+1}}. \tag{2.4}$$

The coefficients f, π, σ and Y can always be chosen so that such a u_0 exists, see [14]. For every $n \geqslant 7$ the implicit function theorem shows that there exists a constant $\eta_0 = \eta_0(n, g, h, f, \pi, \sigma, Y)$ such that, for any X satisfying

$$\|X\|_{L_\infty(M)} = \eta \leqslant \eta_0, \tag{2.5}$$

the Einstein-Lichnerowicz system of equations

$$\begin{cases} \triangle_g u + h u = f u^{2^*-1} + \dfrac{|\mathcal{L}_g T + \sigma|^2 + \pi^2}{u^{2^*+1}} \\ \overrightarrow{\triangle}_g T = u^{2^*} X + Y, \end{cases} \tag{2.6}$$

with h given by (2.3), possesses a solution $(u(X), T(X))$ such that $u(X) \to u_0$ in $C^2(M)$ as η, defined in (2.5), goes to 0. Up to choosing η small enough, it is easily seen that $u(X)$ is again a strictly stable solution of the scalar equation of (2.6). In the following, for a given X, the solution $(u(X), T(X))$ will just be denoted by (u, T).

We endow $H^1(M)$ with the following scalar product

$$\langle u, v \rangle_h = \int_M \left(\langle \nabla u, \nabla v \rangle_g + h u v\right) dv_g, \quad \text{for any } u, v \in H^1(M), \tag{2.7}$$

where h is given by (2.3). Let $(r_k)_k$, $r_k > 0$, $r_k \to 0$ as $k \to \infty$, satisfying

$$\beta_k << r_k << d_g(\xi_k, \xi_{k+1}) \quad \text{and} \quad r_k^N >> \mu_k, \tag{2.8}$$

where β_k is given by (2.2), for some large enough integer N, as $k \to +\infty$. For $t > 0$ we define: $\delta_k(t) = \mu_k t$, where μ_k is as in (2.1). The defects of compactness investigated here are the following ones:

$$W_{k,t,\xi}(x) = \Lambda_\xi(x)\chi\left(\frac{d_{g_\xi}(\xi,x)}{r_k}\right)\delta_k^{\frac{n-2}{2}}\left(\delta_k^2 + \frac{f(\xi)}{n(n-2)}d_{g_\xi}(\xi,x)^2\right)^{1-\frac{n}{2}}, \tag{2.9}$$

where $\chi \in C^\infty(\mathbb{R})$ is a nonnegative, smooth compactly supported function in $[-2,2]$. We also define, for any $x \in M$, any $1 \leqslant i \leqslant n$ and any $\xi \in M$

$$\begin{aligned} Z_{0,k,t,\xi}(x) &= \Lambda_\xi(x)\chi\left(\frac{d_{g_\xi}(\xi,x)}{r_k}\right)\delta_k^{\frac{n-2}{2}}\left(\delta_k^2 + \frac{f(\xi)}{n(n-2)}d_{g_\xi}(\xi,x)^2\right)^{-\frac{n}{2}} \\ &\quad \times \left(\frac{f(\xi)}{n(n-2)}d_{g_\xi}(\xi,x)^2 - \delta_k^2\right) \\ Z_{i,k,t,\xi}(x) &= \Lambda_\xi(x)\chi\left(\frac{d_{g_\xi}(\xi,x)}{r_k}\right)\delta_k^{\frac{n}{2}}\left(\delta_k^2 + \frac{f(\xi)}{n(n-2)}d_{g_\xi}(\xi,x)^2\right)^{-\frac{n}{2}} \\ &\quad \times f(\xi)\left\langle \left(\exp_\xi^{g_\xi}\right)^{-1}(x), e_i(\xi)\right\rangle_{g_\xi(\xi)}, \end{aligned}$$

where the $(e_i)_i$ are a local orthonormal basis for g_ξ around ξ_0. Finally, we let

$$K_{k,t,\xi} = \mathrm{Span}\left\{Z_{i,k,t,\xi}, i = 0\dots n\right\}. \tag{2.10}$$

Then $K_{k,t,\xi}$ is $(n+1)$-dimensional for k large enough and the $Z_{i,k,t,\xi}$ are "almost" orthogonal. We denote by $K_{k,t,\xi}^\perp$ its orthogonal in $H^1(M)$ for the scalar product given by (2.7).

We now define f and X. Let $f_0 > 0$ be a positive constant and define

$$f = f_0 + \sum_{k \geqslant 0} s_k \chi\left(\frac{1}{r_k}\left(\exp_{\xi_k}^{g_{\xi_k}}\right)^{-1}(x)\right), \tag{2.11}$$

where $(s_k)_k$ satisfies $|s_k| = O(\mu_k^N)$ for a sufficiently large $N \in \mathbb{N}^*$. Let X_0 denote any smooth field of 1-forms in M which vanishes in a neighbourhood of ξ_0. Let Z be a fixed smooth 1-form in $\mathbb{R}^n$, compactly supported in $B_0(1)$, and with $|Z_0(0)| > 0$. Define then, for any $x \in M$

$$X(x) = X_0(x) + \sum_{k \geqslant 0} \mu_k^{\frac{n-1}{2}} Z\left(\frac{1}{r_k}\left(\exp_{\xi_k}^{g_{\xi_k}}\right)^{-1}(x)\right), \tag{2.12}$$

where μ_k and r_k are as in (2.1) and (2.8). Up to reducing $\|X_0\|_\infty$ and the τ_k such an X always satisfies (2.5). Again, with (2.1) and (2.8), f and X can always be chosen to belong to $C^r(M)$ for $r \in \mathbb{N}^*$. Finally, let

$$\mathcal{E} = \left\{(\varepsilon_k)_{k\in\mathbb{N}}, \varepsilon_k > 0, \lim_{k\to\infty} \varepsilon_k = 0\right\} \tag{2.13}$$

be the set of sequences of positive real numbers converging to 0. For $(\varepsilon_k)_k \in \mathcal{E}$ and for a given value of $(t,\xi) \in (0,+\infty) \times M$ we define the following sequence of subsets of $C^2(M)$:

$$F_k = F\left(\varepsilon_k, t, \xi\right) = \left\{ v \in C^0(M) \text{ such that } \left\| \frac{v}{u + W_{k,t,\xi}} \right\|_{C^0(M)} \leqslant \varepsilon_k \right\}, \tag{2.14}$$

where $u = u(X)$ is defined after (2.6) and $W_{k,t,\xi}$ is as in (2.9).

3. Semi-decoupling and H^1 reduction.

Let $(\varepsilon_k)_k \in \mathcal{E}$, $(t,\xi) \in (0,+\infty) \times M$ and $v_k \in F_k = F\left(\varepsilon_k, t, \xi\right)$, where $\mathcal{E}$ and F_k are defined in (2.13) and (2.14). Since $\overrightarrow{\triangle}_g$ has no kernel by assumption, there exists a unique 1-form $T_{k,t,\xi}$ in M satisfying

$$\overrightarrow{\triangle}_g T_{k,t,\xi} = \left(u + W_{k,t,\xi} + v_k\right)^{2^*} X + Y. \tag{3.1}$$

Pointwise bounds on $\mathcal{L}_g T_{k,t,\xi}$ follow from the assumption $v_k \in F_k$, see [15]. It turns out that $\mathcal{L}_g T_{k,t,\xi}$ blows up too fast for an H^1 finite-dimensional reduction to apply to the scalar equation of (1.1) with $\mathcal{L}_g T_{k,t,\xi}$ seen as a coefficient: even the very first step (the uniform inversion of the linearized operator) fails. We therefore artificially discard the $|\mathcal{L}_g T_{k,t,\xi} + \sigma|_g^2$ term into a source term and consider instead the equation

$$\triangle_g w + hw = f w^{2^*-1} + \frac{|\mathcal{L}_g T + \sigma|^2 + \pi^2}{\rho(w)^{2^*+1}} + \frac{|\mathcal{L}_g T_{k,t,\xi} + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{\left(u + W_{k,t_k,\xi_k} + v_k\right)^{2^*+1}}, \tag{3.2}$$

where T satisfies $\overrightarrow{\triangle}_g T = u^{2^*} X + Y$ and where we have let $\rho = \rho_{\varepsilon_0}$ for some $\varepsilon_0 > 0$, where

$$\rho_\varepsilon(r) = \begin{cases} \varepsilon \text{ if } r < \varepsilon \\ r \text{ if } r \geqslant \varepsilon. \end{cases}$$

The first step of the proof of Theorem 1.1 is as follows.

PROPOSITION 3.1. Let $D > 0$ and $(\varepsilon_k) \in \mathcal{E}$ and assume that $\varepsilon_k >> \mu_k^{\frac{3}{2}}$ as $k \to +\infty$, where μ_k is as in (2.1). Let $(t_k, \xi_k)_k$ be a sequence in $[1/D, D] \times M$ and, for any k, let $v_k \in F_k = F\left(\varepsilon_k, t_k, \xi_k\right)$. For k large enough, there exists a function $\phi_k = \phi_k\left(t_k, \xi_k, v_k\right) \in K_{k,t_k,\xi_k}^{\perp}$ that satisfies

$$\begin{aligned} &\Pi_{K_{k,t_k,\xi_k}^{\perp}} \left\{ u + W_{k,t_k,\xi_k} + \phi_k - (\triangle_g + h)^{-1} \left(f\left(u + W_{k,t_k,\xi_k} + \phi_k\right)^{2^*-1} \right.\right. \\ &\left.\left. + \frac{|\mathcal{L}_g T + \sigma|_g^2 + \pi^2}{\rho\left(u + W_{k,t_k,\xi_k} + \phi_k\right)^{2^*+1}} \right) - (\triangle_g + h)^{-1} \left(\frac{|\mathcal{L}_g T_{k,t,\xi} + \sigma|_g^2 - |\mathcal{L}_g T + \sigma|_g^2}{\left(u + W_{k,t_k,\xi_k} + v_k\right)^{2^*+1}} \right) \right\} = 0. \end{aligned} \tag{3.3}$$

This ϕ_k is the unique solution of (3.3) in $K^{\perp}_{k,t_k,\xi_k} \cap B_{H^1(M)}(0, C\eta\varepsilon_k)$, where C is independent of k, of the choice of $(t_k, \xi_k)_k$ and of η as in (2.5). Also, in (3.3), K_{k,t_k,ξ_k} is as in (2.10) and $T_{k,t,\xi}$ is as in (3.1).

As an obvious consequence of Proposition 3.1, the function ϕ_k constructed therein satisfies

$$\|\phi_k\|_{H^1(M)} \leqslant C\eta\varepsilon_k, \tag{3.4}$$

for some constant C which is independent of $(t_k, \xi_k)_k$, k and η. In (3.3), the truncation ρ is a technical shortcut required to handle the negative nonlinearity in $H^1(M)$. Lemma 2.1 in [17] shows however that ρ has no influence on the construction process provided ε_0 is small enough. Proposition 3.1 is proven by a Banach-Picard fixed-point method, and crucially relies on the pointwise estimates on $\mathcal{L}_g T_{k,t,\xi}$ directly induced from the a priori *pointwise* control (1.5) on v.

4. Asymptotic pointwise description of the remainder ϕ_k

4.1. Rough pointwise control

In view of an application of a fixed-point argument to the remainders mapping $v_k \mapsto \phi_k$ in F_k defined in (2.14) we need to choose $(\varepsilon_k)_k$ so that the remainder ϕ_k given by Proposition 3.1 belongs to F_k. Proving this is the core of the analysis of [15]. This is far from being obvious: first ϕ_k only comes with an H^1 bound by essence. Then, the criticality of (1.1) does not allow a simple bootstrap procedure to increase regularity. And finally, ϕ_k is only a solution up to some kernel elements. Precisely, letting $u_{k,t_k,\xi_k,v_k} = u + W_{k,t_k,\xi_k} + \phi_k(t_k, \xi_k, v_k)$, there holds

$$\begin{aligned}(\triangle_g + h)u_{k,t_k,\xi_k,v_k} - f u^{2^*-1}_{k,t_k,\xi_k,v_k} - \frac{|\mathcal{L}_g T + \sigma|^2_g + \pi^2}{u^{2^*+1}_{k,t_k,\xi_k,v_k}} - \frac{|\mathcal{L}_g T_k + \sigma|^2_g - |\mathcal{L}_g T + \sigma|^2_g}{\left(u + W_{k,t_k,\xi_k} + v_k\right)^{2^*+1}} \\ = \sum_{i=0}^{n} \lambda^i_k(t_k, \xi_k, v_k)\,(\triangle_g + h)\,Z_{i,k,t,\xi}\end{aligned} \tag{4.1}$$

for some numbers $(\lambda^i_k(t_k, \xi_k, v_k))_{0 \leqslant i \leqslant n}$, where T_{k,t_k,ξ_k} is as in (3.1) and $Z_{i,k,t,\xi}$ as in (2.10). The first step towards a pointwise control on ϕ_k consists in showing that ϕ_k is globally small (in $C^0(M)$) with respect to $W_{k,t,\xi} + u$.

PROPOSITION 4.1. Let $D > 0$ and $(\varepsilon_k)_k \in \mathcal{E}$ and assume that $\varepsilon_k >> \mu_k^{\frac{3}{2}}$ as $k \to +\infty$, where μ_k is as in (2.1). Let $(t_k, \xi_k)_k$ be a sequence of points in $[1/D, D] \times M$, and let $v_k \in F_k = F\left(\varepsilon_k, t_k, \xi_k\right)$. There exists a sequence $(\nu_k)_k$ of positive numbers that goes to zero as $k \to +\infty$ such that

$$|\phi_k(x)| \leqslant \nu_k\Big(u(x) + W_{k,t_k,\xi_k}(x)\Big) \qquad \text{for any } x \in M. \tag{4.2}$$

In (4.2) we have let $\phi_k = \phi_k\big(t_k, \xi_k, v_k\big) \in K_{k,t_k,\xi_k}$ be the solution of (3.3) given by Proposition 3.1.

In the course of the proof the first thing one has to obtain is a control of the $|\lambda_k^i(t_k,\xi_k,v_k)|$ and a lower bound on ϕ_k so as to get rid of the truncation ρ. Then the proof of Proposition 4.1 consists in an adaptation of the methods developed in [5] (see also [8]) to take into account the source term in (3.2). One first obtains a global weak pointwise estimate together with a local rescaled convergence, later refined into a global uniform control by means of successive approximations. The methods of [5] a priori do not apply to nonlinear equations with a source term, but we manage to adapt them here because the source term is *pointwise* controlled.

4.2. Second-order estimates

The main challenge in the proof of Theorem 1.1 is to quantify precisely ν_k in (4.2). The first step towards this is to obtain a local improvement of (4.2) in the region where the bubbling profile is dominant.

PROPOSITION 4.2. Let $D > 0$, $(\varepsilon_k)_k \in \mathcal{E}$ and assume that $\varepsilon_k >> \mu_k^{\frac{3}{2}}$ as $k \to +\infty$, where μ_k is as in (2.1). Let $(t_k, \xi_k)_k$ be a sequence in $[1/D, D] \times M$, let $v_k \in F_k = F\big(\varepsilon_k, t_k, \xi_k\big)$ and let $\phi_k = \phi_k\big(t_k, \xi_k, v_k\big)$ be given by Proposition 3.1. Let $(x_k)_k$ be any sequence of points in $B_{\xi_k}(2\sqrt{\delta_k})$. There holds

(4.3)

$$\theta_k(x_k)\,|\nabla\phi_k(x_k)| + |\phi_k(x_k)| \lesssim \|\phi_k\|_{L^\infty(\Omega_k)} + \sqrt{\delta_k}\|\nabla\phi_k\|_{L^\infty(\Omega_k)} + \delta_k + \Bigg[\delta_k^{\frac{n}{2}} + \delta_k\|\nabla f\|_{L^\infty(2r_k)} + \|h - c_n S_g\|_{L^\infty(2r_k)}\delta_k^2\left|\ln\left(\frac{\theta_k(x_k)}{\delta_k}\right)\right| + \|h - c_n S_g\|_{L^\infty(2r_k)}\theta_k(x_k)^2 + \theta_k(x_k)^4 \mathbb{1}_{nlcf}\Bigg] W_k(x_k) + \left(\frac{\delta_k}{\theta_k(x_k)}\right)^2,$$

where we have let: $\Omega_k = B_{\xi_k}(2r_k)\backslash B_{\xi_k}(\sqrt{\delta_k})$ and $\theta_k(x_k) = \delta_k + d_{g_{\xi_k}}(\xi_k, x_k)$.

Here the notation "$\lesssim$" stands for "$\cdot \leqslant C\cdot$" for a positive constant C independent of k. Estimate (4.3) is obtained by writing a representation formula for ϕ_k (with (4.1)) and estimating precisely every term which appears. Of course, many nonlinear terms to be estimated do depend on ϕ_k: we therefore first obtain a control of $\|\phi_k\|_{L^\infty(B_{\xi_k}(2r_k))}$ that we later iteratively improve into (4.3). The control is the following.

CLAIM 4. There holds

$$\|\phi_k\|_{L^\infty(B_{\xi_k}(2\sqrt{\delta_k}))} \lesssim \max(1, M_k), \tag{4.4}$$

where we have let

$$M_k = \|\phi_k\|_{L^\infty(\Omega_k)} + \sqrt{\delta_k}\|\nabla\phi_k\|_{L^\infty(\Omega_k)} + \delta_k + \delta_k^{2-\frac{n}{2}}\|\nabla f\|_{L^\infty(2r_k)} + \delta_k^{3-\frac{n}{2}}\|h - c_n S_g\|_{L^\infty(2r_k)} + \delta_k^{5-\frac{n}{2}}\mathbb{1}_{nlcf}.$$

To prove (4.4) we go through an involved contradiction argument: assuming that (4.4) does not hold, we localize the maximum point of ϕ_k in M and show that a limiting equation for some suitable rescaling of ϕ_k – denoted $\tilde{\phi}_k$ – can be obtained. The limiting equation ensures that the limit of $\tilde{\phi}_k$ lies in the kernel for the linearized equation of the standard bubble equation in $\mathbb{R}^n$. The contradiction then follows from the nature of ϕ_k constructed in Proposition 3.1, which is by construction almost orthogonal to some approximate rescalings of these kernel elements. Of course these two assertions come at different heights, and the main challenge is to be sure that they can be related after rescaling. In the course of the proof of Proposition 4.2, to iteratively improve the estimates we also derive the following control on the λ_k^i:

$$\sum_{i=0}^{n}|\lambda_k^i| \lesssim \delta_k^{\frac{n-2}{2}}\Big(\|\phi_k\|_{L^\infty(\Omega_k)} + \sqrt{\delta_k}\|\nabla\phi_k\|_{L^\infty(\Omega_k)}\Big) + \delta_k^{\frac{n-2}{2}} + \delta_k\|\nabla f\|_{L^\infty(2r_k)} + \|h - c_n S_g\|_{L^\infty(2r_k)}\delta_k^2 + \delta_k^4\mathbb{1}_{nlcf}.$$

As a second step to quantify ν_k in (4.2) we derive *global* estimates for ϕ_k over M.

PROPOSITION 4.3. Let $D > 0$ and $(\varepsilon_k)_k \in \mathcal{E}$ and assume that $\varepsilon_k >> \mu_k^{\frac{3}{2}}$ as $k \to +\infty$, where μ_k is as in (2.1). Let $(t_k, \xi_k)_k$ be a sequence in $[1/D, D] \times M$, let $v_k \in F_k = F\big(\varepsilon_k, t_k, \xi_k\big)$ and let $\phi_k = \phi_k\big(t_k, \xi_k, v_k\big)$ be given by Proposition 3.1. Let $(x_k)_k$ be any sequence of points in M. There holds

$$|\phi_k(x_k)| \leqslant C(\delta_k + \eta\varepsilon_k)\Big(u(x_k) + W_k(x_k)\Big), \tag{4.5}$$

where η is as in (2.5), for some positive constant C independent of η and k.

The proof of Proposition 4.3 goes again through a global representation formula for ϕ_k. The terms to be estimated are integrals involving again ϕ_k. The contributions of these integrals in the region where $W_{k,t,\xi}$ is dominant – the most problematic – are handled thanks to Proposition 4.2. One of the main subtleties of the proof of Theorem 1.1 is to obtain estimates – as in Propositions 4.2 or 4.3 – which are uniform in the choice of $(\varepsilon_k)_k, (t_k)_k, (\xi_k)_k$ and $(v_k)_k$. Note that the statement of Proposition 4.2 is much more precise than what is required in the proof of Proposition 4.3. But this high precision will turn out to be crucial in section 6 to obtain precise asymptotic expansions of the $\lambda_k^i(t, \xi)$. The a priori analysis techniques used in our proof have been developed in the context of the C^0 theory in [5] Related techniques have independently been developed in the investigation of compactness phenomena for the Yamabe problem (see [11] and the references therein).

5. Global fixed-point argument and resolution of the reduced problem

In this section we explain how a solution of the reduced problem for (1.1) is obtained. By this we mean a function $\varphi_k(t,\xi)$ such that $(u_{k,t,\xi}, W_{k,t,\xi})_k$, with $u_{k,t,\xi} = W_{k,t,\xi} + u + \varphi_k(t,\xi)$, which solves

$$\begin{cases} \triangle_g u_{k,t,\xi} + h u_{k,t,\xi} = f u_{k,t,\xi}^{2^*-1} + \dfrac{\pi^2 + |\sigma + \mathcal{L}_g W_{k,t,\xi}|_g^2}{u_{k,t,\xi}^{2^*+1}} + \displaystyle\sum_{j=0}^{n} \lambda_{k,j}(t,\xi) Z_{j,k,t,\xi}\,, \\ \overrightarrow{\triangle}_g W_{k,t,\xi} = u_{k,t,\xi}^{2^*} X + Y, \end{cases} \tag{5.1}$$

where the $Z_{j,k,t,\xi}$ are defined in (2.10). This amounts to showing that $v_k \mapsto \phi_k$ has a fixed-point in F_k (defined in (2.14)). With Proposition 4.3 we already see that, provided ε_k is suitably chosen and η is small enough, F_k is a stable set for $v_k \mapsto \phi_k$ for any k. Standard elliptic theory together with a Schauder fixed-point theorem would yield, for any k, a solution $\varphi_k(t,\xi)$ of (5.1). However, we need more than this. Schauder's fixed-point theorem comes with no uniqueness statement about the fixed-point it constructs and therefore does not allow to show that such a fixed-point continuously depends in strong spaces in (t,ξ). We therefore apply Banach-Picard's fixed-point theorem to $v_k \mapsto \phi_k$ in F_k.

Proposition 5.1. Let $D > 0$. Assume that η defined in (2.5) is small enough. There exists $k_0 \in \mathbb{N}$ such that for any sequence $(t_k,\xi_k)_k \in [1/D, D] \times M$ and for any $k \geqslant k_0$, there exists a function $\varphi_k = \varphi_k\big(t_k,\xi_k\big) \in K^{\perp}_{k,t_k,\xi_k}$ that satisfies the following system of equations

$$\begin{cases} \Pi_{K^{\perp}_{k,t_k,\xi_k}} \left[u_k - (\triangle_g + h)^{-1} \left(f u_k^{2^*-1} + \dfrac{|\mathcal{L}_g T_k + \sigma|_g^2 + \pi^2}{u_k^{2^*+1}} \right) \right] = 0, \\ \overrightarrow{\triangle}_g T_k = u_k^{2^*} X + Y, \end{cases} \tag{5.2}$$

where we have let $u_k = u + W_{k,t_k,\xi_k} + \varphi_k(t_k,\xi_k)$. Also, for any k, the mapping $(t,\xi) \mapsto \varphi_k(t,\xi) \in C^1(M)$ is continuous and there exists a positive constant C, independent of $(t_k,\xi_k)_k$ such that there holds

$$\|\varphi_k(t_k,\xi_k)\|_{H^1(M)} \leqslant C\delta_k \text{ and } |\varphi_k(t_k,\xi_k)| \leqslant C\delta_k\Big(u + W_{k,t_k,\xi_k}\Big) \text{ in } M, \tag{5.3}$$

and such that $\varphi_k(t_k,\xi_k)$ is the unique solution of (5.2) in $K^{\perp}_{k,t_k,\xi_k}$ satisfying in addition (5.3).

