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A NEW APPROACH TO QUANTUM ORBIT METHOD FOR STANDARD QUANTUM $\mathbb{CP}^N$

Abstract. The purpose of this paper is to show that the topological version of the quantum orbit method for standard complex projective spaces can be easily proven by using groupoid quantization.

1. Introduction.

The well known orbit method in representation theory ([9]) is a way to associate unitary irreducible representations (in short unitary irreps, in what follows) of a given Lie algebra $\mathfrak{g}$ to coadjoint orbits in the dual $\mathfrak{g}^*$. In some cases this correspondence is a bijection and, in fact, even a homeomorphism with respect to suitable topologies.

Irreducible representations of $\mathfrak{g}$ are naturally associated to irreps of its universal enveloping algebras $U(\mathfrak{g})$ and coadjoint orbits can be understood also as symplectic leaves of the Kirillov-Konstant Poisson bracket on $\mathfrak{g}^*$.

In the context of *quantum groups* the universal enveloping algebra is replaced by its quantization $U_q(\mathfrak{g})$: it has become standard to call *quantum orbit method* the correspondence that can be established between unitary irreps of $U_q(\mathfrak{g})$ and symplectic leaves of the dual simply connected Poisson–Lie group G^*. By duality one often calls *quantum orbit method* also a correspondence between unitary irreps of the quantized C^* algebra of functions on G and symplectic leaves of the Poisson-Lie group G itself.

Such correspondence may be clarified considering that the kernel of a unitary irrep of the quantized C^*–algebra of function should tend (in a suitable sense), in the semiclassical limit, to a maximal Poisson ideal I of the classical function algebra $\mathcal{C}^\infty(G)$. The quotient of the Poisson algebra of functions on G by this maximal Poisson ideal should then be isomorphic to the algebra of functions on the symplectic leaf corresponding to the representation.

This *method* was first proven to be a theorem whenever G is a compact, connected, simply connected Poisson-Lie group with its standard structure ([19]). It was later extended to generalized flag manifolds with their Bruhat–Poisson structure, i.e. described as quotient by a Poisson subgroup ([21, 20]). All such results were obtained by separately classifying symplectic leaves of the symplectic foliation and then, via a detailed analysis of the algebraic structure of the quantized C^*–algebra listing all its unitary irreducible representations. They reduced both classifications to the same combinatorial object related to the Weyl group. In such works not much was put on the correspondence being just a "natural" bijective correspondence or preserving some additional structure (e.g. a topology).

*Text of the footnote.

A more complete treatment was recently given in [13] where it was proven that the quantum orbit method holds, at a topological level, for any quotient of the standard compact Poisson–Lie groups by a Poisson stabilizer.

On the other hand it is also known that quantization of so called twisted compact Poisson-Lie groups does not verify the quantum orbit method ([10]).

From the work of Sheu ([17]) it is known that the C^*–algebras of compact quantum groups and of some of their homogeneous spaces can be described as groupoid C^*–algebras. This construction relies on explicitely knowing $*$–irreducible representations of the algebra. Some of Sheu's result were recently obtained in [2] by a different quantization procedure which starts from the original Poisson manifold, passing through its integrating symplectic groupoid as an intermediate object, and produces Sheu's groupoid as the groupoid of Lagrangian Bohr-Sommerfeld leaves (see [7] for the general procedure).

