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SELF-DUALITY AND EXCEPTIONAL GEOMETRY

Preface. This short article is a copy of one that I prepared in advance for a short talk at the conference "Topology and its Applications" held in Baku in October 1987 in the era of Perestroika. Although it remains on my homepage, it has not (to the best of my knowledge) appeared in any other repository or bound volume. It was the first paper I typed in TEX, and spans my career in the sense that one is still working to understand the place that these prototypes assume in the zoo of explicit metrics with exceptional holonomy. I welcome the opportunity to publish it in a collection of papers in memory of Sergio Console, with whom I had frequent scientific discussions about holonomy (both tangential and normal), and with whom I shared editorial work for the *Seminario Matematico*.

1. Introduction

The local isomorphism between the special orthogonal group $SO(4)$ and the product $SO(3)\times SO(3)$ manifests itself in the conformally-invariant decomposition of the bundle of 2-forms

$$\Lambda^2T^*M=\Lambda^2_+T^*M\oplus\Lambda^2_-T^*M$$

over an oriented Riemannian 4-manifold M. There is a corresponding decomposition of the Weyl curvature tensor $W=W_++W_-$, and M is said to be *self-dual* if $W_-=0$. If M is compact, its signature is given by

$$\tau=\frac{1}{3}p_1=\frac{1}{12\pi^2}\int_M(|W_+|^2-|W_-|^2)\upsilon,$$

where υ is the volume form. Consequently, if M is self-dual but not conformally flat, then $\tau>0$.

Self-duality is the integrability condition for a natural almost complex structure on the 6-dimensional sphere bundle of $\Lambda^2_-T^*M$ [1]. Motivated in part by this result, we study the 7-dimensional total space X of $\Lambda^2_-T^*M$, and characterize curvature conditions on M by means of differential relations between invariant forms on X. First though, we define the exceptional Lie group G_2 using the inclusion $SO(4)\subset G_2$, corresponding to a splitting of dimensions $7=3+4$. This enables one to construct a family of G_2-structures on X, which amounts to assigning a metric and vector cross product on each tangent space.

There are only two exceptions in the list of holonomy groups of irreducible non-symmetric Riemannian manifolds, namely G_2 and *Spin*(7) [2, 3, 5, 11]. This explains the importance of G_2-structures, which, in the light of [7], have a richer torsion theory than their *Spin*(7) counterparts. An examination of the structure on X leads us to exhibit there a Riemannian metric with holonomy group G_2, when M is the self-dual Einstein manifold S^4 or $\mathbb{CP}^2$. No such complete metrics were previously known. This, and analogous examples with holonomy G_2 and *Spin*(7), are the subject of a forthcoming joint paper with R. L. Bryant.

2. Definition of G_2

Let V denote an oriented n-dimensional vector space with a positive definite inner product $\langle\ ,\ \rangle$. The inner product extends to one on $\Lambda^k V^*$, and together with the orientation defines a unit volume form $\upsilon \in \Lambda^n V^*$ and an isomorphism $* : \Lambda^k V^* \to \Lambda^{n-k} V^*$, where

$$\sigma(*\tau) = \langle \sigma, \tau \rangle \upsilon, \qquad \sigma, \tau \in \Lambda^k V^*. \tag{1}$$

Here and in the sequel, an exterior product of differential forms is denoted merely by juxtaposition.

Now take $n = 4$ and $k = 2$. Then $*$ is an involution on $\Lambda^2 V^*$, and we consider the 7-dimensional space

$$A = \Lambda^2_- V^* \oplus V^*,$$

where $\Lambda^2_- V^*$ is the -1-eigenspace of $*$. If $\{e^4, e^5, e^6, e^7\}$ is an oriented orthonormal basis of V^*, then $\Lambda^2_- V^*$ is the span of

$$e^1 = e^4 e^5 - e^6 e^7, \quad e^2 = e^4 e^6 - e^7 e^5, \quad e^3 = e^4 e^7 - e^5 e^6. \tag{2}$$

Regarding now $e^1, \dots, e^7$ as all elements of A, rather than $\Lambda^2 A$, we set

$$\begin{aligned} \varphi' &= e^1 e^2 e^3 \\ \varphi'' &= e^1(e^4 e^5 - e^6 e^7) + e^2(e^4 e^6 - e^7 e^5) + e^3(e^4 e^7 - e^5 e^6). \end{aligned}$$

Then $\varphi = \varphi' + \varphi''$ is the sum of 7 simple 3-forms on a 7-dimensional vector space, and has the following well-known property (see [5]).