Proposition 5.1 shows that the estimates on φ_k only depend on the data μ_k and ε_k.We prove it – and Theorem 1.1 – assuming that the L^∞ norm of the coupling field X is small (depending on n, g, h, f, π, σ) . In view of (4.5) this is required to have a stable set for the remainder's mapping, and this assumption is actually necessary since smallness conditions on X are necessary for solutions of (1.1) to exist: see [9, 13, 14]. To prove Proposition 5.1 we prove that $v_k \mapsto \phi_k$ is $\frac{1}{2}$ - contractible in F_k for the norm given by

(2.14). If ϕ_k^i are associated to v_k^i, $i = 1..2$, we estimate the maximum value of $\frac{\phi_k^1-\phi_k^2}{u+W_{k,t,\xi}}$ directly using again representation formulae for (4.1). If this maximum is achieved at a distance from ξ_k comparable to the parameter of the bubble δ_k, we proceed using similar techniques to those used in the proof of Proposition 4.2. Otherwise, it is the smallness of η in (2.5) that allows to conclude.

6. Expansion of the Kernel coefficients and conclusion

The final step in the proof of Theorem 1.1 consists in finding, for any k, a suitable (t_k, ξ_k) which annihilates the $\lambda_k^i(t_k, \xi_k)$ in (5.1). This is achieved through an asymptotic expansion of the $\lambda_{k,j}(t, \xi)$ in C^0 and a limiting degree argument. In standard cases where only H^1 estimates are involved, the precision of such an expansion only depends on the choice of the approximate solution $u + W_{k,t,\xi}$. Here, however, the lack of a variational structure and the strong nonlinear coupling of (1.1) do not give us a better precision than (5.3) on φ_k – no matter the precision of the *ansatz* $u + W_{k,t,\xi}$ –, which is way too rough. We again overcome this by relying on the asymptotic analysis results obtained in Sections 4 and 5. We write the scalar equation in (5.1) as

$$\begin{aligned}
&\sum_{i=0}^{n}\lambda_k^i(t,\xi)\left(\triangle_g+h\right)Z_{i,k,t,\xi} = \left(\triangle_g+h\right)W_{k,t,\xi} - f(\xi)W_{k,t,\xi}^{2^*-1} + \left(f(\xi)-f\right)W_{k,t,\xi}^{2^*-1}\\
&- f\left[\left(u+W_{k,t,\xi}+\varphi_k(t,\xi)\right)^{2^*-1} - \left(u+W_{k,t,\xi}\right)^{2^*-1} - (2^*-1)\left(u+W_{k,t,\xi}\right)^{2^*-2}\phi_k(t,\xi)\right]\\
&- f\left[\left(u+W_{k,t,\xi}\right)^{2^*-1} - u^{2^*-1} - W_{k,t,\xi}^{2^*-1}\right] + \frac{|\mathcal{L}_g T+\sigma|_g^2+\pi^2}{u^{2^*+1}} - \frac{|\mathcal{L}_g T_{k,t,\xi}+\sigma|_g^2+\pi^2}{\left(u+W_{k,t,\xi}+\varphi_k(t,\varphi)\right)^{2^*+1}}\\
&+ \left(\triangle_g+h\right)\varphi_k(t,\xi) - (2^*-1)f(\xi)W_{k,t,\xi}^{2^*-2}\varphi_k(t,p) + (2^*-1)\left(f(\xi)-f\right)W_{k,t,\xi}^{2^*-2}\varphi_k(t,\xi)\\
&- (2^*-1)f\left[\left(u+W_{k,t,\xi}\right)^{2^*-2} - W_{k,t,\xi}^{2^*-2}\right]\varphi_k(t,\xi),
\end{aligned}$$

multiply both sides by $Z_{j,k,t,\xi}$, for a given j, and estimate all the integrals in the right-hand side. At this point we also express ξ as $\xi = \exp_{\xi_k}^{g_{\xi_k}}(\beta_k p)$, with β_k as in (2.2) and $p \in B_0(1) \subset \mathbb{R}^n$. These integrals are directly computed using pointwise a priori estimates on φ_k obtained by our blow-up analysis, and the explicit expression of $W_{k,t,\xi}$ and $Z_{j,k,t,\xi}$ in (2.9) and (2.10). Different contributions in M are estimated differently: when the integration domain is the ball $B_{\xi_k}(\sqrt{\delta_k})$ we use Proposition 4.2, at finite distances from ξ_k we use (4.5) while in the intermediate region we prove that for any sequence $(R_k)_k$, $R_k \geqslant 1$ there holds

$$\|\varphi_k\|_{L^\infty(M\setminus B_{\xi_k}(R_k\sqrt{\delta_k}))} \lesssim \frac{\delta_k}{R_k^2} + R_k^2\delta_k^2 + \delta_k^{\frac{n-2}{2}} r_k^{-n}. \tag{6.1}$$

If for instance (M,g) is locally conformally flat or $7 \leqslant n \leqslant 10$, direct estimations give in the end

(6.2)

$$\Big(I_{n+1}+O(\delta_k)\Big)\begin{pmatrix}\lambda_k^0(t,p)\\ \vdots \\ \lambda_k^n(t,p)\end{pmatrix} = \begin{pmatrix} \mu_k^{\frac{n-2}{2}}\Big[C_1 f(\xi_0)^{-\frac{n}{2}}H(p)t^2 - C_2 f(\xi_0)^{1-\frac{n}{2}}u(\xi_0)t^{\frac{n-2}{2}} \\ \qquad -C_3 f(\xi_0)^{-4}K_{10}^{-10}|W_g(\xi)|_g^2 t^4 \mathbb{1}_{n=10} + R_k^0(t,p)\Big] \\ \dfrac{\mu_k^{\frac{n}{2}}}{\beta_k}\Big[C_4 f(\xi_0)^{-\frac{n}{2}}\nabla_i H(p)t^3 + R_k^i(t,p)\Big]\end{pmatrix},$$

where, for $0 \leqslant i \leqslant n$, $R_k^i(t,p)$ is a function which converges to zero in $C^0\big([1/D,D]\times\overline{B_0(1)}\big)$ as $k \to +\infty$ and $C_1,\dots,C_4$ are positive constant only depending on n. With (6.2), we conclude with a degree argument (see [15]), and the remaining cases (when (M,g) is not locally conformally flat and $n \geqslant 11$) are treated in the same way.

Let us point out again that expansion (6.2) is computed by asymptotic analysis techniques and is not obtained *via* $H^1(M)$ estimates. Estimates (4.3) and (6.1) – which are much more precise than (4.5) – comes crucially into play to estimate the $\lambda_k^i(t,p)$ with a sufficiently high precision. The continuity of the remainders R_k^i – necessary for the concluding degree argument – is a direct consequence of the continuity of φ_k in (t,p) in strong spaces, as given by Proposition 5.1.

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RENORMALIZED SOLUTIONS TO A FOURTH ORDER NLS IN THE MASS SUBCRITICAL REGIME

Abstract. We study the mixed dispersion fourth order nonlinear Schrödinger equation

$$i\partial_t\psi - \gamma\Delta^2\psi + \beta\Delta\psi + |\psi|^{2\sigma}\psi = 0 \text{ in } \mathbb{R}\times\mathbb{R}^N,$$

where $\gamma,\sigma > 0$ and $\beta \in \mathbb{R}$. We focus on standing wave solutions, namely solutions of the form $\psi(x,t) = e^{i\alpha t}u(x)$, for some $\alpha \in \mathbb{R}$. This ansatz yields the fourth-order elliptic equation

$$\gamma\Delta^2 u - \beta\Delta u + \alpha u = |u|^{2\sigma}u.$$

We consider an associated mass constrained minimization problem in the case $\sigma N < 4$. Under suitable conditions, we establish existence of minimizers and we investigate their qualitative properties. Based on a joint work with D. Bonheure, E.M. dos Santos and R. Nascimento [5].

1. Introduction

We consider the following mixed dispersion fourth order nonlinear Schrödinger equation

$$\text{(Mixed 4NLS)} \quad i\partial_t\psi - \gamma\Delta^2\psi + \Delta\psi + |\psi|^{2\sigma}\psi = 0,\ \psi(0,x) = \psi_0(x),\ (t,x) \in \mathbb{R}\times\mathbb{R}^N,$$

for $\sigma > 0$ and some $\gamma > 0$. The fourth order term has been introduced by Karpman and Shagalov (see [15] and the references therein) to regularize and stabilize solutions to the standard nonlinear Schrödinger equation

$$\text{(NLS)} \qquad i\partial_t\psi + \Delta\psi + |\psi|^{2\sigma}\psi = 0,\ \psi(0,x) = \psi_0(x),\ (t,x) \in \mathbb{R}\times\mathbb{R}^N.$$

Indeed, it is well-known that, when $\sigma N < 2$, all solutions to (NLS) exist globally in time and standing waves (solutions of the form $\psi(t,x) = e^{i\alpha t}u(x)$ for some $\alpha \in \mathbb{R}$) are orbitally stable. Whereas if $\sigma N \geqslant 2$, then finite time blow-up may appear and standing wave solutions become unstable. We refer for instance to [10, 19]. Observe that for $N = 2$ and $N = 3$, the Kerr nonlinearity ($\sigma = 1$) is respectively critical and supercritical. Using a combination of stability analysis and numerical simulations, Karpman and Shagalov (see also [12]) showed that when $0 < N\sigma < 4$ and (γ is small enough if $2 \leqslant N\sigma < 4$), standing wave solutions to (Mixed 4NLS) exist globally in time and are stable and when $N\sigma > 4$, they become unstable. Notice that the Kerr nonlinearity is now subcritical in dimension 2 and 3 in this extended model.

The equation (Mixed 4NLS) has attracted less attention than its classical counterpart (NLS) though with an increasing interest more recently. We refer to the works

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by Ben-Artzi-Koch-Saut [2] and Pausader [17] for well-posedness and scattering, see also [13, 18] and to the recent work of Boulenger-Lenzmann [7] and the references therein concerning finite-time blow-up. We also mention that the one-dimensional stationary mixed dispersion NLS has been studied in [1] and [8].

Here, we focus on standing wave solutions of (Mixed 4NLS). The ansatz $\psi(t,x) = e^{i\alpha t}u(x)$ yields the fourth-order semilinear elliptic equation

$$\gamma\Delta^2 u - \Delta u + \alpha u = |u|^{2\sigma}u \text{ in } \mathbb{R}^N. \tag{1.1}$$

Observe that two constrained minimization problems naturally arise as for (NLS). Indeed, if one looks for time independent solutions, it is natural to consider the following problem

$$m = \inf_{u\in\tilde{M}} J_{\gamma,\alpha}(u), \tag{1.2}$$

where $J_{\gamma,\alpha} : H^2(\mathbb{R}^N) \to \mathbb{R}$ is the quadratic form defined by

$$J_{\gamma,\alpha}(u) = \gamma\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \alpha\int_{\mathbb{R}^N} |u|^2\,dx, \tag{1.3}$$

and

$$\tilde{M} = \{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^{2\sigma+2} dx = 1\}. \tag{1.4}$$

Notice that if m is achieved by some $u \in \tilde{M}$, then $v = m^{\frac{1}{2\sigma}}u$ is a solution to (1.1). The following result is proved in [6].

THEOREM 1.1 ([6, Theorem 1.1]). Assume $\alpha,\gamma > 0$ and $2 < 2\sigma+2 < 2N/(N-4)$ if $N \geqslant 5$. Then problem (1.2) has a nontrivial solution. If $\alpha \leqslant 1/(4\gamma)$, then any least energy solution does not change sign, is radially symmetric around some point and strictly radially decreasing.

We now turn to the second natural variational problem associated with (Mixed 4NLS) which will be our main focus. Since the L^2-norm is conserved along the flow for (Mixed 4NLS), it is natural to look for standing waves having a prescribed L^2-norm. Such solutions were built by Cazenave and Lions [11] for (NLS). Their construction consists in minimizing the functional $E_0 : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$E_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2}\,dx \tag{1.5}$$

under the constraint $\|u\|_{L^2}^2 = \mu$. If $0 < N\sigma < 2$, E_0 achieves its infimum and any associated minimizer solves

$$-\Delta u + \alpha u = |u|^{2\sigma}u \text{ in } \mathbb{R}^N, \tag{1.6}$$

with the Lagrange multiplier

$$\alpha = \frac{1}{\mu}\left(\int_{\mathbb{R}^N} |u|^{2\sigma+2}\,dx - \int_{\mathbb{R}^N} |\nabla u|^2\,dx\right). \tag{1.7}$$

Moreover, Cazenave and Lions [11, Theorem II.2] showed that those standing waves minimizing E_0 are orbitally stable for (NLS) whereas standing waves built for instance in [3, 4] are unstable for $N/2 < \sigma < 2/(N-2)$ as arbitrarily close initial conditions lead to blowing up solutions, see [11, Remark II.2] .

For (Mixed 4NLS), we obtain the following counterpart. Define

$$I_\gamma(\mu) = \inf_{u\in M_\mu} E_\gamma(u) \tag{1.8}$$

where

$$M_\mu = \{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2\,dx = \mu\} \tag{1.9}$$

and

$$E_\gamma(u) = \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2}\,dx. \tag{1.10}$$

If $I_\gamma(\mu)$ is achieved, then any associated minimizer solves (1.1) with the Lagrange multiplier

$$\alpha = \frac{1}{\mu}\left(\int_{\mathbb{R}^N} |u|^{2\sigma+2}\,dx - \gamma\int_{\mathbb{R}^N} |\Delta u|^2\,dx - \int_{\mathbb{R}^N} |\nabla u|^2\,dx\right). \tag{1.11}$$

Our main result is the following:

THEOREM 1.2. Assume $\gamma > 0$. If $0 < \sigma < 2/N$, then $I_\gamma(\mu)$ is achieved for every $\mu > 0$. If $2/N \leqslant \sigma < 4/N$, then there exists a critical mass $\mu_c(\gamma,\sigma)$ such that

1. $I_\gamma(\mu)$ is not achieved if $\mu < \mu_c$;
2. $I_\gamma(\mu)$ is achieved if $\mu > \mu_c$ and $\sigma = 2/N$;
3. $I_\gamma(\mu)$ is achieved if $\mu \geqslant \mu_c$ and $\sigma \neq 2/N$;

As far as we know, it is the first result in the literature concerning the existence of standing waves of (Mixed 4NLS) with a prescribed L^2 mass. Observe that the mass threshold for existence is due to a lack of homogeneity. Indeed, all the terms of the functional to be minimized scale differently. Such a behaviour is present in other models like the Schrödinger-Poisson equation, see [9, 14].

The plan of this paper is the following : in a first time, we sketch the proof of Theorem 1.2. Then, we discuss briefly the qualitative properties of solutions such as positivity, symmetry and stability.

2. Existence of standing waves with a prescribed mass

In all the following, to avoid technical issues and simplify notations, we restrict ourselves to the case $N > 4$ and we assume that $\sigma \neq 2/N$.

2.1. Gagliardo-Nirenberg interpolation inequalities

We begin by recalling two well-known Gagliardo-Nirenberg interpolation inequalities for functions $u \in H^2(\mathbb{R}^N)$, namely, for $0 \leqslant \sigma < \frac{4}{N-4}$,

$$\|u\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant B_N(\sigma)\|\Delta u\|_{L^2}^{\frac{\sigma N}{2}}\|u\|_{L^2}^{2+2\sigma-\frac{\sigma N}{2}}, \tag{2.1}$$

and, for $0 \leqslant \sigma < \frac{2}{N-2}$,

$$\|u\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant C_N(\sigma)\|\nabla u\|_{L^2}^{\sigma N}\|u\|_{L^2}^{2+\sigma(2-N)}. \tag{2.2}$$

The constants $B_N(\sigma)$ and $C_N(\sigma)$ depend on σ and N. Thanks to these inequalities, we can prove a 2-parameters Gagliardo-Nirenberg interpolation type inequality involving the L^2 norms of $u, \nabla u$ and Δu.

LEMMA 2.1. Assume $0 < \sigma < 4/(N-4)$ if $N > 4$. Let $0 < \delta < \sigma < \tau$ and assume $\tau < 4/(N-4)$ and $\delta < 2/(N-2)$. Then, there exists $C > 0$ such that, for any $u \in H^2(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |u|^{2\sigma+2}dx \leqslant C\left(\int_{\mathbb{R}^N} u^2dx\right)^p\left(\int_{\mathbb{R}^N} |\nabla u|^2dx\right)^q\left(\int_{\mathbb{R}^N} |\Delta u|^2dx\right)^r, \tag{2.3}$$

where $p = 1-\frac{\sigma(N-4)+N\delta(1-\lambda)}{4}$, $q = \frac{\delta N}{2}(1-\lambda)$, $r = \frac{\tau N}{4}\lambda$ and $\lambda = (\sigma-\delta)/(\tau-\delta)$. Moreover, we have $C \leqslant (B_N(\sigma))^{\lambda}(C_N(\sigma))^{1-\lambda}$.

As a direct consequence of this lemma, when $2/N < \sigma < 4/N$, there exists a constant $C_{\sigma,N} > 0$ such that

$$\int_{\mathbb{R}^N} |u|^{2\sigma+2}dx \leqslant C_{\sigma,N}\left(\int_{\mathbb{R}^N} |u|^2dx\right)^{\sigma}\left(\int_{\mathbb{R}^N} |\Delta u|^2dx\right)^{\frac{\sigma N}{2}-1}\left(\int_{\mathbb{R}^N} |\nabla u|^2dx\right)^{\frac{4-\sigma N}{2}}. \tag{2.4}$$

This inequality will be very useful in the following.

2.2. Estimates of the energy

This subsection is devoted to energy estimates on the functional E_γ. The main aim is to deduce the sign of $I_\gamma(\mu)$ as a function of μ. We begin by showing the coercivity of E_γ.

LEMMA 2.2. The energy E_γ is bounded from below and coercive over M_μ when $0 < \sigma < 4/N$. Moreover, for $\sigma \in (0, 4/N)$ the map $\mu \mapsto I_{\gamma,1}(\mu)$ is non-increasing, $I_{\gamma,1}(\mu) \leqslant 0$ for all $\mu > 0$. When $\sigma > 4/N$, we have $I_{\gamma,1}(\mu) = -\infty$ for every $\mu > 0$.

Proof. First, we infer from the Gagliardo-Nirenberg inequality (2.1) that

$$\begin{aligned} E_\gamma(u) &= \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2}\,dx \\ &\geqslant \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2\,dx - \frac{B_N(\sigma)\mu^{1+\sigma-\frac{\sigma N}{4}}}{2\sigma+2}\left(\int_{\mathbb{R}^N} |\Delta u|^2\,dx\right)^{\frac{\sigma N}{4}}. \end{aligned}$$

This shows that the functional E_γ is bounded from below and coercive over M_μ when $0<\sigma<4/N$. Now, let $u \in M_\mu$ and consider $u_\lambda(x) = \lambda^{\frac{N}{2}} u(\lambda x)$ for $\lambda>0$ so that $u_\lambda \in M_\mu$. Then,

$$I_\gamma(\mu) \leqslant E_\gamma(u_\lambda) = \frac{\gamma\lambda^4}{2}\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \frac{\lambda^2}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{\lambda^{\sigma N}}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2}\,dx \tag{2.5}$$

for all $\lambda > 0$. Letting λ go to zero, we get $I_\gamma(\mu) \leqslant 0$. Note that the so-called *large inequalities*

$$I_\gamma(\mu) \leqslant I_\gamma(\theta) + I_\gamma(\mu-\theta), \qquad \text{for all } \theta \in]0,\mu[, \tag{2.6}$$

always hold true. Indeed, for any $\varepsilon > 0$ we may choose test functions $u_\varepsilon \in M_\theta$ and $v_\varepsilon \in M_{\mu-\theta}$ with compact supports such that

$$I_\gamma(\theta) \leqslant E_\gamma(u_\varepsilon) \leqslant I_\gamma(\theta)+\varepsilon, \qquad I_\gamma(\mu-\theta) \leqslant E_\gamma(v_\varepsilon) \leqslant I_\gamma(\mu-\theta)+\varepsilon.$$

Then, if $e \in \mathbb{R}^N$ is a unit vector, we have that for k large enough the supports of u_ε and $v_\varepsilon(\cdot+ke)$ are disjoint. So, using the translation invariance of E_γ and M_μ we have $u_\varepsilon + v(\cdot+ke) \in M_\mu$ for k large and therefore

$$I_\gamma(\mu) \leqslant \limsup_{k\to\infty} E_\gamma(u_\varepsilon + v_\varepsilon(\cdot+ke)) \leqslant I_\gamma(\theta) + I_\gamma(\mu-\theta) + 2\varepsilon.$$

Hence, (2.6) holds and as a consequence we infer that $\mu \mapsto I_\gamma(\mu)$ is non-increasing since $I_\gamma(\mu) \leqslant 0$ for all μ. We finally observe that the last claim follows by letting $\lambda\to\infty$ in (2.5) when $\sigma > 4/N$. □

Using scaling arguments, it is possible to show that $I_\gamma(\mu)$ is strictly negative when σ is H^1-subcritical or the mass is large.

LEMMA 2.3. Let $0<\sigma<2/N$. For any given $\mu>0$, we have $I_\gamma(\mu)<0$. Moreover, if $2/N<\sigma<4/N$, we have that $I_\gamma(\mu)<0$ for large enough μ.

Proof. Take $u \in M_\mu$ and set $u_\lambda(x) = \lambda^{\frac{N}{2}} u(\lambda x)$ for $\lambda > 0$. We have

$$\frac{E_\gamma(u_\lambda)}{\lambda^{\sigma N}} = \frac{\gamma\lambda^{4-\sigma N}}{2}\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \frac{\lambda^{2-\sigma N}}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2}\,dx.$$

Taking λ small enough, we deduce that $I_\gamma(\mu)<0$ when $0<\sigma<2/N$. Next, we assume $2/N<\sigma<4/N$. Taking $u\in M_1$, it is easy to see that $E_\gamma(\sqrt{\mu}u)<0$ when μ is large enough. □

Using the extended Gagliardo-Nirenberg interpolation inequality (2.4), it is possible to deduce some refined estimates on the sign of $I_\gamma(\mu)$ when $2/N < \sigma < 4/N$.

LEMMA 2.4. Let $2/N < \sigma < 4/N$. There exists an (explicit) constant μ_c depending on $C_{\sigma,N}$ and γ such that $I_\gamma(\mu) = 0$ if and only if $\mu \leqslant \mu_c$.

2.3. Proof of Theorem 1.2

We begin by sketching the proof of point 3 when $\mu > \mu_c$ (we will not discuss the equality case). Recall that we always have the following inequality

$$I_\gamma(\mu) \leqslant I_\gamma(\theta) + I_\gamma(\mu - \theta), \qquad \text{for all } \theta \in]0,\mu[.$$

It is standard that the Concentration-Compactness method [16] yields that the minimizing sequences, up to translations, are relatively compact if and only if the *strict subaddivity condition* holds, namely

$$I_\gamma(\mu) < I_\gamma(\theta) + I_\gamma(\mu - \theta), \qquad \text{for all } \theta \in]0,\mu[. \tag{2.7}$$

In fact, arguing as in [16], the inequality (2.7) is easily obtained provided $I_\gamma(\mu) < 0$ which holds true thanks to the two previous lemma.

Next we sketch the proof of point 1. Assume by contradiction that there exists $\tilde{\mu} \in (0,\mu_c)$ such that $I_\gamma(\tilde{\mu})$ has a minimizer $u_{\tilde{\mu}}$. From the definition of μ_c, we have that $I_\gamma(\mu_c) = 0$. It is easy to check that, for $t > 1$, we have

$$I_\gamma(t\tilde{\mu}) \leqslant E_\gamma(\sqrt{t}u_{\tilde{\mu}}) < tE_\gamma(u_{\tilde{\mu}}) = tI_\gamma(\tilde{\mu}),$$

which implies that $I_\gamma(\mu_1) < 0$ for any $\mu_1 > \tilde{\mu}$. Hence, a contradiction with the definition of μ_c.

