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Gudlaugur Thorbergsson

CLASSICAL SYMMETRIC *R*-SPACES

Dedicated to the memory of Sergio Console

Abstract. We will give a survey on the classical symmetric *R*-spaces from the point of view of projective and polar geometry. We will show that these spaces are all implicitly discussed in Chow's paper [Ch] and Dieudonné's book [Di].

1. Introduction

W.-L. Chow writes at the beginning of his paper [Ch] that its object is a 'study of those symmetric homogeneous spaces (in the sense of E. Cartan) which can be represented as algebraic varieties.' In the paper, four classes of spaces over general fields are studied; one of these classes consists of the Grassmannians and the other three are related to skew-symmetric and symmetric bilinear forms. Following Dieudonné's exposition of Chow's work in [Di], Chapitre III, one can also consider Hermitian and skew-Hermitian sesquilinear forms over a possibly noncommutative field; see also the remarks on p. 50 of [Ch]. Now assuming the ground field to be the reals $\mathbb{R}$, the complex numbers $\mathbb{C}$, or the quaternions $\mathbb{H}$, our goal will be to show that these four classes of spaces taken together are precisely the symmetric *R*-spaces of classical type. The only references to [Ch] in papers on differential geometry that we are aware of are in [Na2], [Pe], and [Ta2] where it is pointed out that the classical compact Hermitian symmetric spaces are among the spaces considered by Chow. We will discuss these papers in Section 5.

One of the goals of Chow in [Ch] is to generalize the *fundamental theorem of projective geometry* to the four classes of spaces he is considering. Let V be a right vector space of dimension at least three over a field $\mathbb{F}$, which can be noncommutative, and let $P(V)$ be the corresponding projective space whose *points* are the one-dimensional subspaces of V. A *line* in $P(V)$ is the set of points in $P(V)$ consisting of one-dimensional subspaces contained in a given two-dimensional subspace. A *collineation* of $P(V)$ is a line preserving bijection of $P(V)$ to itself. The fundamental theorem of projective geometry says in a formulation that is sufficient for our purposes that a collineation of $P(V)$ is induced by a semilinear automorphism of V, i.e., a linear automorphism of V composed with an automorphism of $\mathbb{F}$; see [Ar], Chapter II, §10 or [Di], Chapitre III, §1. We will write $P^n(\mathbb{F})$ instead of $P(\mathbb{F}^{n+1})$ when $\mathbb{F}^{n+1}$ is the standard right vector space over the field $\mathbb{F}$.

Here we are mostly interested in projective spaces over the fields $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ and collineations that are diffeomorphisms (or at least homeomorphisms). The real field $\mathbb{R}$ has no nontrivial automorphisms, the *continuous* automorphism of $\mathbb{C}$ are the identity and the conjugation (but there are uncountably many *discontinuous* automorphisms of $\mathbb{C}$), and, finally, the automorphisms of $\mathbb{H}$ are all inner. If the automorphisms of the ground field are inner, then the collineations of $P(V)$ are induced by the linear automorphisms of V. Hence the collineation groups of the projective spaces $P^n(\mathbb{R})$ and $P^n(\mathbb{H})$ for $n \geqslant 2$ are the projective general linear groups $\mathrm{PGL}(n+1,\mathbb{R})$ and $\mathrm{PGL}(n+1,\mathbb{H})$ respectively, where $\mathrm{PGL}(n+1,\mathbb{F})$ is by definition the quotient of the general linear group $\mathrm{GL}(n+1,\mathbb{F})$ by the kernel of its action on $P^n(\mathbb{F})$. The group of *continuous* collineations of $P^n(\mathbb{C})$ is the semi-direct product $\mathrm{PGL}(n+1,\mathbb{C}) \rtimes \{\mathrm{id}, ^{-}\}$, where $^{-}$ is the bijection of $P^n(\mathbb{C})$ induced by the conjugation in $\mathbb{C}$. These collineation groups are noncompact Lie groups.

We would like to stress that the projective spaces $P^n(\mathbb{R})$, $P^n(\mathbb{C})$, and $P^n(\mathbb{H})$ have two geometric structures that are of interest to us. From the point of view of differential geometry, they are Riemannian symmetric spaces with compact isometry groups, which we will denote by G. Then they are also projective spaces (as their name indicates) with noncompact automorphism groups, denoted by L, the groups of continuous (or differentiable) collineations. Note that L contains G.

In Section 3, we will define *(generalized) lines*. It will turn out that these generalized lines exist in all classical symmetric R-spaces with the exception of the spheres. In Section 3, we will also define the *arithmetic distance* between two points as the minimal length of a chain of lines needed to connect the points. The group of continuous line or arithmetic distance preserving transformations of symmetric R-spaces will turn out to be a noncompact Lie group containing the isometry group of the symmetric space. We will discuss this in Section 4.

The spheres were an exception in the above discussion. Still they have an additional geometric structure with a noncompact automorphism group. More precisely, *Möbius geometry* is the study of the action on S^n of the Möbius group, which by definition is the projective orthogonal group $\mathrm{PO}(1,n+1)$ acting on S^n considered as a quadric in $P^{n+1}(\mathbb{R})$.* One can now prove that a not necessarily continuous circle preserving bijection of S^n belongs to $\mathrm{PO}(1,n+1)$; see [Je] for a proof. Another such result is Liouville's theorem that a conformal diffeomorphism between connected open sets in S^n for $n \geqslant 3$ is the restriction of a Möbius transformation; see [Her], p. 52, for a proof.

The question arises when a compact symmetric space admits the action of a noncompact Lie group that contains its isometry group. More precisely, does a compact symmetric space G/K, where (G,K) is an almost effective symmetric pair, admit the action of a noncompact Lie group L containing G. Here we mean by an *almost effective* symmetric pair (G,K) that the action of G on G/K has a discrete kernel. Nagano answered this question in [Na1]. His result is as follows.

Assume we have a compact almost effective symmetric pair (G,K) and a noncompact Lie group L containing G and acting on $M = G/K$. We assume furthermore that the action of L on M is *indecomposable* in the sense that there is no nontrivial splitting of M into a Riemannian product $M_1 \times M_2$ and a splitting of L into a product

*The equation of S^n as a quadric in $P^{n+1}(\mathbb{R})$ is $x_1^2 - x_2^2 - \cdots - x_{n+1}^2 = 0$ in homogeneous coordinates. Hence S^n is invariant under the action of the orthogonal group $\mathrm{O}(1,n+1)$. The quotient of $\mathrm{O}(1,n+1)$ by the kernel of its action on $P^{n+1}(\mathbb{R})$ is the projective orthogonal group $\mathrm{PO}(1,n+1)$. See 4.3 and 4.4 for more details.

$L_1 \times L_2$ such that L_1 acts on M_1 and L_2 acts on M_2. We also assume that the center of G is at most one-dimensional. Then the main result of [Na1] is that L is simple and G a maximal compact subgroup of L. In particular, L/G is a symmetric space of noncompact type into which M is G-equivariantly embedded.

If L is a noncompact simple Lie group and G a maximal compact subgroup of L, then it is true that L acts on all G-orbits in L/G. More precisely, the G-orbits in L/G are precisely the quotients L/P where P is a parabolic subgroup of L. Quotients of the type L/P are called *R-spaces*† or *generalized flag manifolds.* If $L/P = G/K$ has the property that (G,K) is a symmetric pair, we will refer to it as a *symmetric R-space*. The symmetric R-space G/K will be said to be *indecomposable* if L is simple. An indecomposable symmetric R-space is not necessarily an irreducible symmetric space; see the tables in Section 4 for several examples.

The paper is organized as follows. In Section 2, we review the definition of (σ,ε)-Hermitian forms and make some remarks on determinants over the quaternions. In Section 3, we review projective and polar geometries. In Section 4, we come to the main goal of this paper, which is to discuss the classical symmetric R-spaces from the point of view of Chow's paper. In Section 5, we discuss the contributions in [Pe], [Na2], and [Ta2] to this circle of ideas. Finally, in Section 6, I explain in a few lines how Sergio Console and I intended in an unfinished project to generalize some of the results explained in this paper.

2. Some linear algebra

In this section, we first explain some basic facts about bilinear and sesquilinear forms. Then we make some remarks on the determinant over the quaternions.

2.1. (σ,ε)-Hermitian forms

We will give a short review of basic facts on bilinear and sesquilinear forms, which we expect to be known over $\mathbb{R}$ and $\mathbb{C}$, but maybe less so over $\mathbb{H}$. A reference that stresses the three fields we are interested in, is [Br], Kapitel VI; see also [Di], Chapitre I, for a more general discussion.

We will let V denote a right vector space over $\mathbb{F}$ where $\mathbb{F}$ is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. We will let $\bar{\alpha}$ denote the conjugate of α if $\mathbb{F}$ is $\mathbb{C}$ or $\mathbb{H}$. We recall that $\overline{zw} = \bar{w}\bar{z}$ holds in $\mathbb{H}$, i.e., the conjugation is an antiautomorphism of $\mathbb{H}$. Let $\sigma : \mathbb{F} \to \mathbb{F}$ either be the identity or the conjugation in $\mathbb{F}$ (in the latter case $\mathbb{F}$ is $\mathbb{C}$ or $\mathbb{H}$).

A map

$$f : V \times V \to \mathbb{F}$$

†The terminology '*R*-space' was, at least as far as we have been able to verify, introduced by Tits in the paper [Ti1] where these spaces are considered from the point of view of incidence geometry, assuming that L is a complex simple group. It is of course in the spirit of Tits' incidence geometry to call these spaces 'generalized flag manifolds.' Chow's point of view is of course also incidence geometric.

that is additive in both arguments is said to be a σ-*sesquilinear form* if

$$f(x\alpha, y\beta) = \sigma(\alpha) f(x,y)\beta$$

for all x and y in V and all α and β in $\mathbb{F}$. If σ is the identity, we call f a *bilinear form*, and if σ is the conjugation in $\mathbb{F}$, we call f a *sesquilinear form*.

It is easy to see that σ must be the conjugation if $\mathbb{F} = \mathbb{H}$, i.e., there are no bilinear forms on vector spaces over $\mathbb{H}$. We therefore have the following four cases: bilinear forms if $\mathbb{F} = \mathbb{R}$, bilinear and sesquilinear forms if $\mathbb{F} = \mathbb{C}$, and sesquilinear forms if $\mathbb{F} = \mathbb{H}$.

We will only be interested in forms that satisfy the following symmetry property. Let f be a σ-sesquilinear form on V and ε be either 1 or -1. Then f is said to be (σ, ε)-*Hermitian* if

$$f(x,y) = \varepsilon\sigma(f(y,x))$$

for all x and y in V.