PROPOSITION 2.1. *The set* $G_2 = \{g \in GL(A) : g^*\varphi = \varphi\}$ *is a compact Lie group of dimension* 14.

Proof. As defined above, G_2 is a closed subgroup of $GL(A)$ containing $SO(4)$. Decreeing $\{e^1, \dots, e^7\}$ to be an oriented orthonormal basis of A defines an action of $SO(7)$ with Lie algebra

$$\begin{aligned} \mathfrak{so}(7) \cong \Lambda^2 A &\cong \Lambda^2(\Lambda^2_- V^*) \oplus (\Lambda^2_- V^* \otimes V^*) \oplus \Lambda^2 V^* \\ &\cong \Lambda^2_- V^* \oplus (V^* \oplus K) \oplus (\Lambda^2_+ V^* \oplus \Lambda^2_- V^*). \end{aligned} \tag{3}$$

Here K denotes the 8-dimensional subspace of $\Lambda^2_- V^* \otimes V^*$ of elements with zero contraction; for example K contains $e^1 \otimes e^4 + e^2 \otimes e^7$ which defines a skew-symmetric endomorphism of V annihilating φ. Hence the Lie algebra $\mathfrak{g}_2$ of G_2 contains K, not to mention $\Lambda^2_+ V^*$ and one copy of $\Lambda^2_- V^*$. Now $S^2 A \cong \mathbb{R} \oplus S^2_0 A$, where

$$S^2_0 A \cong S^2_0(\Lambda^2_- V^*) \oplus \mathbb{R} \oplus V^* \oplus K \oplus S^2_0 V^*$$

is the space of traceless symmetric endomorphisms of A, decomposed into $SO(4)$-modules. Consideration of the action of $K \subset \mathfrak{g}_2$ shows that $S^2_0 A$ is G_2-irreducible. Thus

$$\mathfrak{g}_2 = \mathfrak{so}(4) \oplus K,$$

and it is not hard to check that $G_2 \subset SO(7)$. □

The form φ defines by contraction a two-fold vector cross product

$$m\colon\ \Lambda^2 A \longrightarrow A, \tag{4}$$

of the sort that exists only on a space of dimension 3 or 7 [4]. Using m, the direct sum $\mathbb{O} = \mathbb{R}\oplus A$ can be identified with the alternative algebra of Cayley numbers, to give the description of G_2 as the group of automorphisms of $\mathbb{O}$. The subspace $\mathbb{H} = \mathbb{R}\oplus\Lambda^2_- V^*$ corresponds to a quaternionic subalgebra, and K may be identified with the tangent space of the quaternionic symmetric space $G_2/SO(4)$, parametrizing all quaternionic subalgebras in $\mathbb{O}$ [9].

Like $S^2_0 A$, the G_2-modules A and $\mathfrak{g}_2$ are irreducible, and from (4), the orthogonal complement $\mathfrak{g}_2^\perp$ of $\mathfrak{g}_2$ in $\mathfrak{so}(7)$ must be isomorphic to A. The derivative

$$\delta\colon\ \operatorname{End} A \cong A\otimes A \ \hookrightarrow\ \Lambda^3 A$$

of the action of the general linear groups $GL(V)$ on φ has kernel $\mathfrak{g}_2$. It follows that the orbit (isomorphic to $GL(V)/G_2$) containing φ is open in $\Lambda^3 A$; in fact there is just one other open orbit, containing the form $\varphi' - \varphi''$, with stabilizer the non-compact form G^* [5]. Anyway, the above remarks establish

PROPOSITION 2.2. *There are G_2-invariant decompositions*

$$\Lambda^2 A \cong \mathfrak{g}_2\oplus A, \qquad \Lambda^3 A \cong \mathbb{R}\oplus S^2_0 A\oplus A.$$

3. Four-dimensional Riemannian geometry

Let M be an oriented Riemannian 4-dimensional manifold. We shall now use the symbols e^4, e^5, e^6, e^7 to denote elements of an oriented orthonormal basis of 1-forms on an open set U of M. Accordingly e^1, e^2, e^3 defined by (2) form a basis of sections over U of $\Lambda^2_- T^*M$. The Levi-Civita connection on M induces a covariant derivative ∇ on this vector bundle, and we set

$$\nabla e^i = \Sigma\,\omega^i_j\otimes e^j, \qquad \Omega^i_j = d\omega^i_j - \Sigma\,\omega^i_k\omega^k_j.$$

Summations here and below are exclusively over the range of indices 1,2,3.