3. Qualitative properties

3.1. Existence of positive standing waves with a prescribed mass

In this section, we consider the slightly modified minimization problem

$$\tilde{I}_\gamma(\mu) = \inf_{u \in M_\mu} \tilde{E}_\gamma(u) \tag{3.1}$$

where M_μ is defined as in (1.9) and

$$\tilde{E}_\gamma(u) = \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |u^+|^{2\sigma+2}\,dx. \tag{3.2}$$

The proof of Theorem 1.2 applies to problem (3.1) with straightforward modifications. Let us recall that if u is a solution of problem (3.1), then u solves

$$\gamma\Delta^2 u - \Delta u + \alpha(\mu)u = |u|^{2\sigma}u^+, \tag{3.3}$$

where

$$-\alpha(\mu)\mu = \int_{\mathbb{R}^N} (\gamma|\Delta u|^2 + |\nabla u|^2 - |u|^{2\sigma+2})\,dx = 2E_\gamma(u) - \frac{\sigma}{\sigma+1}\int_{\mathbb{R}^N} |u^+|^{2\sigma+2}\,dx. \tag{3.4}$$

It is immediate to see that $\alpha(\mu) \geqslant 0$. We establish the positivity of solutions to (3.1) provided that $\alpha(\mu)$ is small enough.

PROPOSITION 3.1. Suppose that $\alpha(\mu) \leqslant 1/(4\gamma)$. Then, any solution to (3.1) is strictly positive (or strictly negative).

Proof. Using that $\alpha(\mu) \leqslant 1/(4\gamma)$, we can rewrite the equation satisfied by u as

$$\begin{cases} -\sqrt{\gamma}\Delta u + \lambda_1 u &= v, \\ -\sqrt{\gamma}\Delta v + \lambda_2 v &= |u|^{2\sigma}u^+, \end{cases}$$

for some positive constants λ_i, $i = 1,2$ satisfying $\lambda_2\lambda_1 = \alpha(\mu)$ and $\lambda_1 + \lambda_2 = 1/\sqrt{\gamma}$. It is then standard to see that $u > 0$. □

We next estimate the Lagrange multiplier of problem (3.1) by the L^2 mass, namely we will prove

$$\alpha(\mu) \leqslant C\mu^{\frac{\sigma}{1-\sigma N/4}}$$

for some $C > 0$. This estimate enables us to apply the previous theorem when the mass is small enough.

COROLLARY 3.1. Assume $0 < \sigma < 4/N$. There exists $\mu_0 > 0$ such that for all $\mu \leqslant \mu_0$ then

$$\alpha(\mu) \leqslant 1/(4\gamma),$$

and therefore any solution to (3.1) is strictly positive (or strictly negative).

Proof. First, we recall that any solution to (3.1) satisfies the Pohozaev identity

$$2\gamma\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \int_{\mathbb{R}^N} |\nabla u|^2\,dx = \frac{\sigma N}{2\sigma+2}\int_{\mathbb{R}^N} |u^+|^{2\sigma+2}\,dx.$$

Thanks to this equality, we deduce that

$$\alpha(\mu) \leqslant \frac{1}{\mu}\left(2 - \frac{\sigma N}{2\sigma+2}\right)\int_{\mathbb{R}^N} |u^+|^{2\sigma+2}\,dx.$$

The Gagliardo-Nirenberg inequality (2.1) then implies that

$$\int_{\mathbb{R}^N} |u^+|^{2\sigma+2}\,dx \leqslant B_N(\sigma)\left(\int_{\mathbb{R}^N} |u^+|^{2\sigma+2}\,dx\right)^{\frac{\sigma N}{4}}\left(\int_{\mathbb{R}^N} |u|^2\,dx\right)^{1+\sigma-\frac{\sigma N}{4}}.$$

Combining the two previous lines, we conclude that

$$\alpha(\mu) \leqslant B_N(\sigma)\left(2 - \frac{\sigma N}{2\sigma+2}\right)\mu^{\frac{\sigma}{1-\sigma N/4}}.$$

□

3.2. Radial symmetry of at least one minimal standing wave with prescribed mass

Using the method of [7], one can show that at least one solution of (1.8) is radially symmetric if $2\sigma \in \mathbb{N}_0$.

PROPOSITION 3.2. Suppose that problem (1.8) has a minimizer and assume $2\sigma \in \mathbb{N}_0$. Then there exists at least one radially symmetric minimizer for (1.8).

Proof. The proof is a direct adaptation of [7, Appendix A.2]. The main ingredient of their proof is the *Fourier rearrangement*. Namely, for $u \in L^2(\mathbb{R}^N)$ we set its Fourier rearrangement by $u^\sharp = \mathcal{F}^{-1}\{(\mathcal{F}u)^*\}$, where $\mathcal{F}$ stands for the Fourier transform and f^* denotes the symmetric-decreasing rearrangement of a measurable function $f : \mathbb{R}^N \to \mathbb{R}$ that vanishes at infinity. Observe that $\left\|u^\sharp\right\|_{L^2} = \|u\|_{L^2}$. Then, assuming that $u \in H^2$, we have [7]

$$\left\|\Delta u^\sharp\right\|_{L^2} \leqslant \|\Delta u\|_{L^2}\,,\ \left\|\nabla u^\sharp\right\|_{L^2} \leqslant \|\nabla u\|_{L^2}\ \textit{and}\ \|u\|_{L^{2m}} \leqslant \left\|u^\sharp\right\|_{L^{2m}}, \tag{3.5}$$

for any $m \in \mathbb{N}_0$. Therefore, if u is a minimizer for (1.8), then $u^\sharp$ is a minimizer as well. □

It is an open problem to extend the previous proposition for $2\sigma \notin \mathbb{N}_0$. Observe also that we do not know whether or not all solutions of (1.8) are radially symmetric even if $2\sigma \in \mathbb{N}_0$. Indeed, Boulenger and Lenzmann proved that equality holds in (3.5) if and only if $|\mathcal{F}u| = |\mathcal{F}u|^*$.

3.3. Orbital stability

Finally, using the method of Cazenave and Lions [11], we prove the orbital stability of the set of solutions to (1.8). Let us begin with a definition.

DEFINITION 3.1. Let $\mathcal{G} := \{U \in H^2(\mathbb{R}^N) :\ U \textit{ is a solution of } (1.8)\}$. We say that the set $\mathcal{G}$ is stable in $\mathbb{H}^2 = H^2(\mathbb{R}^N) \times H^2(\mathbb{R}^N)$ if, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $\Psi_0 \in H^2(\mathbb{R}^N)$ satisfies $\|\Psi_0 - U\|_{H^2} < \delta$ for some $U \in \mathcal{G}$, then the solution $\Psi(t)$ of (Mixed 4NLS) with initial data Ψ_0 exists for all $t \geqslant 0$ and satisfies, for some $V \in \mathcal{G}$,

$$d(\Psi(t), V) < \varepsilon,\ \textit{for all } t \geqslant 0,$$

where, for any $f, g \in \mathbb{H}^2$, $d(f,g) := \inf\left\{\|f(.) - e^{i\theta}g(. - r)\|_{\mathbb{H}^2} : \theta, r \in \mathbb{R}\right\}$.

As a direct consequence of Theorem 1.2, we have:

THEOREM 3.1. The set $\mathcal{G}$ is stable.

Proof. Let $U \in \mathcal{G}$. Assume by contradiction that there exists a sequence of solutions $(\Psi_k)_k$ of (Mixed 4NLS) with $\Psi_k(0) = \varphi_k$, for some $(\varphi_k)_k$ such that $\lim_{k\to\infty} \|\varphi_k - U\|_{H^2} = 0$ and such that there exists $(t_k)_k \subset \mathbb{R}^+$ with $d(\Psi_k(t_k), U) \geqslant \varepsilon$, for some $\varepsilon > 0$ fixed. Using the conservation of the energy and the mass, it is easy to see that $\|\Psi_k(t_k)\|_{L^2} \to$

$\|U\|_{L^2}$ and $E_\gamma(\Psi_k(t_k)) \to E_\gamma(U) = I_\gamma$ as $k \to \infty$. Therefore, using Theorem 1.2, we get that $d(\Psi_k(t_k), V) \to 0$, for some $V \in \mathcal{G}$ which gives a contradiction. □

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PARTIAL SYMMETRY OF SOLUTIONS TO PARABOLIC PROBLEMS VIA REFLECTION METHODS

Abstract. We survey several results on radial and axial symmetry of solutions to parabolic and elliptic systems and equations with Neumann and Dirichlet boundary conditions. As a particular case, these results describe the qualitative properties of solutions to cooperative and competitive systems coming from models in physics and biology.

1. Introduction.

The relationship between the symmetry of a solution to a nonlinear partial differential equation (PDE) and the symmetries of the data has been a very active field of research in the past years. Sometimes the physical motivation of a problem suggests strongly that any solution should inherit the symmetry of the data. Whenever this is true, the proofs are often far from trivial and are a source of new, powerful, and elegant methods and techniques. Surprisingly, in many situations the phenomenon of *symmetry breaking* can occur, even in a strongly symmetric setting. In this short survey we recall and comment on some of the symmetry results that are available for elliptic and parabolic systems and equations. The proofs are mostly based on reflection methods, which means that a nonlinear problem is linearized using reflections, and then these linear problems are studied with a perturbation argument using different types of maximum principles. The main challenge is to adapt this scheme to each of the different problems. As we comment below in more detail, a proof via reflection methods varies substantially between Neumann and Dirichlet boundary conditions, for example, or between scalar equations and systems. Furthermore, in the case of systems, proofs of symmetry results and the kind of symmetry that is obtained, vary according to the way components interact (cooperatively or competitively).

The main goal of this survey is to focus on *parabolic problems*, and we obtain straightforward corollaries for elliptic problems. For more results on reflection methods focused on elliptic equations we refer to [18].

This survey is organized as follows. We begin presenting one of the most well-known techniques to obtain symmetry via reflection methods and discuss its limitations. Then we comment on some variants which yield partial symmetry results. In Section 2.2 we briefly discuss the case of Neumann boundary conditions in scalar equations and finally, in Section 2.3, we present some applications to models coming from ecology and physics.

2. Moving plane method and radial symmetry

One of the most versatile and robust methods to show symmetry of solutions is the well-known *moving-plane method* (MPM). This technique has its roots in the work of Alexandrov [1], who investigated surfaces with constant mean curvature; then Serrin [16] elaborated it in order to analyze overdetermined boundary value problems associated with elliptic PDEs. In the seminal paper [5], Gidas, Ni, and Nirenberg developed a powerful variant of Serrin's argument to derive, in particular, radial symmetry of positive solutions to some elliptic problems in balls.

To be precise, consider the elliptic problem

$$-\Delta u = f(u,|x|) \quad \text{in } B, \qquad u = 0 \quad \text{on } \partial B, \tag{2.1}$$

or its parabolic version

$$\begin{aligned} u_t - \Delta u &= f(u,|x|,t), && x \in B,\ t > 0, \\ u(x,t) &= 0, && x \in \partial B,\ t > 0, \\ u(x,0) &= u_{0,i}(x), && x \in B \end{aligned} \tag{2.2}$$

where $\Delta u = \sum_{i=1}^{N} \partial_{ii} u$ is the *Laplacian*, f is a smooth nonlinearity, and $B \subset \mathbb{R}^N$ ($N \geqslant 1$) is a ball or an annulus. In this setting, the MPM relies strongly on the following hypothesis:

(C) *Convexity of the domain, in particular, B cannot be an annulus.*

(P) *Positivity of the solution, in particular, u cannot be sign-changing.*

(M) *Monotonicity of the nonlinearity f in the $|x|$ variable.*

Note that a solution of (2.1) is also a (stationary) solution of (2.2), and therefore all the symmetry results that we present below for parabolic problems have immediate corollaries for the associated elliptic problem.

If u is a classical solution of (2.1) and assumptions (C), (P), and (M) are satisfied, then the MPM yields that u must be radially symmetric and monotone decreasing in the radial variable [5]. On the other hand, if u is a uniformly bounded classical solution of (2.2) and assumptions (C), (P), and (M) are satisfied, then a parabolic version of the MPM yields that u is *asymptotically radially symmetric and monotone decreasing*, that is, all elements in the *omega limit set of u* given by

$$\omega(u) := \{z \in C(\overline{B}) \,:\, \exists\, t_n \to \infty \quad \text{such that} \quad \lim_{n\to\infty} \|u(\cdot,t_n) - z\|_{L^\infty} = 0\},$$

are radially symmetric functions and decreasing in the radial variable [11]. These results extend to much more general situations, including fully nonlinear elliptic and parabolic problems, and domains which are only symmetric and convex in one direction (in this case one may only obtain reflectional symmetry and monotonicity with respect to a symmetry hyperplane). We refer to the survey [10] and its references for an account of these results.

2.1. Symmetry breaking and partial symmetries

If one removes any of the assumptions (C), (P), and (M), one cannot expect results on radial symmetry in general and explicit counterexamples are known to exist, see the discussion in [14] in this regard. However, even without (C), (P), and/or (M), one can expect that *some* symmetry should be inherited to solutions if the data of the problem is symmetric. This can be studied by suitably adapting the MPM. For example, a variant of the MPM, known as *rotating-plane method* (RPM), was used in [9] to prove a weaker notion of symmetry of solutions of elliptic problems without assuming (C), (P), and/or (M), but at the cost of some extra stability and convexity assumptions. The symmetry involved in the results from [9] is called *foliated Schwarz symmetry*. Since this symmetry plays an important role in the following results we give a precise definition. Let $N \geqslant 2$, $B \subset \mathbb{R}^N$ be a ball or an annulus, $\mathbb{S}^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$ be the unit sphere in $\mathbb{R}^N$, and let $p \in \mathbb{S}^{N-1}$. We say that a function $u \in C(B)$ is *foliated Schwarz symmetric with respect to* p if u is axially symmetric with respect to the axis $\mathbb{R}p$ and nonincreasing in the polar angle $\theta := \arccos(\frac{x}{|x|} \cdot p) \in [0, \pi]$.

The results from [9] already show that the MPM can be adjusted to study a variety of problems, however, the stability assumptions needed in [9] do not have a direct analogue for parabolic problems and therefore new ideas were needed to extend these results to problem (2.2).

In [14] a different kind of assumption was explored to substitute the extra stability and convexity assumptions required in [9]. This new kind of assumption is a weak condition imposed directly on the shape of the solution (in the case of elliptic problems) or on the initial profile (in the parabolic case), and refers to the existence of a *dominant half domain*. To state this hypothesis in a precise way we introduce some notation. For a vector $e \in \mathbb{S}^{N-1}$, consider the half domain $B(e) := \{x \in B : x \cdot e > 0\}$ and the reflection $\sigma_e : \overline{B} \to \overline{B}$ given by $\sigma_e(x) := x - 2(x \cdot e)e$ for each $x \in B$. For a function $v \in C(B)$, consider the following condition.

(G) There is $e \in \mathbb{S}^{N-1}$ such that $v(x) \geqslant v(\sigma_e(x))$ for all $x \in B(e)$ and $v \not\equiv v \circ \sigma_e$ in $B(e)$.

Geometrically, assumption (G) may be understood as requiring that v is slightly more concentrated on a half domain than in the other (with respect to reflections). The main result in [14] reads as follows.

THEOREM 1. *Let B be a ball or an annulus, $f : \mathbb{R} \times \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be a continuous function locally Lipschitz in the first variable uniformly with respect to the other two variables and such that $f(0, \cdot, \cdot)$ is bounded. Moreover, let u be a classical uniformly bounded solution of* (2.2) *such that the initial profile $u(\cdot, 0)$ satisfies (G). Then u is asymptotically foliated Schwarz symmetric, i.e., there is $p \in \mathbb{S}^{N-1}$ such that all elements of $\omega(u)$ are foliated Schwarz symmetric with respect to p.*

We stress that Theorem 1 applies to sign-changing solutions, to annular domains, and to non-monotone nonlinearities.

Condition (G) is sharp, in the sense that solutions of (2.2) which are not foliated Schwarz symmetric can be constructed if (G) is not satisfied. For precise references on how to build these counterexamples, we refer to [14].

2.2. Neumann boundary conditions

Although many (existence) results hold regardless of the boundary conditions (Neumann or Dirichlet, for instance), the Neumann boundary conditions are much less restrictive than the Dirichlet counterpart, and the symmetry is coerced to a lesser degree by the influence of the boundary. This is exemplified by the fact that the Gidas-Ni-Nirenberg result from [5] on radial symmetry of solutions to elliptic problems is no longer true under Neumann boundary conditions, see [8]. However, the technique used in Theorem 1 can be used to show that solutions satisfying Neumann boundary conditions are foliated Schwarz symmetric, although the proof requires some major adjustments, in particular, a quantitative Harnack-Hopf type lemma, which estimates the boundary values of solutions, is used. Since the statement is very similar to Theorem 1 (only changing the boundary conditions) we do not quote this result here, but we refer to [12, 15] for the precise statements and the proofs.

2.3. Applications

In this last section we would like to show how these ideas can be used to study some well-known models coming from biology and physics. To begin with, let us consider the *Lotka-Volterra system*,

$$\begin{aligned}
&(u_1)_t - \mu_1(t)\Delta u_1 = a_1(t)u_1 - b_1(t)u_1^2 - \alpha_1(t)u_1u_2 \quad \text{in } B\times(0,\infty),\\
&(u_2)_t - \mu_2(t)\Delta u_2 = a_2(t)u_2 - b_2(t)u_2^2 - \alpha_2(t)u_1u_2 \quad \text{in } B\times(0,\infty),\\
&\frac{\partial u_i}{\partial \nu} = 0 \quad \text{on } \partial B\times(0,\infty), \qquad u_i(x,0) = u_{0,i}(x) \geqslant 0 \quad \text{for } x\in B,\ i=1,2,
\end{aligned} \tag{2.3}$$

where B is a ball or an annulus in $\mathbb{R}^N$, ν denotes the outward-pointing normal vector field, and $u_{0,1}$ and $u_{0,2}$ are continuous functions. This system is commonly used to model two different species that compete for food, nesting sites, or resources in general. The coefficients μ_i, a_i, b_i, and α_i represent diffusion, birth, saturation, and competition rates respectively [2]; and the functions u_1 and u_2 represent the population density of each species. The time-dependence of the coefficients can be used to model the effect of different time periods (e.g. seasons) on the birth rates, the movement, or the aggressiveness of the species. Furthermore, the Neumann boundary conditions are a no-flux condition, which means that the species are isolated in a compound and individuals are not allowed to go in or out of the domain B.

In this setting, loosely speaking, the following asymptotic symmetry result is available: *if the initial profiles $u_{0,1}$ and $u_{0,2}$ satisfy a reflectional inequality, then the resulting population densities of the species become increasingly symmetric as the time variable t goes to infinity; in particular, they tend to be foliated Schwarz symmetric functions with a common symmetry axis but with respect to antipodal points.*

A precise statement is given by studying the symmetry and monotonicity properties of elements in the associated omega limit set of a classical solution (u_1,u_2) of

(2.3), which is defined as

$$\omega(u_1,u_2) := \{(z_1,z_2) \in C(\overline{B}) \times C(\overline{B}) : \max_{i=1,2} \lim_{n\to\infty} \|u_i(\cdot,t_n) - z_i\|_{L^\infty(B)} = 0 \text{ for some sequence } t_n \to \infty\}.$$

For global solutions which are uniformly bounded and have equicontinuous semiorbits $\{u_i(\cdot,t) : t \geqslant 1\}$, $i = 1,2$, the set $\omega(u_1,u_2)$ is nonempty, compact, and connected. The equicontinuity can be obtained under mild boundedness and regularity assumptions on the equation and using boundary and interior Hölder estimates, see [2, 15]. We are ready to give a precise statement taken from [15].

THEOREM 2. *Suppose that*

$$\begin{aligned} &a_i, b_i, \alpha_i \in L^\infty((0,\infty)) \text{ satisfy} \\ &a_i(t), b_i(t) \geqslant 0 \text{ for } t > 0 \quad \text{and} \quad \inf_{t>0} \alpha_i(t) > 0 \text{ for } i = 1,2. \end{aligned} \tag{2.4}$$

and let $u = (u_1,u_2)$ *be a classical solution of* (2.3) *such that* $\|u_i\|_{L^\infty(B\times(0,\infty))} < \infty$ *for* $i = 1,2$*. Moreover, assume that*

$$(G_s) \quad u_{0,1} \geqslant u_{0,1}\circ\sigma_e,\, u_{0,2} \leqslant u_{0,2}\circ\sigma_e \quad \text{in } B(e) \quad \begin{cases} \text{for some } e \in \mathbb{S}^{N-1} \text{ with} \\ u_{0,i} \not\equiv u_{0,i}\circ\sigma_e \text{ for } i = 1,2. \end{cases}$$

Then there is some $p \in \mathbb{S}^{N-1}$ *such that every* $(z_1,z_2) \in \omega(u_1,u_2)$ *has the property that* z_1 *is foliated Schwarz symmetric with respect to* p *and* z_2 *is foliated Schwarz symmetric with respect to* $-p$.

As far as we know, this is hitherto the only result available regarding symmetry for the Lotka-Volterra problem with competition, even in the stationary case with constant coefficients, i.e., the elliptic version of problem (2.3). We remark that the long-time dynamics of this system have a very rich structure and depend strongly on the relationships between the coefficients, see [2] for a broad discussion on these kind of systems.

As in the scalar case, if the assumption (G_s) is not satisfied, then it is possible to construct the following counterexample to the symmetry results.

THEOREM 3 (Theorem 7.1 in [15]). *Let* $k \in \mathbb{N}$*. Then there exists* $\varepsilon > 0$ *and* $\lambda > 0$ *such that*

$$\begin{aligned} -\Delta u_1 &= \lambda u_1 - u_1u_2 && \text{in } B, \\ -\Delta u_2 &= \lambda u_2 - u_1u_2 && \text{in } B, \\ \partial_\nu u_1 &= \partial_\nu u_2 = 0 && \text{on } \partial B. \end{aligned} \tag{2.5}$$

admits a positive classical solution (u_1,u_2) *in* $B := B_\varepsilon = \{x \in \mathbb{R}^2 : 1-\varepsilon < |x| < 1\} \subset \mathbb{R}^2$ *such that the angular derivatives* $\frac{\partial u_i}{\partial\theta}$ *of the components change sign at least* k *times on every circle contained in* $\overline{B}_\varepsilon$.

For cooperative systems, that is, if the coefficients α_1 and α_2 in (2.3) are *negative*, then much more is known for positive solutions on balls, since in this case a maximum principle for small domains can be used to perform a moving plane method and obtain radial symmetry and monotonicity, see [4]. In population dynamics, cooperative systems model species which help each other to survive.

We remark that the Neumann boundary conditions in (2.3) are essential in the proof of Theorem 2, since a reflection across the boundary preserving positivity of the solutions is done, which is only possible with zero normal derivatives at the boundary. The boundary conditions also play an important role in a normalization argument involved in the proofs. Therefore, a symmetry result for the Dirichlet version of (2.3) requires some major changes in the proof. This was studied in [13], where one of the key ingredients in the proofs is a new parabolic version of Serrin's boundary point lemma [16, Lemma 1] that provides bounds which depend *only* on given quantities. In ecology, Dirichlet boundary conditions can be used to model a very harmful environment at the boundary of the domain for the modeled species, due to lack of food, for example, or the presence of predators.

Finally, as a further example for applications, we mention the cubic system

$$
\begin{aligned}
(u_1)_t - \Delta u_1 &= \lambda_1 u_1 + \gamma_1 u_1^3 - \alpha_1 u_2^2 u_1 && \text{in } B\times(0,\infty),\\
(u_2)_t - \Delta u_2 &= \lambda_2 u_2 + \gamma_2 u_2^3 - \alpha_2 u_1^2 u_2 && \text{in } B\times(0,\infty),\\
\partial_\nu u_1 = \partial_\nu u_2 &= 0 && \text{on } \partial B\times(0,\infty),\\
u_i(x,0) = u_{0,i}(x) &\geqslant 0 && \text{for } x\in B,\ i=1,2,
\end{aligned}
\tag{2.6}
$$

where λ_i,γ_i, and α_i are positive constants. The elliptic counterpart of this system is being studied extensively due to its relevance in the study of binary mixtures of Bose-Einstein condensates, see [3]. The asymptotic symmetry of uniformly bounded classical solutions of this problem satisfying (G_s) can be characterized in a similar way as in Theorem 2. To see this, minor adjustments are needed in the proof of Theorem 2 to deal with a slightly different linearized system. Details can be found in [12]. Symmetry aspects of the elliptic counterpart of (2.6) have been studied in [17], exploiting the fact that this system has a variational structure.