We have the following four cases.

1. σ is the identity and $\varepsilon = 1$. Then f is referred to as a *symmetric bilinear form*.
2. σ is the identity and $\varepsilon = -1$. Then f is referred to as a *skew-symmetric bilinear form*.
3. σ is the conjugation and $\varepsilon = 1$. Then f is referred to as an *Hermitian sesquilinear form*.
4. σ is the conjugation and $\varepsilon = -1$. Then f is referred to as a *skew-Hermitian sesquilinear form*.

If $\mathbb{F} = \mathbb{C}$, a skew-Hermitian form becomes Hermitian after multiplying it by i and vice versa. We will therefore assume that sesquilinear forms on vector spaces over $\mathbb{C}$ are Hermitian.

We therefore have the following seven cases: symmetric and skew-symmetric bilinear forms over $\mathbb{R}$, symmetric, skew-symmetric, and Hermitian forms over $\mathbb{C}$, and Hermitian and skew-Hermitian sesquilinear forms over $\mathbb{H}$.

A (σ, ε)-Hermitian form f is said to be *nondegenerate* if $f(x,y) = 0$ for all y in V implies that $x = 0$. In the following we will always assume that f is nondegenerate. Nondegenerate skew-symmetric bilinear forms are usually said to be *symplectic*.

Let f be a nondegenerate (σ, ε)-Hermitian form on V. An *automorphism of* (V, f) is a linear automorphism $A : V \to V$ such that

$$f(Ax, Ay) = f(x,y)$$

for all x and y in V. The automorphisms form a group that we will denote by $\mathrm{Aut}(V, f)$. It is clear that $\mathrm{Aut}(V, f)$ is a closed subgroup in the general linear group $\mathrm{GL}(V)$ and hence a Lie group.

Let f be a nondegenerate (σ,ε)-Hermitian form on V. A subspace W in V is said to be *totally isotropic* if $f(x,y)=0$ for all x and y in W. It is a consequence of Witt's Theorem, see [Br], p. 373, that given totally isotropic subspaces V_1 and V_2 of (V,f) with the same dimension, there is an automorphism A in $\mathrm{Aut}(V,f)$ that maps V_1 to V_2. It follows that all maximal totally isotropic subspaces of (V,f) have the same dimension. The *Witt index of* (V,f) is now defined to be the dimension of a maximal totally isotropic subspace of (V,f).

We will denote by $N_i(V,f)$ for $i \leqslant r$ the space of totally isotropic subspaces of (V,f) with dimension i where r denotes the Witt index of (V,f). By Witt's theorem $\mathrm{Aut}(V,f)$, acts transitively on $N_i(V,f)$. It follows that $N_i(V,f)$ is a differentiable manifold that can be represented as a coset space.

2.2. Determinants over $\mathbb{H}$ and the groups $\mathbf{SL}(n,\mathbb{H})$ and $\mathbf{SU}^*(2n)$

In books on linear algebra, the determinant is usually only defined for matrices with entries in a commutative field. There is an extension due to Dieudonné of the theory of determinants to noncommutative fields that is explained in Chapter IV, §1 of the book [Ar] by E. Artin. In the case of quadratic matrices with quaternion entries, the image of the determinant is in $\mathbb{R}_{\geqslant 0}$, the set of nonnegative real numbers. The determinant is multiplicative, vanishes if and only if applied to a singular matrix, and is equal to one on the identity matrix. We can now define $\mathrm{SL}(n,\mathbb{H})$ to be the group of quaternionic matrices with determinant equal to one. One can avoid the Dieudonné determinant by embedding the $n\times n$ quaternionic matrices into the vector space of $2n\times 2n$ complex matrices and define $\mathrm{SU}^*(2n)$ to be the group of matrices in the image whose determinant over $\mathbb{C}$ is equal to one. The groups $\mathrm{SL}(n,\mathbb{H})$ and $\mathrm{SU}^*(2n)$ turn out to be isomorphic.

There is an interesting survey on quaternionic determinants in [As].

3. Projective and polar geometry

3.1. Geometries

We set $I=\{0,1,\ldots,n-1\}$ and define following Tits [Ti3] a *geometry over I* as a triple $\Gamma=(V,\tau,*)$ consisting of a set V, a surjective map $\tau: V\to I$, and a binary symmetric relation $*$ on V such that $x*y$ holds for elements $x,y\in V$ with $\tau(x)=\tau(y)$ if and only if $x=y$. The relation $*$ is called the *incidence relation* of the geometry Γ, the image of x under τ is the *type of x*, and the cardinality n of I is called the *rank of* Γ.

We denote the set of elements of V of type i by V_i and think of V_0 as the space of points, V_1 as the space of lines, V_2 as the space of 2-planes, and so on.

If $x\in V$, we define the *shadow of x on V_i* to be the set of elements of V_i that are incident to x.

A *flag of* Γ is a set of pairwise incident elements. The set $\Delta(\Gamma)$ of all flags of Γ is called the *flag complex of* Γ. It is clear that $\Delta(\Gamma)$ is an (abstract) simplicial complex in the sense that every subset of a set in $\Delta(\Gamma)$ is contained in $\Delta(\Gamma)$.

A *point-line geometry* is a geometry of rank two with the property that any two points (elements of type 0) are incident with at most one line (element of type 1). Another way to say this is that the shadows of different lines on the space of points meet in at most one point or, equivalently, that the shadows of two different points on the space of lines meet in at most one line.

3.2. Projective geometry

Let $\mathbb{F}$ be a field that can be noncommutative. Let $G_k(\mathbb{F}^{n+1})$ denote the space of k-planes in $\mathbb{F}^n$, the *Grassmannian of k-planes in* $\mathbb{F}^{n+1}$, where we consider $\mathbb{F}^{n+1}$ to be a right vector space.

We set $V(\mathbb{F}^{n+1}) = G_1(\mathbb{F}^{n+1}) \cup \cdots \cup G_n(\mathbb{F}^{n+1})$ and define a type map $\tau : V(\mathbb{F}^{n+1}) \to I$ by setting $\tau(R) = i-1$ for $R \in G_i(\mathbb{F}^{n+1})$. We set $R * S$ for $R, S \in V(\mathbb{F}^{n+1})$ if $R \subset S$ or $S \subset R$.

The geometry $\Gamma(\mathbb{F}^{n+1}) = (V(\mathbb{F}^{n+1}), \tau, *)$ is called *projective geometry.* We set $P^n(\mathbb{F}) = G_1(\mathbb{F}^{n+1})$ and call it the n-dimensional projective space over $\mathbb{F}$ or the point space of the projective geometry $\Gamma(\mathbb{F}^{n+1})$.

The flag complex $\Delta(\Gamma(\mathbb{F}^{n+1}))$ of projective geometry over $\mathbb{F}$ satisfied the axioms of a *building* in the sense of Tits; see [Ti2], p. 38. As such it has a Coxeter group attached to it. The Coxeter diagram of the Coxeter group of $\Delta(\Gamma(\mathbb{F}^{n+1}))$ is of type A_n.

We will now define a point-line geometry with $G_k(\mathbb{F}^n)$ as a point space for every $k \in I$.

Let X be a $(k-1)$-plane and Y a $(k+1)$-plane in $\mathbb{F}^{n+1}$. Let $L_{X,Y}$ denote the set of k-planes in $G_k(\mathbb{F}^{n+1})$ containing X and contained in Y. We will call $L_{X,Y}$ a *(generalized) line in* $G_k(\mathbb{F}^{n+1})$. Let $\mathcal{L}_k(\mathbb{F}^{n+1})$ denote the set of all lines $L_{X,Y}$ in $G_k(\mathbb{F}^{n+1})$. Then we obviously have a point-line geometry $\Gamma_k(\mathbb{F}^{n+1})$ with $G_k(\mathbb{F}^n)$ as a point space and $\mathcal{L}_k(\mathbb{F}^{n+1})$ as a space of lines.

If $k = 1$ or $k = n$, then $L_{X,Y}$ is nothing but a projective line in the projective space $P^n(\mathbb{F}) = G_1(\mathbb{F}^{n+1})$ or in its dual projective space $G_n(\mathbb{F}^{n+1})$.

Let V and W be elements in the Grassmannian $G_k(\mathbb{F}^{n+1})$. We say that V and W are *adjacent* if $\dim\,(V \cap W) = k-1$. It is clear that V and W are adjacent if and only if there is a generalized line containing both of them. We define the *arithmetic distance* $d_a(V,W)$ *between* V *and* W to be $k - \dim\,(V \cap W)$. The arithmetic distance between V and W can be characterized as the shortest length of a chain of generalized lines in $G_k(\mathbb{F}^{n+1})$ joining V and W in which consecutive lines intersect. Clearly, $d_a(V,W)$ can also be characterized as the shortest length of a chain of k-planes in $G_k(\mathbb{F}^{n+1})$ joining V and W in which consecutive planes are adjacent.

It is clear that $\mathrm{PGL}(n+1,\mathbb{F})$ acts transitively on $G_k(\mathbb{F}^{n+1})$, leaves the arithmetic distance invariant, and maps lines to lines. The following theorem addresses the question to which extend the converse holds; see Theorem I in [Ch].

THEOREM 3.1. *An adjacency preserving bijection of* $G_k(\mathbb{F}^n)$ *is induced by a semilinear automorphism of* $\mathbb{F}^n$ *if* $n-1 > k > 1$.

Theorem 3.1 cannot hold if $k = 1$ or $k = n-1$, since then $G_k(\mathbb{F}^n)$ is a projective space or its dual in which any two points are adjacent. Combining the fundamental theorem of projective geometry and Theorem 3.1, one sees that a line preserving bijection of $G_k(\mathbb{F}^n)$ is induced by a semilinear automorphism of $\mathbb{F}^n$ if $n \geqslant 3$ and $n > k \geqslant 1$.

3.3. Polar geometry

We will assume that $\mathbb{F}$ is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and that f is a (σ, ε)-Hermitian form on a vector space W over $\mathbb{F}$ with Witt index $r \geqslant 2$. We will phrase the results from [Ch] and [Di] in terms of *polar geometry*, which was only introduced later by Veldkamp in [Ve]. We will also refer to *oriflamme geometry*, which was as well introduced later by Tits; see [Ti2], 7.12.

As in 2.1, we let $N_i(W, f)$ denote the space of i-dimensional totally isotropic subspaces of (W, f) where $1 \leqslant i \leqslant r$. We set $V(W, f) = N_1(W, f) \cup \cdots \cup N_r(W, f)$. We have a type map $\tau : V(W, f) \to \{0, \ldots, r-1\}$ defined by setting $\tau(R) = i-1$ for $R \in N_i(W, f)$. We set $R * S$ for $R, S \in V(W, f)$ if one of the spaces is a subspace of the other. This gives rise to a geometry $\Gamma(W, f) = (V(W, f), \tau, *)$, which we call a *polar geometry of rank r*.