In this paper we will show how unitary irreducible representations can be recovered purely by the groupoid C^*–algebra, at least for the case of so–called standard projective space, in such a way that the correspondence between unitary irreps and symplectic leaves of the classical Poisson manifold is naturally a homeomorphism of topological space, with respect to the Jacobson topology on irreps and to the quotient topology on the space of leaves. In this way the groupoid quantization procedure produces the *quantum orbit method* in a natural way and without having to rely, for its conceptual properties, on explicit enumeration of such irreps. Irreps for groupoid C^*–algebras were widely studied (see [15, 16]) under the general purpose of establishing the nicest possible connection between the orbit space of the groupoid and a suitably topological version of the space of irreducible representations (or of primitive ideals). We will use results from some recent work on unitary irreps of C^*–algebras: [18]; such results are much stronger than needed and it is quite possible that they will be of help in understanding exactly for which Poisson homogeneous spaces it is reasonable to expect the quantum orbit method to hold. The aim is to generalize results of [13] to cases in which the stabilizer is coisotropic. Let us remark, again, that the specific cases that will be dealt with in this paper could also be analyzed in the context of graph C^*–algebra representations, as studied in [8]. It is, on the other hand, still unclear whether nonstandard quantum projective spaces are graph C^*–algebras and it is quite clear that more general quantum Hermitian projective spaces will not be, if we want the quantum orbit method to hold (due to the fact that the underlying symplectic foliation carries $\mathbb{T}^k$–families of symplectic leaves of a given dimension with, generically, $k > 1$).

In future work our aim is to obtain the same results for the 1-parameter family of nonstandard complex projective spaces, where validity of the *quantum orbit method* was never proved (except for the $n = 1$ case, which will be commented upon in this paper). In such situation, however, the kind of groupoid underlying quantization is more involved. It has to be remarked here that explicit groupoid quantization of other symmetric spaces is still open (see [1] for a preliminary discussion).

2. Poisson standard $\mathbb{CP}^n$

The Bruhat or standard Poisson structure on $\mathbb{CP}^n$ is obtained via projection from the standard multiplicative Poisson stucture on $SU(n+1)$. The latter is the one induced by the classical Cartan decomposition

$$SL(n+1;\mathbb{C}) = SU(n+1)SB(n+1;\mathbb{C})$$

interpreted, at the infinitesimal level, as a Manin triple

$$(\mathfrak{sl}(n+1;\mathbb{C}), \mathfrak{su}(n+1), \mathfrak{sb}(n+1;\mathbb{C}))$$

with respect to the standard invariant bilinear form on $\mathfrak{sl}(n+1;\mathbb{C})$ (see [10] for additional details). One of the properties of the standard Poisson structure on $SU(n)$ is that the embedding

$$U(n) \hookrightarrow SU(n+1)\,; \qquad A \mapsto \begin{pmatrix} A & 0 \\ 0 & D(A)^{-1} \end{pmatrix}$$

defines a Poisson-Lie subgroup and therefore $SU(n)/U(n-1) \simeq \mathbb{CP}^n$ inherits a Poisson structure preserved by the action map

$$SU(n+1) \times \mathbb{CP}^n \to \mathbb{CP}^n\,.$$

It is an interesting fact that the symplectic foliation on the Poisson homogeneous space is essentially determined by the images under the projection of the Poisson subgroups of $SU(n+1)$ (see [4]). For each $k = 1, \ldots, n$, let

$$G_k = S(U(k) \times U(n+1-k)) \subset SU(n+1)\,; \quad P_k = p(G_k) \subset \mathbb{CP}^n\,.$$

Each G_k is a Poisson subgroup so that P_k is a Poisson submanifold of $\mathbb{CP}^n$. As can be easily deduced from [21], Proposition 2.1, this exhausts the list of maximal (with respect to inclusion) connected Poisson subgroups of $SU(n+1)$.

The Poisson submanifolds $P_k \sim \mathbb{CP}^{k-1}$ are contained one inside the other giving rise to the following chain of Poisson embeddings:

$$\{\infty\} = \mathbb{CP}^0 \subset \mathbb{CP}^1 \subset \ldots \subset \mathbb{CP}^n \tag{1}$$

where $\mathbb{CP}^k$ corresponds to $X_j = 0$ for $j > k$ so that $\infty = [1, 0, \ldots, 0]$. The symplectic foliation, then, corresponds to the Bruhat decomposition (hence its name):