Let X denote the total space of the rank 3 vector bundle $\Lambda^2_- T^*M$; its cotangent space at x admits a splitting

$$T^*_x X = V^{\mathrm{o}}\oplus H^{\mathrm{o}}, \tag{5}$$

where H^{o} is the annihilator of the horizontal subspaces defined by the connection, and $V^{\mathrm{o}} = \pi^* T^*_m M$ with $m = \pi(x)$. A local section $\Sigma a^i e^i$ of $\Lambda^2_- T^*M$ is covariant constant if and only if $\Sigma(da^i + \Sigma a^j\omega^j_i)\otimes e^i = 0$, so H^{o} is spanned by 1-forms

$$f^i = da^i + \Sigma a^j\pi^*\omega^j_i,$$

where a^1, a^2, a^3 are now interpreted as fibre coordinate functions on X. Of course, V^o is spanned by $\pi^*e^4, \pi^*e^5, \pi^*e^6, \pi^*e^7$, 1-forms that annihilate the fibres.

Omitting the symbol π^*, consider the following invariant forms that are defined globally on X independently of the choice of basis:

$$r = \Sigma (a^i)^2$$

$$dr = 2\Sigma a^i f^i$$

$$\alpha = \Sigma a^i e^i$$

$$d\alpha = \Sigma e^i f^i, \quad \beta = f^1 f^2 f^3$$

$$\gamma = e^1 f^2 f^3 + e^2 f^3 f^1 + e^3 f^1 f^2, \quad \upsilon = -\tfrac{1}{6}\Sigma e^i e^i$$

For example r is simply the radius squared, α is the tautological 2-form on X, and $\upsilon = e^4 e^5 e^6 e^7$ is the pullback of the volume form on M.

PROPOSITION 3.1. *(i) M is self-dual if and only if $d\gamma = 2t\upsilon dr$ for (the pull-back of) some scalar function t on M; (ii) M is self-dual and Einstein if and only if $d\beta = \frac{1}{2}t d\alpha dr$, for some constant t. If t exists in either case, it equals $\frac{1}{12}$ of the scalar curvature of M.*

Proof. We refer the reader to [1] for the basic properties of the curvature tensor of a Riemannian 4-manifold. The curvature of the induced connection on the bundle $\Lambda^2_- T^*M$ is determined by the Ricci tensor, and the half W_- of the Weyl tensor which may be regarded as a section of $\Lambda^2_- T^*M \otimes \Lambda^2_- T^*M$. Moreover M is self-dual and Einstein if and only if

$$\Omega^1_2 = te^3, \quad \Omega^2_3 = te^1, \quad \Omega^3_1 = te^2, \tag{6}$$

where $t = \frac{1}{12}$(scalar curvature). Since the trace-free Ricci tensor essentially belongs to $\Lambda^2_- T^*M \otimes \Lambda^2_+ T^*M$, M is self-dual if and only if (6) holds modulo elements of $\Lambda^2_+ T^*M$. The proposition is now the result of a computation involving the formulae

$$de^i = \Sigma \omega^i_j e^j, \qquad df^i = \Sigma (f^j \omega^j_i + a^j \Omega^j_i)$$

for exterior derivatives. □

Motivated by section 1, we next consider the 3-form

$$\varphi = \lambda^3 \beta + \lambda \mu^2 d\alpha, \tag{7}$$

where λ and μ are scalar functions on X. Observe that

$$\varphi = E^1E^2E^3 + E^1E^4E^5 - E^1E^6E^7 + E^2E^4E^6 - E^2E^7E^5 + E^3E^4E^7 - E^3E^5E^6,$$

where E^i equals λf^i for $i=1,2,3$ and $\mu\pi^* e^i$ for $i=4,5,6,7$, and forms an oriented orthonormal basis of 1-forms for the underlying $SO(7)$-structure on X. In view of (1), we also have

$$\begin{aligned} *\varphi &= E^4E^5E^6E^7+E^2E^3E^6E^7-E^2E^3E^4E^5+E^3E^1E^7E^5 \\ &\qquad -E^3E^1E^4E^6+E^1E^2E^5E^6-E^1E^2E^4E^7 \\ &= \mu^4\upsilon-\lambda^2\mu^2\gamma. \end{aligned} \tag{8}$$