One can show that a polar geometry of rank r satisfies one of the following two conditions.

(i) Every plane R in $N_{r-1}(W, f)$ is contained in at least three different maximal isotropic subspaces in $N_r(W, f)$. We say that the polar geometry is *thick* if this condition is satisfied.

(ii) Every plane R in $N_{r-1}(W, f)$ is contained in precisely two different maximal isotropic subspaces in $N_r(W, f)$. This case gives rise to *oriflamme geometry*, which we will discuss at the end of this section.

It turns out that $N_r(W, f)$ is connected when the polar geometry is thick and that it consists of precisely two components when it is not thick.

Our next goal is to define the dual space of $\Gamma(W, f)$ as a point-line geometry. The definition will depend on whether $\Gamma(W, f)$ is a thick polar space or not.

The dual space of a thick polar geometry

We consider a thick polar geometry $\Gamma(W, f)$ of rank r. In this case the flag complex of $\Gamma(W, f)$ is a thick building of type C_r.

The point space of the dual geometry will be $N_r(W, f)$. A *(generalized) line L_T in $N_r(V, f)$* is the set of all R in $N_r(V, f)$ containing T where $T \in N_{r-1}(W, f)$. We denote the set of generalized lines in $N_r(W, f)$ by $\mathcal{L}_r(W, f)$. Then it is clear that we have a point-line geometry with $N_r(W, f)$ as space of points and $\mathcal{L}_r(W, f)$ as space of lines. Let R and S be elements of $N_r(W, f)$. Then R and S are said to be *adjacent* if $\dim\,(R \cap S) = r-1$. The *arithmetic distance $d_a(R, S)$ between R and S* is defined by setting $d_a(R, S) = r - \dim\,(R \cap S)$.

The arithmetic distance between R and S can be characterized as the shortest length of a chain of lines in $N_r(V,f)$ joining R and S in which consecutive lines intersect, or, equivalently, as the shortest length of a chain of r-planes in $N_r(V,f)$ joining R and S in which consecutive planes are adjacent.

It is clear that $\mathrm{Aut}(V,f)$ leaves the arithmetic distance invariant and maps lines to lines. The following theorem can be found in [Di], p. 82; see also Theorem II in [Ch].

THEOREM 3.2. *A bijection of $N_r(W,f)$ for $r \geqslant 3$ that is adjacency preserving in both directions is induced by elements of $Aut(V,f)$ composed with an automorphism of $\mathbb{F}$.*

Oriflamme geometry and its dual

Now we assume that $\Gamma(W,f)$ is a polar geometry that is not thick. In this case the flag complex of $\Gamma(W,f)$ is a only a weak building in the sense of [Ti2], p. 38. We will now explain a modification of the the geometry $\Gamma(W,f) = (V(W,f),\tau,*)$ due to Tits in [Ti2], 7.12. This geometry leads to a building of type D_n.

We now divide $N_r(W,f)$ into two parts that correspond to its connected components. To this end, we choose an element R in $N_r(W,f)$. Let $N_r^+(W,f)$ be the subset of all $S \in N_r(W,f)$ such that $r - \dim\,(S \cap R)$ is an even number. The set $N_r^-(W,f)$ is now defined to be the complement of $N_r^+(W,f)$ in $N_r(W,f)$.

We now set $O(W,f) = N_1(W,f) \cup \cdots \cup N_{r-2}(W,f) \cup N_r^+(W,f) \cup N_r^-(W,f)$. If we compare this with the definition of $V(W,f)$, then we have skipped $N_{r-1}(W,f)$ and split $N_r(W,f)$ into two sets. We define a type map $\tau : O(W,f) \to \{0,\ldots,r-1\}$ by setting $\tau(R) = i-1$ if $R \in N_i^-(W,f)$ for $i \leqslant r-2$, $\tau(R) = r-2$ if $R \in N_r^-(W,f)$, and $\tau(R) = r-1$ if $R \in N_r^+(W,f)$. If R is of type $i \leqslant r-3$ and S is of type j where $i \leqslant j \leqslant r-1$, then we define the incidence relation by setting $R * S$ if $R \subset S$. If R is of type $r-2$ and S of type $r-1$ we set $R * S$ if $\dim\,(R \cap S) = r-1$. We will refer to $\Gamma_{\mathrm{or}}(W,f) = (O(W,f),\tau,*)$ as *oriflamme geometry*. The flag complex of an oriflamme geometry is called an *oriflamme complex*. It is a building of type D_r. A maximal flag $(R_0,\ldots,R_{r-1})$ in an oriflamme complex can be schematically represented as in the following diagram.[‡]

$$
\begin{array}{ll}
 & R_{r-1} \\
 & \rotatebox{45}{$\subset$} \\
R_0 \subset \cdots \subset R_{r-3} & \\
 & \rotatebox{-45}{$\subset$} \\
 & R_{r-2}
\end{array}
$$

[‡]The oriflamme (golden flame) was a sacred banner used by the kings of France in the Middle Ages. The diagram is supposed to remind us of its elongated swallow tailed form.

We now define the dual oriflamme space. Its point space will be $N_r^+(W,f)$. Let $S \in N_{r-2}(W,f)$ be given. Then we define the *(generalized) line* L_S as the subset of those $R \in N_r^+(W,f)$ that contain S. We let $\mathcal{L}_r^+(W,f)$ denote the set of generalized lines in $N_r^+(W,f)$. Then $(N_r^+(W,f), \mathcal{L}_r^+(W,f))$ gives rise to a point-line geometry that we will refer to as the *dual oriflamme geometry*. We say that that two elements in $N_r^+(W,f)$ are *adjacent* if there is line passing through them. Again, we define the *arithmetic distance* between two elements in $N_r^+(W,f)$ as the minimal length of a chain of lines joining one to the other. These definitions agree with those in [Ch], p. 52, and [Di], p. 86.

Clearly, $\mathrm{Aut}(W,f)$ preserves lines and arithmetic distance in the dual oriflamme geometry. The following result is Theorem VII on p. 55 in [Ch]; see also [Di], p. 86.

THEOREM 3.3. *A bijection of $N_r^+(W,f)$ for $r \geqslant 5$ that is adjacency preserving in both directions is induced by elements of $Aut(V,f)$ composed with an automorphism of $\mathbb{F}$.*

There is also a fundamental theorem when $r = 4$, but it is more complicated to state since it involves triality; see [Ch], p. 55 and [Di], p. 87. We will therefore not explain it in detail.

4. Classical symmetric *R*-spaces

The triples (L,G,K) in Nagano's theorem that we mentioned in the introduction are completely classified. In [Na1], p. 445, there is a list in which some of the spaces $M = G/K$ have been replaced by locally isometric ones. One finds a discussion of all the symmetric *R*-spaces in [Ta1], but they are not listed in one table. There is a classification of an equivalent problem in [KN], albeit in a somewhat hidden form. A complete list of the symmetric *R*-spaces with L simple can be found in the table on p. 41 in [Oh]. A symmetric *R*-space is indecomposable if and only if L is simple.

In this section, we will discuss those triples (L,G,K) in which all three groups are classical. We will assume that L is a connected simple Lie group and that the symmetric pairs (L,G) and (G,K) are almost effective.

It turns out that the triples (L,G,K) of classical groups giving rise to indecomposable symmetric *R*-spaces are either related to projective or polar geometry, and that the type of the geometry depends on the Coxeter group of the restricted root system of L that can be found in the tables in Appendix C of [Kn] or in the table on p. 119 in [Lo]. This Coxeter group is equivalently the Coxeter group of the symmetric space L/G. Since we are only interested in the Coxeter group and not in the Weyl group of the restricted root system, the cases B_n, C_n, and $(BC)_n$ in Appendix C of [Kn] all coincide and the type of the corresponding Coxeter group will be given by the symbol C_n.

We will divide the triples into four classes that are more and less the same as the four classes of Chow in [Ch]. The main difference is that we allow sesquilinear forms in (II).

(I) The first class corresponds to triples (L,G,K) with Coxeter group of L/K of type A_n. These triples are related to n-dimensional projective geometry; see 3.2. The corresponding symmetric R-spaces G/K are the Grassmannians $G_k(\mathbb{F}^n)$ where $1 \leqslant k \leqslant n-1$.

(II) The second class corresponds to triples (L,G,K) where the Coxeter group of L/K is of type C_n. These triples are related to *thick* n-dimensional polar geometries; see 3.3. The corresponding symmetric R-spaces are then the Grassmannians of hyperplanes in a thick polar geometry, or, equivalently, Grassmannians of maximal isotropic subspaces with respect to a (σ,ε)-Hermitian form f on $\mathbb{F}^N$ that is not symmetric. It turns out that the Grassmannians of maximal isotropic subspaces are symmetric R-spaces if and only if $N = 2n$, where n is the Witt index of f; see the classification in 4.2. The symmetric R-spaces in this class can be seen as the point spaces of the dual of thick polar geometries; see 3.3.

(III) The third class corresponds to triples (L,G,K) where the Coxeter group of L/K is of type D_n. These triples are related to n-dimensional oriflamme geometry; see 3.3. Analogous to what we saw in class (II), the connected components of the Grassmannians of maximal isotropic subspaces with respect to a symmetric bilinear form f on $\mathbb{F}^N$ with Witt index equal to n are symmetric if and only if $N = 2n$. Here the symmetric R-spaces are the point spaces of the dual of an oriflamme geometry; see 3.3.

(IV) In the forth class, the symmetric R-spaces are nondegenerate quadrics containing projective lines in $P^N(\mathbb{R})$ and $P^N(\mathbb{C})$ and hence the point spaces of certain polar geometries. The point spaces of polar geometries defined by (σ,ε)-Hermitian forms that are not symmetric do not give rise to symmetric R-spaces.

We now start the discussion of these four classes of symmetric R-spaces. Low dimensional examples are typically spheres and quadrics that we will exclude in the tables. We make some remarks on these excluded cases. To facilitate the reading for those who are not interested, we put these remarks in square brackets.

4.1. Class (I). Grassmannians of k-planes in $\mathbb{F}^n$

The triples (L,G,K) giving rise to Grassmannians are listed in the following table.