$$\mathbb{CP}^n = \bigcup_{i=1}^{n+1} \mathcal{S}_i\,, \tag{2}$$

where each $\mathcal{S}_i$ can be described as $\mathcal{S}_i = \{[X_1, \ldots, X_i, 0, \ldots 0],\ X_i \neq 0\} \subset \mathbb{CP}^{i-1}$. There exists, therefore, one contractible symplectic leaf in each even dimension, which turns out to be symplectomorphic to standard $\mathbb{C}^i$. The maximal symplectic leaf (the maximal cell) $\mathcal{S}_{n+1}$ is open and dense in $\mathbb{CP}^n$. In terms of suitable coordinates the Bruhat Poisson structure restricted to $\mathcal{S}_{n+1}$ reads as follows:

$$\pi_0|_{\mathcal{S}_{n+1}} = i\sum_{i=1}^{n}(1+|y_i|^2)\partial_{y_i}\wedge\partial_{\bar{y}_i}\,. \tag{3}$$

Let us now describe the source simply connected symplectic groupoid integrating $(\mathbb{CP}^n,\pi)$. It can be constructed as a symplectic reduction of the Lu-Weinstein symplectic groupoid structure on $SL(n+1,\mathbb{C})$ integrating the Poisson Lie group $(SU(n+1),\pi)$, along the lines described in [3]. As a manifold it can be characterized as:

$$\Sigma(\mathbb{CP}^n,\pi) = \{[g\gamma], g\in SU(n+1), \gamma\in SB(n+1,\mathbb{C}), {}^{g}\gamma\in U(n)^{\perp}\}\,, \tag{4}$$

where we denoted with $[g\gamma]$ the class of $g\gamma\in SL(n+1,\mathbb{C})$ under the quotient map $SL(n+1,\mathbb{C})\to U_t(n)\backslash SL(n+1,\mathbb{C})$ and with $U(n)^{\perp}$ the coisotropic subgroup of $SB(n+1;\mathbb{C})$ which integrates the Lie subalgebra

$$\mathfrak{u}(n)^{\perp} = \left\{\xi\in\mathfrak{sb}(n+1;\mathbb{C})\,\middle|\,\langle\xi,X\rangle = 0, \forall X\in\mathfrak{u}(n)\right\}.$$

As a smooth manifold, the symplectic groupoid is nothing but a fibre bundle over $\mathbb{CP}^n$ with contractible fibre $U(n)^{\perp}$ associated to the homogeneous principal bundle with respect to the dressing action of $U(n)$ on $U(n)^{\perp}$. We will denote its symplectic form as Ω_Σ, and with l and r the left and right surjective submersions onto the unit space $\Sigma^0\simeq\mathbb{CP}^n$. Since the l (or r) fibers are diffeomorphic to the contractible subgroup $U(n)^{\perp}$ the restriction of the symplectic form Ω_Σ to the l-fibres is exact; by applying Corollary 5.3 of [6] we can conclude that the 2-form Ω_Σ is exact as well.

Let us conclude this section with the following statement, the proof of which is completely general (see [5], Corollary at page 22).

PROPOSITION 1. *Let (Σ,Ω_Σ) be a simply connected groupoid integrating the Poisson manifold (M,π). Then there is a canonical homemorphism*

$$\Psi : S_M \to {}_{\Sigma}\backslash\Sigma^0$$

between the space of symplectic leaves of M and the orbit space of Σ (each endowed with the quotient topology).

In the $\mathbb{CP}^n$ case such space can be abstractly described as $\{1,\dots,n\}$ with (non Hausdorff) topology given by the family of open sets

$$\{\varnothing,\{1\},\{1,2\},\{1,2,3\},\dots,\{1,\dots,n\}\}.$$

Such space is manifestly not Hausdorff and not even T_1. It satisfies the weaker separation axiom T_0.