We close this survey by mentioning that reflection methods have also been used to describe symmetry properties of solutions to parabolic problems involving pseudodifferential operators, such as the fractional Laplacian, which is an example of a nonlocal operator that has received a lot of attention in the recent years, and is used to model *long-distance* interactions. For more results in this direction we refer to [6, 7] and the references therein.

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A p-LAPLACIAN NEUMANN PROBLEM WITH A POSSIBLY SUPERCRITICAL NONLINEARITY

Abstract. We look for nonconstant, positive, radial, radially nondecreasing solutions of the quasilinear equation $-\Delta_p u + u^{p-1} = f(u)$ with $p > 2$, in the unit ball B of $\mathbb{R}^N$, subject to homogeneous Neumann boundary conditions. The assumptions on the nonlinearity f are very mild and allow it to be possibly supercritical in the sense of Sobolev embeddings. The main tools used are the truncation method and a mountain pass-type argument. In the pure power case, i.e., $f(u) = u^{q-1}$, we detect the limit profile of the solutions of the problems as $q \to \infty$. These results are proved in [3], in collaboration with B. Noris.

1. Introduction and main results

In [3], we study the existence of nonconstant, radially nondecreasing solutions of the following quasilinear problem

$$\begin{cases} -\Delta_p u + u^{p-1} = f(u) & \text{in } B, \\ u > 0 & \text{in } B, \\ \partial_\nu u = 0 & \text{on } \partial B, \end{cases} \tag{1.1}$$

where B is the unit ball of $\mathbb{R}^N$, $N \geqslant 1$, ν is the outer unit normal of ∂B, and $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, with $p > 2$. We require very mild assumptions on the nonlinearity f on the right-hand side, namely $f \in C^1([0,\infty))$ and satisfies the following hypotheses

(f_1) $\lim_{s\to 0^+} \frac{f(s)}{s^{p-1}} \in [0,1)$;

(f_2) $\liminf_{s\to +\infty} \frac{f(s)}{s^{p-1}} > 1$;

(f_3) $\exists$ a constant $u_0 > 0$ such that $f(u_0) = u_0^{p-1}$ and $f'(u_0) > (p-1)u_0^{p-2}$.

Our main results in [3] read as follows.

THEOREM 1. *If f satisfies (f_1)-(f_3), there exists a nonconstant, radial, radially nondecreasing solution of* (1.1). *If furthermore there exist n different positive constants $u_0^{(1)} \neq \cdots \neq u_0^{(n)}$ for which (f_3) holds, then* (1.1) *admits at least n distinct nonconstant, radial, radially nondecreasing solutions.*

THEOREM 2. *Let $f(u) = u^{q-1}$, with $q > p$. Denote by u_q the solution found in Theorem 1, corresponding to such f. Then, as $q \to \infty$,*

$$u_q \to G \textit{ in } W^{1,p}(B) \cap C^{0,\mu}(\overline{B}) \quad \textit{for any } \mu \in (0,1),$$

where G is the unique solution of the Dirichlet problem

$$\begin{cases} -\Delta_p G + |G|^{p-2}G = 0 & \textit{in } B, \\ G = 1 & \textit{on } \partial B. \end{cases}$$

REMARKS.

• We observe that f is allowed to be supercritical in the sense of Sobolev embeddings, which will be the most interesting case.

• The model f is the pure power function $f(u) = u^{q-1}$, with $q > p$. In this case, problem (1.1) admits the constant solution $u \equiv 1$ for every $q > p$, including the supercritical case $q > p^*$, where $p^* := Np/(N-p)$ if $p < N$ and $p^* := +\infty$ otherwise. Therefore, the natural question that arises is whether (1.1) admits any *nonconstant* solutions. It is worth stressing a remarkable difference between problem (1.1) and the analogous problem under homogeneous Dirichlet boundary conditions. Indeed, it is well-known that, as a consequence of the Pohožaev identity (cf. [5, Section 2]), the Dirichlet problem does not admit any nonzero solutions when $q \geqslant p^*$.

• We remark that condition (f_3) is absolutely natural under (f_1) and (f_2). Indeed, by the regularity of f and by (f_1)-(f_2), there must exist an intersection point u_0 between f and the power s^{p-1} such that $f'(u_0) \geqslant (s^{p-1})'(u_0) = (p-1)u_0^{p-2}$. Hence, (f_3) is only meant to exclude the possibility of a degenerate situation in which f is tangent to s^{p-1} at u_0.

• We can always think f to satisfy also

(f_0) $f \geqslant 0$ and $f' \geqslant 0$.

Indeed, if this is not the case, we can replace f by $g(s) := f(s) + (m-1)s^{p-1}$ for a suitable $m > 1$ such that $g \geqslant 0$ and $g' \geqslant 0$, and study the equivalent problem

$$\begin{cases} -\Delta_p u + m u^{p-1} = g(u) & \text{in } B, \\ u > 0 & \text{in } B, \\ \partial_\nu u = 0 & \text{on } B. \end{cases}$$

Therefore, without loss of generality, *from now on in the paper we assume f to satisfy (f_0) as well.*

Since f is possibly supercritical, the energy functional I associated to the problem is not well defined in the whole of $W^{1,p}(B)$, and so a priori we cannot use variational techniques to solve the problem. This issue is overcome for the first time in [6] for the semilinear case ($p = 2$) and then in [7] for any $1 < p < \infty$, by working in the closed and convex cone

$$\mathcal{C} := \left\{ u \in W^{1,p}_{\mathrm{rad}}(B) : u \geqslant 0 \text{ and } u(r) \leqslant u(s) \text{ for } r \leqslant s \right\},$$

where we have denoted by $W^{1,p}_{\mathrm{rad}}(B)$ the space of $W^{1,p}(B)$-functions which are radially symmetric and with abuse of notation we have written $u(x) = u(r)$ for $|x| = r$. Indeed, this cone has the property that all its functions are bounded, i.e.,

$$(1.2) \qquad \|u\|_{L^\infty(B)} \leqslant C(N)\|u\|_{W^{1,p}(B)} \quad \text{for some } C(N) > 0 \text{ independent of } u \in \mathcal{C},$$

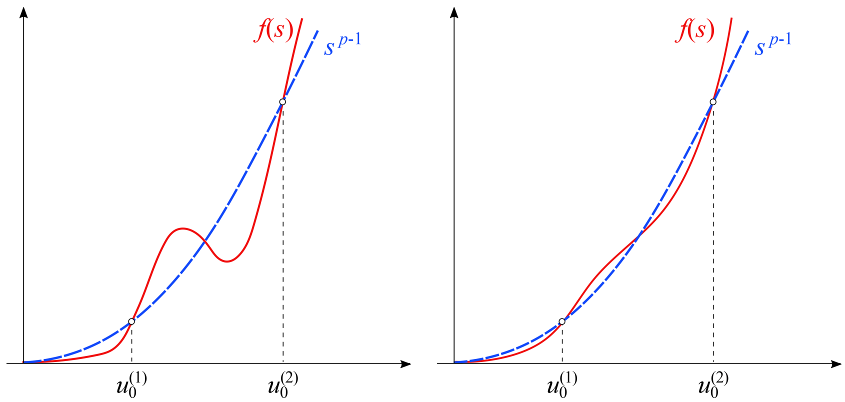


Figure 1: *Left*: Graph of a sample nonlinearity f satisfying (f_1)-(f_3). *Right*: Graph of a sample nonlinearity f satisfying (f_0)-(f_3).

see e.g. [3, Lemma 2.2]. Due to (1.2), it makes sense to define an energy functional I in $\mathcal{C}$, associated to the equation. On the other hand, the main disadvantage for working in this cone is the fact that it has empty interior in the $W^{1,p}$-topology. As a consequence, in general, critical points of $I : \mathcal{C} \to \mathbb{R}$ are not solutions of (1.1). In [6, 7], the authors require additional assumptions on f to prove that the critical point of I, found via variational techniques, is indeed a weak solution of the problem. While in [2], in order to weaken the hypotheses on f, a different strategy based on the truncation method is proposed.

The techniques that we use in [3] to prove Theorem 1 are essentially in the spirit of [2]. The scheme of the proof can be split into five steps.

STEP 1. We first obtain, in [3, Lemma 2.5], the following a priori estimate

$$\|u\|_{L^\infty(B)} \leqslant K_\infty \quad \text{ for all } u \in \mathcal{C} \text{ that solve } (1.1),$$

for some $K_\infty > 0$ independent of u. Clearly, $K_\infty \geqslant u_0$, being $u \equiv u_0$ a solution of (1.1) belonging to $\mathcal{C}$.

STEP 2. This allows us to truncate the nonlinearity f, in order to deal with a subcritical nonlinearity $\tilde{f}$, and so in [3, Lemma 3.1], we prove that

$$\textit{For all } \ell \in (p, p^*) \textit{ there exists } \tilde{f} \in C^1([0,\infty)) \textit{ satisfying } (f_0)\text{-}(f_3),$$
$$\lim_{s\to\infty} \frac{\tilde{f}(s)}{s^{\ell-1}} = 1, \quad \textit{and} \quad \tilde{f} = f \textit{ in } [0, K_\infty].$$

We introduce the following auxiliary problem

$$\begin{cases} -\Delta_p u + u^{p-1} = \tilde{f}(u) & \text{in } B, \\ u > 0 & \text{in } B, \\ \partial_\nu u = 0 & \text{on } \partial B. \end{cases} \tag{1.3}$$

As a consequence of the previous two steps, it is immediate to see that

In the cone $\mathcal{C}$, the two problems (1.1) and (1.3) are equivalent.

STEP 3. Thanks to the subcriticality of $\tilde{f}$, we can define the energy functional associated to (1.3) in the whole of $W^{1,p}(B)$ as follows

$$\tilde{I}(u) := \frac{1}{p}\int_B (|\nabla u|^p + |u|^p)dx - \int_B \tilde{F}(u)dx, \quad \text{where } \tilde{F}(u) := \int_0^u \tilde{f}(s)ds$$

for all $u \in W^{1,p}(B)$. All critical points of $\tilde{I}$ are weak solutions of (1.3).

REMARK 1. Since $p > 2$, $\tilde{I}$ is of class C^2, while if $1 < p < 2$, the functional $\tilde{I}$ is only of class C^1. This lack of regularity prevents either the use of second order Taylor expansions as done in [2, 3] (see also Section 3 below) or the use of a generalized Morse Lemma when looking for *nonconstant* solutions. Moreover, when $1 < p < 2$, Simon's inequalites relating $\tilde{I}'$ and the pseudo-differential gradient are weaker than the ones found for the case $p > 2$, this makes harder the construction of a descending flow and consequently the proof of a deformation-type lemma.

STEP 4. We find a critical point u of $\tilde{I}$ belonging to $\mathcal{C}$ via a mountain pass-type argument. We localize the solution in such a way that, if we have n different positive constants $u_0^{(i)}$ verifying (f_3), we get "for free" also the multiplicity result stated in Theorem 1.

STEP 5. We prove that the solution found in Step 4. is nonconstant, by using a second order Taylor expansion of $\tilde{I}$.

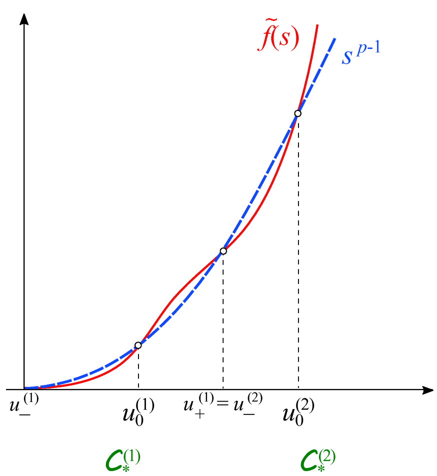
In the next two sections we give some details about Steps 4. and 5., respectively. While in the last section we sketch the proof of Theorem 2.

2. Step 4: A nonconstant solution of (1.1) belonging to $\mathcal{C}$

Due to the subcriticality of $\tilde{f}$, it is standard to prove the following compactness result (cf. [3, Lemma 3.3]):

The functional $\tilde{I}$ satisfies the Palais-Smale condition.

The restricted cone $\mathcal{C}_*$



Let $n \in \mathbb{N}$ be the number of positive constants $u_0^{(i)}$ satisfying (f_3). For every $i = 1,\dots,n$, we set

$$u_-^{(i)} := \sup\left\{s \in [0, u_0^{(i)}) \,:\, \tilde f(s) = s^{p-1}\right\},$$
$$u_+^{(i)} := \inf\left\{s \in (u_0^{(i)}, \infty) \,:\, \tilde f(s) = s^{p-1}\right\}.$$

For every i, we further introduce the following subset of $\mathcal{C}$

$$\mathcal{C}_*^{(i)} := \left\{u \in \mathcal{C} \,:\, u_-^{(i)} \leqslant u \leqslant u_+^{(i)}\right\}$$

which turns out to be itself a closed convex cone of $W^{1,p}(B)$.

REMARKS.

- Thanks to (f_3), each $u_0^{(i)}$ is an isolated zero of $\tilde f(s) - s^{p-1}$, hence $u_-^{(i)} \neq u_0^{(i)} \neq u_+^{(i)}$ for every $i = 1,\dots,n$.
- We observe that $u_+^{(n)}$ can be possibly $+\infty$. For instance, for the pure power function $f(u) = u^{q-1}$ with $q > p$, it results $n = 1$, $u_- = 0$, $u_0 = 1$, $u_+ = +\infty$, and $\mathcal{C} = \mathcal{C}_*$.
- All and only the zeros of $\tilde f(s) - s^{p-1}$ are constant solutions of (1.3), and so of (1.1). Hence, the only constant solutions of (1.1) belonging to $\mathcal{C}_*^{(i)}$ are $u_-^{(i)}$, $u_0^{(i)}$, and $u_+^{(i)}$.
- If we prove the existence of a nonconstant solution u belonging to $\mathcal{C}_*^{(i)}$, we know at once that $u_-^{(i)} \leqslant u \leqslant u_+^{(i)}$ and that $u_-^{(i)} \not\equiv u \not\equiv u_+^{(i)}$. This implies that nonconstant solutions of (1.1) belonging to different $\mathcal{C}_*^{(i)}$'s are different.

As a consequence of the last two remarks, we can see that the advantage of working in $\mathcal{C}_*^{(i)}$ instead of $\mathcal{C}$ is twofold. Firstly, it helps avoiding constant solutions: it is enough to prove that the solution found is none of the three constant solutions in $\mathcal{C}_*^{(i)}$. Secondly, the restricted cone $\mathcal{C}_*^{(i)}$ allows us to localize our solution, so that the multiplicity part of Theorem 1 follows immediately by the existence part.

Hereafter, we assume for simplicity $n = 1$ and we omit all the superscripts (i). Clearly, if $n > 1$, it is possible to repeat the same arguments in each cone $\mathcal{C}_*^{(i)}$.

A deformation lemma

This is the most technical part of the proof. Since the space $W^{1,p}(B)$ in which the energy functional $\tilde I$ is defined is bigger than the set $\mathcal{C}_*$ in which we want to find a minimax solution, we need a slightly different version of the deformation lemma.

LEMMA 1 (Lemma 3.8 of [3]). *Let $c \in \mathbb{R}$ be such that $\tilde I'(u) \neq 0$ for all $u \in \mathcal{C}_*$,* *with $\tilde I(u) = c$. Then, there exist a positive constant $\bar\varepsilon$ and a function $\eta : \mathcal{C}_* \to \mathcal{C}_*$ satisfying the following properties:*

(i) η *is continuous with respect to the topology of* $W^{1,p}(B)$*;*

(ii) $\tilde I(\eta(u)) \leqslant \tilde I(u)$ *for all* $u \in \mathcal{C}_*$*;*

(iii) $\tilde I(\eta(u)) \leqslant c - \bar\varepsilon$ *for all* $u \in \mathcal{C}_*$ *such that* $|\tilde I(u) - c| < \bar\varepsilon$*;*

(iv) $\eta(u) = u$ *for all* $u \in \mathcal{C}_*$ *such that* $|\tilde I(u) - c| > 2\bar\varepsilon$*.*

REMARKS.

• We stress here that we build a deformation η not only for regular values c of $\tilde I$ (i.e., such that $\tilde I'(u) \neq 0$ *for all* $u \in W^{1,p}(B)$ with $\tilde I(u) = c$), but also for all $c \in \mathbb{R}$ for which $\tilde I'(u) \neq 0$ *for all* $u \in \mathcal{C}_*$ with $\tilde I(u) = c$.

• In this version of the deformation lemma, we need to prove that the η preserves the cone $\mathcal{C}_*$. This is the most delicate point of the proof. It requires the existence of a pseudo-gradient vector field K of $\tilde I$ which is not only locally Lipschitz continuous, but which satisfies also the following property

$$K\left(\mathcal{C}_* \setminus \{\text{critical points of } \tilde I\}\right) \subset \mathcal{C}_*. \tag{2.1}$$

Indeed, for every $u \in \mathcal{C}_*$, the deformation $\eta(u)$ is built as the unique solution $\mu(t,u)$ of the Cauchy problem

$$\begin{cases} \frac{d}{dt}\mu(t,u(x)) = -\Phi(\mu(t,u(x))) & (t,x) \in (0,\infty)\times B,\\ \partial_\nu \mu(t,u(x)) = 0 & (t,x) \in (0,\infty)\times \partial B,\\ \mu(0,u(x)) = u(x) & x \in B, \end{cases} \tag{2.2}$$

$$\text{where } \Phi(u) := \begin{cases} \chi_1(\tilde I(u))\chi_2(u)\frac{u-K(u)}{\|u-K(u)\|} & \text{if } |\tilde I(u)-c| \leqslant 2\bar\varepsilon,\\ 0 & \text{otherwise,} \end{cases} \qquad \chi_1, \chi_2 \text{ cutoff}$$

for t (fixed) sufficiently large (i.e., $\eta(u) := \mu(\bar t, u)$). The existence of such operator K and of its properties are proved in [3, Proposition 3.2 and Lemmas 3.4-3.7] (see also [1] for the case of an open cone) and passes through the study of an auxiliary operator $\tilde T$ related to the inverse of $-\Delta_p(\cdot) + |\cdot|^{p-2}(\cdot)$. In particular, property (2.1) is a consequence of the fact that $\tilde T(\mathcal{C}_*) \subseteq \mathcal{C}_*$, that is proved –by hands– in [3, Lemma 3.4]. Finally, thanks to (2.1), the convexity, and the closedness of $\mathcal{C}_*$, we are able to prove that $\eta(\mathcal{C}_*) \subseteq \mathcal{C}_*$.

• Condition (iv) is an immediate consequence of the fact that μ solves the Cauchy problem (2.2). While, (ii) and (iii) rely essentially on Simon-type inequalities, that is to say relations between $\tilde I'$ and K, see [3, Proposition 3.2 and Lemmas 3.5-3.7].

A mountain pass-type geometry

LEMMA 2 (Lemma 3.9 and formula (44) of [3]). *Let* τ *be a constant such that* $0 < \tau < \min\{u_0 - u_-, u_+ - u_0\}$. *Then there exists* $\alpha > 0$ *such that* *(i)* $\tilde{I}(u) \geqslant \tilde{I}(u_-) + \alpha$ *for every* $u \in \mathcal{C}_*$ *with* $\|u - u_-\|_{L^\infty(B)} = \tau$*;*

(ii) if $u_+ < \infty$*, then* $\tilde{I}(u) \geqslant \tilde{I}(u_+) + \alpha$ *for every* $u \in \mathcal{C}_*$ *with* $\|u - u_+\|_{L^\infty(B)} = \tau$*.*

Furthermore,

(iii) $\tilde{I}(t \cdot 1) \to -\infty$ *as* $t \to +\infty$*.*

REMARKS.

• If $u_+ = +\infty$, then (i) and (iii) are pretty much the classical conditions required for the mountain pass geometry centered at u_-.

• If $u_+ < +\infty$, then the roles played by u_- and u_+ are interchangeable, hence we prove that the points on the sphere $\partial B_\tau(u_-) := \{u \in \mathcal{C}_* \,:\, \|u - u_-\|_{L^\infty(B)} = \tau\}$ and those on $\partial B_\tau(u_+) := \{u \in \mathcal{C}_* \,:\, \|u - u_+\|_{L^\infty(B)} = \tau\}$ satisfy the same condition with respect to u_- and to u_+, respectively. In this case, since $u_0 - u_- > \tau$ and $u_+ - u_0 > \tau$, then the two closed balls $\overline{B}_\tau(u_-)$ and $\overline{B}_\tau(u_+)$ are disjoint. Therefore, suppose –to fix ideas– that $\tilde{I}(u_-) \leqslant \tilde{I}(u_+)$. By (ii), for all $u \in \partial B_\tau(u_+)$ it results $\tilde{I}(u) \geqslant \tilde{I}(u_+) + \alpha$ and there exists u_-, for which

$$\|u_- - u_+\|_{L^\infty(B)} > \tau \quad \text{and} \quad \tilde{I}(u_-) \leqslant \tilde{I}(u_+).$$

• We remark that in (i) and (ii) it is possible to use the L^∞-norm instead of the $W^{1,p}$-norm, because $\mathcal{C}_*$-functions are bounded by (1.2). In particular, the use of the L^∞-norm allows us to simplify the constants.

Existence of a solution of (1.1) in $\mathcal{C}_*$

Let τ and α be the constants introduced in the previous subsection,

$$U_- := \left\{u \in \mathcal{C}_* \,:\, \tilde{I}(u) < \tilde{I}(u_-) + \frac{\alpha}{2},\, \|u - u_-\|_{L^\infty(B)} < \tau\right\},$$

$$U_+ := \begin{cases} \left\{u \in \mathcal{C}_* \,:\, \tilde{I}(u) < \tilde{I}(u_+) + \dfrac{\alpha}{2},\, \|u - u_+\|_{L^\infty(B)} < \tau\right\}, & \text{if } u_+ < \infty, \\[2ex] \left\{u \in \mathcal{C}_* \,:\, \tilde{I}(u) < \tilde{I}(u_-),\, \|u - u_-\|_{L^\infty(B)} > \tau\right\}, & \text{if } u_+ = \infty \end{cases}$$

the sets from/to which the admissible paths used to define the minimax level start/arrive,

$$\Gamma := \{\gamma \in C([0,1];\mathcal{C}_*) \,:\, \gamma(0) \in U_-,\, \gamma(1) \in U_+\}$$

the set of admissible paths, and

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \tilde{I}(\gamma(t)) \tag{2.3}$$

the minimax level.

By combining together the compactness condition, the mountain pass-type geometry of $\tilde{I}$, and the deformation lemma presented above, we are able to prove the following result.

PROPOSITION 1 (Proposition 3.10 of [3]). *The value c defined in* (2.3) *is finite and there exists a critical point $u \in \mathcal{C}_* \setminus \{u_-, u_+\}$ of $\tilde{I}$ such that $\tilde{I}(u) = c$ and $u > 0$. In particular, u is a weak solution of* (1.1).

REMARKS.

• We observe that, since every admissible path $\gamma \in \Gamma$ starts from $B_\tau(u_-)$ and arrives in $B_\tau(u_+)$, due to its continuity, it must cross the sphere $\partial B_\tau(u_-)$ (and also $\partial B_\tau(u_+)$ if $u_+ < +\infty$). Then, by Lemma 2-(i) (and also by (ii) if $u_+ < +\infty$),

$$\tilde{I}(u_-) < c < +\infty \quad \text{(and also } \tilde{I}(u_+) < c \text{ if } u_+ < +\infty).$$

This immediately excludes the possibility that the solution u is the constant u_- (or the constant u_+ when this latter is finite).

• By the maximum principle [8, Theorem 5], u is positive.

3. Step 5: The solution found is nonconstant.

In this section we conclude the proof of Theorem 1. As already observed in Section 3, the multiplicity part of the theorem follows easily when one works in the restricted cone $\mathcal{C}_*$. Concerning the nonconstancy of the solution, we already know by Proposition 1 that the solution $u \in \mathcal{C}_*$, at level c, is neither the constant u_- nor the constant u_+. It remains to show that $u \not\equiv u_0$. In particular, we prove that $c = \tilde{I}(u) < \tilde{I}(u_0)$.