L	G/K	Symbol	Description of G/K
$\mathrm{SL}(n,\mathbb{R})$	$\mathrm{SO}(n)/\mathrm{S}(\mathrm{O}(k)\times\mathrm{O}(n-k))$, $n\geqslant 3$, $n>k\geqslant 1$	$G_k(\mathbb{R}^n)$	Grassmannian of k-planes in $\mathbb{R}^n$
$\mathrm{SL}(n,\mathbb{C})$	$\mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(k)\times\mathrm{U}(n-k))$ $n\geqslant 3$, $n>k\geqslant 1$	$G_k(\mathbb{C}^n)$	Grassmannian of k-planes in $\mathbb{C}^n$
$\mathrm{SL}(n,\mathbb{H})$	$\mathrm{Sp}(n)/\mathrm{Sp}(k)\times\mathrm{Sp}(n-k))$ $n\geqslant 3$, $n>k\geqslant 1$	$G_k(\mathbb{H}^n)$	Grassmannian of k-planes in $\mathbb{H}^n$

The group $\mathrm{SL}(n,\mathbb{H})$ in the above table is explained in 2.2.

[We have restricted n to be at least three in the above table since G/K is a one dimensional projective space for $n=2$ and hence trivial from our point of view having only the space itself as a generalized line. As manifolds these spaces are S^1, S^2, and S^4. The group L acts on S^1 by projective transformations and on S^2 and S^4 by Möbius transformations.]

Theorem 3.1 now applies to the spaces in the table. We are assuming that $\mathbb{F}$ is either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Hence we are in the same situation as when explaining the fundamental theorem of projective geometry for these fields in the introduction. It follows from the remark after Theorem 3.1 that the group of line preserving bijections of $G_k(\mathbb{F}^n)$ if $n\geqslant 3$ and $n>k\geqslant 1$ (continuous or not) is the projective linear group $\mathrm{PGL}(n,\mathbb{F})$ if $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{H}$. The group of such bijections of $G_k(\mathbb{C}^n)$ that are continuous is the semidirect product $\mathrm{PGL}(n,\mathbb{C})\rtimes\{\mathrm{id},\bar{\ }\}$, where $\bar{\ }$ is the bijection of $G_k(\mathbb{C}^n)$ induced by the conjugation in $\mathbb{C}$.

4.2. Class (II). Grassmannians of hyperplanes in thick polar spaces.

We saw in 2.1 that there are the following seven classes of (σ,ε)-Hermitian forms. Symmetric and skew-symmetric bilinear forms over $\mathbb{R}$, symmetric, skew-symmetric, and Hermitian forms over $\mathbb{C}$, and Hermitian and skew-Hermitian sesquilinear forms over $\mathbb{H}$.

The symmetric forms over $\mathbb{R}$ and $\mathbb{C}$ fall under class (III); see 4.3. The remaining five cases belong to class (II).

Hermitian forms over $\mathbb{C}$ and $\mathbb{H}$.

Let f be an Hermitian sesquilinear form on a right vector space V over $\mathbb{F}$ where $\mathbb{F}$ is either $\mathbb{C}$ or $\mathbb{H}$. We will assume that f is nondegenerate. There is a basis $(e_1,\ldots,e_N)$ of

V and numbers p and q with $p+q=N$ such that

$$f(x,y)=\sum_{i=1}^{p}\bar{x}_i y_i-\sum_{i=p+1}^{p+q}\bar{x}_i y_i.$$

The numbers p and q do not depend on the choice of such a basis and we will say that f is of *type* (p,q). The Witt index of (V,f) is $\min\{p,q\}$.

If $\mathbb{F}=\mathbb{C}$, we denote the automorphism group of (V,f) by $\mathrm{U}(p,q)$ and call it the *unitary group of type* (p,q); the *special unitary group of type* (p,q) is its subgroup $\mathrm{SU}(p,q)$ of automorphisms with determinant equal to one.

If $\mathbb{F}=\mathbb{H}$, we denote the automorphism group of (V,f) by $\mathrm{Sp}(p,q)$ and call it the *quaternionic unitary group of type* (p,q). We made a remark on the determinant over $\mathbb{H}$ in 2.2. It turns out that all elements of $\mathrm{Sp}(p,q)$ have determinant equal to one.

If $(p,q)=(N,0)$, we denote the above groups by $\mathrm{U}(N)$, $\mathrm{SU}(N)$, and $\mathrm{Sp}(N)$, respectively.

By the classification of symmetric R-spaces, we only have to consider the Hermitian forms of type (n,n), i.e., $N=2n$. We have the following table in which $\Delta(G)$ denotes the diagonal in $G\times G$. In the third column, we have the usual symbol for G/K.

L	G/K	Symbol	Description of G/K
$\mathrm{SU}(n,n)$	$\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(n))/\Delta(\mathrm{SU}(n)),\ n\geqslant 3$	$\mathrm{U}(n)$	unitary group
$\mathrm{Sp}(n,n)$	$\mathrm{Sp}(n)\times\mathrm{Sp}(n)/\Delta(\mathrm{Sp}(n)),\ n\geqslant 2$	$\mathrm{Sp}(n)$	quaternionic unitary group

[We assume that $n\geqslant 3$ in the first line of the table. If $n=1$, then G/K is S^1 on which L acts by projective transformations. If $n=2$, then G/K is the quadric $Q_{1,3}(\mathbb{R})$ on which L acts as $\mathrm{SO}_0(2,4)$.[§] This example belongs to class (IV); see 4.4.

In the second line of the table, we assume that $n\geqslant 2$. If $n=1$, G/K is S^3 on which L acts by Möbius transformations.[¶] We do not exclude $n=2$ in the second line although it is an exception since we do not have a fundamental theorem for it as we will see further down.]

We use the notation in 2.1 and let $N_n(\mathbb{F}^{2n},f)$ denote the Grassmannian of maximal totally isotropic subspaces in (V,f) where f has Witt index n and and we have identified V with $\mathbb{F}^{2n}$. Our goal is to identify $N_n(\mathbb{F}^{2n},f)$ with $\mathrm{U}(n)$ if $\mathbb{F}$ is $\mathbb{C}$ and $\mathrm{Sp}(n)$ if $\mathbb{F}=\mathbb{H}$.

[§] Here we are using that $\mathrm{SU}(2,2)$ and $\mathrm{SO}_o(2,4)$ are locally isomorphic; see 4.3 for the definition of the latter group and 4.4 for the quadric.

[¶] Here we are using that $\mathrm{Sp}(2,2)$ and $\mathrm{SO}_o(4,1)$ are locally isomorphic.

We first consider the case $\mathbb{F} = \mathbb{C}$ and identify V with $\mathbb{C}^{2n} = \mathbb{C}^n + \mathbb{C}^n$ in such a way that

$$f((x_1,x_2),(y_1,y_2)) = \langle x_1,y_1\rangle - \langle x_2,y_2\rangle$$

where $\langle x,y\rangle$ is the usual Hermitian scalar product on $\mathbb{C}^n$.

Let W be a maximal totally isotropic subspace in $N_n(\mathbb{C}^{2n},f)$ and let (x,y) be an element in W. Then $\|x\| = \|y\|$ which implies that the projections π_1 and π_2 of W onto the first and the second factor of $\mathbb{C}^{2n} = \mathbb{C}^n + \mathbb{C}^n$, respectively, are bijections. Furthermore, $W \in N_n(\mathbb{C}^{2n},f)$ induces the map $A_W = \pi_2 \circ \pi_1^{-1}$ from the first to the second factor of $\mathbb{C}^{2n} = \mathbb{C}^n + \mathbb{C}^n$ that we identify with an endomorphism of $\mathbb{C}^n$. It is clear that $A_W \in \mathrm{U}(n)$.

Conversely, let $A \in \mathrm{U}(n)$ be given and let $V_A = \{(x,Ax) \mid x \in \mathbb{C}^n\}$ be the graph of A. Clearly, $V_A \in N_n(\mathbb{C}^{2n},f)$ and the map that sends A to V_A is the inverse of the map that sends V to A_V. We have thus identified $N_n(\mathbb{C}^{2n},f)$ with $\mathrm{U}(n)$. Note that $\mathrm{U}(n)$ is not an irreducible symmetric space.

The automorphism group of (V,f) is $\mathrm{U}(n,n)$. The action of $\mathrm{U}(n,n)$ on $\mathrm{U}(n) = N_n(\mathbb{C}^{2n},f)$ is not effective. On can either replace it by its quotient by the kernel of the action, the projective unitary group $\mathrm{PU}(n,n)$, or by the special unitary group $\mathrm{SU}(n,n)$ as in the above table, whose action on $\mathrm{U}(n) = N_n(\mathbb{C}^{2n},f)$ is almost effective.

The continuous bijections of $\mathrm{U}(n) = N_n(\mathbb{C}^{2n},f)$ for $n \geqslant 3$ that are adjacency preserving in both directions are induced by elements of $\mathrm{SU}(n,n)$ possibly composed with the conjugation; see Theorem 3.2.

The quaternionic case $\mathbb{F} = \mathbb{H}$ is completely analogous to the complex case we have been discussing, and we can identify $N_n(\mathbb{H}^{2n},f)$ with $\mathrm{Sp}(n)$. Again by Theorem 3.2, the group of bijections of $\mathrm{Sp}(n) = N_n(\mathbb{H}^{2n},f)$ for $n \geqslant 3$ that are adjacency preserving in both directions is $\mathrm{Sp}(n,n)$ modulo the kernel of its action on $\mathrm{Sp}(n) = N_n(\mathbb{H}^{2n},f)$. Theorem 3.2 does not apply to the case $n = 2$ in the table.

Symplectic forms

Let f be a symplectic form on a vector space V over $\mathbb{F}$ where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. It follows that the dimension of V is an even number $2n$. There is a basis $(e_1,\dots,e_{2n})$ of V such that f can be written in the form

$$f(x,y) = \sum_{i=1}^{n} (x_i y_{n+i} - x_{n+i} y_i).$$

We identify V with $\mathbb{F}^{2n}$ with help of this basis. The Witt index of f is equal to n. The maximal totally isotropic subspaces of (V,f) are said to be *Lagrangian*. Let $G_L(\mathbb{F}^{2n})$ denote the set of all Lagrangian subspaces in $(\mathbb{F}^{2n},\omega)$; i.e., $G_L(\mathbb{F}^{2n}) = N_n(V,f)$ in the notation of 2.1. The automorphism group of (V,f) is called the *symplectic group over* $\mathbb{F}$ and denoted by $\mathrm{Sp}(2n,\mathbb{F})$. As we remarked in 2.1, the action of $\mathrm{Sp}(2n,\mathbb{F})$ on $G_L(\mathbb{F}^{2n})$ is transitive by Witt's Theorem.

The possibilities for the triples (L,G,K) according to the classification of symmetric R-spaces when $L = \mathrm{Sp}(2n,\mathbb{F})$ is given in the following table.

