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CLIFFORD SYSTEMS IN OCTONIONIC GEOMETRY

Dedicated to the memory of Sergio Console

Abstract. We give an inductive construction for irreducible Clifford systems on Euclidean vector spaces. We then discuss how this notion can be adapted to Riemannian manifolds, and outline some developments in octonionic geometry.

1. Introduction

The notion of Clifford system, as formalized in 1981 by D. Ferus, H. Karcher and H. F. Münzner, has been used in the last decades both in the study of isometric hypersurfaces and of Riemannian foliations [6, 18, 8]. In particular, Clifford systems have been used by Sergio Console and Carlos Olmos [4] to give an alternative proof of a Theorem of E. Cartan stating that a compact isoparametric hypersurface of a sphere with three distinct principal curvatures is a tube around the Veronese embedding of the projective planes $\mathbb{R}P^2, \mathbb{C}P^2, \mathbb{H}P^2, \mathbb{O}P^2$ over the reals, complex numbers, quaternions and Cayley numbers, respectively.

In this statement, the Veronese embedding of the four projective planes goes into spheres S^4, S^7, S^{13}, S^{25} and these embeddings admit an analogy in complex projective geometry. Namely, the (so-called) projective planes

$$\mathbb{C}P^2,\ (\mathbb{C}\otimes\mathbb{C})P^2,\ (\mathbb{C}\otimes\mathbb{H})P^2,\ (\mathbb{C}\otimes\mathbb{O})P^2$$

over complex numbers and over the other three composition algebras of complex complex numbers, complex quaternions and complex octonions, admit an embedding into complex projective spaces $\mathbb{C}P^5$, $\mathbb{C}P^8$, $\mathbb{C}P^{14}$, $\mathbb{C}P^{26}$. These latter embeddings are also named after Veronese and give rise to projective algebraic varieties of degrees 4, 6, 14, 78, respectively. Very interesting properties of the mentioned two series of Veronese embeddings have been pointed out in [1].

The following Table A collects "projective planes" $(\mathbb{K}\otimes\mathbb{K}')P^2$ over composition algebras $\mathbb{K}\otimes\mathbb{K}'$, where $\mathbb{K},\mathbb{K}' \in \{\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}\}$. Here notations V_2^4, V_4^6, V_8^{14}, V_{16}^{78} (with lower and upper indices being the complex dimension and the degree, respectively) are for the so-called *Severi varieties*, smooth projective algebraic varieties with nice characterizations realizing the mentioned embeddings [20]. Table A will give a general reference for our discussion. In particular, the fourth Severi variety $\mathrm{E}_6/\mathrm{Spin}(10)\cdot\mathrm{U}(1) \cong V_{16}^{78} \subset \mathbb{C}P^{26}$ has been recently studied both with respect to the

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structure given by its holonomy and in the representation of the differential forms that generate its cohomology [17].

Table A: Projective planes

$\mathbb{K}=\backslash^{\mathbb{K}'=}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathbb{R}P^2$	$\mathbb{C}P^2 \cong V_2^4$	$\mathbb{H}P^2$	$\mathbb{O}P^2 \cong \mathrm{F}_4/\mathrm{Spin}(9)$
$\mathbb{C}$	$\mathbb{C}P^2 \cong V_2^4$	$\mathbb{C}P^2 \times \mathbb{C}P^2 \cong V_4^6$	$Gr_2(\mathbb{C}^6) \cong V_8^{14}$	$\mathrm{E}_6/\mathrm{Spin}(10)\cdot\mathrm{U}(1) \cong V_{16}^{78}$
$\mathbb{H}$	$\mathbb{H}P^2$	$Gr_2(\mathbb{C}^6) \cong V_8^{14}$	$Gr_4^{or}(\mathbb{R}^{12})$	$\mathrm{E}_7/\mathrm{Spin}(12)\cdot\mathrm{Sp}(1)$
$\mathbb{O}$	$\mathbb{O}P^2 \cong \mathrm{F}_4/\mathrm{Spin}(9)$	$\mathrm{E}_6/\mathrm{Spin}(10)\cdot\mathrm{U}(1) \cong V_{16}^{78}$	$\mathrm{E}_7/\mathrm{Spin}(12)\cdot\mathrm{Sp}(1)$	$\mathrm{E}_8/\mathrm{Spin}(16)^+$