By the very definition of c, it is enough to find an admissible path $\bar{\gamma}$ such that

$$\max_{t\in[0,1]} \tilde{I}(\bar{\gamma}(t)) < \tilde{I}(u_0). \tag{3.1}$$

We sketch below the construction of such curve $\bar{\gamma} \in \Gamma$, see [3, Lemma 4.2] for more details.

• It is easy to see that there exist two positive numbers t_- and t_+ ($t_- < 1 < t_+$), such that $t_- u_0 \in U_-$ and $t_+ u_0 \in U_+$.

• By (f_3), the function $t \in [t_-, t_+] \mapsto \tilde{I}(tu_0)$ has a unique strict maximum point at $t = 1$. Hence,

$$\tilde{I}(tu_0) < \tilde{I}(u_0) \quad \text{for all } t \in [t_-, t_+] \setminus \{1\}.$$

• Let $v \in W^{1,p}_{\mathrm{rad}}(B) \setminus \{0\}$ be nondecreasing and such that $\int_B v\,dx = 0$. For every $t \in [t_-, t_+]$, the function $s \in \mathbb{R} \mapsto \tilde{I}(t(u_0 + sv))$ is continuous. Therefore, by the previous step, we get for s in a neighborhood of 0

$$\tilde{I}(t(u_0 + sv)) < \tilde{I}(u_0) \quad \text{for all } t \in [t_-, t_+] \setminus [1 - \varepsilon, 1 + \varepsilon],$$

where $\varepsilon > 0$ is a sufficiently small constant.

• In order to have the same inequality also for t close to 1, we use condition (f_3), the C^2-regularity of $\tilde{I}$ and the Implicit Function Theorem, see [3, Lemma 4.1]. This allows us to prove that u_0 is not a local minimum of the Nehari-type set

$$\mathcal{N}_* := \{u \in \mathcal{C}_* : \tilde{I}'(u)[u] = 0\}.$$

In particular, we prove that for all $s \in \mathbb{R}$ there exists a unique $\bar{t}_s > 0$ such that $\bar{t}_s(u_0 + sv) \in \mathcal{N}_*$ and $\bar{t}_s$ is the unique maximum point of the map $t \in [1-\varepsilon, 1+\varepsilon] \mapsto \tilde{I}(t(u_0 + sv))$. Furthermore, by using a second order Taylor expansion of the energy functional and (f_3), we obtain that for s in a neighborhood of 0

$$\tilde{I}(\bar{t}_s(u_0+sv)) - \tilde{I}(u_0) = \frac{s^2}{2}\int_B [(p-1)u_0^{p-2} - \tilde{f}'(u_0)]v^2 dx + o(s^2) < 0.$$

Therefore, we get for s close to 0

$$\tilde{I}(t(u_0+sv)) \leqslant \tilde{I}(\bar{t}_s(u_0+sv)) < \tilde{I}(u_0) \quad \text{for all } t \in [1-\varepsilon, 1+\varepsilon].$$

- Clearly, for $\bar{s} > 0$ small enough, $t_-(u_0+\bar{s}v) \in U_-$ and $t_+(u_0+\bar{s}v) \in U_+$.
- By the convexity of $\mathcal{C}_*$, keeping in mind that $U_-, U_+ \subset \mathcal{C}_*$,

$$t(u_0+\bar{s}v) \in \mathcal{C}_* \quad \text{for every } t \in [t_-, t_+].$$

- Hence, the curve $\bar{\gamma} : t \in [0,1] \mapsto ((1-t)t_- + tt_+)(u_0+\bar{s}v) \in \mathcal{C}_*$ belongs to Γ and satisfies (3.1).

4. Sketch of the proof of Theorem 2

We denote by $u_q \in \mathcal{C}$ the nonconstant solution of

$$\begin{cases} -\Delta_p u + u^{p-1} = u^{q-1} & \text{in } B, \\ u > 0 & \text{in } B, \\ \partial_\nu u = 0 & \text{on } \partial B \end{cases} \tag{4.1}$$

at minimax level c_q and by $\tilde{I}_q$ the energy functional associated to the corresponding truncated problem. We describe below the main steps to prove Theorem 2, see for reference [3, Theorem 1.3] and also [4].

- In [3, Lemma 5.5], we find an a priori bound on u_q, uniform in q. Namely,

$$\|u_q\|_{C^1(\overline{B})} \leqslant C, \text{ with } C > 0 \text{ independent of } q \geqslant p+1.$$

Here we use the special form of f.

- This ensures the existence of a limit profile u_∞ for which

$$u_q \rightharpoonup u_\infty \text{ in } W^{1,p}(B) \quad \text{and} \quad u_q \to u_\infty \text{ in } C^{0,\mu}(\overline{B})\, \forall \mu \in (0,1) \quad \text{as } q \to \infty.$$

Furthermore, $u_\infty(1) = 1$, see [3, Lemma 5.6].

REMARK 2. By integrating over B the first equation of problem (4.1), we get $\int_B u_q^{p-1}(1-u_q^{q-p})dx = 0$. Since $u_q > 0$, $u_q \not\equiv 1$, and $u_q' \geqslant 0$, it results

$$u_q(0) < 1 \quad \text{and} \quad u_q(1) > 1 \quad \text{for all } q \geqslant p+1.$$

Heuristically, where $u_q \leqslant$ Const. < 1 (i.e., near the center of the ball B), $\lim_{q\to\infty} u_q^{q-1} = 0$. So, it is natural to expect that u_∞ solves $-\Delta_p u + u^{p-1} = 0$ at least in a neighborhood of the origin. On the other hand, in the region where $u_q \geqslant 1$ (i.e., in a neighborhood of ∂B), the same limit is an indeterminate form. This is somehow responsible of the fact that the boundary condition is not preserved in the limit. We further remark that, by Hopf's lemma, $\partial_\nu G > 0$ on ∂B, hence the $C^{0,\mu}(\overline{B})$-convergence is optimal.

• We introduce the quantity $c_\infty := \inf\left\{\frac{1}{p}\|u\|^p_{W^{1,p}(B)} : u \in \mathcal{C}, u\big|_{\partial B} = 1\right\}$ and we show that $c_\infty = \inf\left\{\frac{1}{p}\|u\|^p_{W^{1,p}(B)} : u \in W^{1,p}(B), u\big|_{\partial B} = 1\right\}$. Furthermore, this infimum is *uniquely* achieved at G (via the Direct Method of the Calculus of Variations), see [3, Lemma 5.7].

• We show in [3, Lemma 5.8 and Theorem 1.3] that $c_\infty = \lim_{q\to\infty} c_q$. The proof relies mainly on the fact that the minimax level c_q in the cone coincides with a Nehari-type level in the cone (also here we use the fact that f is a pure power function), cf. [3, Lemma 5.4]. As a consequence, we get that c_∞ is attained at u_∞ and $\|u_q\|_{W^{1,p}(B)} \to \|u_\infty\|_{W^{1,p}(B)}$.

• By uniqueness, $u_\infty = G$ a.e. in B. Finally, the weak convergence ($u_q \rightharpoonup G$ in $W^{1,p}(B)$) together with the convergence of the norms ($\|u_q\|_{W^{1,p}(B)} \to \|G\|_{W^{1,p}(B)}$) guarantee that $u_q \to G$ in $W^{1,p}(B)$, by the uniform convexity of the space.

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ON THE UNIQUENESS OF SOLUTIONS TO A CLASS OF SEMILINEAR ELLIPTIC EQUATIONS BY SERRIN'S SWEEPING PRINCIPLE

Abstract. We present an alternative proof of the uniqueness of saddle-shaped solutions to equations of Allen-Cahn type that vanish on the Simons cone, a property which was originally shown by X. Cabré in 2012. An interesting feature of our approach is that we combine Serrin's sweeping principle with the maximum principle in domains with small volume to deal with the singularity at the vertex of the cone. Exploiting further this approach, we revisit some other related uniqueness results. Moreover, we can establish new uniqueness properties for positive solutions to a class of defocusing Gross-Pitaevskii equations.

1. Introduction

We consider the semilinear elliptic equation

$$-\Delta u = f(u), \tag{1.1}$$

with f a reasonably smooth odd function such that

$$f > 0 \text{ in } (0,1),\ f(1) = 0. \tag{1.2}$$

The typical example is $f(u) = u - u^3$ which gives rise to the well known Allen-Cahn equation from phase transition models. Entire solutions of such equations and their connections to *minimal surface theory* have been the subject of intense investigations over the last years (see [19]).

An important class of entire solutions for such Allen-Cahn type equations in even dimensions $\mathbb{R}^{2m}$, $m \geqslant 1$, are the *saddle-shaped solutions* that we briefly recall (see [12, 13, 14] for more details). For $x = (x_1, \cdots, x_{2m}) \in \mathbb{R}^{2m}$, consider the two radial variables:

$$s = \left(x_1^2 + \cdots + x_m^2\right)^{\frac{1}{2}},\ \ t = \left(x_{m+1}^2 + \cdots + x_{2m}^2\right)^{\frac{1}{2}}.$$

A saddle-shaped solution of (1.1) is a solution u which depends only on s and t, satisfies $|u| < 1$, is positive in $\{s > t\}$, and is odd with respect to $\{s = t\}$, i.e., $u(t,s) = -u(s,t)$ in $\mathbb{R}^{2m}$. In particular, every saddle-shaped solution vanishes on the *Simons cone*

$$\mathcal{C} = \{s = t\} = \partial \mathcal{O} \ \text{ where } \ \mathcal{O} = \{s > t\}. \tag{1.3}$$

Existence of a saddle-shaped solution $u \in C^{2,\alpha}(\mathbb{R}^{2m})$ for all $0 < \alpha < 1$ to (1.1) was established by variational methods in [12], assuming that $f \in C^1(\mathbb{R})$ and conditions

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for f in the interval $(0,1)$ that are even more general than (1.2). If in addition $f \in C^{2,\alpha}(\mathbb{R})$ for some $0<\alpha<1$, and

$$f'' < 0 \text{ in } (0,1), \tag{1.4}$$

uniqueness of a saddle-shaped solution was established in [14]. This was accomplished by proving that a maximum principle holds in $O=\{s>t\}$ for the linearized operator at every saddle-shaped solution, and by combining this with the common asymptotic behaviour of all saddle-shaped solutions at infinity which was established in [13] (see (1.5) below for more details). Interestingly enough, a key ingredient for showing this maximum principle in O, which was also used for showing various important qualitative properties of the solution, was a maximum principle in the "narrow" domain $\{t<s<t+\varepsilon\}$ with $\varepsilon>0$ small. The aforementioned asymptotic behaviour at infinity of saddle-shaped solutions is the following: Denoting

$$U(x)=u_0\left(\frac{s-t}{\sqrt{2}}\right) \text{ for } x\in\mathbb{R}^{2m} \quad \left(|s-t|=\sqrt{2}\mathrm{dist}(x,\partial O)\right),$$

where u_0 is uniquely determined by the one-dimensional problem

$$-u_0''=f(u_0),\ z\in\mathbb{R};\ u_0(0)=0,\ u_0(\pm\infty)=\pm 1,$$

it holds

$$(u-U)(x)\to 0 \text{ and } (\nabla u-\nabla U)(x)\to 0 \text{ uniformly as } |x|\to\infty. \tag{1.5}$$

Under slightly weaker regularity conditions, this uniqueness result was shown much earlier if $m=1$ in [18]. Roughly, this was achieved by considering the quotient of two saddle-shaped solutions and showing that it is equal to one using integrations by part. Remarkably, still on the plane, the same uniqueness property was established without assuming the concavity assumption (1.4) away from $u=1$ in [24] by the sliding method (see also [6]). As was observed in [14], the aforementioned techniques for $m=1$ do not apply, at least directly, to the case $m\geqslant 2$.

In this short note we will provide an alternative proof of the aforementioned uniqueness result of [14]. Our strategy will be to adapt *Serrin's sweeping principle* [35], that has already been used successfully since the early seventies to show uniqueness of positive solutions for the corresponding problem with Dirichlet boundary conditions in bounded and smooth domains (see for example [37, pg. 40-41]). The unboundedness feature of the cone will be taken care of by the common asymptotic behaviour (1.5) of saddle-shaped solutions at infinity. On the other hand, to deal with the singularity at its vertex we will apply in small balls around it the *maximum principle for domains with small volume*, see [5] (see also [17], the review [11] and the references therein). In the aforementioned reference, this type of maximum principle was incorporated in the *method of moving planes* to allow for non-smooth domains, where Hopf's boundary point lemma that was used in the original approach of [22] does not apply in general. Actually, as is explained in [36], the method of moving planes can be put in the general framework of Serrin's sweeping principle. It is therefore natural to adopt the point of view of [5] to extend the method for showing uniqueness by the sweeping principle to the case of non-smooth domains. In fact, using this approach, we can also give an alternative proof of the maximum principle of [14] for the associated linearized operator at the saddle-shaped solution. All of this will be carried out in Section 2. We will exploit further this approach in Section 3 to revisit a classical uniqueness result for positive solutions to (1.1) in bounded domains with Dirichlet boundary conditions, *without any smoothness assumption on the boundary*. Moreover, we will consider the uniqueness of positive solutions to a class of Gross-Pitaevskii equations in the whole space.

2. The main result

Our main result is the following, which clearly implies the aforementioned uniqueness result for saddle-shaped solutions.

THEOREM 1. *Suppose that f is locally Lipschitz continuous on $[0,\infty)$, $f>0$ in $(0,1)$, $f(1)=0$, and*

$$\frac{f(u)}{u} \text{ is nonincreasing in } (0,\infty). \tag{2.1}$$

Then, problem (1.1) has at most one positive solution $u \in C^2(O)\cap C(\bar{O})$ with $u<1$, which vanishes on ∂O, where the latter domain is as in (1.3).

Proof. We begin by obtaining some rough estimates for such solutions near ∂O and at infinity. Under our assumptions on f, it is well known that there exists a large radius R such that the radial problem

$$-\Delta u = f(u) \text{ in } B_R;\ u=0 \text{ on } \partial B_R,$$

has a unique positive solution u_R, which is radial and decreasing in $r=|x|$ (see for instance [15] for the existence part, and the references in Subsection 3.1 herein for the uniqueness). Actually, the main property that we will use is that by Hopf's boundary point lemma we have $\partial_r u_R<0$ at $r=R$ (recall that $f(0)\geqslant 0$). Now let u be a solution as in the assertion of the theorem and let $B_R(y)$ be any ball of radius R that is contained in the closure of O (possibly touching ∂O). Using Serrin's sweeping principle (we will demonstrate this later in more generality), it follows that

$$u(x)>u_R(x-y),\ x\in B_R(y). \tag{2.2}$$

In fact, since the sum of the squares of the principal curvatures of the cone tends to zero at infinity, each point on ∂O sufficiently far from the origin can be touched by a ball $B_R(y)\subset \bar{O}$. Hence, u satisfies

$$u(x)\to 1 \text{ as } \mathrm{dist}(x,\partial O)\to\infty \tag{2.3}$$

and

$$u(x)\geqslant c\,\mathrm{dist}(x,\partial O) \text{ if } \mathrm{dist}(x,\partial O)\leqslant 1 \text{ as } |x|\to\infty, \tag{2.4}$$

for some $c > 0$. Note that the above two relations hold uniformly for all such solutions u. On the other hand, the following estimate which follows directly from the above relation and Hopf's boundary point lemma depends on each solution u: Given any $\delta > 0$ small, there exist numbers $c_\delta > 0$, $r_\delta > 0$ such that

$$u(x) \geqslant c_\delta \mathrm{dist}(x, \partial O) \text{ if } \mathrm{dist}(x, \partial O) \leqslant r_\delta \text{ and } x \in O \backslash (B_\delta \cap O), \tag{2.5}$$

where $B_\delta = B_\delta(0)$. On the other side, since by standard elliptic regularity estimates all such solutions are uniformly bounded in C^1 away from the vertex, we get the following crude upper bound:

$$u(x) \leqslant C_\delta \mathrm{dist}(x, \partial O) \text{ if } x \in O \backslash (B_\delta \cap O), \tag{2.6}$$

for some $C_\delta > 0$.

We can now proceed to the main part of the proof. Let u, v be two solutions as in the assertion of the theorem. We claim that

$$\lambda u > v \text{ in } O, \tag{2.7}$$

provided that $\lambda > 1$ is sufficiently large. In light of (2.3), (2.4), (2.5) and (2.6), given $\delta > 0$, we deduce that there exists $\lambda_\delta \gg 1$ such that

$$\lambda_\delta u > v \text{ in } O \backslash (B_\delta \cap O).$$

We will show next that the above relation remains true in $B_\delta \cap O$, provided that $\delta > 0$ is chosen sufficiently small. To this aim, let us suppose to the contrary that $\lambda_\delta u - v$ is negative somewhere in $B_\delta \cap O$. Then, in each connected component of the set $\{x \in B_\delta \cap O : \lambda_\delta u < v\}$ the function $v - \lambda_\delta u$ is a positive lower solution to a linear equation of the form

$$\Delta\varphi + q(x)\varphi = 0 \text{ with } |q| \textit{ bounded uniformly in } \delta, \tag{2.8}$$

and vanishes on the boundary (the bound for $\|q\|_{L^\infty}$ can be chosen to be the Lipschitz constant of f on $[0, 1]$). By choosing $\delta > 0$ sufficiently small (from now on fixed), we find that this is impossible by the maximum principle for domains with small volume (see [5, 11, 17]).

We will show next that property (2.7) holds for all $\lambda > 1$. For this purpose, we define

$$\lambda_* = \inf\{\mu > 1 \ : \ \text{property (2.7) holds for } \lambda \in (\mu, \infty)\},$$

and we will show that $\lambda_* = 1$ (we know that $1 \leqslant \lambda_* \leqslant \lambda_\delta$). Let us suppose to the contrary that

$$\lambda_* > 1.$$

Firstly, since $\lambda_* u$ is an upper solution to (1.1) and v is a solution, we deduce by the strong maximum principle that $\lambda_* u - v > 0$ in O. From (1.5) we obtain for $x \in \partial O$:

$$\lim_{|x|\to\infty} |\nabla(\lambda_* u - v)| = (\lambda_* - 1)u_0'(0) > 0.$$

We point out that the proof of (1.5) in [13] does not require the solutions to be doubly radial in s,t. Pushing further the validity of the above estimate with the help of Hopf's boundary point lemma, we obtain that

$$|\nabla(\lambda_* u - v)| > c_1 \text{ on } \partial\mathcal{O} \cap B_\delta^c,$$

for some $c_1 > 0$. In turn, since u, v are bounded in $C^{1,\alpha}$ for all $\alpha \in (0,1)$ away from the origin, we get

$$\lambda_* u - v \geqslant c_2 \mathrm{dist}(x, \partial\mathcal{O}) \text{ in } \{t < s < t+d\} \cap B_\delta^c$$

for some $c_2, d > 0$ (at this point we used again that the cone becomes flat at infinity). Then, thanks to (2.6), we must have that

$$\lambda u - v > 0 \text{ in } \{t < s < t+d\} \cap B_\delta^c \text{ for } \lambda \in [\lambda_* - \varepsilon, \lambda_*] \tag{2.9}$$

for some small $\varepsilon > 0$.

By the definition of λ_* and the above relation, there should be

$$\lambda_n < \lambda_* \text{ with } \lambda_n \to \lambda_* \text{ and } x_n \in \{s > t+d\} \cup (B_\delta \cap \mathcal{O})$$

such that

$$(\lambda_n u - v)(x_n) < 0. \tag{2.10}$$

We claim that

$$x_n \in B_\delta \cap \mathcal{O} \text{ for large } n. \tag{2.11}$$

Indeed, if not, by virtue of (2.3), passing to a subsequence if needed, we may assume that $\mathrm{dist}(x_n, \partial\mathcal{O}) \to d'$ with $d' \in \left[\frac{d}{\sqrt{2}}, \infty\right)$, and that $|x_n| \to \infty$ (by the strong maximum principle for $\lambda_* u - v$). Then, we infer from (1.5) that

$$(\lambda_n u - v)(x_n) \to (\lambda_* - 1)u_0(d') > 0,$$

which contradicts (2.10). Therefore, relation (2.11) holds. However, this is impossible by (2.9) and the maximum principle for domains with small volume as before (applied to $v - \lambda_n u$).

Consequently, it holds $\lambda_* = 1$ which implies that $v \leqslant u$ in $\mathcal{O}$. Analogously we have that $u \leqslant v$ in $\mathcal{O}$, which completes the proof of the theorem. □

REMARK 1. It is easy to check from the proof of (1.5) in [13] and that of the above theorem that the assertion of the latter continues to hold if $\partial\mathcal{O}$ is a cone which is smooth away from its vertex and has zero mean curvature.

REMARK 2. The behaviour near the origin of all saddle-shaped solutions to (1.1), with f as in (1.2), to leading order should be that of a harmonic function that vanishes on $\mathcal{C}$ and is positive in $\mathcal{O}$. Clearly, the function $s^2 - t^2$ is such a harmonic function. Moreover, by a result of [2] all such harmonic functions behave in the same way near the origin. Therefore, all saddle-shaped solutions should behave as a positive multiple of $s^2 - t^2$ near the origin. If so, as is indeed the case when $m = 1$ (see [24, Sec. 4]), then the use of the maximum principle in domains with small volume in the above theorem may be avoided (at least for the proof of (2.7)) for showing the uniqueness of a saddle-shaped solution.

Analogously, we can also prove the maximum principle of [14] for the linearized operator at the saddle-shaped solution.

PROPOSITION 1. *Under the assumptions of Theorem 1 with $f \in C^1$, let $\Omega \subseteq O = \{s > t\}$ be an open set and $c \in C(\Omega)$ with $c \leqslant 0$ in Ω. Then, the maximum principle holds in Ω for the linear operator*

$$M = \Delta + \left[f'(u(x)) + c(x)\right] I.$$

That is, if $\varphi \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies

$$M(\varphi) \geqslant 0 \text{ in } \Omega,\ \varphi \leqslant 0 \text{ on } \partial\Omega \text{ and } \limsup_{|x|\to\infty,\ x\in\Omega} \varphi(x) \leqslant 0, \tag{2.12}$$

then $\varphi \leqslant 0$ in Ω.

Proof. We will show that $\varphi^+ = \max\{\varphi, 0\}$ is identically equal to zero. Note that φ^+ is a nonnegative weak lower solution of $M = 0$ in O with c extended as zero in $O \backslash \Omega$ (see [4]). More precisely, it holds

$$M(\varphi^+) \geqslant 0 \text{ weakly in } O,\ \varphi^+ = 0 \text{ on } \partial O \text{ and } \lim_{|x|\to\infty,\ x\in O} \varphi^+(x) = 0.$$

We will show that $\varphi^+ \equiv 0$ by sweeping from above with the family of positive upper solutions $\{\lambda u,\ \lambda > 0\}$. In view of the proof of Theorem 1, the only point to pay attention to is on the behaviour of φ^+ on ∂O far away from the origin, as it could be that φ^+ crosses it with a non-zero slope there. Fortunately, this is not the case because it holds

$$\frac{\varphi^+(x)}{\operatorname{dist}(x, \partial O)} \to 0 \text{ as } |x| \to \infty,\ x \in O, \tag{2.13}$$

as we will show in the remaining part of the proof.

Let $x_0 \in \partial O$ be far away from the origin. We consider the unique solution ψ of the problem:

$$-\Delta\psi = f'(u(x))\,\varphi^+ \text{ in } B_2(x_0) \cap O;\ \ \psi = \varphi^+ \text{ on } \partial B_2(x_0) \cup \partial O.$$

Since φ^+ is a weak lower solution to the above problem (because $(M - cI)\varphi^+ \geqslant 0$ weakly), we deduce by the maximum principle in the weak setting (see [39]) that

$$0 \leqslant \varphi^+ \leqslant \psi \ \text{ on } \overline{B_2(x_0) \cap O}. \tag{2.14}$$

On the other side, since $\varphi^+ \to 0$ uniformly on $\overline{B_2(x_0)\cap O}$ as $|x_0| \to \infty$, by standard elliptic regularity estimates, we infer that

$$\frac{\partial \psi}{\partial \nu} \to 0 \text{ on } B_1(x_0)\cap \partial O \text{ uniformly as } |x_0| \to \infty,$$

where ν denotes the outer unit normal to ∂O (this works because the curvatures of ∂O do not blow-up at infinity, see [23, Ch. 3]). Consequently, the desired relation (2.13) follows at once from (2.14) and the above relation. The proof of the proposition is complete. □

REMARK 3. Besides for the purpose of showing uniqueness of a saddle-shaped solution, the above proposition was applied in [14] to the equations satisfied by u_s, u_t, u_{st} and others for establishing various monotonicity and convexity properties of the saddle-shaped solution. In those cases, the coefficient c satisfied $c(x) \to -\infty$ as $x \to 0$. Therefore, the assertion of Proposition 1 in these cases can be shown by using the usual maximum principle near the vertex of the cone instead of that for domains with small volume.