Proposition 1 implies

PROPOSITION 3.2. *If λ and μ are strictly positive everywhere, (7) determines a G_2-structure on X, i.e. a G_2-subbundle P of the principal frame bundle of X, whose underlying Riemannian metric has the form $\lambda^2 g^V+\mu^2 g^H$ in terms of the splitting (5).*

4. Torsion considerations

If D denotes the Levi-Civita connection of the Riemannian metric in Proposition 4, the quantity $D\varphi$ measures the failure of the holonomy group to reduce to G_2, i.e. the extent to which parallel transport does not preserve the principal subbundle P. Its properties were studied by Fernández and Gray in [7], and we first summarize their approach.

Choose any connection $\widetilde{D}$ that reduces to P, so that $\widetilde{D}\varphi=0$. Fix a frame $p\in P$ at the point $x=\pi(p)\in X$, and a vector $v\in T_xX$. The difference $D_v-\widetilde{D}_v$ defines, relative to p, an element of the Lie algebra $\mathfrak{so}(7)$. The same is true of $D_v\varphi=(D_v-\widetilde{D}_v)\varphi$, but since this is independent of the choice of $\widetilde{D}$, it actually belongs to the subspace $\mathfrak{g}_2^\perp$. Therefore $(D\varphi)_x$ may be regarded as an element of

$$T_x^*X\otimes\mathfrak{g}_2^\perp \;\cong\; A\otimes A \;\cong\; \mathbb{R}\oplus\mathfrak{g}_2\oplus S_0^2A\oplus A, \tag{9}$$

a real vector space of dimension 49.

Let $W_1X\cong X\times\mathbb{R}$, W_2X, W_3X, $W_4X\cong TX\cong T^*X$ denote the vector bundles associated to P with fibre $\mathbb{R}$, $\mathfrak{g}_2$, S_0^2A, A respectively. Corresponding to (9), there is a decomposition

$$D\varphi=w_1+w_2+w_3+w_4,$$

in which w_i is a section of W_iX. Now D is torsion-free, and there exist surjective homomorphisms

$$\begin{aligned} a\colon\; T^*X\otimes\Lambda^3T^*X &\longrightarrow \Lambda^4T^*X \;\cong\; W_1X\oplus W_3X\oplus W_4X \\ a^*\colon\; T^*X\otimes\Lambda^3T^*X &\longrightarrow \Lambda^5T^*X \;\cong\; W_2X\oplus W_4X, \end{aligned}$$

such that $d\varphi=a(D\varphi)$ and $d{*\varphi}=a^*(D\varphi)$ (cf. Proposition 2). Thus

PROPOSITION 4.1 ([7]). *With the above identifications, $d\varphi=(w_1,w_3,w_4)$, and $d{*\varphi}=(w_2,w_4)$, so $D\varphi=0$ if and only if $d\varphi=0=d{*\varphi}$.*

Call a differential form on X of type (p,q) if, at each point, it is built up from forms on the base of degree p and forms of degree q involving f^i. Endow X with the G_2-structure of Proposition 4, with λ and μ arbitrary positive scalar functions on X. Then $d\varphi$, unlike $*\varphi$, has no component of type $(4,0)$. Moreover $\varphi d\varphi = 0$, whence $d\varphi$ has no component in the subbundle $W_1 X \subset \Lambda^4 T^* X$, and we always have $w_1 = 0$. Further components of $D\varphi$ can be eliminated by a suitable choice of λ and μ.