L	G/K	Symbol	Description of G/K
$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{U}(n)/\mathrm{O}(n), \quad n \geqslant 3$	$G_L(\mathbb{R}^{2n})$	Grassmannian of Lagrangians in $\mathbb{R}^{2n}$
$\mathrm{Sp}(2n,\mathbb{C})$	$\mathrm{Sp}(n)/\mathrm{U}(n), \quad n \geqslant 3$	$G_L(\mathbb{C}^{2n})$	Grassmannian of Lagrangians in $\mathbb{C}^{2n}$

[If $\mathbb{F} = \mathbb{R}$ and $n = 1$, then $G_L(\mathbb{R}^2)$ coincides with S^1 on which L acts by projective transformations. If $\mathbb{F} = \mathbb{C}$ and $n = 1$, then $G_L(\mathbb{C}^2)$ coincides with S^2. The group $L = \mathrm{Sp}(2,\mathbb{C})$ is isomorphic to $\mathrm{SL}(2,\mathbb{C})$, which acts on S^2 by Möbius transformations.

If $n = 2$, then the spaces in the table are quadrics and therefore belong to class (IV); see 4.4. More precisely, if $\mathbb{F} = \mathbb{R}$ and $n = 2$, then $G_L(\mathbb{R}^4)$ coincides with the three-dimensional quadric $Q_{2,1}(\mathbb{R})$ in $P^4(\mathbb{R})$, which has $S^2 \times S^1$ as a double cover. The group $L = \mathrm{Sp}(2n,\mathbb{R})$ is a double cover of the connected component $\mathrm{SO}_o(3,2)$ of $\mathrm{SO}(3,2)$, which acts on $P^4(\mathbb{R})$ leaving the quadric $Q_{2,1}(\mathbb{R})$ invariant; see 4.3 for definitions. If $\mathbb{F} = \mathbb{C}$ and $n = 2$, then $G_L(\mathbb{C}^4)$ coincides with the quadric $Q_3(\mathbb{C})$ in $P^4(\mathbb{C})$. The group $L = \mathrm{Sp}(4,\mathbb{C})$ is locally isomorphic to $\mathrm{SO}(5,\mathbb{C})$, which acts on $P^4(\mathbb{C})$ leaving the quadric $Q_3(\mathbb{C})$ invariant.]

Our goal is now to identify $G_L(\mathbb{F}^{2n})$ with G/K as in the table.

We first consider the real case $(\mathbb{R}^{2n}, f)$. We identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$ by setting $(x,y) = x + iy = z$. If $\langle z,w\rangle$ is the usual Hermitian scalar product on $\mathbb{C}^n$, then $f(z,w) = \mathrm{Im}\langle z,w\rangle$, i.e. f is the imaginary part of the Hermitian scalar product. Furthermore, $\mathrm{Re}\langle z,w\rangle$ is the standard real scalar product on $\mathbb{R}^{2n}$.

Let $e_1,\dots,e_n$ be the standard basis of $\mathbb{C}^n$ and W be its real span. It is clear that W is Lagrangian. Let $\widehat{W}$ be an other Lagrangian subspace and $\widehat{e}_1,\dots,\widehat{e}_n$ an orthonormal basis of $\widehat{W}$. Then $\widehat{e}_1,\dots,\widehat{e}_n$ is clearly a unitary basis of $\mathbb{C}^n$ and there is a unitary matrix $A \in \mathrm{U}(n)$ that maps W to $\widehat{W}$. The stabilizer of W under the action of $\mathrm{U}(n)$ is clearly $\mathrm{O}(n)$. Hence we see that $G_L(\mathbb{R}^{2n}) = \mathrm{U}(n)/\mathrm{O}(n)$. The space $\mathrm{U}(n)/\mathrm{O}(n)$ is not an irreducible symmetric space.

The complex case is very similar to the real case. We identify $\mathbb{C}^{2n}$ with $\mathbb{H}^n = \mathbb{C}^n + j\mathbb{C}^n$ by setting $(x,y) = x + jy = z$. Now f is the j-part of the standard quaternionic scalar product on $\mathbb{H}^{2n}$; see [Che], Chapter I, §VIII. Let $e_1,\dots,e_n$ be the standard basis of $\mathbb{H}^n$ and W be its complex span. Then W is in $G_L(\mathbb{C}^{2n})$. Let $\widehat{W}$ be another element in $G_L(\mathbb{C}^{2n})$ and let $\widehat{e}_1,\dots,\widehat{e}_n$ be a unitary basis of $\widehat{W}$. Then $\widehat{e}_1,\dots,\widehat{e}_n$ is also a quaternionic unitary basis of $\mathbb{H}^n$ since $\widehat{W}$ is Lagrangian. Hence there is an element $A \in \mathrm{Sp}(n)$ that maps W to $\widehat{W}$. The stabilizer of W under the action of $\mathrm{Sp}(n)$ is clearly $\mathrm{U}(n)$. Hence we see that $G_L(\mathbb{C}^{2n}) = \mathrm{Sp}(n)/\mathrm{U}(n)$.

Theorem 3.2 now says that the bijections of $G_L(\mathbb{F}^{2n})$ for $n \geqslant 3$ that are adjacency preserving in both directions are induced by elements of $\mathrm{Sp}(2n,\mathbb{F})$ composed with an automorphism of $\mathbb{F}$. Hence the group of continuous bijections of $G_L(\mathbb{F}^{2n})$ that are adjacency preserving in both directions is $\mathrm{Sp}(2n,\mathbb{R})$ in the real case and $\mathrm{Sp}(2n,\mathbb{C}) \rtimes \{\mathrm{id}, \bar{\ }\}$ in the complex case.

Skew-Hermitian forms over $\mathbb{H}$

Let now V be a right vector space over $\mathbb{H}$ with a skew-Hermitian form f. Then there is a basis $(e_1,\dots,e_N)$ of V such that f can be written as

$$f(x,y)=\sum_{k=1}^{N}\bar{x}_k j y_k$$

where j is the the third element in the standard basis of $\mathbb{H}$ over $\mathbb{R}$. We will actually work with a different normal form below, since it is more practical for our purposes, although it might look more complicated. The automorphism group of (V,f) is called the *quaternionic anti-unitary group* and denoted by $\mathrm{U}_\alpha(N,\mathbb{H})$. One can show that the quaternionic determinant of an endomorphism in $\mathrm{U}_\alpha(N,\mathbb{H})$ is equal to one. The Witt index of (V,f) is $[\frac{N}{2}]$.

One frequently finds the group $\mathrm{U}_\alpha(N,\mathbb{H})$ in the guise of $\mathrm{SO}^*(2N)$. The reason for this is that one can define $\mathrm{U}_\alpha(N,\mathbb{H})$ over $\mathbb{C}$ instead of $\mathbb{H}$. The group is then the intersection of $\mathrm{SO}(2N,\mathbb{C})$ with $\mathrm{SU}(N,N)$ as can be seen from the normal form for f that we introduce below. This is similar to the two different notations $\mathrm{SL}(n,\mathbb{H})$ and $\mathrm{SU}^*(2n)$ in 2.2.

Only the case $N=2n$ gives rise to a symmetric R-space.

We will set $\mathrm{OG}(\mathbb{H}^{2n})=N_n(\mathbb{H}^{2n},f)$ and call it the *quaternionic orthogonal Grassmannian*. Our goal ist to identify $\mathrm{OG}(\mathbb{H}^{2n})$ with the symmetric space $\mathrm{U}(2n)/\mathrm{Sp}(n)$ and thus verify the following table.

L	G/K	Symbol	Description of G/K
$\mathrm{U}_\alpha(2n,\mathbb{H})$	$\mathrm{U}(2n)/\mathrm{Sp}(n),\ n\geqslant 3$	$\mathrm{OG}(\mathbb{H}^{2n})$	quaternionic orthogonal Grassmannian

[We first look at the values of n excluded in the table. The space $\mathrm{U}(2n)/\mathrm{Sp}(n)$ is S^1 if $n=1$ on which $L=\mathrm{U}_\alpha(2,\mathbb{H})$ acts by projective transformations. If $n=2$, it is the quadric $Q_{1,5}(\mathbb{R})$ in $P^7(\mathbb{R})$ on which $L=\mathrm{U}_\alpha(4,\mathbb{H})$ acts as the locally isomorphic group $\mathrm{SO}(2,6)$. Hence the case $n=2$ belongs to class (IV); see 4.4.]

We will write $\mathbb{H}^{2n}=\mathbb{C}^{2n}+j\mathbb{C}^{2n}$. We consider $u+jv$ and $w+jz$ in $\mathbb{H}^{2n}=\mathbb{C}^{2n}+j\mathbb{C}^{2n}$ and the form f on $\mathbb{H}^n$ defined by setting

$$f(u+jv,w+jz)=i\sum_{k=1}^{2n}(\bar{u}_k w_k-\bar{v}_k z_k)+j\sum_{k=1}^{2n}(u_k z_k+v_k w_k).$$

One easily checks that f is nondegenerate and skew-Hermitian over $\mathbb{H}$. It is not equal to the form f defined at the beginning of this subsection, but it can be brought in that form by changing the basis. The first sum in the definition of f is a nondegenerate Hermitian form with Witt index $2n$ on $\mathbb{C}^{4n} = \mathbb{C}^{2n} + \mathbb{C}^{2n}$ and the second sum is a nondegenerate symmetric form on $\mathbb{C}^{4n} = \mathbb{C}^{2n} + \mathbb{C}^{2n}$.

We will now show that $\mathrm{OG}(\mathbb{H}^{2n})$ can be identified with $\mathrm{U}(2n)/\mathrm{Sp}(n)$. Let S be a maximal isotropic subspace in $\mathrm{OG}(\mathbb{H}^{2n})$ and let $z_1 = u_1 + jv_1, \ldots, z_n = u_n + jv_n$ be a basis in S such that

$$\langle z_i, z_j \rangle = 2\delta_{ij}$$

where $\langle z_i, z_j \rangle$ denotes the standard quaternionic inner product in $\mathbb{H}^{2n}$ and δ_{ij} is the Kronecker delta. Splitting $\langle u + jv, w + jz \rangle$ into its complex and j-part, we get

$$\langle u + jv, w + jz \rangle = \sum_{k=1}^{2n} (\bar{u}_k w_k + \bar{v}_k z_k) + j \sum_{k=1}^{2n} (u_k z_k - v_k w_k).$$

The equation $\langle z_i, z_i \rangle = 2\delta_{ij}$ is therefore equivalent to

$$(u_i, u_i) + (v_i, v_i) = 2$$

where (u, v) is the standard Hermitian scalar product in $\mathbb{C}^{2n}$.