3. Related results

3.1. Uniqueness of positive solutions for (1.1) in bounded domains

The following theorem is well known when the domain is smooth, as in that case Hopf's boundary point lemma applies and the sweeping argument goes through in the standard way (see [33, 37]). Actually, still in a smooth domain, this result can be traced back to Krasnoselskii [30] from the early sixties. Since then, there have been several different proofs and extensions (see for instance [3, 9, 10, 26]). To the best of our knowledge, the strongest result is contained in [28], where Theorem 2 is proven without even assuming that f is locally Lipschitz, thus covering the model sub-linear nonlinearity $f(u) = u^p$ with $0 < p < 1$ (strict monotonicity in (2.1) is assumed however). The proof in [28] is rather involved and proceeds by establishing a weak comparison principle for (1.1), making use of Green's formula and Sard's theorem with the help of a mollifier. On the other hand, at the sole expense of assuming that f is locally Lipschitz continuous, we can give a simple direct proof analogously to Theorem 1. Furthermore, we can show that the maximum principle holds for the associated linearized operator.

THEOREM 2. *Assume that $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, is a bounded domain with continuous boundary. Then, problem (1.1) admits at most one positive solution $u \in C^2(\Omega)\cap C(\bar{\Omega})$ with Dirichlet boundary conditions on $\partial\Omega$, provided that f satisfies the assumptions of Theorem 1. Moreover, the analogous assertion of Proposition 1 holds if f is C^1.*

Proof. The proof of this result is completely analogous to those of Theorem 1 and Proposition 1, but simpler since we do not have to worry about things at infinity. Therefore, we just indicate the main difference which is the following. Letting u, v be two solutions as in the assertion of the theorem, we now take a domain $D \subset \Omega$ so that the set $\bar{\Omega}\backslash D \supseteq \partial\Omega$ has sufficiently small volume for the maximum principle to apply to the linear equation corresponding to (2.8) in subdomains of $\Omega\backslash\bar{D}$ where the function $|q|$ is bounded by the Lipschitz constant of f on $\left[0, \max\left\{\|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)}\right\}\right]$. □

REMARK 4. The uniqueness proof in the aforementioned reference [28] makes use at some points of the fact that the Laplacian is self-adjoint. On the other hand, our approach works equally well if we replace $-\Delta$ in (1.1) by a general second order elliptic operator.

3.2. Uniqueness of positive solutions to the Gross-Pitaevskii equation

A Bose-Einstein condensate is described by a ground state which is a positive solution of

$$\begin{cases} -\Delta u + V(x)u + gu^3 = 0, \ \ x \in \mathbb{R}^n, \ n \geqslant 1, \\ \\ u \to 0 \ \text{ as } \ |x| \to \infty, \end{cases} \tag{3.1}$$

where V is a smooth trapping potential such that

$$\liminf_{|x|\to\infty} V(x) = +\infty, \tag{3.2}$$

and $g > 0$ is a constant (see [27] and the references therein for more details). It was shown in the aforementioned reference that, under the above assumptions, problem (3.1) admits at most one positive solution. The proof was based on the approach in [10]. More precisely, they considered the quotient of two solutions and showed that it is equal to one. However, as it was not clear how such a quotient behaves at infinity, they also had to multiply the equation satisfied by the quotient by some cutoff function and argue in the spirit of a well known Liouville type theorem of [7] (they showed this for $n = 2$ but it is easy to check that their arguments carry over to arbitrary dimensions because u, v decay exponentially fast at infinity). A simple proof can be given in analogy to Theorem 2, by taking a large ball such that the maximum principle for the corresponding linear problem (2.8) applies outside of it. More generally, using this approach we can show the following result.

THEOREM 3. *If $V, G \in C(\mathbb{R}^n)$, $n \geqslant 1$, and $G \geqslant 0$ is non-trivial, the following problem*

$$\begin{cases} -\Delta u + V(x)u + G(x)u^3 = 0, \ \ x \in \mathbb{R}^n, \\ \\ u \to 0 \ \ as \ \ |x| \to \infty, \end{cases} \tag{3.3}$$

has at most one positive solution $u \in C^2(\mathbb{R}^n)$ such that

$$3Gu^2 + V \geqslant 0$$

for sufficiently large $|x|$. Moreover, if such a solution u exists, the associated linearized operator

$$M = -\Delta + (3Gu^2 + V)I$$

satisfies the maximum principle (2.12) in any domain $\Omega \subseteq \mathbb{R}^n$.

REMARK 5. Establishing the above theorem using Serrin's sweeping principle in the "traditional" way would require knowledge of the sharp decay rate of positive solutions to (3.3). In fact, this property plays the role of Hopf's lemma in the case of domains with nonempty boundary (recall the proof of Theorem 1 and the references preceding Theorem 2). However, to the best of our knowledge, this information is only available in the case of (3.1) for V radial and diverging at infinity as some power of $|x|$. This was shown recently in [9] (in the context of coupled Gross-Pitaevskii systems) based on a useful result of [32] for the decay of solutions of a class of linear Schrödinger equations.

REMARK 6. Problem (3.3) with $V \equiv 1$ and $G > 0$ such that $G(x) \to \infty$ as $|x| \to \infty$ has been studied from the existence point of view in the recent paper [31] when the diffusion is small (i.e. when there is a small positive parameter in front of the Laplacian conveying a singular perturbation).

REMARK 7. It follows from the proof of [21, Prop. 1.1] that the differential equation in (3.1) with V radial and continuous admits at most one radial positive solution in $L^2(\mathbb{R}^n)$.

REMARK 8. We observed in [29] that problem (3.1) in even dimensions with a smooth radial potential V satisfying (3.2), in the small diffusion regime mentioned in Remark 6, admits saddle-shaped solutions that vanish on the Simons cone. By combining ingredients from the proofs of Theorems 1, 2 we can now deduce their uniqueness (and also of the validity of the analogous maximum principle of Proposition 1).

REMARK 9. Similarly to Theorem 3, one can give a simple proof of the uniqueness of a positive solution to the following problem:

$$-u'' + xu + u^3 + \alpha = 0,\ x \in \mathbb{R};\ \ |x|^{-\frac{1}{2}}u \to 1 \text{ as } x \to -\infty,\ u \to 0 \text{ as } x \to +\infty, \tag{3.4}$$

where $\alpha \leqslant 0$. When $\alpha = 0$, it is well known that the above problem (3.4) is solved uniquely by the so called the Hastings-McLeod solution [25] (note that the ODE in that case is non other than the second Painlevé transendent). On the other hand, when $\alpha < 0$ the above problem came up in the recent paper [16] where, in particular, existence of a positive solution is proven.

REMARK 10. The following boundary value problem in the half-line

$$u'' + \frac{n-1}{r}u' - \frac{k}{r^2}u + u - u^3 = 0;\ u(0) = 0,\ u(\infty) = 1, \tag{3.5}$$

for some $k > 0$ (note that the first two terms make the radial Laplacian in $\mathbb{R}^n$), arises in the study of vortices to the Ginzburg-Landau system (see [8, 20]). The uniqueness of a positive solution $u \in C^2(0,\infty) \cap C[0,\infty)$ to the above problem is contained in the aforementioned references and proceeds along the lines of [10] once the behaviour of solutions near $r = 0$ is understood (it can be shown that $u(r) \sim ar^b$ near $r = 0$ for some $a, b > 0$ depending on k). On the other hand, using Serrin's sweeping principle as before, the uniqueness of a positive solution follows without worrying about the behaviour at the origin. Indeed, the third term blows-up at the origin with the correct sign so that the maximum principle holds for the corresponding linear equation to (2.8) in a small interval $(0, \delta)$. We can similarly show that the associated linearized operator of (3.6) satisfies the maximum principle in any interval.

Similar comments apply to the ODE problem arising in the study of "trapped" vortices in BECs:

$$u'' + \frac{n-1}{r}u' - V(r)u - \frac{k}{r^2}u - u^3 = 0;\ u(0) = 0,\ u(\infty) = 0, \tag{3.6}$$

(see [34, 38] and the references therein; recall also the preamble to our Theorem 3). A uniqueness result for this problem, following the result of [21] discussed in Remark 7, can be found in [1].

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NONLOCAL PHASE TRANSITIONS: RIGIDITY RESULTS AND ANISOTROPIC GEOMETRY

Abstract. We provide a series of rigidity results for a nonlocal phase transition equation. The prototype equation that we consider is of the form

$$(-\Delta)^{s/2}u = u - u^3,$$

with $s \in (0,1)$. More generally, we can take into account equations like

$$Lu = f(u),$$

where f is a bistable nonlinearity and L is an integro-differential operator, possibly of anisotropic type.

The results that we obtain are an improvement of flatness theorem and a series of theorems concerning the one-dimensional symmetry for monotone and minimal solutions, in the research line dictated by a classical conjecture of E. De Giorgi in [10].

Here, we collect a series of pivotal results, of geometric type, which are exploited in the proofs of the main results in [12].

1. Introduction and main results

In phase coexistence models, a classical question, which was posed in [10], is whether or not "typical solutions possess one-dimensional symmetry. In the models driven by semilinear partial differential equations, this type of problems has a long history, see e.g. [17, 2, 1, 18, 9] and the references therein. Related problems arise in the theory of quasilinear equations, see e.g. [8, 13, 15], and find applications in dynamical systems, see [16]. We refer to [14] for a review on this topic.

Recently, similar questions have been posed for a phase transition model in which the long-range particle interaction is described by a nonlocal operator of fractional type, see [6, 21, 5, 3, 4]. Similar models describe also the atom dislocation in some crystals, see e.g. Section 2 in [11], and some phenomena in mathematical biology, see e.g. [7]. The goal of this paper is to present a series of rigidity and symmetry results for semilinear problems driven by nonlocal operators. The results are so general that they can be applied also in a non-isotropic medium (but, as far as we know, they are also new in the isotropic case).

More precisely, we consider a nonlocal Allen-Cahn equation of the type

$$Lu = f(u) \quad \text{in } \mathbb{R}^n,$$

where L is an operator of the form

$$Lu(x) := \int_{\mathbb{R}^n} \big(u(x) - u(x+y)\big)\frac{\mu\big(y/|y|\big)}{|y|^{n+s}}\,dy, \qquad x \in \mathbb{R}^n,$$

with $s \in (0,1)$. The typical example of operator comprised by our setting is the fractional Laplacian (in this case $L := (-\Delta)^{s/2}$). The basic nonlinearity f that we take into account is when f is "bistable", i.e. it is minus the derivative of a double-well potential (e.g., $f(u) = u - u^3$). We assume that the measure μ (which is often called in jargon the "spectral measure") satisfies

$$\mu(z) = \mu(-z) \quad \text{and} \quad \lambda \leqslant \mu(z) \leqslant \Lambda \quad \text{for all } z \in S^{n-1},$$

for some $\Lambda \geqslant \lambda > 0$. Given a bounded $\psi \in C^2(\mathbb{R})$ we define

$$A\psi(z) := \int_{-\infty}^{+\infty} \frac{\psi(z) - \psi(z+\zeta)}{|\zeta|^{1+s}}\, d\zeta, \qquad z \in \mathbb{R}. \tag{1.1}$$

Roughly speaking, the operator A plays a role of the one-dimensional fractional Laplacian. In order to take into account the possible anisotropy of the operator L, we need to scale A appropriately in any fixed direction. To this aim, for ψ as above, $\omega \in S^{n-1}$, and $h > 0$ we define, for $x \in \mathbb{R}^n$,

$$\bar{\psi}_{\omega,h}(x) := \psi\left(\omega \cdot \frac{x}{h}\right).$$

We set $h_L(\omega) := h$ where $h > 0$ satisfies

$$L\bar{\psi}_{\omega,h}(x) = A\psi\left(\omega \cdot \frac{x}{h}\right) \text{ for all } \psi \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R}).$$

We also define

$$\mathcal{C} = \mathcal{C}_L := \bigcap_{\omega \in S^{n-1}} \left\{ x \in \mathbb{R}^n \, : \, x \cdot \omega \leqslant h_L(\omega) \right\} \tag{1.2}$$

and assume that

$$\partial \mathcal{C}_L \text{ is } C^{1,1} \text{ and strictly convex.}$$

More quantitatively, we assume that there exist $\rho' > \rho > 0$ such that

$$\text{the curvatures of } \partial \mathcal{C}_L \text{ are bounded above by } \frac{1}{\rho} \text{ and below by } \frac{1}{\rho'}. \tag{H1}$$

Concerning the nonlinearity f, we assume that $f \in C^1\big([-1,1]\big)$ and, for some $\kappa > 0$ and $c_\kappa > 0$,

$$f(-1) = f(1) = 0 \quad \text{and} \quad f'(t) < -c_\kappa \quad \text{for } t \in [-1,-1+\kappa] \cup [1-\kappa, 1]. \tag{H2}$$

Moreover, recalling the setting in (1.1), we assume that

$$\text{there exists } \phi_0 \text{ satisfying } \begin{cases} A\phi_0 = f(\phi_0) & \text{in } \mathbb{R}, \\ \phi_0' > 0 & \text{in } \mathbb{R}, \\ \phi_0(0) = 0, \\ \lim\limits_{x \to \pm\infty} \phi_0 = \pm 1. \end{cases} \tag{H3}$$

The main result obtained in [12] is the following improvement of flatness:

THEOREM 1. *Assume that L satisfies* (H1) *and that f satisfies* (H2) *and* (H3)*. Then there exist universal constants $\alpha_0 \in (0,s/2)$, $p_0 \in (2,\infty)$ and $a_0 \in (0,1/4)$ such that the following statement holds.*

Let $a \in (0,a_0)$ and $\varepsilon \in (0,a^{p_0})$. Let $u : \mathbb{R}^n \to (-1,1)$ be a solution of

$$Lu = \varepsilon^{-s} f(u) \quad \text{in} \quad B_{j_a},$$

with

$$j_a := \left\lfloor \frac{\log a}{\log(2^{-\alpha_0})} \right\rfloor.$$

Assume that $0 \in \{-1+\kappa \leqslant u \leqslant 1-\kappa\}$ and that

$$\{\omega_j \cdot x \leqslant -a2^{j(1+\alpha_0)}\} \subset \{u \leqslant -1+\kappa\} \subset \{u \leqslant 1-\kappa\} \subset \{\omega_j \cdot x \leqslant a2^{j(1+\alpha_0)}\} \quad \text{in } B_{2^j},$$

for any $j = \{0,1,2,\dots,j_a\}$ and for some $\omega_j \in S^{n-1}$.

Then,

$$\left\{\omega \cdot x \leqslant -\frac{a}{2^{1+\alpha_0}}\right\} \subset \{u \leqslant -1+\kappa\} \subset \{u \leqslant 1-\kappa\} \subset \left\{\omega \cdot x \leqslant \frac{a}{2^{1+\alpha_0}}\right\} \quad \text{in } B_{1/2},$$

for some $\omega \in S^{n-1}$.

Theorem 1 says that if the level sets of the solution are C^{1,α_0}-flat from infinity up to B_1, then they are also C^{1,α_0}-flat up to $B_{1/2}$, and so one can dilate the picture once again and repeat the argument at any small scale towards the origin (as a matter of fact, suitable scaled iterations of Theorem 1 are given in Corollaries 7.1 and 7.2 of [12]). An important consequence of Theorem 1 is related to the one-dimensional symmetry properties of the solutions. For this, we say that a function $u : \mathbb{R}^n \to \mathbb{R}$ is 1D if there exist $\bar{u} : \mathbb{R} \to \mathbb{R}$ and $\bar{\omega} \in S^{n-1}$ such that $u(x) = \bar{u}(\bar{\omega} \cdot x)$ for any $x \in \mathbb{R}^n$.

Then, we have the following consequences of Theorem 1:

THEOREM 2 (One-dimensional symmetry for asymptotically flat solutions). *Assume that L satisfies* (H1) *and that f satisfies* (H2) *and* (H3)*.*

Let u be a solution of $Lu = f(u)$ in $\mathbb{R}^n$.

Assume that there exists $a : (1,\infty) \to (0,1]$ such that $a(R) \searrow 0$ as $R \nearrow +\infty$ and such that, for all $R > 1$, we have that

$$\{\omega \cdot x \leqslant -a(R)R\} \subset \{u \leqslant -1+\kappa\} \subset \{u \leqslant 1-\kappa\} \subset \{\omega \cdot x \leqslant a(R)R\} \quad \text{in } B_R,$$

for some $\omega \in S^{n-1}$, which may depend on R. Then, u is 1D.

We stress that all these results, as far as we know, are new even for the equation $(-\Delta)^{s/2} u = u - u^3$, with $s \in (0,1)$, which is a particular case of our setting.

As a matter of fact, we can consider the concrete case of minimal solutions of the nonlocal Allen-Cahn equation $(-\Delta)^{s/2} u = u - u^3$, with $s \in (0,1)$. We remark that the energy functional related to such equation is

$$\mathcal{E}(u,\Omega) := \mathcal{E}^{\mathrm{D}ir}(u,\Omega) + \int_{\Omega} (1-u^2(x))^2\,dx,$$

where

$$\mathcal{E}^{\mathrm{D}ir}(u,\Omega) := C_{n,s} \iint\limits_{\mathbb{R}^{2n}\setminus(\mathbb{R}^n\setminus\Omega)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+s}}\,dx\,dy, \tag{1.3}$$

for a suitable normalization constant $C_{n,s} > 0$. In this setting, we say that a solution u of $(-\Delta)^{s/2}u = u - u^3$ is a *minimizer* of $\mathcal{E}$ in $\mathbb{R}^n$ if

$$\mathcal{E}(u,B) \leqslant \mathcal{E}(u+\varphi,B),$$

for any ball $B \subset \mathbb{R}^n$ and any $\varphi \in C_0^\infty(B)$. In this setting, the following results hold true:

THEOREM 3 (One-dimensional symmetry in the plane). *Let u be a minimizer of $\mathcal{E}$ in $\mathbb{R}^2$. Then, u is 1D.*

THEOREM 4 (One-dimensional symmetry for monotone solutions in $\mathbb{R}^3$). *Let $n \leqslant 3$ and u be a solution of $(-\Delta)^{s/2}u = u - u^3$ in $\mathbb{R}^n$.*

Suppose that

$$\frac{\partial u}{\partial x_n}(x) > 0 \quad \textit{for any } x \in \mathbb{R}^n \qquad \textit{and} \qquad \lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1.$$

Then, u is 1D.

THEOREM 5 (One-dimensional symmetry when s is close to 1). *Let $n \leqslant 7$. Then, there exists $\eta_n \in (0,1)$ such that for any $s \in [1-\eta_n, 1)$ the following statement holds true.*

Let u be a minimizer of $\mathcal{E}$ in $\mathbb{R}^n$. Then, u is 1D.

THEOREM 6 (One-dimensional symmetry for monotone solutions in $\mathbb{R}^8$ when s is close to 1). *Let $n \leqslant 8$. Then, there exists $\eta_n \in (0,1)$ such that for any $s \in [1-\eta_n, 1)$ the following statement holds true.*

Let u be a solution of $(-\Delta)^{s/2}u = u - u^3$ in $\mathbb{R}^n$.

Suppose that

$$\frac{\partial u}{\partial x_n}(x) > 0 \quad \textit{for any } x \in \mathbb{R}^n \qquad \textit{and} \qquad \lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1.$$

Then, u is 1D.

The full proofs of the results mentioned above are given in [12]. Here, we present the details of a series of pivotal results of geometric type that will be exploited in [12].

Related results on symmetry problems with possible applications to nonlocal phase transitions have been recently announced in [19] and obtained in [20] (we stress that the range of fractional parameter dealt with in [19, 20] is complementary to the one of this paper and [12]).

2. Some useful facts on the distance function

We collect here some ancillary results of elementary nature from the theory of convex sets and anisotropic distance functions. For the convenience of the reader, we give full details of the results we need, that are stated in a convenient form for their use in the forthcoming paper [12].

For this, we consider a continuous and degree 1 positively homogeneous function $h:\mathbb{R}^n \to [0,+\infty)$ and the convex set in (1.2). We assume that the boundary of $\mathcal{C}$ is of class C^1. The set $\mathcal{C}$ in our setting plays the role of an anisotropic ball, and so, for any $r>0$, we set

$$\mathcal{C}_r(y) := y + r\mathcal{C}. \tag{2.1}$$

This anisotropic ball induces naturally a norm, defined, for any $p\in\mathbb{R}^n$, by the following formula:

$$\|p\|_{\mathcal{C}} := \frac{1}{\sup\{\tau>0 \text{ s.t. } \tau p\in\mathcal{C}\}}. \tag{2.2}$$

We observe that, in view of (1.2), (2.1), and (2.2), for any $R>0$ and $z_0\in\mathbb{R}^n$,

$$\mathcal{C}_R(z_0) = \big\{x\in\mathbb{R}^n \text{ s.t. } \|x-z_0\|_{\mathcal{C}} \leqslant R\big\}. \tag{2.3}$$

Also, we have the following elementary inequality of Cauchy-Schwarz type:

LEMMA 1. *For any $x,\, y\in\mathbb{R}^n$, we have that $x\cdot y\leqslant h(y)\,\|x\|_{\mathcal{C}}$.*

Proof. If either $x=0$ or $y=0$ we are done, so we can suppose that $y\neq 0$ and $x\neq 0$. We set $\omega := \frac{y}{|y|}\in S^{n-1}$ and $\eta := \frac{x}{\|x\|_{\mathcal{C}}}$. Then $\eta\in\mathcal{C}$, thus by (1.2)

$$\frac{x}{\|x\|_{\mathcal{C}}}\cdot y = \eta\cdot\omega\,|y| \leqslant h(\omega)\,|y| = h\left(\frac{y}{|y|}\right)|y| = h(y),$$

which gives the desired result. □

Now we show that, in the terminology of convex geometry, the function h is the "support function" of the convex body $\mathcal{C}$.

LEMMA 2. *For any $\omega\in S^{n-1}$,*

$$h(\omega) = \sup_{x\in\mathcal{C}} x\cdot\omega. \tag{2.4}$$

Proof. From (1.2), we have that, for any $x \in \mathcal{C}$ and any $\omega \in S^{n-1}$, $x \cdot \omega \leqslant h(\omega)$, and so

$$\sup_{x \in \mathcal{C}} x \cdot \omega \leqslant h(\omega), \tag{2.5}$$

for any $\omega \in S^{n-1}$. In particular, this implies that $\mathcal{C}$ is bounded in any direction. Therefore, to check the opposite inequality to the one in (2.5), and thus to complete the proof of the desired result, we can fix $\omega \in S^{n-1}$ and slide a hyperplane with normal direction ω till it touches $\mathcal{C}$ at some point $P \in \partial\mathcal{C}$. That is, we have that for any $x \in \mathcal{C}$ it holds that $\omega \cdot (x - P) \leqslant 0$ and so

$$\sup_{x \in \mathcal{C}} \omega \cdot x = \omega \cdot P. \tag{2.6}$$

Also, since $P \in \partial\mathcal{C}$, we deduce from (1.2) that there exists $\varpi \in S^{n-1}$ for which

$$\varpi \cdot P = h(\varpi). \tag{2.7}$$

Notice that $\{\varpi \cdot (x - P) = 0\}$ is a supporting hyperplane for $\mathcal{C}$, since, for any $x \in \mathcal{C}$,

$$\varpi \cdot (x - P) = \varpi \cdot x - h(\varpi) \leqslant 0,$$

thanks to (1.2).