Recall that a *Clifford system* on the Euclidean vector space $\mathbb{R}^N$ is the datum of an $(m+1)$-ple

$$C_m = (P_0, \dots, P_m)$$

of symmetric transformations P_α such that:

$$P_\alpha^2 = \text{ Id for all } \alpha, \qquad P_\alpha P_\beta = -P_\beta P_\alpha \text{ for all } \alpha \neq \beta.$$

A Clifford system on $\mathbb{R}^N$ is said to be *irreducible* if $\mathbb{R}^N$ is not direct sum of two positive dimensional subspaces that are invariant under all the P_α.

From representation theory of Clifford algebras one recognizes (cf. [6, page 483], [10, page 163]) that $\mathbb{R}^N$ admits an irreducible Clifford system $C = (P_0, \dots, P_m)$ if and only if

$$N = 2\delta(m),$$

where $\delta(m)$ is given by the following

Table B: Clifford systems

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...	$8+h$
$\delta(m)$	1	2	4	4	8	8	8	8	16	32	64	64	128	128	128	128	...	$16\delta(h)$

One can discuss uniqueness as follows. Given on $\mathbb{R}^N$ two Clifford systems $C_m = (P_0, \dots, P_m)$ and $C'_m = (P'_0, \dots, P'_m)$, they are said to be *equivalent* if there exists $A \in O(N)$ such that $P'_\alpha = A^t P_\alpha A$ for all α. Then for $m \not\equiv 0 \bmod 4$ there is a unique equivalence class of irreducible Clifford systems, and for $m \equiv 0 \bmod 4$ there are two, classified by the two possible values of $\mathrm{tr}(P_0 P_1 \dots P_m) = \pm 2\delta(m)$.

In the approach by Sergio Console and Carlos Olmos to the mentioned E. Cartan theorem on isoparametric hypersurfaces with three distinct principal curvatures in spheres, the Clifford systems are related with the Weingarten operators of their focal manifolds, and the possible values of m turn out to be here only $1,2,4$ or 8, the multiplicities of the eigenvalues of the Weingarten operators.

In the present paper we outline an inductive construction for all Clifford systems on real Euclidean vector spaces $\mathbb{R}^N$, by pointing out how the four Clifford systems C_1,C_2,C_4,C_8 considered in [4] correspond to structures given by the groups $\mathrm{U}(1)$, $\mathrm{U}(2)$, $\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)$, $\mathrm{Spin}(9)$. We also develop, following ideas contained in [15, 16, 14, 17], the intermediate cases as well as some further cases appearing in Table B. We finally discuss the corresponding notion on Riemannian manifolds and relate it with the notion of even Clifford structure and with the octonionic geometry of some exceptional Riemannian symmetric spaces.

We just mentioned the even Clifford structures, a kind of unifying notion proposed by A. Moroianu and U. Semmelmann [13]. It is the datum, on a Riemannian manifold M, of a real oriented Euclidean vector bundle (E,h), together with an algebra bundle morphism $\varphi : \mathrm{Cl}^0(E) \to \mathrm{End}(TM)$ mapping $\Lambda^2 E$ into skew-symmetric endomorphisms. Indeed, a Clifford system gives rise to an even Clifford structure, but there are some even Clifford structures on manifolds that cannot be constructed, even locally, from Clifford systems. This will be illustrated by examples in Sections 6 and 8.

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2. From $\mathbb{R}$ to $\mathbb{C}$ and to $\mathbb{H}$: the Clifford systems C_1,C_2,C_3,C_4

We examine here the first four columns of Table B, describing with some details irreducible Clifford systems $C_1,C_2.C_3,C_4$, acting on $\mathbb{R}^2,\mathbb{R}^4,\mathbb{R}^8,\mathbb{R}^8$, respectively.