On the other hand $f(z_i, z_i) = 0$ is equivalent to

$$(u_i, u_i) - (v_i, v_i) = 0 \text{ and } \phi(u_i, v_i) = 0$$

where ϕ is the standard symmetric form $\phi(u, v) = \sum_{k=1}^{2n} u_k v_k$ on $\mathbb{C}^{2n}$. Note that $\phi(u, v) = 0$ is equivalent to $(u, \bar{v}) = 0$. Hence we get

$$(u_i, u_i) = (v_i, v_i) = 1 \text{ and } (u_i, \bar{v}_i) = 0.$$

Furthermore, $\langle z_i, z_j \rangle = 0$ für $i \neq j$ is equivalent to

$$(u_i, u_j) + (v_i, v_j) = 0 \text{ and } (u_i, \bar{v}_j) - (v_i, \bar{u}_j) = 0$$

and $f(z_i, z_j) = 0$ is equivalent to

$$(u_i, u_j) - (v_i, v_j) = 0 \text{ and } (u_i, \bar{v}_j) + (v_i, \bar{u}_j) = 0.$$

As a consequence of these considerations, we see that $u_1, \ldots, u_n, \bar{v}_1, \ldots, \bar{v}_n$ is a unitary basis of basis of $\mathbb{C}^{2n}$. Conversely, every unitary basis of $\mathbb{C}^{2n}$ gives rise to a maximal isotropic subspace S in $\mathbb{H}^n$. In fact, if we write the basis in the form $u_1, \ldots, u_n, \bar{v}_1, \ldots, \bar{v}_n$, then S is the subspace of $\mathbb{H}^{2n}$ spanned by $z_1 = u_1 + jv_1, \ldots, z_n = u_n + jv_n$.

We would like to show that $\mathrm{U}(2n)$ acts transitively on $\mathrm{OG}(\mathbb{H}^{2n})$ where we have embedded $\mathrm{U}(2n)$ into $\mathrm{U}_\alpha(2n, \mathbb{H})$ by letting $A \in \mathrm{U}(2n)$ send $u + jv$ in $\mathbb{H}^{2n} = \mathbb{C}^{2n} + j\mathbb{C}^{2n}$ to $Au + j\bar{A}v$.

Let S and S^* be in $\mathrm{OG}(\mathbb{H}^{2n})$. We choose as above bases $z_1 = u_1 + jv_1, \ldots, z_n = u_n + jv_n$ of S and $z_1^* = u_1^* + jv_1^*, \ldots, z_n^* = u_n^* + jv_n^*$ of S^*. We would like to find an A in $\mathrm{U}(2n)$ that maps the basis of S to the basis of S^*, or more precisely such that $Au_1 = u_1^*, \ldots, Au_n = u_n^*, \bar{A}v_1 = v_1^*, \ldots, \bar{A}v_n = v_n^*$. This is equivalent to finding an $A \in \mathrm{U}(2n)$ that maps the unitary basis $u_1, \ldots, u_n, \bar{v}_1, \ldots, \bar{v}_n$ of $\mathbb{C}^{2n}$ to the unitary basis $u_1^*, \ldots, u_n^*, \bar{v}_1^*, \ldots, \bar{v}_n^*$. Such an A clearly exists.

Finally, we have to determine which A in $\mathrm{U}(2n)$ leave a given S in $\mathrm{OG}(\mathbb{H}^{2n})$ invariant. Let $z_1 = u_1 + jv_1, \ldots, z_n = u_n + jv_n$ be a quaternionic unitary basis of S. Then A sends this basis into $Az_1 = Au_1 + j\bar{A}v_1, \ldots, \bar{A}z_n = \bar{A}u_n + j\bar{A}v_n$, which is an another quaternionic unitary basis of S. It follows that $A \in \mathrm{Sp}(n)$. Hence we have proved that $\mathrm{OG}(\mathbb{H}^{2n}) = \mathrm{U}(2n)/\mathrm{Sp}(n)$ as we wanted to do.

4.3. Class (III). Grassmannians of hyperplanes in oriflamme geometries.

We have two such geometries related to symmetric forms over $\mathbb{R}$ and $\mathbb{C}$. We discuss the two cases separately.

Symmetric forms over $\mathbb{R}$

This case is very similar to the one of Hermitian forms over $\mathbb{C}$ and $\mathbb{H}$ in 4.2. Let f be a nondegenerate symmetric form on a real vector space V. There is then a basis $(e_1, \ldots, e_N)$ of V and numbers p and q with $p + q = N$ such that

$$f(x,y) = \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{p+q} x_i y_i.$$

As in 4.2, the Witt index of (V, f) is $\min\{p,q\}$ and p and q do not depend on the basis.

We denote the automorphism group of (V, f) by $\mathrm{O}(p,q)$ and call it the *orthogonal group of type* (p,q). The subgroup of $\mathrm{O}(p,q)$ consisting of elements with determinant equal to one is denoted by $\mathrm{SO}(p,q)$ and called the *special orthogonal group of type* (p,q). The group $\mathrm{SO}(p,q)$ has two connected components when $p,q \geqslant 1$. We denote its identity component by $\mathrm{SO}_o(p,q)$. If $(p,q) = (N,0)$ or $(0,N)$, we denote the automorphism group by $\mathrm{O}(N)$. The group $\mathrm{O}(N)$ has two components; its identity component consists of automorphism with determinant equal to one and will be denoted by $\mathrm{SO}(N)$.

According to the classification of symmetric R-spaces, only symmetric forms of type (n,n) give rise to such spaces. Hence $N = 2n$. In this case the polar geometry of (V, f) is not thick, and we have a corresponding oriflamme geometry. Our goal is to show that $N_n^+(V, f)$ can be identified with $\mathrm{SO}(n)$ as in the following table where $\Delta(\mathrm{SO}(n))$ denotes the diagonal in $\mathrm{SO}(n) \times \mathrm{SO}(n)$.

L	G/K	Symbol	Description of G/K
$\mathrm{SO}_o(n,n)$	$\mathrm{SO}(n) \times \mathrm{SO}(n)/\Delta(\mathrm{SO}(n)),\ n \geqslant 4$	$\mathrm{SO}(n)$	special orthogonal group

[We assume $n \geqslant 4$ for the following reasons. If $n = 2$, then $\mathrm{SO}_o(2,2)$ is not simple, contradicting our assumption on L. If $n = 3$, then $\mathrm{SO}_o(3,3)$ is locally isomorphic to $\mathrm{SL}(4,\mathbb{R})$ and the space $N_3(V,f) = \mathrm{O}(3)$ corresponds to the union of the $\mathrm{SL}(4,\mathbb{R})$ orbits $G_1(\mathbb{R}^4) = P^3(\mathbb{R})$ and its dual projective space $G_3(\mathbb{R}^4)$. This example belongs more to class (I) than class (III).[‖] There cannot be a fundamental theorem for oriflamme geometry when $n = 3$ since any two elements in $N_3^+(V,f)$ are adjacent making the hypothesis of Theorem 3.3 vacuous, but the fundamental theorem of projective geometry applies to this case. We allow $n = 4$ since the action of $\mathrm{SO}_o(4,4)$ on $\mathrm{SO}(4)$ is indecomposable, and $\mathrm{SO}(4)$ is an indecomposable R-space, although $\mathrm{SO}(4)$ is reducible as a Lie group and a symmetric space. Theorem 3.3 does not apply when $n = 4$, but there is a fundamental theorem for this case that involves triality.]

We first identify $N_n(\mathbb{R}^{2n}, f)$ with $\mathrm{O}(n)$ only sketching the arguments since they are very similar to those for the Hermitian forms over $\mathbb{C}$ and $\mathbb{H}$ in 4.2.

We are dealing with a symmetric bilinear form f of type (n,n) on $\mathbb{R}^{2n}$. We consider $\mathbb{R}^{2n}$ as a direct sum $\mathbb{R}^{2n} = \mathbb{R}^n + \mathbb{R}^n$ where each factor is endowed with the usual Euclidean scalar product in such a way that $f((x_1,x_2),(y_1,y_2)) = \langle x_1,y_1\rangle - \langle x_2,y_2\rangle$. The automorphism group of $(\mathbb{R}^{2n}, f)$ is $\mathrm{O}(n,n)$.

Let W be a maximal totally isotropic subspace in $N_n(\mathbb{R}^{2n}, f)$. Then the projections π_1 and π_2 of W onto the first and the second factor of $\mathbb{R}^{2n} = \mathbb{R}^n + \mathbb{R}^n$, respectively, are bijections. Furthermore, $W \in N_n(\mathbb{R}^{2n}, f)$ induces the map $A_W = \pi_2 \circ \pi_1^{-1}$ from the first to the second factor of $\mathbb{R}^{2n} = \mathbb{R}^n + \mathbb{R}^n$ that we identify with an endomorphism of $\mathbb{R}^n$, which is clearly in $\mathrm{O}(n)$. Conversely, let $A \in \mathrm{O}(n)$ be given and let $W_A = \{(x,Ax) | x \in \mathbb{R}^n\}$ be the graph of A. Clearly, $W_A \in N_n(\mathbb{R}^{2n}, f)$ and the map that sends A to W_A is the inverse of the map that sends W to A_W. We have thus identified $N_n(\mathbb{R}^{2n}, f)$ with $\mathrm{O}(n)$. In particular, we have confirmed that $N_n(\mathbb{R}^{2n}, f)$ consists of two components. One of these components corresponds to $\mathrm{SO}(n)$ and will be our choice of $N_n^+(\mathbb{R}^{2n}, f)$. It is now clear that the connected component $\mathrm{SO}_o(n,n)$ acts transitively on $\mathrm{SO}(n) = N_n^+(\mathbb{R}^{2n}, f)$.

We now come to the adjacency preserving automorphisms of $\mathrm{SO}(n) = N_n^+(\mathbb{R}^{2n}, f)$. According to Theorem 3.2, a bijection of $N_n^+(\mathbb{R}^{2n}, f)$ for $n \geqslant 5$ that is adjacency preserving in both directions is induced by an element of $\mathrm{SO}(n,n)$. The situation is more complicated and involves triality if $n = 4$.

Symmetric forms over $\mathbb{C}$

This case is similar to the one for the skew-Hermitian forms over $\mathbb{H}$ in 4.2, but somewhat easier since we are dealing with complex numbers instead of quaternions.

Let V be a complex vector space with a nondegenerate symmetric form f. Then there is a basis $(e_1, \ldots, e_N)$ of V such that f can be written as

$$f(x,y) = \sum_{i=1}^{N} x_i y_i.$$

[‖]This ambiguity reflects the fact that the Coxeter diagrams D_3 and A_3 coincide.