Since $\partial\mathcal{C}$ has been assumed to be a C^1 manifold, the two supporting hyperplanes at P, namely $\{\omega \cdot (x - P) = 0\}$ and $\{\varpi \cdot (x - P) = 0\}$, must coincide, and so $\omega = \varpi$.

As a consequence of this, recalling (2.6) and (2.7), we obtain that

$$\sup_{x \in \mathcal{C}} \omega \cdot x = \omega \cdot P = \varpi \cdot P = h(\varpi) = h(\omega),$$

as desired. □

As a counterpart of Lemma 1, we also have

LEMMA 3. *Let $z_0 \in \mathbb{R}^n$, $R > 0$ and $z \in \partial\mathcal{C}_R(z_0)$. Let $\omega_0 \in S^{n-1}$ be the inner normal of $\partial\mathcal{C}_R(z_0)$ at the point z. Then*

$$\omega_0 \cdot (z_0 - z) = R\,h(\omega_0).$$

Proof. Let

$$\zeta := \frac{z - z_0}{R}$$

and notice that we know that

$$\zeta \in \partial\mathcal{C}. \tag{2.8}$$

Also, since $\mathcal{C}_R(z_0)$ is convex, we know that $\mathcal{C}_R(z_0) \subset \{x \in \mathbb{R}^n \text{ s.t. } \omega_0 \cdot (x - z) \geqslant 0\}$ and so

$$\mathcal{C} \subset \{y \in \mathbb{R}^n \text{ s.t. } \omega_0 \cdot (y - \zeta) \geqslant 0\}.$$

Hence, by (2.4),

$$h(\omega_0) = h(-\omega_0) = \sup_{y\in C}(-y\cdot\omega_0) \leqslant -\zeta\cdot\omega_0.$$

On the other hand, by Lemma 1,

$$-\zeta\cdot\omega_0 \leqslant h(\omega_0)\,\|-\zeta\|_C = h(\omega_0),$$

and so

$$h(\omega_0) = -\zeta\cdot\omega_0 = \frac{(z_0-z)\cdot\omega_0}{R},$$

as desired. □

Given a nonempty, closed and convex set $K\subset\mathbb{R}^n$, we define the anisotropic signed distance function from K as

$$\begin{aligned} d_K(x) := \inf\big\{\ell(x)\,:\,\ell(x)=\omega\cdot x+c,\quad h(\omega)=1,\\ c\in\mathbb{R}\quad\text{and}\quad \ell\geqslant 0\ \text{ in all of } K\big\}. \end{aligned} \tag{2.9}$$

Notice that d_K is a concave function, since it is the infimum of affine functions. Also, we have that d_K is a Lipschitz function, with Lipschitz constant 1 with respect to the anisotropic norm, as stated in the following result:

LEMMA 4. *For any $p,\,q\in\mathbb{R}^n$,*

$$|d_K(p)-d_K(q)| \leqslant \|p-q\|_C.$$

Proof. Up to exchanging p and q, we suppose that $d_K(p)\geqslant d_K(q)$. Fixed $\delta>0$, we let $\ell_\delta(x)=\omega_\delta\cdot x+c_\delta$ be such that $h(\omega_\delta)=1$, $c_\delta\in\mathbb{R}$, $\ell_\delta(x)\geqslant 0$ for any $x\in K$, and with $d_K(q)\geqslant \ell_\delta(q)-\delta$. Then, we have that $d_K(p)\leqslant\ell_\delta(p)$ and so, by Lemma 1,

$$\begin{aligned} |d_K(p)-d_K(q)| &= d_K(p)-d_K(q) \leqslant \ell_\delta(p)-\ell_\delta(q)+\delta\\ &= \omega_\delta\cdot(p-q)+\delta \leqslant h(\omega_\delta)\,\|p-q\|_C+\delta = \|p-q\|_C+\delta. \end{aligned}$$

Hence, taking δ arbitrarily close to 0 we obtain the desired result. □

It is also useful to observe that the infimum in (2.9) is attained, namely:

LEMMA 5. *For any $p\in\mathbb{R}^n$ there exists an affine function ℓ_p, of the form $\ell_p(x)=\omega_p\cdot x+c_p$, with $h(\omega_p)=1$, $c_p\in\mathbb{R}$, such that $\ell_p\geqslant 0$ in K and $d_K(p)=\ell_p(p)$.*

Moreover, if $t_0\in\mathbb{R}$ and $z_0\in\{d_K>t_0\}$ are such that $p\in\partial\mathcal{C}_R(z_0)\cap\{d_K=t_0\}$, with $\mathcal{C}_R(z_0)\subset\{d_K\geqslant t_0\}$, and ω_0 is the interior normal of $\mathcal{C}_R(z_0)$ at p, we have that

$$\omega_p = \frac{\omega_0}{h(\omega_0)} \tag{2.10}$$

$$\textit{and}\qquad c_p = t_0 - \frac{\omega_0}{h(\omega_0)}\cdot p. \tag{2.11}$$

Proof. The existence of the optimal affine function ℓ_p follows from the direct methods of the calculus of variations, so we focus on the proof of the second claim. We have that, for any $x \in \mathcal{C}_R(z_0)$,

$$\omega_p \cdot p + c_p = d_K(p) = t_0 \leqslant d_K(x) \leqslant \omega_p \cdot x + c_p,$$

that is

$$\min_{x \in \mathcal{C}_R(z_0)} \omega_p \cdot x = \omega_p \cdot p.$$

Hence, by Lagrange multipliers, the gradient of the map $\omega_p \cdot x$ is parallel to (and in the same direction of) ω_0, that is

$$\omega_p = c\omega_0, \tag{2.12}$$

for some $c \geqslant 0$. Hence, since h is homogeneous,

$$1 = h(\omega_p) = c\,h(\omega_0).$$

This gives that $c = \frac{1}{h(\omega_0)}$, which, combined with (2.12), proves (2.10). Then, we write $t_0 = d_K(p) = \omega_p \cdot p + c_p$ and we obtain (2.11). □

In case of tangent anisotropic spheres to level sets of the anisotropic distance function, a useful comparison occurs with respect to Euclidean hyperplanes, as stated in the following result:

LEMMA 6. *Let K be convex, $z_0 \in \{d_K > t_0\}$, $t_0 \in \mathbb{R}$. Suppose that $\mathcal{C}_R(z_0) \subset \{d_K \geqslant t_0\}$ and let $z \in \partial\mathcal{C}_R(z_0) \cap \{d_K = t_0\}$.*

Let ω_0 be the interior normal of $\mathcal{C}_R(z_0)$ at z and $\{d_K \geqslant t_0\} \subset \{x \in \mathbb{R}^n$ s.t. $\omega_0 \cdot (x - z) \geqslant 0\}$. Then, for any $x \in \mathbb{R}^n$ it holds that

$$d_K(x) \leqslant \frac{\omega_0}{h(\omega_0)} \cdot (x - z) + t_0.$$

Proof. We let

$$\tilde{d}(x) := \frac{\omega_0}{h(\omega_0)} \cdot (x - z) + t_0. \tag{2.13}$$

We claim that

$$\tilde{d}(x) \geqslant 0 \text{ for any } x \in K. \tag{2.14}$$

For this, we use Lemma 5 (with $p = z$), according to which the affine function

$$\ell_z(x) := \omega_z \cdot x + c_z,$$

with $\omega_z = \frac{\omega_0}{h(\omega_0)}$ and $c_z = t_0 - \frac{\omega_0}{h(\omega_0)} \cdot z$ satisfies $\ell_z(z) = d_K(z)$ and $\ell_z(x) \geqslant 0$ for any $x \in K$. In particular, for any $x \in K$, we have that $\tilde{d}(x) = \ell_z(x) \geqslant 0$, which proves (2.14).

In addition, from the homogeneity of h we have that

$$h\left(\frac{\omega_0}{h(\omega_0)}\right) = \frac{h(\omega_0)}{h(\omega_0)} = 1.$$

Using this and (2.14), we obtain the desired result from (2.9). $\square$

Now we show that the function d_K, as defined in (2.9), coincides with the signed distance from the boundary of K, namely:

PROPOSITION 1. *Let $K \subset \mathbb{R}^n$ be nonempty, closed and convex. Then it holds that*

$$d_K(x) = \begin{cases} +\inf\big\{\|z-x\|_C \ : \ z \in \partial K\big\} & \textit{for } x \in K \\ -\inf\big\{\|z-x\|_C \ : \ z \in \partial K\big\} & \textit{for } x \in \mathbb{R}^n \backslash K. \end{cases} \tag{2.15}$$

Proof. First, we show that

$$d_K \geqslant 0 \text{ in } K. \tag{2.16}$$

For this, let ℓ be any affine function in (2.9). Since $\ell \geqslant 0$ in K, the claim in (2.16) plainly follows.

Now we show that

$$d_K \leqslant 0 \text{ in } \mathbb{R}^n \backslash K. \tag{2.17}$$

To this aim, let $p \in \mathbb{R}^n \backslash K$. Since K is convex, we can separate it from p, namely there exists an affine function $\ell_o(x) = \omega_o \cdot x + c_o$, for suitable $\omega_o \in \mathbb{R}^n \backslash \{0\}$ and $c_o \in \mathbb{R}$, such that $\ell_o \geqslant 0$ in K and $\ell_o(p) \leqslant 0$. So, we define

$$\omega := \frac{\omega_o}{h(\omega_o)}, \qquad c := \frac{c_o}{h(\omega_o)} \qquad \text{and} \qquad \ell(x) := \omega \cdot x + c.$$

In this way, we have that $\ell(x) = \frac{\ell_o(x)}{h(\omega_o)} \geqslant 0$ for any $x \in K$, and $\ell(p) \leqslant 0$. In addition, we have that $h(\omega) = 1$ and so ℓ is an admissible affine function in (2.9). This implies that $d_K(p) \leqslant \ell(p) \leqslant 0$, which gives (2.17).

From (2.16), (2.17) and the continuity of d_K (recall Lemma 4), it follows that $d_K = 0$ along ∂K. Hence, to complete the proof of (2.15), we can restrict to the case in which $x \notin \partial K$. Hence, it suffices to check that, for any $P \notin \partial K$,

$$|d_K(P)| = \inf\big\{\|z-P\|_C \ : \ z \in \partial K\big\}. \tag{2.18}$$

To check this, we first observe that, from Lemma 4, for any $z \in \partial K$,

$$|d_K(P)| = |d_K(P) - d_K(z)| \leqslant \|z-P\|_C$$

and therefore

$$|d_K(P)| \leqslant \inf\big\{\|z-P\|_C \ : \ z \in \partial K\big\}.$$

Thus, to complete the proof of (2.18), we only need to show that

$$|d_K(P)| \geqslant \inf\big\{\|z-P\|_C \ : \ z \in \partial K\big\}. \tag{2.19}$$

For this, we set $R(P) := \inf\big\{\|z-P\|_C \ : \ z \in \partial K\big\} > 0$ and we notice that $C_{R(P)}(P)$ is contained either in K (if $P \in K$) or in the closure of the complement of K (if $P \in \mathbb{R}^n \setminus K$), and there exists $p \in \partial K$ with $\|p-P\|_C = R(P)$.

So, if $P \in K$, we use Lemma 5 (with $z_0 = P$ and $t_0 = 0$) to find that the affine function $\ell_p(x) := \omega_p \cdot x + c_p$, with $\omega_p = \frac{\omega_0}{h(\omega_0)}$ and $c_p = -\frac{\omega_0}{h(\omega_0)} \cdot p$, satisfies $\ell_p \geqslant 0$ in K and $d_K(p) = \ell_p(p)$. Moreover, we have that

$$d_K(P) \geqslant \ell_p(P) \tag{2.20}$$

(and, in fact, equality holds, due to (2.9)). To check (2.20), let $\omega \in \mathbb{R}^n \setminus \{0\}$, and set $\tilde\omega := \frac{h(\omega)\,\omega}{|\omega|^2}$. We observe that $\tilde\omega \cdot \omega = h(\omega)$ and thus $\tilde\omega \in C$, thanks to (1.2). In consequence of this and of (2.2), we have that

$$\frac{1}{\|\tilde\omega\|_C} = \sup\{\tau > 0 \text{ s.t. } \tau\tilde\omega \in C\} \geqslant 1$$

and so

$$1 \geqslant \|\tilde\omega\|_C = \frac{h(\omega)\,\|\omega\|_C}{|\omega|^2}.$$

Therefore, recalling Lemma 3,

$$\left\| \frac{h(\omega)}{|\omega|^2} \frac{\omega_0 \cdot (P-p)}{h(\omega_0)} \omega \right\|_C = \frac{|\omega_0 \cdot (P-p)|}{h(\omega_0)} \frac{h(\omega)\,\|\omega\|_C}{|\omega|^2} = R(P) \frac{h(\omega)\,\|\omega\|_C}{|\omega|^2} \leqslant R(P),$$

which implies that

$$P - \frac{h(\omega)}{|\omega|^2} \frac{\omega_0 \cdot (P-p)}{h(\omega_0)} \omega \in C_{R(P)}(P). \tag{2.21}$$

Now, let $\ell(x) := \frac{\omega}{h(\omega)} \cdot x + c$ and suppose that $\ell \geqslant 0$ in K, and so in particular in $C_{R(P)}(P)$. Then, from (2.21), we have that

$$\begin{aligned} 0 \leqslant \ell\left(P - \frac{h(\omega)}{|\omega|^2} \frac{\omega_0 \cdot (P-p)}{h(\omega_0)} \omega\right) &= \frac{\omega}{h(\omega)} \cdot \left(P - \frac{h(\omega)}{|\omega|^2} \frac{\omega_0 \cdot (P-p)}{h(\omega_0)} \omega\right) + c \\ &= \frac{\omega}{h(\omega)} \cdot P - \frac{\omega_0 \cdot (P-p)}{h(\omega_0)} + c = \ell(P) - \ell_p(P). \end{aligned}$$

Given the validity of such inequality for every ℓ, we have established (2.20).

Accordingly, by (2.20) and Lemma 3,

$$|d_K(P)| = d_K(P) \geqslant \ell_p(P) = \frac{\omega_0}{h(\omega_0)} \cdot (P-p) = \|P-p\|_C = \inf\big\{\|z-P\|_C \ : \ z \in \partial K\big\}.$$

This proves (2.19) when P lies inside K, so we now deal with the case in which P lies in $\mathbb{R}^n \backslash K$.

For this, we let $p \in K \cap \partial \mathcal{C}_{R(P)}(P)$ and we denote by $\omega_0 \in S^{n-1}$ the inner normal of $\partial \mathcal{C}_{R(P)}(P)$ at p. Then, since K is convex, we have that $\omega_0 \cdot (x-p) \leqslant 0$ for any $x \in K$. Hence, the affine function

$$\ell(x) := \frac{-\omega_0}{h(\omega_0)} \cdot (x-p)$$

satisfies $\ell \geqslant 0$ in K and so it is admissible in (2.9). Consequently, by Lemma 3,

$$-|d_K(P)| = d_K(P) \leqslant \ell(P) = -\frac{\omega_0}{h(\omega_0)} \cdot (P-p) = -R(P)$$

and so

$$|d_K(P)| \geqslant R(p) \geqslant \inf\left\{ \|z-P\|_{\mathcal{C}} \ : \ z \in \partial K \right\}.$$

This completes the proof of (2.19), as desired. $\square$

3. The distance function from a graph

For convenience, we give here two results on the Euclidean distance function from a graph (the anisotropic case follows also from this results directly, up to changing constants, thanks to the equivalency of the norms). For this, we denote by d_* the distance function d_K in (2.9) when h is identically 1 (hence $\mathcal{C}$ in (1.2) is the Euclidean unit ball B_1) and K is the portion of space lying above a function $\zeta \in C^1(\mathbb{R}^{n-1})$, that is $K := \{x_n \geqslant \zeta(x')\}$.

Notice that, in this case, the anisotropic norm $\|\cdot\|_{\mathcal{C}}$ in (2.2) is simply the Euclidean norm and, by (2.15), d_* is simply the signed distance function from the graph of ζ.

Then we have the following results:

LEMMA 7. *Let $b \in \left(0, \frac{1}{2}\right)$. Assume that $\zeta(0) = 0$ and that $|\nabla \zeta(x')| \leqslant b$ for every $x' \in \mathbb{R}^{n-1}$ with $|x'| \leqslant 2$. Then, for any $x \in B_1$ with $x_n \geqslant \zeta(x')$*

$$d_*(x) \geqslant \frac{1}{2}\left(x_n - \zeta(x')\right).$$

Proof. We let $R := d_*(x) \geqslant 0$ and we observe that $B_R(x)$ lies above the graph of ζ and it is tangent to it at some point $z = (z', z_n) \in \partial B_R(x)$ with $z_n = \zeta(z')$. We also denote by ω the interior normal of $B_R(x)$ at z. Then, by construction,

$$\frac{x-z}{R} = \omega = \frac{\left(-\nabla \zeta(z'),\, 1\right)}{\sqrt{1+|\nabla \zeta(z')|^2}}. \tag{3.1}$$

Also, since the origin belongs to the graph of ζ, we have that $R = d_*(x) \leqslant |x| \leqslant 1$. Therefore $|z| \leqslant |z-x| + |x| = R + |x| \leqslant 2$. Accordingly, we deduce from (3.1) that

$$\frac{|x'-z'|}{R} = \frac{|\nabla \zeta(z')|}{\sqrt{1+|\nabla \zeta(z')|^2}} \leqslant |\nabla \zeta(z')| \leqslant b.$$

Thus, using again (3.1),

$$1 \geqslant \frac{1}{\sqrt{1+|\nabla\zeta(z')|^2}} = \frac{x_n - z_n}{R} = \frac{x_n - \zeta(x') + \zeta(x') - \zeta(z')}{R}$$
$$\geqslant \frac{x_n - \zeta(x') - b\,|x'-z'|}{R} \geqslant \frac{x_n - \zeta(x') - b^2 R}{R}.$$

Therefore

$$x_n - \zeta(x') \leqslant (1+b^2)R \leqslant 2R = 2d_*(x). \qquad \square$$

Given $r \in \mathbb{R}$, we use the notation $r_- := \max\{-r, 0\}$.

LEMMA 8. *Let* $\alpha \in (0,1)$*,* $b \in \left(0, \frac{1}{2}\right)$ *and* $r \geqslant 1$*, with* $br^\alpha \leqslant \frac{1}{2}$*. Assume that* $\zeta(0) = 0$ *and that* $|\nabla\zeta(x')| \leqslant br^\alpha$ *for every* $x' \in \mathbb{R}^{n-1}$ *with* $|x'| \leqslant 3r$*.*

Suppose also that

$$\zeta(x') \geqslant 0 \text{ for every } x' \in \mathbb{R}^{n-1}. \tag{3.2}$$

Let $x \in \mathbb{R}^n$ *with* $|x'| \leqslant r$*. Then*

$$\big(d_*(x)\big)_- \geqslant \frac{1}{2}\big(x_n - \zeta(x')\big)_-.$$

Proof. We can suppose that $x_n < \zeta(x')$, otherwise $\big(x_n - \zeta(x')\big)_- = 0$ and the desired claim is obvious.

Then, we take $R := -d_*(x) > 0$ and we consider the ball $B_R(x)$. By construction, $B_R(x)$ lies below the graph of ζ and it is tangent to it at some point $z = (z', z_n) \in \partial B_R(x)$ with $z_n = \zeta(z')$. Notice that, in view of (3.2),

$$z_n \geqslant 0. \tag{3.3}$$

We also denote by ω the interior normal of $B_R(x)$ at z, and so

$$\frac{x-z}{R} = \omega = \frac{\big(\nabla\zeta(z'), -1\big)}{\sqrt{1+|\nabla\zeta(z')|^2}}. \tag{3.4}$$

We claim that

$$z_n x_n \leqslant br^{\alpha+1} z_n. \tag{3.5}$$

Indeed, if $x_n \leqslant 0$, then (3.5) follows from (3.3). If instead $x_n > 0$, then we know that

$$\xi(x') = \xi(x') - \xi(0) \leqslant br^\alpha |x'| \leqslant br^{\alpha+1}$$

and thus $x_n \in (0, \xi(x')) \subseteq (0, br^{\alpha+1})$, which gives (3.5).

From (3.3) and (3.5) we obtain that

$$z_n^2 - 2x_n z_n \geqslant z_n^2 - 2br^{\alpha+1} z_n \geqslant \inf_{t \geqslant 0} t^2 - 2br^{\alpha+1} t = -b^2 r^{2(\alpha+1)}.$$

Consequently, using that the origin lies on the graph of ζ,

$$\begin{aligned} r^2+x_n^2 &\geqslant |x'|^2+x_n^2 = |x|^2 \geqslant |d_*(x)|^2 = |x-z|^2 \\ &= |x'-z'|^2+|x_n-z_n|^2 = |x'-z'|^2+x_n^2+z_n^2-2x_nz_n \\ &\geqslant |x'-z'|^2+x_n^2-b^2r^{2(\alpha+1)}, \end{aligned}$$

and thus $|x'-z'|^2 \leqslant r^2+b^2r^{2(\alpha+1)} \leqslant 2r^2$.

Therefore $|z'| \leqslant |x'|+|x'-z'| \leqslant r+\sqrt{2}\,r \leqslant 3r$. Hence, we deduce from (3.4) that

$$\frac{|x'-z'|}{R} = \frac{|\nabla\zeta(z')|}{\sqrt{1+|\nabla\zeta(z')|^2}} \leqslant |\nabla\zeta(z')| \leqslant br^\alpha$$

and thus

$$\begin{aligned} 1 = \frac{|x-z|}{R} &\geqslant \frac{z_n-x_n}{R} = \frac{\zeta(z')-\zeta(x')+\zeta(x')-x_n}{R} \\ &\geqslant \frac{-br^\alpha|z'-x'|+\zeta(x')-x_n}{R} \geqslant -b^2r^{2\alpha}+\frac{\zeta(x')-x_n}{R}. \end{aligned}$$

That is,

$$2\big(d_*(x)\big)_- = 2R \geqslant (1+b^2r^{2\alpha})R \geqslant \zeta(x')-x_n = \big(x_n-\zeta(x')\big)_-,$$

which gives the desired result. □

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ON THE EXTENSIONS OF THE DE GIORGI APPROACH TO NONLINEAR HYPERBOLIC EQUATIONS

Abstract. In this talk we present an overview on the extensions of the De Giorgi approach to general second order nonlinear hyperbolic equations. We start with an introduction to the original conjecture by E. De Giorgi ([1, 2]) and to its solution by E. Serra and P. Tilli ([4]). Then, we discuss a first extension of this idea (Serra&Tilli, [5]) aimed at investigating a wide class of homogeneous equations. Finally, we announce a further extension to nonhomogeneous equations, obtained by the author in [9] in collaboration with P. Tilli.

1. De Giorgi's conjecture.

In 1996, E. De Giorgi stated the following conjecture on weak solutions of the *defocusing* NLW equation.

CONJECTURE 1 (De Giorgi, [1, 2]). Let $w_0, w_1 \in C_0^\infty(\mathbb{R}^n)$, let $k > 1$ be an integer; for every positive real number ε, let $w_\varepsilon = w_\varepsilon(t,x)$ be the minimizer of the functional

$$F_\varepsilon(u) := \int_0^\infty \int_{\mathbb{R}^n} e^{-t/\varepsilon} \left(|u''(t,x)|^2 + \tfrac{1}{\varepsilon^2}|\nabla u(t,x)|^2 + \tfrac{1}{\varepsilon^2}|u(t,x)|^{2k}\right) dx\,dt \tag{1.1}$$

in the class of all u satisfying the initial conditions

$$u(0,x) = w_0(x), \qquad u'(0,x) = w_1(x). \tag{1.2}$$

Then, there exists $\lim\limits_{\varepsilon\downarrow 0} w_\varepsilon(t,x) = w(t,x)$, satisfying the equation

$$w'' = \Delta w - k w^{2k-1}. \tag{1.3}$$

REMARK 1. In the statement of the conjecture, we maintained the original formulation of [1, 2] and we only changed notation, according to that we use in the sequel. The same thing holds for all the results we mention in this paper. In addition, we recall that $u'(t,x)$ denotes $\frac{\partial u}{\partial t}(t,x)$ and that, for the sake of simplicity, we always omit the dependence of the functional spaces on $\mathbb{R}^n$, i.e. $H^1 = H^1(\mathbb{R}^n)$, $L^p = L^p(\mathbb{R}^n)$ and so on.