A Clifford system C_1 on $\mathbb{R}^2$ (here $m=1$ and $\delta(m)=1$) is given by matrices

$$N_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad N_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

representing in $\mathbb{C} \cong \mathbb{R}^2$ the involutions $z \in \mathbb{C} \to i\bar{z}$ and $z \in \mathbb{C} \to \bar{z}$, whose composition

$$N_{01} = N_0 N_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is the complex structure on $\mathbb{C} \cong \mathbb{R}^2$.

Going to the next case, the Clifford system C_2 (now $m=2$ and $\delta(m)=2$) is the prototype example of the Pauli matrices:

$$P_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

that we will need in their real representation:

$$P_0 = \left(\begin{array}{c|c} 0 & \mathrm{Id} \\ \hline \mathrm{Id} & 0 \end{array}\right), \quad P_1 = \left(\begin{array}{c|c} 0 & -N_{01} \\ \hline N_{01} & 0 \end{array}\right), \qquad P_2 = \left(\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline 0 & -\mathrm{Id} \end{array}\right).$$

The compositions $P_{\alpha\beta} = P_\alpha P_\beta$, for $\alpha < \beta$, yield as complex structures on $\mathbb{R}^4$ the multiplication on the right $R^{\mathbb{H}}_i, R^{\mathbb{H}}_j, R^{\mathbb{H}}_k$ by unit quaternions i, j, k:

$$P_{01} = R^{\mathbb{H}}_i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \; P_{02} = R^{\mathbb{H}}_j = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \; P_{12} = R^{\mathbb{H}}_k = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Multiplication $L^{\mathbb{H}}_i$ on the left by i coincides with

$$P_{012} = P_0 P_1 P_2 = L^{\mathbb{H}}_i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and to complete $R^{\mathbb{H}}_i, R^{\mathbb{H}}_j, R^{\mathbb{H}}_j, L^{\mathbb{H}}_i$ to a basis of the Lie algebra $\mathfrak{so}(4) \cong \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ one has to add the two further left multiplications

$$L^{\mathbb{H}}_j = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad L^{\mathbb{H}}_k = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Thus:

PROPOSITION 2.1. *Orthogonal linear transfomations in* $\mathbb{R}^4$ *preserving the individual* P_0, P_1, P_2 *are the ones in* $\mathrm{U}(1) = \mathrm{SO}(2)_\Delta \subset \mathrm{SO}(4)$*, those preserving the vector space* $E^3 =< P_0, P_1, P_2 >$ *are the ones in* $\mathrm{U}(2) = \mathrm{Sp}(1) \cdot \mathrm{U}(1)$.

The next Clifford systems C_3 and C_4 act on $\mathbb{R}^8$. They can be defined by the following 4×4 block matrices

$$C_3: \qquad Q'_0 = \left(\begin{array}{c|c} 0 & \mathrm{Id} \\ \hline \mathrm{Id} & 0 \end{array}\right), \qquad Q'_1 = \left(\begin{array}{c|c} 0 & -P_{01} \\ \hline P_{01} & 0 \end{array}\right),$$

$$Q'_2 = \left(\begin{array}{c|c} 0 & -P_{02} \\ \hline P_{02} & 0 \end{array}\right), \qquad Q'_3 = \left(\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline 0 & -\mathrm{Id} \end{array}\right),$$

and

$$C_4: \qquad Q_0 = Q'_0,\ \ Q_1 = Q'_1,\ \ Q_2 = Q'_2,\ \ Q_3 = \left(\begin{array}{c|c} 0 & -P_{12} \\ \hline P_{12} & 0 \end{array}\right),\ \ Q_4 = Q'_3.$$

The following characterizations of the structures on $\mathbb{R}^8$ associated with C_3 and C_4 are easily seen.