3. $C^*_q(\mathbb{CP}^n)$ as a groupoid C^*–algebra

The first step in groupoid geometric quantization is the choice of a multiplicative Lagrangian polarization. Multiplicativity here refers to the fact that the set of leaves inherits from the symplectic groupoid a topological groupoid structure (with respect to the quotient topology). While this choice is certainly not unique, and quantization may well depend on it, for the standard Poisson structure on $\mathbb{CP}^n$ there exists a reasonally preferred choice. We will not describe this step in details, since it is was adressed in [2], but rather we will describe its outcome.

The quotient groupoid $\mathcal{G}^L(\mathbb{CP}^n)$ consisting of Lagrangian leaves can be identified with the restriction of the action groupoid $\mathbb{R}^n \ltimes \mathbb{R}^n$ to the n–dimensional standard simplex Δ_n, thus:

$$\mathcal{G}^L(\mathbb{CP}^n) = \mathbb{R}^n \ltimes \mathbb{R}^n\big|_{\Delta_n} = \{(x,y) \in \mathbb{R}^n \,\big|\, x, x+y \in \Delta_n \,,\, x_i = 0 \Rightarrow y_i = 0\}\,,$$

where, as usual, the groupoid structure is defined by

$$\begin{cases} l(x,y) &= x \\ r(x,y) &= x+y \end{cases} \qquad (x,y)\cdot(x+y,z) = (x,z)$$

As a second step in groupoid geometric quantization we want to select a subgroupoid of Bohr-Sommerfeld Lagrangian leaves. The Bohr-Sommerfeld condition is an integrality type condition that guarantees that the holonomy group of the restricted prequantization connection vanishes along a leaf. This condition allows existence of covariantly costant sections of the prequantum line bundle.

The ($\hbar$-dependent) *BS* subgroupoid of $\mathbb{CP}^n$ can be described, in our case, as follows (see section 6.2 of [2]):

$$\mathcal{G}^{bs}(\mathbb{CP}^n) = \{(x,y) \in \mathcal{G}^L(\mathbb{CP}^n) \,\big|\, \log|x_i| \in \hbar\mathbb{Z}, y_i \in \hbar\mathbb{Z}\,, i = 1,\ldots,n\}\,.$$

With respect to the quotient topology this groupoid is easily seen to be Hausdorff and étale. It can be furthermore remarked that $\mathcal{G}^L(\mathbb{CP}^n)$ and $\mathcal{G}^{bs}(\mathbb{CP}^n)$ have homeomorphic orbit spaces.

The analysis of these groupoids is much simplified by putting them in Sheu's form [17], as shown in Section 7.1 of [2]. Let

$$\mathcal{T}_n = \mathbb{Z}^n \times \overline{\mathbb{Z}}\big|_{\overline{\mathbb{N}}^n} = \{(j,k) \in \mathbb{Z}^n \times \overline{\mathbb{Z}} \,\big|\, k_i, j_i + k_i \geqslant 0\}$$

(here $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{\infty\}$) and let $\tilde{\mathcal{T}}_n$ be the subgroupoid determined by

$$k_i = \infty \Rightarrow \begin{cases} \sum_{k=1}^{i} j_k &= 0 \\ j_{i+1} = \cdots = j_n &= 0 \end{cases}\,.$$

Finally let $\mathfrak{T}_m$ be the quotient groupoid of $\tilde{\mathcal{T}}_n$ with respect to the equivalence relation that identifies

$$(j,k) \simeq (j,k_1,\ldots,k_{i-1},\infty\ldots,\infty)$$

if $k_i = \infty$. This groupoid is isomorphic, as topological groupoid, to $\mathcal{G}^{bs}(\mathbb{CP}^n)$.