In order to better understand the meaning of the conjecture, it is worth stressing some characteristic features of the functional F_ε.

First we note that it involves second order time derivatives. Thus, a minimizer of F_ε solves a fourth order PDE. However, if one computes the formal *Euler–Lagrange*

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equation satisfied by a minimizer w_ε, then one obtains

$$\varepsilon^2 (e^{-t/\varepsilon} w_\varepsilon'')'' = e^{-t/\varepsilon} (\Delta w_\varepsilon - k w_\varepsilon^{2k-1})$$

and thus, expanding and dropping $e^{-t/\varepsilon}$,

$$\varepsilon^2 w_\varepsilon'''' - 2\varepsilon w_\varepsilon''' + w_\varepsilon'' = \Delta w_\varepsilon - k w_\varepsilon^{2k-1}. \tag{1.4}$$

Consequently, if one assumes that $w_\varepsilon \to w$ in some suitable sense and lets $\varepsilon \downarrow 0$, then one formally obtains (1.3).

On the other hand, we also remark that, as F_ε is defined through integrals over the “space–time” $[0,\infty) \times \mathbb{R}^n$, the initial conditions of the Cauchy problem are in fact *boundary conditions* for the minimization problem.

In addition, it is convenient to stress the singular nature of the *integration weight* $e^{-t/\varepsilon}$. More precisely, one can see that $\varepsilon^{-1} e^{-t/\varepsilon} dt$ is an approximate *Dirac delta* measure and hence, at least formally,

$$\varepsilon F_\varepsilon(u) \approx \int_{\mathbb{R}^n} \left(|\nabla w_0(x)|^2 + |w_0(x)|^{2k} \right) dx, \qquad \text{as} \quad \varepsilon \downarrow 0.$$

Hence, this prevents a straightforward application of classical techniques of variational convergence, such as Γ–*convergence*. The previous asymptotic expansion, indeed, shows that this technique does not provide useful information on the limit behavior of the sequence of the minimizers.

Finally, one can note that F_ε is convex (for fixed $\varepsilon > 0$) and that therefore, up to some suitable technical adaptation, the proof of the existence and uniqueness of the minimizers is not a demanding issue.

We also recall that the existence of *global* solutions for the Cauchy problem (1.3)&(1.2) is not new (see e.g. [7] and the references therein). Actually, as highlighted in [3, 4], the originality of the strategy hinted by De Giorgi lies in *how* he intended to exploit techniques from the Calculus of Variations. The variational approaches to the wave equation $w'' = \Delta w$ and its nonlinear variants that can one can find in the literature (see e.g. [7, 8] and references therein) are based on the interpretation of $w'' = \Delta w$ as the Euler–Lagrange equation of the functional

$$I(w) := \int_0^\infty \int_{\mathbb{R}^n} \left(|w'(t,x)|^2 - |\nabla w(t,x)|^2 \right) dx\, dt$$

(with possibly lower order terms like $|w|^{2k}$). However, since I is neither convex nor bounded from below, one is forced to search for *critical points* rather than *global minimizers*. Unfortunately, functionals like I behave badly also for the application of Critical Point Theory, so that only partial results can be proved. De Giorgi, on the contrary, introduces a new functional F_ε that is quite easy to minimize (regardless of the magnitude of k) and thus moves the problem to the investigation of the limit behavior of the sequence of the minimizers.

2. The proof of the conjecture.

In 2012, E. Serra and P. Tilli showed that Conjecture 1 is in fact true. Precisely, in [4], they proved the following theorem.

THEOREM 1 (Serra&Tilli, [4]). *For $p \geqslant 2$ and $\varepsilon > 0$, let $w_\varepsilon(t,x)$ denote the unique minimizer of the strictly convex functional*

$$F_\varepsilon(u) = \int_0^\infty \int_{\mathbb{R}^n} e^{-t/\varepsilon} \left(|u''(t,x)|^2 + \tfrac{1}{\varepsilon^2}|\nabla u(t,x)|^2 + \tfrac{1}{\varepsilon^2}|u(t,x)|^p\right) dx\,dt$$

under the boundary conditions (1.2), where w_0 and w_1 are given functions such that

$$w_0, w_1 \in H^1 \cap L^p.$$

Then:

(a) Estimates. There exists a constant C (which depends only on w_0, w_1, p and n) such that, for every $\varepsilon \in (0,1)$,

$$\int_0^T \int_{\mathbb{R}^n} \left(|\nabla w_\varepsilon(t,x)|^2 + |w_\varepsilon(t,x)|^p\right) dx\,dt \leqslant CT, \qquad \forall T > \varepsilon,$$

$$\int_{\mathbb{R}^n} |w'_\varepsilon(t,x)|^2\,dx \leqslant C \quad \text{and} \quad \int_{\mathbb{R}^n} |w_\varepsilon(t,x)|^2\,dx \leqslant C(1+t^2), \qquad \forall t \geqslant 0,$$

and, for every function $h \in H^1 \cap L^p$

$$\left|\int_{\mathbb{R}^n} w''_\varepsilon(t,x)h(x)\,dx\right| \leqslant C\left(\|h\|_{L^p} + \|\nabla h\|_{L^2}\right), \qquad \text{for a.e.} \quad t > 0.$$

(b) Convergence. Every sequence w_{ε_i} (with $\varepsilon_i \downarrow 0$) admits a subsequence which is convergent, in the strong topology of $L^q((0,T)\times A)$ for every $T > 0$ and every bounded open set $A \subset \mathbb{R}^n$ (with arbitrary $q \in [2,p)$ if $p > 2$ and $q = p$ if $p = 2$), almost everywhere in $\mathbb{R}^+ \times \mathbb{R}^n$ and in the weak topology of $H^1((0,T)\times\mathbb{R}^n)$ for every $T > 0$, to a function w such that

$$w \in L^\infty(\mathbb{R}^+; L^p), \quad \nabla w \in L^\infty(\mathbb{R}^+; L^2),$$

$$w' \in L^\infty(\mathbb{R}^+; L^2), \quad w \in L^\infty((0,T); H^1) \quad \forall T > 0,$$

which solves in $\mathbb{R}^+ \times \mathbb{R}^n$ the nonlinear wave equation

$$w'' = \Delta w - \tfrac{p}{2}|w|^{p-2}w \tag{2.1}$$

with initial conditions as in (1.2).

(c) *Energy inequality. Letting*

$$\mathcal{E}(t) := \int_{\mathbb{R}^n} \left(|w'(t,x)|^2 + |\nabla w(t,x)|^2 + |w(t,x)|^p\right) dx,$$

the function $w(t,x)$ satisfies the energy inequality

$$\mathcal{E}(t) \leqslant \mathcal{E}(0) = \int_{\mathbb{R}^n} \left(|w_1(x)|^2 + |\nabla w_0(x)|^2 + |w_0(x)|^p\right) dx, \qquad \textit{for a.e.} \quad t > 0.$$

REMARK 2. We stress the fact that, in *(b)*, the limit function w solves (2.1) in a *distributional* (or *weak*) sense, namely

$$\int_0^\infty \int_{\mathbb{R}^n} w'(t,x)\varphi'(t,x)\,dx\,dt = \int_0^\infty \int_{\mathbb{R}^n} \nabla w(t,x) \cdot \nabla \varphi(t,x)\,dx\,dt + \\ + \int_0^\infty \int_{\mathbb{R}^n} \tfrac{p}{2}|w(t,x)|^{p-2} w(t,x)\varphi(t,x)\,dx\,dt$$

for every $\varphi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$. In the sequel we only deal with this type of solutions.

Some comments are in order. First, CONJECTURE 1 deals with the nonlinearity $|w|^{2k}$ with *k integer*, while THEOREM 1 treats $|w|^p$ without the assumption of p integer. Another relevant feature of THEOREM 1, is that the assumptions on the initial data w_0, w_1 are much weaker than those of the conjecture.

On the other hand, the convergence of the sequence of the minimizers is obtained up to extracting subsequences, thus "losing" the *uniqueness* claimed in the conjecture. In particular, it is an open problem to avoid the extraction of subsequences when p is large.

In addition, THEOREM 1 establishes an estimate for the *mechanical energy* $\mathcal{E}$ usually associated with (2.1), which proves that the obtained solutions are *of energy class* in the sense of Struwe (see [8]). When p is "sufficiently" small, the inequality is in fact an equality, whereas, when p is large, energy *conservation* is still open.

For the sake of completeness, we mention that [6] discusses a simplified version of the conjecture on bounded intervals. However, that paper only deals with the proof of (2.1) and does not treat the fulfillment of the initial condition $w'(0,x) = w_1(x)$.

3. Extension to homogeneous equations.

Now, one can easily see that, setting

$$\mathcal{W}(v) = \int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla v|^2 + \frac{1}{p}|v|^p\right) dx,$$

up to some multiplicative constants equation (2.1) reads

$$w''(t,x) = -\nabla \mathcal{W}(w(t,\cdot))(x), \tag{3.1}$$

where $\nabla\mathcal{W}$ denotes the *Gâteaux derivative* of the functional $\mathcal{W}$. Therefore, it is natural to wonder if the sequence of the minimizers of the functional F_ε, that here is defined by

$$F_\varepsilon(u) := \int_0^t e^{-t/\varepsilon}\left(\int_{\mathbb{R}^n} \frac{\varepsilon^2|u''(t,x)|}{2}\,dx + \mathcal{W}(u(t,\cdot))\right)dt, \tag{3.2}$$

converges to a solution of the Cauchy problem associated with (3.1), even for different choices of $\mathcal{W}$.

REMARK 3. In (3.2) one uses a different scaling in ε, with respect to (1.1). This is due to the fact that in the *abstract* framework this choice simplifies computations. However, this does not yield significant differences.

This problem has been solved again by E. Serra and P. Tilli, in [5]. Before showing the statements of the main results, it is necessary to point out under which assumptions on the functional $\mathcal{W}$ (that we refer to as assumption **(H)** in the following), they are valid.

(H) The functional $\mathcal{W}: L^2 \to [0,\infty]$ is lower semi–continuous in the weak topology of L^2, i.e

$$\mathcal{W}(v) \leqslant \liminf_k \mathcal{W}(v_k), \qquad \text{whenever} \quad v_k \rightharpoonup v \quad \text{in} \quad L^2.$$

Moreover, we assume that the set of functions

$$\mathrm{W} = \{v \in L^2 : \mathcal{W}(v) < \infty\}$$

is a Banach space such that

$$C_0^\infty \hookrightarrow \mathrm{W} \hookrightarrow L^2 \qquad \text{(dense embeddings)}.$$

Finally, $\mathcal{W}$ is Gâteaux differentiable on W and its derivative $\nabla\mathcal{W}: \mathrm{W} \to \mathrm{W}'$ satisfies

$$\|\nabla\mathcal{W}(v)\|_{\mathrm{W}'} \leqslant C(1+\mathcal{W}(v)^\theta), \qquad \forall v \in \mathrm{W},$$

for suitable constants $C \geqslant 0$ and $\theta \in (0,1)$.

REMARK 4. Assumption **(H)** is typically satisfied by standard functionals like

$$\mathcal{W}(v) = \frac{1}{p}\int_{\mathbb{R}^n} |\nabla^k v|^p\,dx, \qquad p > 1,$$

(with possibly lower order terms) where W is the space of the L^2 functions v with $\nabla^k v \in L^p$.

THEOREM 2 (Serra&Tilli, [5]). *Given $w_0, w_1 \in \mathrm{W}$ and $\varepsilon \in (0,1)$, under assumption **(H)** the functional F_ε defined in* (3.2) *has a minimizer w_ε in the space $H^2_{loc}([0,\infty); L^2)$ subject to* (1.2). *Moreover:*

(a) *Estimates. There exists a constant C, independent of* ε*, such that*

$$\int_\tau^{\tau+T} \mathcal{W}(w_\varepsilon(t,\cdot))\,dt \leqslant CT, \qquad \forall\tau \geqslant 0, \quad \forall T \geqslant \varepsilon,$$

$$\int_{\mathbb{R}^n} |w'_\varepsilon(t,x)|^2\,dx \leqslant C \quad \text{and} \quad \int_{\mathbb{R}^n} |w_\varepsilon(t,x)|^2\,dx \leqslant C(1+t^2), \qquad \forall t \geqslant 0,$$

$$\|w_\varepsilon\|_{L^\infty(\mathbb{R}^+;\mathrm{W}')} \leqslant C.$$

(b) *Convergence. Every sequence* w_{ε_i} *(with* $\varepsilon_i \downarrow 0$*) admits a subsequence which is convergent, in the weak topology of* $H^1((0,T);L^2)$ *for every* $T > 0$*, to a function* w *such that*

$$w \in H^1_{loc}([0,\infty);L^2), \qquad w' \in L^\infty(\mathbb{R}^+;L^2), \qquad w'' \in L^\infty(\mathbb{R}^+;\mathrm{W}').$$

Moreover, w satisfies the initial conditions (1.2).

(c) *Energy inequality. Letting*

$$\mathcal{E}(t) := \frac{1}{2}\int_{\mathbb{R}^n} |w'(t,x)|^2\,dx + \mathcal{W}(w(t,\cdot)), \tag{3.3}$$

the function $w(t,x)$ *satisfies the energy inequality*

$$\mathcal{E}(t) \leqslant \mathcal{E}(0) = \frac{1}{2}\int_{\mathbb{R}^n} |w_1(x)|^2\,dx + \mathcal{W}(w_0) \qquad \textit{for a.e. } t > 0.$$

Unfortunately, under these assumptions, it is not known whether w satisfies (3.1). Anyway, Serra&Tilli, still in [5], provided a *sufficient* condition on $\mathcal{W}$ that allows to obtain (3.1).

THEOREM 3 (Serra&Tilli, [5]). *Assume that, for some real number* $m > 0$*,*

$$\mathcal{W}(v) = \frac{1}{2}\|v\|^2_{\dot{H}^m} + \sum_{0\leqslant k<m} \frac{\lambda_k}{p_k}\int_{\mathbb{R}^n} |\nabla^k v(x)|^{p_k}\,dx \qquad (\lambda_k \geqslant 0,\ p_k > 1). \tag{3.4}$$

*Then, assumption **(H)** is fulfilled if* W *is the space of those* $v \in H^m$ *with* $\nabla^k v \in L^{p_k}$ $(0 \leqslant k < m)$ *endowed with its natural norm.*

Moreover, the limit function w obtained via Theorem 2 solves, in the sense of distributions, the hyperbolic equation (3.1).

REMARK 5. We recall that, as usual, $\|v\|_{\dot{H}^m}$ is the L^2 norm of $|\xi|^m\,\hat{v}(\xi)$, where $\hat{v}$ is the Fourier transform of v. The typical case is when m is an integer, so that $\|v\|_{\dot{H}^m}$ reduces to $\|\nabla^m v\|_{L^2}$.

3.1. Examples.

In addition to (2.1) there are many other second order hyperbolic equations that can be investigated using the approach suggested by THEOREM 2&THEOREM 3. We briefly recall here some of the most significant ones (for a complete discussion we refer the reader to [5]):

1. *Nonlinear vibrating–beam equation*:
$$w'' = -\Delta^2 w + \Delta_p w - |w|^{q-2} w \qquad (p, q > 1).$$
Here $\mathcal{W}$ is defined by
$$\mathcal{W}(v) = \int_{\mathbb{R}^n} \left(\frac{1}{2} |\Delta v|^2 + \frac{1}{p} |\nabla v|^p + \frac{1}{q} |v|^q \right) dx$$
and $\mathrm{W} = \{v \in H^2 : \nabla v \in L^p, v \in L^q\}$.

2. *Wave equation with fractional Laplacian*:
$$w'' = -(-\Delta)^s \qquad (0 < s < 1).$$
Here $\mathcal{W}$ is defined by
$$\mathcal{W}(v) = c_{n,s} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} \, dx\, dy$$
(which is, for a proper choice of $c_{n,s}$, the natural energy associated to the fractional Laplacian) and $\mathrm{W} = H^s$.

3. *Sine–Gordon equation*:
$$w'' = \Delta w - \sin w.$$
Here $\mathcal{W}$ is defined by
$$\mathcal{W}(v) = \int_{\mathbb{R}^n} \left(\frac{1}{2} |\nabla v|^2 + 1 - \cos v \right) dx$$
and $\mathrm{W} = H^1$.

4. *Wave equation with p–Laplacian* :
$$w'' = \Delta_p w.$$
Here $\mathcal{W}$ is defined by
$$\mathcal{W}(v) = \frac{1}{p} \int_{\mathbb{R}^n} |\nabla v|^p \, dx$$
and $\mathrm{W} = \{v \in L^2 : \nabla v \in L^p\}$.

Note that in the cases of the Sine–Gordon and the p–Laplacian equation, the functional $\mathcal{W}$ satisfies assumption **(H)**, but not assumption (3.4). Consequently, one could not apply THEOREM 3 to this cases. In the Sine–Gordon case, however, since the functional is quadratic in the higher order space derivatives, one can prove an analogous of THEOREM 3. On the contrary, it is an open problem whether this can be done also for the case of the p–Laplacian.

4. Extension to nonhomogeneous equations.

The natural further extension is the addition of a general forcing term at the right–hand side of (3.1), that is, the study of the Cauchy problem associated with the nonhomogeneous equation

$$w''(t,x) = -\nabla \mathcal{W}(w(t,\cdot))(x) + f(t,x). \tag{4.1}$$

The proper choice for the functional F_ε in this case is given by

$$F_\varepsilon(u) = \int_0^t e^{-t/\varepsilon}\left(\int_{\mathbb{R}^n} \frac{\varepsilon^2|u''(t,x)|}{2}\,dx + \mathcal{W}(u(t,\cdot)) - \int_{\mathbb{R}^n} f_\varepsilon(t,x)u(t,x)\,dx\right)dt, \tag{4.2}$$

where (f_ε) is a sequence suitably converging to f.

This issue has been the topic of the doctoral dissertation of the author and is extensively investigated in [9]. Here we just announce the result.

THEOREM 4 (Tentarelli&Tilli, [9]). *Let $\mathcal{W}$ be a functional satisfying assumption **(H)** and $w_0, w_1 \in \mathrm{W}$. Let also $f \in L^2_{loc}([0,\infty), L^2)$. Then, there exists a sequence (f_ε), converging to f in $L^2([0,T];L^2)$ for all $T > 0$, such that:*

- *(a) Minimizers. For every $\varepsilon \in (0,1)$, the functional F_ε defined by (4.2) has a minimizers w_ε in the class of functions in $H^2_{loc}([0,\infty);L^2)$ that are subject to (1.2).*
- *(b) Estimates. There exist two positive constants $C_t, C_{\tau,T}$, depending on t, τ and T (in a continuous way), but independent of ε, such that*

$$\int_{\mathbb{R}^n} |w'_\varepsilon(t,x)|^2\,dx \leqslant C_t, \qquad \int_{\mathbb{R}^n} |w_\varepsilon(t,x)|^2\,dx \leqslant C_t, \qquad \forall t \geqslant 0,$$

$$\int_\tau^{\tau+T} \mathcal{W}(w_\varepsilon(t,\cdot))\,dt \leqslant C_{\tau,T}, \qquad \forall \tau \geqslant 0, \quad \forall T > \varepsilon,$$

$$\int_0^t \|w''_\varepsilon(s)\|^2_{\mathrm{W}'}\,ds \leqslant C_t, \qquad \forall t \geqslant 0.$$

- *(c) Convergence. Every sequence w_{ε_i} (with $\varepsilon_i \downarrow 0$) admits a subsequence which is convergent in the weak topology of $H^1([0,T];L^2)$, for every $T > 0$, to a function w that satisfies (1.2) (where the latter is meant as an equality in W'). In addition,*

$$w' \in L^\infty_{loc}([0,\infty);L^2) \qquad \text{and} \qquad w'' \in L^2_{loc}([0,\infty);\mathrm{W}').$$

- *(d) Energy inequality. Letting $\mathcal{E}$ be again the mechanical energy defined by (3.3), there results*

$$\mathcal{E}(t) \leqslant \left(\sqrt{\mathcal{E}(0)} + \sqrt{\frac{t}{2}\int_0^t\int_{\mathbb{R}^n} |f(s,x)|^2\,dx\,ds}\right)^2, \qquad \text{for a.e.} \quad t \geqslant 0. \tag{4.3}$$

- *(e) Solution of (4.1). Assuming, furthermore, that for some real numbers $m > 0$, $\lambda_k \geqslant 0$ and $p_k > 1$, $\mathcal{W}$ satisfies (3.4), then the limit function w solves (4.1).*

Some comments are in order. First, we point out that the estimate on the mechanical energy established by (4.3) is the same that one can find applying a formal *Grönwall–type* argument to (4.1). In addition, setting $f \equiv 0$, the results of THEOREM 4 recover exactly those of THEOREM 2&THEOREM 3, thus showing that our extension to nonhomogeneous equations is consistent.

On the other hand, the Euler–Lagrange equation satisfied by the minimizers of F_ε (which is analogous to (1.4)) suggests to work directly with f in place of f_ε in (4.2). However, this gives rise to several issues in establishing the requested a priori estimates on $F_\varepsilon(w_\varepsilon)$. On the contrary, a proper choice of (f_ε) allows one to adapt the De Giorgi approach under the sole assumption $f \in L^2_{loc}([0,\infty);L^2)$, which is the usual one in the search of solutions of *finite energy* for (4.1). In particular, the detection of a proper (topology and) "speed of convergence" for f_ε to f is one of the main issues in the extension to nonhomogeneous problems.

Finally, it is worth to outline briefly the main difference between the homogeneous and the nonhomogeneous case: the estimates on the sequence (w_ε) are no longer *global* in time. This occurs since the presence of f drops all the uniform bounds deduced in [5] and allows to establish estimates that are either independent of ε or independent of t. In particular, the presence of the forcing term entails that the quantity

$$E_\varepsilon(t) := \frac{1}{2}\int_{\mathbb{R}^n} |w'_\varepsilon(t,x)|^2\,dx + \int_t^\infty \varepsilon^{-2}\,e^{-(s-t)/\varepsilon}\,(s-t)\,\mathcal{W}(w_\varepsilon(s,\cdot))\,ds$$

is not decreasing (as in the homogeneous case) and not even uniformly bounded with respect to both ε and t. This function, that we call *approximate energy*, is a *formal* approximation of the mechanical energy $\mathcal{E}$ and the investigation of its behavior is the main point of our approach, since it provides the a priori estimates on the minimizers w_ε. Consequently, the fact that it admits only estimates on bounded intervals is the reason for which the inequalities in *(b)* are no longer global.

This transition "from global to local" of the a priori estimates affects the regularity of the limit function w, but fortunately does not rule out the possibility of extending the De Giorgi approach. Actually, the proofs of the energy inequality, the initial conditions and (4.1) do not require any global estimate on the sequence of minimizers (even in the homogeneous case).

Moreover, we point out that the choice of the sequence (f_ε) is crucial also for establishing the proper estimate on $E_\varepsilon(t)$; in particular, for establishing *causal* estimates for a quantity which is *a–causal* by definition.

5. A further extension: dissipative equations.

Finally, it worth recalling that [5] also shows that an approach *à la De Giorgi* is available also for *dissipative* homogeneous wave equations of the type

$$w''(t,x) = -\nabla\mathcal{W}(w(t,\cdot))(x) - \nabla\mathcal{G}(w'(t,\cdot))(x),$$

where $\mathcal{G}$ is a *quadratic form* defined on a suitable Hilbert space. A typical example is given by the *Telegraph* equation

$$w'' = \Delta w - |w|^{p-2}w - w' \qquad (p > 1)$$

(just setting $\mathcal{W}(v) = \int_{\mathbb{R}^n}\left(\frac{1}{2}|\nabla v| + \frac{1}{p}|v|^p\right)dx$ and $\mathcal{G}(v) = \frac{1}{2}\int_{\mathbb{R}^n}|v|^2\,dx$).

As in the non–dissipative case, also here it is natural to wonder if an extension to the nonhomogeneous case, namely

$$w''(t,x) = -\nabla\mathcal{W}(w(t,\cdot))(x) - \nabla\mathcal{G}(w'(t,\cdot))(x) + f(t,x),$$

is possible. The answer is again positive and this issue will be treated in a forthcoming paper by the author, as well.

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