PROPOSITION 2.2. *The structure defined in* $\mathbb{R}^8$ *by the datum of the vector space* $E^4 = < Q'_0, Q'_1, Q'_2, Q'_3 > \subset \mathrm{End}(\mathbb{R}^8)$ *can be described as follows. Matrices*

$$B = \left(\begin{array}{c|c} B' & B'' \\ \hline B''' & B'''' \end{array}\right)$$

commuting with the single endomorphisms Q'_0, Q'_1, Q'_2, Q'_3 *are characterized by the conditions* $B' = B'''' \in \mathrm{Sp}(1) \subset \mathrm{SO}(4)$ *and* $B'' = B''' = 0$. *Accordingly, matrices of* $\mathrm{SO}(8)$ *that preserve the vector space* E^4 *belong to a subgroup* $\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)\cdot\mathrm{Sp}(1) \subset \mathrm{SO}(8)$.

To prepare the next Clifford systems, namely C_5, C_6, C_7, C_8 on $\mathbb{R}^{16}$, we need to look at the complex structures $Q_{0\alpha} = Q_0 Q_\alpha$ on $\mathbb{R}^8$. They indeed coincide with R_i, R_j, R_k, R_e, the right multiplication on $\mathbb{R}^8 \cong \mathbb{O}$ by octonions i, j, k, e, respectively:

$$Q_{01} = R_i = \left(\begin{array}{c|c} R_i^{\mathbb{H}} & 0 \\ \hline 0 & -R_i^{\mathbb{H}} \end{array}\right), \qquad Q_{02} = R_j = \left(\begin{array}{c|c} R_j^{\mathbb{H}} & 0 \\ \hline 0 & -R_j^{\mathbb{H}} \end{array}\right),$$

$$Q_{03} = R_k = \left(\begin{array}{c|c} R_k^{\mathbb{H}} & 0 \\ \hline 0 & -R_k^{\mathbb{H}} \end{array}\right), \qquad Q_{04} = R_e = \left(\begin{array}{c|c} 0 & -\mathrm{Id} \\ \hline \mathrm{Id} & 0 \end{array}\right).$$

Associated with the Clifford system $C_4 = (Q_0, Q_1, Q_2, Q_3, Q_4)$, we have ten complex structures $Q_{\alpha\beta} = Q_\alpha Q_\beta$ with $\alpha < \beta$, a basis of the Lie algebra $\mathfrak{sp}(2) \subset \mathfrak{so}(8)$.

Their Kähler forms $\theta_{\alpha\beta}$, written in the coordinates of $\mathbb{R}^8$, and using short notations like $12 = dx_1 \wedge dx_2$, read:

$$\begin{aligned}
&\theta_{01} = -12+34+56-78, && \theta_{02} = -13-24+57+68, && \theta_{03} = -14+23+58-67,\\
&\theta_{12} = -14+23-58+67, && \theta_{13} = 13+24+57+68, && \theta_{23} = -12+34-56+78,\\
&\theta_{04} = -15-26-37-48, && \theta_{14} = -16+25+38-47, &&\\
&\theta_{24} = -17-28+35+46, && \theta_{34} = -18+27-36+45. &&
\end{aligned}$$

Thus, the second coefficient of the characteristic polynomial of their skew-symmetric matrix $\theta = (\theta_{\alpha\beta})$ turns out to be:

$$\tau_2(\theta) = \sum_{\alpha<\beta} \theta^2_{\alpha\beta} = -12\,1234 - 4\,1256 - 4\,1357 + 4\,1368 - 4\,1278 - 4\,1467 - 4\,1458 + \star = -2\Omega_L,$$

where $\star$ denotes the Hodge star of what appears before it, and where

$$\Omega_L = \omega^2_{L^{\mathbb{H}}_i} + \omega^2_{L^{\mathbb{H}}_j} + \omega^2_{L^{\mathbb{H}}_k}$$

is the left quaternionic 4-form.