The unit space of $\mathcal{G}^{bs}(\mathbb{CP}^n)$ can then be identified with a subspace of $\overline{\mathbb{Z}}^n$. The infinity extension $\overline{\mathbb{Z}}$ is given the natural topology in which $\{n \geqslant n_0\}$ are a neighbourhood basis of ∞. Then:

$$\mathcal{G}^{bs}(\mathbb{CP}^n)^{(0)} = \{(k_1, \ldots, k_n) \in \overline{\mathbb{Z}}^n \,\big|\, k_i = \infty \Rightarrow k_{i+1} = \ldots = k_n = \infty\}.$$

Orbits and their topology are described as follows:

$$\Delta^i = \{(k_1, \ldots, k_i, \infty \ldots, \infty)\}; \quad \overline{\Delta_i} = \bigcup_{j \geqslant i} \Delta^j.$$

LEMMA 1. *The space of symplectic leaves of standard $\mathbb{CP}^n$ is homeomorphic to the orbit space of the groupoid $\mathcal{G}^{bs}(\mathbb{CP}^n)$ via the map that to each orbit of BS Lagrangian leaves associates the symplectic leaf on which it projects:*

$$\Phi : \mathcal{L} \mapsto \mathcal{S}_L.$$

Let's now compute the isotropy groups of $\mathcal{G}^{bs}(\mathbb{CP}^n)$. Let $\underline{k} \in \Delta^i$. The corresponding isotropy group is given by the set of $(\underline{j}, \underline{k})$ such that $\underline{j} = \overline{\underline{j} + \underline{k}}$. By considering what happens to the j_i's when $k_i = \infty$ it is easily proved that all isotropy groups are abelian and trivial. This, together with the fact that the orbit space is not T_1 but satisfies the weaker separation axiom T_0, implies via theorem 1.3 of [14] that the corresponding C^*–algebra is GCR (i.e. postliminal) and type I. In particular the unitary dual of the algebra is homeomorphic the the space of primitive ideals with the Jacobson topology. On the other hand the C^*–algebra is not a continuous trace C^*–algebra. By a direct analysis of the topology involved the following statement is clear.

LEMMA 2. *The subsets $\overline{\Delta_i}$ of $(\mathcal{G}^{bs})^{(0)}$ exhausts the list of all closed invariant subsets on the space of units.*

Let us recall that a groupoid is called *topologically principal* if the space of units with trivial isotropy is dense. Then the preceding discussion is summarized in what follows:

PROPOSITION 2. *The topological groupoid $\mathcal{G}^{bs}(\mathbb{CP}^n)$ is an amenable, étale, Hausdorff groupoid such that $\mathcal{G}^{bs}(\mathbb{CP}^n)\big|_X$ is topologically principal for every closed invariant subset X of the unit space.*

We will use Lemma 4.6 of [18], which says that under the conditions listed in the previous proposition there is a homeomorphism between the primitive ideal space of the C^*–algebra and the space of quasi–orbits of the groupoid. Let us remark that for our groupoid the quasi-orbit spaces does not differ from the orbit space since there are no pairs of distinct points sharing the same orbit closure. Since the previous proposition guarantees that all hypothesis are satisfied we can conclude that the map

$$\Xi : x \mapsto \ker \omega_x$$

is a homeomorphism from the space of orbits of the Bohr-Sommerfeld groupoid onto $Prim(C^*(\mathcal{G}^{bs}(\mathbb{CP}^n)))$ endowed with the Jacobson topology, and the latter can be identified with the unitary dual $C^*(\widehat{\mathcal{G}^{bs}(\mathbb{CP}^n)})$. This can be put together with Lemma 1 to conclude

PROPOSITION 3 (Quantum orbit method for $\mathbb{CP}^n$). *The composition of the above maps* $\Xi \circ \Phi \circ \Psi$ *establishes a homeomorphism between the space of symplectic leaves* $S_{\mathbb{CP}^n}$ *and the space of unitary irreps* $(C^*(\widehat{\mathcal{G}^{bs}(\mathbb{CP}^n)}))$.

4. Non standard $\mathbb{C}P^1_t$

In this section we will briefly comment upon the non standard case for the projective line, to show how it can be treated under the same footing although in this case the isotropy groupoid is non trivial.