The automorphism group of (V,f) is called the *complex orthogonal group* and denoted by $\mathrm{O}(N,\mathbb{C})$. The determinant of an endomorphism in $\mathrm{O}(N,\mathbb{C})$ is 1 or -1. The *special complex orthogonal group* $\mathrm{SO}(N,\mathbb{C})$ consists by definition of the elements in $\mathrm{O}(N,\mathbb{C})$ with determinant equal to one. One can show that $\mathrm{SO}(N,\mathbb{C})$ is one of the two connected components of $\mathrm{O}(N,\mathbb{C})$. The Witt index of (V,f) is $[\frac{N}{2}]$.

By the classification of symmetric R-spaces, only the case $N=2n$ gives rise to such a space.

We will from now assume that $N=2n$. Hence f has Witt index n and we are in the situation of oriflamme geometry; see 3.3. The space $N_n(\mathbb{C}^{2n},f)$ of maximal isotropic subspaces is homogeneous under the action of $\mathrm{O}(2n,\mathbb{C})$ and consists of two connected components $N_n^+(\mathbb{C}^{2n},f)$ and $N_n^-(\mathbb{C}^{2n},f)$, each of which is homogeneous under the action of $\mathrm{SO}(2n,\mathbb{C})$. We will denote the space $N_n^+(\mathbb{C}^{2n},f)$ by $\mathrm{OG}^+(\mathbb{C}^{2n})$ and call it the *orthogonal Grassmannian*.

By the classification of symmetric R-spaces, we have the following table.

L	G/K	Symbol	Description of G/K
$\mathrm{SO}(2n,\mathbb{C})$	$\mathrm{SO}(2n)/\mathrm{U}(n),\ n\geqslant 4$	$\mathrm{OG}^+(\mathbb{C}^{2n})$	orthogonal Grassmannian

[We first look at the values excluded in the table. If $n=2$, then $\mathrm{SO}(4,\mathbb{C})$ is not simple, contradicting our assumptions on L. If $n=3$, then the situation is as explained after the previous table in this section, and we are more in class (I) than in class (III). More precisely, $\mathrm{OG}^+(\mathbb{C}^6)$ coincides with the complex projective space $P^3(\mathbb{C})$ on which $\mathrm{SO}(6,\mathbb{C})$ acts as the locally isomorphic group $\mathrm{SL}(4,\mathbb{C})$ by projective transformations. Finally, if $n=4$, then $\mathrm{OG}^+(\mathbb{C}^{2n})$ coincides with the quadric $\mathbb{Q}_6(\mathbb{C})$ in $P^7(\mathbb{C})$ that we will again encounter in class (IV); see 4.4. This last case is not excluded, although it might contradict the principles of taxonomy to allow things to belong to two different classes. The fundamental theorem of oriflamme geometry as stated in Theorem 3.3 does not apply to this case; see the remark after Theorem 3.3.]

We would now like to identify $\mathrm{OG}_n^+(\mathbb{C}^{2n})$ with the compact Hermitian symmetric space $\mathrm{SO}(2n)/\mathrm{U}(n)$. We only sketch the arguments since they are very similar to those for the skew-Hermitian forms over $\mathbb{H}$ in 4.2.

We write $\mathbb{C}^{2n}=\mathbb{R}^{2n}+i\mathbb{R}^{2n}$. Let $u+iv$ and $w+iz$ be elements in $\mathbb{C}^{2n}=\mathbb{R}^{2n}+i\mathbb{R}^{2n}$. Then

$$f(u+iv,w+iz)=\sum_{i=1}^{2n}(u_iw_i-v_iz_i)+i\sum_{i=1}^{2n}(u_iz_i+v_iw_i).$$

Let S be a maximal isotropic subspace contained in $\mathrm{OG}_n^+(\mathbb{C}^{2n})$. We choose an orthogonal basis $z_1=u_1+iv_1,\dots,z_n=u_n+iv_n$ in S with respect to the standard Hermitian scalar product in $\mathbb{C}^{2n}$ where u_i and v_i are elements in $\mathbb{R}^{2n}$ and assume that

$\|z_i\|^2 = 2$ for all i. Then the equations $\|z_i\|^2 = 2$ and $f(z_i, z_i) = 0$ imply

$$\|u_i\|^2 = \|v_i\|^2 = 1 \text{ and } \langle u_i, v_i\rangle = 0.$$

Furthermore, the equations $\langle z_i, z_j\rangle = 0$ and $f(z_i, z_j) = 0$ for $i \neq j$ imply

$$\langle u_i, u_j\rangle = \langle v_i, v_j\rangle = 0 \text{ and } \langle u_i, v_j\rangle = 0.$$

As a consequence, we see that $u_1, \dots, u_n, v_1, \dots, v_n$ is an orthonormal basis of $\mathbb{R}^{2n}$. Conversely, we see that every such basis $u_1^*, \dots, u_n^*, v_1^*, \dots, v_n^*$ of $\mathbb{R}^{2n}$ gives rise to a maximal isotropic subspace in $(\mathbb{C}^{2n}, f)$ spanned by $z_1^* = u_1^* + iv_1^*, \dots, z_n^* = u_n^* + iv_n^*$ that is contained in $\mathrm{OG}_n^+(\mathbb{C}^{2n})$ if and only if it induces the same orientation on $\mathbb{R}^{2n}$ as $u_1, \dots, u_n, v_1, \dots, v_n$. It follows that the compact subgroup $\mathrm{SO}(2n)$ of $\mathrm{SO}(2n, \mathbb{C})$ acts transitively on $\mathrm{OG}_n^+(\mathbb{C}^{2n})$. We now need to determine the subgroup of $\mathrm{SO}(2n)$ that stabilizes a subspace S in $\mathrm{OG}_n^+(\mathbb{C}^{2n})$. We choose the subspace S in $\mathbb{C}^{2n}$ that is spanned by $e_1 + ie_{n+1}, \dots, e_n + ie_{2n}$ where $e_1, \dots, e_{2n}$ is the standard basis of $\mathbb{R}^{2n}$. Let A in $\mathrm{SO}(2n)$ be such that $A(S) = S$. Then A is complex linear since it is belongs to $\mathrm{SO}(2n, \mathbb{C})$. The above considerations show that A maps a unitary basis of S to another such basis of S. It follows that A belongs to $\mathrm{U}(n)$. This finishes the proof that $\mathrm{OG}_n^+(\mathbb{C}^{2n})$ is the symmetric space $\mathrm{SO}(2n)/\mathrm{U}(n)$.

We now discuss the adjacency preserving continuous bijections of $\mathrm{OG}_n^+(\mathbb{C}^{2n})$. According to Theorem 3.3, a continuous bijection of $\mathrm{OG}_n^+(\mathbb{C}^{2n})$ with $n \geqslant 5$ that is adjacency preserving in both directions is induced by an element of $\mathrm{SO}(2n, \mathbb{C})$ possibly composed with the conjugation in $\mathbb{C}$. If $n = 4$, there is a fundamental theorem for oriflamme geometry involving triality.

4.4. Class (IV). Quadrics

We are left with the following two examples of symmetric R-spaces, the real quadric $Q_{p,q}(\mathbb{R})$ and the complex quadric $Q_n(\mathbb{C})$. These quadrics lie in the projective space $P^{n+1}(\mathbb{F})$ where $n = p + q$ in the real case.

L	G/K	Symbol
$\mathrm{SO}_o(p+1, q+1)$	$\mathrm{SO}(p+1) \times \mathrm{SO}(q+1)/\mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q))$, $1 \leqslant p \leqslant q$ and $2 < p + q$	$Q_{p,q}(\mathbb{R})$
$\mathrm{SO}(n+2, \mathbb{C})$	$\mathrm{SO}(n+2)/\mathrm{SO}(n) \times \mathrm{SO}(2)$, $n \geqslant 3$	$Q_n(\mathbb{C})$

[In the first line, we have excluded $L = \mathrm{SO}_o(2,2)$ since it is not simple. The assumption $1 \leqslant p \leqslant q$ is to guarantee that $Q_{p,q}(\mathbb{R})$ contains projective lines. In the second line we exclude $n = 1$ since $Q_1(\mathbb{C})$ is S^2. The group $\mathrm{SO}(3, \mathbb{C})$ is locally isomorphic to $\mathrm{SL}(2, \mathbb{C})$ and acts on S^2 by Möbius transformations. We exclude $n = 2$ since $\mathrm{SO}(4, \mathbb{C})$ is not simple.]

Let f be a nondegenerate symmetric bilinear form on $\mathbb{F}^{n+2}$.

If $\mathbb{F} = \mathbb{R}$, we saw in 4.3 that the normal form of f is

$$f(x,y) = \sum_{i=1}^{p+1} x_i y_i - \sum_{i=p+2}^{p+q+2} x_i y_i$$

where $p+q=n$. The Witt index of f is equal to $p+1$ and hence at least two by the assumption that $p \geqslant 1$. The corresponding quadric $Q_{p,q}(\mathbb{R})$ in $P^{n+1}(\mathbb{R})$ is by definition

$$Q_{p,q}(\mathbb{R}) = \{x = (x_1 : \ldots : x_{n+2}) \,|\, f(x,x) = 0\}.$$

where $(x_1 : \ldots : x_{n+2})$ denotes homogeneous coordinates. It follows that $Q_{p,q}(\mathbb{R})$ contains projective lines since the Witt index is at least two. One can show that the action of $\mathrm{SO}_o(p+1,q+1)$ on $Q_{p,q}(\mathbb{R})$ is transitive and that $Q_{p,q}(\mathbb{R})$ coincides with the symmetric space $\mathrm{SO}(p+1)\times\mathrm{SO}(q+1)/\mathrm{S}(\mathrm{O}(p)\times\mathrm{O}(q))$. In fact $Q_{p,q}(\mathbb{R})$ has $S^p \times S^q$ as a double cover. It is therefore not an irreducible symmetric space, but it is an indecomposable symmetric R-space under our assumptions on p and q.

If $\mathbb{F} = \mathbb{C}$, we saw in 4.3 that the normal form of f is

$$f(x,y) = \sum_{i=1}^{n+2} x_i y_i.$$

with Witt index $[\frac{n+2}{2}]$. We are assuming that $n \geqslant 2$. Hence the Witt index of f is at least two. The corresponding quadric $Q_n(\mathbb{C})$ in $P^{n+1}(\mathbb{C})$ is defined by

$$Q_n(\mathbb{C}) = \{x = (x_1 : \ldots : x_{n+2}) | f(x,x) = 0\}.$$

The quadric $Q_n(\mathbb{C})$ contains projective lines since the Witt index of f is at least two. The group $\mathrm{SO}(n+2,\mathbb{C})$ acts transitively on $Q_n(\mathbb{C})$. As a compact symmetric space, $Q_n(\mathbb{C})$ coincides with $\mathrm{SO}(n+2)/\mathrm{SO}(n)\times\mathrm{SO}(2)$, which might be more familiar as the Grassmannian $G_2^+(\mathbb{R}^n)$ of oriented 2-planes in $\mathbb{R}^n$. The quadric $Q_n(\mathbb{C})$ is an irreducible symmetric space since $n \geqslant 3$ (but $Q_2(\mathbb{C}) = S^2 \times S^2$).