One can check that matrices $B = \left(\begin{array}{c|c} B' & B'' \\ \hline B''' & B'''' \end{array}\right) \in \mathrm{SO}(8)$ commuting with each of the Q_α are again the ones satisfying $B'' = B''' = 0$ and $B' = B'''' \in \mathrm{Sp}(1) \subset \mathrm{SO}(4)$. Hence the stabilizer of all individual Q_α is the diagonal $\mathrm{Sp}(1)_\Delta \subset \mathrm{SO}(8)$, and the stabilizer of their spanned vector space E^5 is the quaternionic group $\mathrm{Sp}(2)\cdot\mathrm{Sp}(1) \subset \mathrm{SO}(8)$. Thus:

PROPOSITION 2.3. *The vector space E^5 spanned by the Clifford system $C_4 = (Q_0, Q_1, Q_2, Q_3, Q_4)$ gives rise to the quaternion Hermitian structure of $\mathbb{R}^8$, and it is therefore equivalent to the datum either of the reduction to* $\mathrm{Sp}(2)\cdot\mathrm{Sp}(1) \subset \mathrm{SO}(8)$ *of its structure group, or to the quaternionic 4-form Ω_L. The complex structures $Q_{\alpha\beta} = Q_\alpha Q_\beta$ are for* $\alpha < \beta$ *a basis of the Lie subalgebra* $\mathfrak{sp}(2) \subset \mathfrak{so}(8)$.

REMARK 2.1. As mentioned in the Introduction, when $m \equiv 0 \bmod 4$, there are two equivalence classes of Clifford systems. It is clear from the construction of C_4 that a representative of the other class is just $\widetilde{C}_4 = (Q_0, \widetilde{Q}_1, \widetilde{Q}_2, \widetilde{Q}_3, Q_4)$, where:

$$\widetilde{Q}_1 = \left(\begin{array}{c|c} 0 & -L^{\mathbb{H}}_i \\ \hline L^{\mathbb{H}}_i & 0 \end{array}\right), \qquad \widetilde{Q}_2 = \left(\begin{array}{c|c} 0 & -L^{\mathbb{H}}_j \\ \hline L^{\mathbb{H}}_j & 0 \end{array}\right), \qquad \widetilde{Q}_3 = \left(\begin{array}{c|c} 0 & -L^{\mathbb{H}}_k \\ \hline L^{\mathbb{H}}_k & 0 \end{array}\right).$$

3. Statement on how to write new Clifford systems and representation theory

The Clifford systems C_3 and C_4 have been obtained from C_2 through the following procedure. Similarly for the step $C_1 \to C_2$.

THEOREM 3.1. *[Procedure to write new Clifford systems from old] Let $C_m = (P_0, P_1, \dots, P_m)$ be the last (or unique) Clifford system in $\mathbb{R}^N$. Then the first (or unique) Clifford system*

$$C_{m+1} = (Q_0, Q_1, \dots, Q_m, Q_{m+1})$$

in $\mathbb{R}^{2N}$ has as first and as last endomorphisms respectively

$$Q_0 = \left(\begin{array}{c|c} 0 & \mathrm{Id} \\ \hline \mathrm{Id} & 0 \end{array}\right), \qquad Q_{m+1} = \left(\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline 0 & -\mathrm{Id} \end{array}\right),$$

where the blocks are $N \times N$. The remaining matrices are

$$Q_\alpha = \left(\begin{array}{c|c} 0 & -P_{0\alpha} \\ \hline P_{0\alpha} & 0 \end{array}\right) \qquad \alpha = 1, \dots, m.$$

Here $P_{0\alpha}$ are the complex structures given by compositions $P_0 P_\alpha$ in the Clifford system C_m. When the complex structures $P_{0\alpha}$ can be viewed as (possibly block-wise) right multiplications by some of the unit quaternions i, j, k or unit octonions i, j, k, e, f, g, h, and if the dimension permits, similarly defined further endomorphisms Q_β can be added by using some others among i, j, k or i, j, k, e, f, g, h.