The non standard Poisson structure on $\mathbb{C}P^1 \simeq \mathbb{S}^2$ is zero on a parallel circle $z=t$, with $t \in]-1,1[$, and this zero set separates two symplectic emispheres, each of which is symplectomorphic to standard $\mathbb{R}^2$. After quantization the corresponding groupoid of Bohr-Sommerfeld leaves can be described as the following subgroupoid of the action groupoid $\mathbb{Z}^2 \times \overline{\mathbb{Z}}^2\big|_{\overline{\mathbb{N}}}$:

$$\mathfrak{C} = \{(j,j,k_1,k_2)\,\big|\,k_1 \vee k_2 = \infty\}\,.$$

As such its unit space has the three orbits:

$$\Delta^{(0,0)} = \{(\infty,\infty)\}\,, \qquad \begin{array}{rcl} \Delta^{(1,0)} & = & \{(k,\infty), k \in \mathbb{N}\} \\ \Delta^{(0,1)} & = & \{(\infty,k), k \in \mathbb{N}\} \end{array} .$$

Isotropy groups are easily computed to be:

$$\mathcal{G}^{(0,0)}_{(0,0)} = \{(j,j,\infty,\infty)\} \simeq \mathbb{Z}\,, \qquad \begin{array}{rcl} \mathcal{G}^{(k,\infty)}_{(k,\infty)} & = & \{(0,0,k,\infty)\} \\ \mathcal{G}^{(\infty,k)}_{(\infty,k)} & = & \{(0,0,\infty,k)\} \end{array} .$$

Just as in Proposition 2, the corresponding topological groupoid is still an amenable, étale, Hausdorff groupoid. Since, however, there appears non trivial isotropy over the one point-orbit $\Delta^{(0,0)}$ we cannot expect an homeomorphism between the primitive ideal space of the C^*–algebra and the space of orbits of the corresponding groupoid. At a geometric level orbits can be put in correspondence with the set of T-leaves ([**?**]) and the corresponding isotropy allows the orbit method to hold true.

5. Conclusions

In this brief note we have seen how groupoid C^*–algebra quantization may be employed to recover the topological version of quantum orbit method for the standard Bruhat–Poisson structure on $\mathbb{C}P^n$. Such result is mainly based on general theory rather than on explicit computations.

The correspondence between symplectic leaves and unitary irreps can be seen as the composition of three maps of different nature. The first map goes from the space of symplectic leaves of a Poisson manifold M to the space of orbits of its symplectic groupoid; it exists for every integrable Poisson manifold and can be considered to be of purely geometrical nature. The second map connects orbits of the (smooth) symplectic groupoid to orbits the topological groupoid of BS leaves. This map, in general, depends on the choice of a Lagrangian polarization. In fact, its existence and property may also be used as a signal of having performed a well chosen polarization. It is this map that has to be considered as the core of quantization. The last map links orbit of the groupoid $\mathcal{G}$ to unitary irreps of its convolution C^*–algebra $C(\mathcal{G})$. General theory suggests us that the target space of a reasonable quantum orbit procedure should rather be the space of primitive ideals $\mathrm{Prim}(C(\mathcal{G}))$ then the unitary dual $\widehat{C(\mathcal{G})}$. It is known that for non GCR C^*–algebras the two are not necessarily homeomorphic. The properties of this map are strongly dependent on the topological properties of the space of groupoid quasi-orbits. It can both fail to be bijective and homeomorphism. Furthermore the topological space $\mathrm{Prim}(C(\mathcal{G}))$, in the non T_0 case may be sensibly smaller than the unitary dual $\widehat{C(\mathcal{G})}$ This is the case, for example, for the quantum torus and it is probably at this point that quantum orbit method for twisted compact quantum groups fails: it is reasonable to expect that the correspondence between symplectic leaves and primitive ideals can still hold in this case and this will be the subject of forthcoming researches. It would eventually ask for a clearer explanation on which is the semiclassical object describing the space of unitary irreps. This problem is again postponed to further investigations.

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