THEOREM 4.1. *(i) If M is self-dual, an open set of X admits a G_2-structure with $D\varphi = w_3$; (ii) if M is self-dual and Einstein, an open set of X admits a G_2-structure with $D\varphi = 0$.*

Proof. We apply Proposition 3. If M is self-dual, we seek λ, μ such that

$$d{*}\varphi = d(\mu^4)\upsilon - d(\lambda^2\mu^2)\gamma - \lambda^2\mu^2 2t\,\upsilon\, dr,$$

vanishes. Taking $\lambda\mu = c =$ constant, we obtain a solution

$$\mu = (2c^2 tr + d)^{1/4}, \quad \lambda = c(2c^2 tr + d)^{-1/4}, \tag{10}$$

where d is another constant. If M is also Einstein, then $dt = 0$ and

$$d\varphi = d(\lambda^3)\beta + \lambda^3 \tfrac{1}{2} t d\alpha dr + d(\lambda\mu^2) d\alpha = 0.$$

Note that the functions λ, μ can only be strictly positive on all of X if t is everywhere non-negative. $\square$

In [7] it is shown that any minimally embedded hypersurface of $\mathbb{R}^8$ also has a G_2-structure with $D\varphi = w_3$. A contrasting example with $D\varphi = w_2 \neq 0$ has been found in [6]. We remark that, in general, w_2 is the obstruction to the existence of a short elliptic complex

$$0 \to C^\infty(X) \xrightarrow{\text{grad}} C^\infty(X, TX) \xrightarrow{\text{curl}} C^\infty(X, TX) \xrightarrow{\text{div}} C^\infty(X) \to 0,$$

on X whose operators are manufactured using D and and the cross product (4) in analogy with the 3-dimensional case. Indeed, if $f \in C^\infty(X)$ is a smooth function, and $v \in C^\infty(X, TX)$ is a vector field, $\text{curl}(\text{grad} f) = m(D \wedge (\text{grad} f))$ vanishes identically, but $\text{div}(\text{ curl} v)$ equals the contraction of Dv with w_2. We conjecture that a complex of this sort can be defined on X, using only the self-dual conformal structure of M. We note that topological consequences of the existence of a self-dual metric with t non-negative have been given by LeBrun [10].

Self-dual Einstein metrics have been generated by quaternion-Kähler reduction, see [8]. However a theorem of Hitchin states that a complete Riemannian 4-manifold which is self-dual, Einstein and of positive scalar curvature is necessarily isometric to the sphere S^4, or the complex projective plane $\mathbb{CP}^2$ [3, 13.30]. In either of these two cases, the Riemannian metric

$$(2tr+1)^{-1/2} g^V + (2tr+1)^{1/2} g^H$$

on X corresponding to the solution (10) with $c = d = 1$ is complete, essentially because $\int_0^\infty (2tr+1)^{-\frac{1}{4}} d(r^{1/2})$ diverges. Moreover, since $D\varphi = 0$, the holonomy group H is contained in G_2, which in turn implies that the Ricci tensor is zero [3]. Furthermore, the respective groups $SO(5)$, $SU(3)$ act as isometries on X with generic orbits of codimension 1. Consideration of the induced action on a hypothetical space of covariant constant 1-forms shows that X is locally irreducible, and it follows that H actually equals G_2. In conclusion:

COROLLARY 4.1. *The 7-dimensional total spaces $\Lambda^2_- T^*S^4$ and $\Lambda^2_- T^*\mathbb{CP}^2$ each admit a complete Ricci-flat Riemannian metric with holonomy equal to G_2.*

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Postscript. Although the realization of explicit metrics with holonomy G_2 is nowadays seen as straightforward, readers should bear in mind that only months before the appearance of [5] and this paper, many experts believed that they did not exist or were at least difficult to write down. On the other hand, the four-dimensional approach to the problem was a natural one for someone like myself who had been a graduate student in Oxford during the development in [1] of self-duality and twistor theory. Even the coset space $G_2/SO(4)$ relevant to this paper fits into the quaternionic generalization of the theory.

Now we are in an internet era, readers will have no difficulty in updating the references above, and in accessing my joint paper with Robert Bryant, my Pitman book on Riemannian Geometry and Holonomy, and subsequent papers and books of Dominic Joyce and many others. To complete that list, I choose merely to adjoin my survey [13] and recent preprints [14, 15] that generalize the 4-dimensional vision of this particular article.

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