Proof. Since $P_{0\alpha} P_{0\alpha} = -\mathrm{Id}$, it is straightforward that C_{m+1} is a Clifford system. As for the statement concerning the further Q_β, its proof follows as in the cases of C_4 (already seen), the further cases of C_6, C_7, C_8 in the next Section, and of $C_{12}, C_{14,}, C_{15}, C_{16}$ in Section 7. □

We now discuss some aspects of Clifford systems and of even Clifford structures (defined in the Introduction) related with representation theory of Clifford algebras. As pointed out in in [6, pages 482–483], any irreducible Clifford system $C_m = (P_0, \dots, P_m)$ in $\mathbb{R}^N$, $N = 2\delta(m)$, gives rise to an irreducible representation of the Clifford algebra $\mathrm{Cl}_{0,m-1}$ in $\mathbb{R}^{\delta(m)}$. This latter is given by skew-symmetric matrices

$$E_1, \dots, E_{m-1} \in \mathfrak{so}(\delta(m))$$

satisfying

$$E_\alpha E_\beta + E_\beta E_\alpha = -2\delta_{\alpha\beta} I.$$

To get such matrices E_α from C_m consider the $\delta(m)$-dimensional subspace E_+ of $\mathbb{R}^N$ defined as the $(+1)$-eigenspace of P_0, and observe that E_+ is also invariant under the compositions $P_1 P_{\alpha+1}$. Then define the skew-symmetric endomorphisms on $\mathbb{R}^{\delta(m)}$

$$E_\alpha = P_\alpha P_{\alpha+1}|_{E_+} \quad (\alpha = 1, \dots, m-1).$$

This gives the system of $E_1, \dots, E_{m-1} \in \mathfrak{so}(\delta(m))$, thus a representation of the Clifford algebra $\mathrm{Cl}_{0,m-1}$ in $\mathbb{R}^{\delta(m)}$. Conversely, given the anti-commuting $E_1, \dots, E_{m-1} \in \mathfrak{so}(\delta(m))$, define on $\mathbb{R}^N$ the Clifford system C_m given by the symmetric involutions

$$P_0(u, v) = (v, u), \dots, P_\alpha(u, v) = (-E_\alpha v, E_\alpha u), \dots, P_m(u, v) = (u, -v).$$

As a consequence, the procedure given by Theorem 3.1 can be seen as rephrasing the way to get irreducible representations of Clifford algebras. For these latter one can see [12, pages 30–40], and more explicitly the construction of Clifford algebra representations in [19, pages 18–20].

REMARK 3.1. An alternative aspect of Clifford systems is to look at C_m in $\mathbb{R}^N$, $N = 2\delta(m)$, as a representation of the Clifford algebra $\mathrm{Cl}_{0,m+1}$ in $\mathbb{R}^N$ such that any vector of the pseudo-euclidean $\mathbb{R}^{0,m+1} \subset \mathrm{Cl}_{0,m+1}$ acts as a symmetric endomorphism in $\mathbb{R}^N$.

Recall now from the structure of Clifford algebras the following periodicity relations

$$\mathrm{Cl}_{m+8} \cong \mathrm{Cl}_m \otimes \mathbb{R}(16), \qquad \mathrm{Cl}_{0,m+8} \cong \mathrm{Cl}_{0,m} \otimes \mathbb{R}(16),$$

where $\mathbb{R}(16)$ denotes the algebras of all real matrices of order 16.

Look now at the even Clifford structures, mentioned in the Introduction. First observe that a natural notion of irreducibility can be given for them, by requiring the Euclidean vector bundle (E,h) not to be a direct sum. Then, by definition an irreducible even Clifford structure of rank $m+1$ is equivalent to an irreducible representation of the even Clifford algebra $\mathrm{Cl}^0_{m+1} \cong \mathrm{Cl}_m$ in $\mathbb{R}^N$, $N = 2\delta(m)$, mapping $\Lambda^2\mathbb{R}^{m+1}$ into skew-symmetric endomorphisms of $\mathbb{R}^N$.