In the following theorem, we will let Q refer to either $Q_{p,q}(\mathbb{R})$ or $Q_n(\mathbb{C})$ assuming p, q, and n to satisfy the conditions in the table. The theorem is a fundamental theorem for these quadrics. Tits proved a much more general result in Theorem 8.6 (II) on p. 135 in [Ti2], which we only state in our special case.

THEOREM 4.1. *Let Q be a quadric in $P^{n+1}(\mathbb{F})$ defined with help of a nondegenerate symmetric form f with Witt index at least two where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. Let $\phi : Q \to Q$ be a bijection that preserves the set of projective lines contained in Q. If $n \geqslant 3$, then the map ϕ extends in a unique way to a collineation of $P^{n+1}(\mathbb{F})$.*

The theorem does not hold if $n=2$; see the counterexample on p. 520 in [Ve]. This is not surprising since the corresponding L is not simple and the quadrics decomposable into two factors of S^1 in the real case and two factors of S^2 in the complex case.

In the theorem on p. 526 in [Ve], a weaker version of the theorem of Tits is stated under the assumption that the Witt index of f is at least three.

Other special cases of Theorem 4.1 were known. Theorem VI in [Ch] implies Theorem 4.1 for the complex quadrics $Q_n(\mathbb{C})$, $n \geqslant 3$, and for the real quadrics $Q_{p,p+1}(\mathbb{R})$ for $p \geqslant 1$ and $Q_{p,p}(\mathbb{R})$ for $p \geqslant 2$.

Chow points out that this result for $Q_{1,2}(\mathbb{R})$ was already proved by Lie and is what is known as the fundamental theorem of the Lie geometry of circles. In the Lie geometry of circles, there is a one to one correspondence between the points of the *Lie quadric* $Q_{1,2}(\mathbb{R})$ and the oriented circles in $S^2 = \mathbb{R}^2 \cup \{\infty\}$. The line in $P^4(\mathbb{R})$ through two different points in $Q_{1,2}(\mathbb{R})$ lies in the Lie quadric if and only if the corresponding oriented circles are in oriented contact. The fundamental theorem of the Lie geometry of circles is therefore a description of the bijections of the space of oriented circles that preserve oriented contact; see [Li], p. 437, where this is explained with references to papers of Lie from the years 1871 and 1872.

All of this has been generalized to the space of oriented spheres in $S^n = \mathbb{R}^n \cup \{\infty\}$ by Pinkall in [Pi] where a one to one correspondence is defined between the oriented spheres in S^n and the *Lie quadric* $Q_{1,n}(\mathbb{R})$. Again two oriented spheres in S^n are in oriented contact if the line through the corresponding points in $Q_{1,n}(\mathbb{R})$ is contained in the Lie quadric. Pinkall then proves independently of [Ti2] and with different methods the 'fundamental theorem of Lie sphere geometry,' which is Theorem 4.1 for the quadric $Q_{1,n}(\mathbb{R})$, $n \geqslant 2$; see Lemma 4 in [Pi]. A good introduction to this material and Lie sphere geometry in general is the book [Ce] by Cecil.

5. Maximally curved spheres in symmetric spaces

We would like to make some comments on the papers [Pe], [Na2], and [Ta2], which all refer to Chow's work in [Ch].

Let M be an irreducible symmetric space of compact type. We denote the maximum of the sectional curvature on M by κ. Then it is proved in [He1] (see also [He2], Chapter VII, §11) that M contains totally geodesics submanifolds of constant curvature κ and that any two such submanifolds of the same dimension are conjugate under the isometry group of M. The maximal dimension d of such submanifolds is $d = 1 + m(\alpha)$ where $m(\alpha)$ is the multiplicity of a longest (restricted) root α of M. Hence $d \geqslant 2$. We will refer to the d-dimensional totally geodesic submanifolds of constant curvature κ in M as *Helgason spheres*. It is remarked on the first page of [He1] that these submanifolds are actually diffeomorphic to spheres except when M is a real projective space, where they obviously coincide with M itself.

If M is a projective space over $\mathbb{C}$ or $\mathbb{H}$, or the projective plane over the octonions $\mathbb{O}$, then it turns out that the Helgason spheres S^d in M are precisely the projective lines. It is also not difficult to see that the Helgason spheres in the Grassmannians $G_k(\mathbb{C}^n)$ and $G_k(\mathbb{H}^n)$ coincide with their (generalized) lines as we defined them in 3.2. In $P^n(\mathbb{R})$ and $G_k(\mathbb{R}^n)$, this is not true since the (generalized) lines in these spaces are one-dimensional.

Peterson mentions in [Pe] the action of $\mathrm{SL}(n,\mathbb{C})$ on $G_k(\mathbb{C}^n)$ and writes that Chow gives in [Ch] 'a 'geometric' characterization of this action in the case of classical hermitian symmetric spaces.' He proves in Theorem 2 that there is for any given points p and q in a compact irreducible symmetric space M a chain of length $k \leqslant \mathrm{rank}(M)$ connecting the two points. Motivated by Chow's work he defines the *arithmetic distance* between p and q to be the shortest chain of Helgason spheres connecting p and q.

Peterson defines L to be the group of diffeomorphisms of M that preserve the arithmetic distance. In the main theorem of the paper the (identity component) of L is determined for the Grassmannians $G_k(\mathbb{R}^n)$, $G_k(\mathbb{C}^n)$, $G_k(\mathbb{H}^n)$ for $k \geqslant 2$, and the space $\mathrm{SU}(n)/\mathrm{SO}(n)$. His result is then that L is the special linear group over the corresponding field in the case of the Grassmannians and $\mathrm{SU}(n)$ in the last case. This result agrees with the one of Chow (see Theorem 3.1) in the case of the Grassmannians over $\mathbb{C}$ and $\mathbb{H}$, but is different for $G_k(\mathbb{R})$ since the Helgason spheres do not in that case coincide with the (generalized) lines as we have pointed out. The space $\mathrm{SU}(n)/\mathrm{SO}(n)$ is not a symmetric R-space and Chow's results do therefore not apply to it. It is of course related to the symmetric R-space $\mathrm{U}(n)/\mathrm{O}(n)$ that we considered in 4.2.

Nagano, who was the advisor of the doctoral thesis of Peterson on which [Pe] is based, continuous this study in [Na2]. He also writes that Chow defined arithmetic distance on 'every classical hermitian symmetric space M' and then says that 'Peterson generalized this by dropping 'hermitian'.' Nagano then determines L for the symmetric space $F_4/\mathrm{Sp}(3) \times \mathrm{SU}(2)$, which is not an R-space.

Note that the result of Nagano from [Na1] that we quoted in the introduction would answer the question about the group L if we can prove that L is a Lie group. Then L must be the isometry group of M if M is not a symmetric R-space.

The paper [Ta2] of Takeuchi was a major breakthrough. He restricts his attention to symmetric R-spaces and changes the definition of Helgason spheres in these spaces. To avoid misunderstanding, we will refer to the objects in this new definition as maximally curved spheres.

Let M be a symmetric R-space. A *maximally curved sphere in M* is a Helgason sphere if M is simply connected and a shortest nonconstant closed geodesic otherwise. Now it is not difficult to see that the maximally curved spheres in the Grassmannians $G_k(\mathbb{R}^n)$, $G_k(\mathbb{C}^n)$ and $G_k(\mathbb{H}^n)$ coincide with their (generalized) lines as we defined them in 3.2. The same is clearly true for the quadrics $Q_n(\mathbb{C})$ and $Q_{p,q}(\mathbb{R})$ that we considered in 4.4. This is also very likely to be true for all classical symmetric R-spaces, but it has not been verified in all cases as far as we know. Takeuchi does not say that the spaces considered by Chow are the classical symmetric R-spaces, but he points out that if 'the ground field is the complex number field, these manifolds are the irreducible compact Hermitian symmetric spaces M of classical type.' For these spaces he says on p. 260 that the Helgason spheres coincide with the (generalized) lines of Chow. There is no proof of this claim in [Ta2], but there is a hint in Example 5.11 on p. 291. Note that the compact Hermitian symmetric spaces are simply connected symmetric R-spaces in which the maximally curved spheres are Helgason spheres.

Takeuchi proves in Lemma 6.1 that there is for any two points in an indecomposable symmetric R-space** a finite chain of of maximally curved spheres joining the points. He then defines arithmetic distance between two points in M as the length of a shortest such chain that is needed to connected the points.

Now let the symmetric R-space $M = G/L$ belong to the triple (L, G, K). If M is an indecomposable symmetric R-space with rank at least two, then the the main result of Takeuchi in [Ta2] is that the (identity component of the) group of diffeomorphisms of M preserving the arithmetic distance is L.

We believe that this theorem generalizes the results of Chow that we have been explaining to all symmetric R-spaces (if one is satisfied with diffeomorphisms instead of homeomorphisms or even more general line preserving bijections). To see this, one needs to identify the maximally curved spheres with the lines in the classical symmetric R-spaces. This is clear in many cases as we have pointed out. Takeuchi's theorem has for example Theorem 4.1 as a corollary if one assumes that the line preserving bijection is a diffeomorphism.

6. An unfinished project

Some of the results in Dieudonné's book [Di] that we have been quoting, apply to more general R-spaces than those that are symmetric. Theorem 3.2 gives many examples of such nonsymmetric R-spaces if the Witt index of f is not equal to half the dimension of W.

If we introduce the usual partial order on the orbit types of the G-action on the symmetric space L/G, then the symmetric R-spaces are all of minimal type. The R-spaces of minimal type play a similar role as the Grassmannians among the flag manifolds. Most of the R-spaces of minimal type are not symmetric. The R-spaces to which the results in [Di] apply are all of minimal type, also those that are not symmetric.

Sergio Console and I were working on a fundamental theorem for these more general R-spaces of minimal type in an unfinished project. Our approach was differential geometric and to some extend in the spirit of the theory of isoparametric submanifolds.

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**Our 'indecomposable symmetric R-spaces' are called 'irreducible symmetric R-spaces' in [Ta2]; see the introduction for our definition.

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Gudlaugur Thorbergsson
Mathematisches Institut, Universität zu Köln, Weyertal 86–90, 50931 Köln, Germany
gthorber@mi.uni-koeln.de

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