The mentioned representations are listed, for low values of m, in the following:

Table C: Representations of $\mathrm{Cl}_{0,m-1}$ in $\mathbb{R}^{\delta(m)}$ and of Cl^0_{m+1} in $\mathbb{R}^{2\delta(m)}$

m	1	2	3	4	5	6	7	8	9
$\delta(m)$	1	2	4	4	8	8	8	8	16
$\mathrm{Cl}_{0,m-1}$	$\mathbb{R}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
$2\delta(m)$	2	4	8	8	16	16	16	16	32
$\mathrm{Cl}^0_{m+1} \cong \mathrm{Cl}_m$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$

Of course, a Clifford system $C_m = (P_0, \dots, P_m)$ in $\mathbb{R}^N$ gives rise to an even Clifford structure on the same $\mathbb{R}^N$ just by requiring the vector bundle E to be the vector sub-bundle of the endomorphism bundle generated by $P_0, \dots, P_m$. Not every irreducible even Clifford structure can be obtained in this way, and not only by dimensional reasons, as we will see on manifolds, cf. Section 6. We call *essential* an irreducible even Clifford structure that is not induced by an irreducible Clifford system. Thus, to see whether an irreducible even Clifford structure is essential, Table C and the mentioned periodicity relation can be used. This gives the following:

PROPOSITION 3.1. *Irreducible even Clifford structures of rank $m+1$ on $\mathbb{R}^{2\delta(m)}$ are essential when $m \equiv 3,5,6,7$ mod. 8, and non essential when $m = 0,4$ mod 8. For*

$m = 1,2$ *mod* 8 *both possibilities are open.*

We will come back to this point on manifolds, see the last Sections.

4. From $\mathbb{H}$ to $\mathbb{O}$: the Clifford systems C_5, C_6, C_7, C_8

According to Table B and to Theorem 3.1, the next Clifford system to consider is

$$C_5 = (S'_0, S'_1, S'_2, S'_3, S'_4, S'_5)$$

in $\mathbb{R}^{16}$, where:

$$S'_0 = \left(\begin{array}{c|c} 0 & \mathrm{Id} \\ \hline \mathrm{Id} & 0 \end{array}\right), \quad S'_1 = \left(\begin{array}{c|c} 0 & -Q_{01} \\ \hline Q_{01} & 0 \end{array}\right), \quad S'_2 = \left(\begin{array}{c|c} 0 & -Q_{02} \\ \hline Q_{02} & 0 \end{array}\right),$$

$$S'_3 = \left(\begin{array}{c|c} 0 & -Q_{03} \\ \hline Q_{03} & 0 \end{array}\right), \quad S'_4 = \left(\begin{array}{c|c} 0 & -Q_{04} \\ \hline Q_{04} & 0 \end{array}\right), \quad S'_5 = \left(\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline 0 & -\mathrm{Id} \end{array}\right).$$

A computation shows that:

PROPOSITION 4.1. *The orthogonal transformations in $\mathbb{R}^{16}$ commuting with the individual $S'_0, \dots, S'_5$ are the ones in the diagonal $\mathrm{Sp}(1)_\Delta \subset \mathrm{SO}(16)$. The orthogonal transformations preserving the vector subspace $E^6 =< C_5 >\subset \mathrm{End}(\mathbb{R}^{16})$ are the ones in the subgroup $\mathrm{SU}(4) \cdot \mathrm{Sp}(1) \subset \mathrm{SO}(16)$.The complex structures $S'_{\alpha\beta} = S'_\alpha S'_\beta$ are for $\alpha < \beta$ a basis of a Lie subalgebra $\mathfrak{su}(4) \subset \mathfrak{so}(16)$.*

By reminding that

$$Q_{01} = R_i, \;\; Q_{02} = R_j, \;\; Q_{03} = R_k, \;\; Q_{04} = R_e,$$

are the right multiplications on $\mathbb{O}$ by i, j, k, e, one completes C_5 to the Clifford system

$$C_8 = (S_0, S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8)$$

with

$$S_0 = S'_0, \;\; S_1 = S'_1, \;\; S_2 = S'_2, \;\; S_3 = S'_3, \;\; S_4 = S'_4, \;\; S_8 = S'_5$$

and

$$S_5 = \left(\begin{array}{c|c} 0 & -R_f \\ \hline R_f & 0 \end{array}\right), \quad S_6 = \left(\begin{array}{c|c} 0 & -R_g \\ \hline R_g & 0 \end{array}\right), \quad S_7 = \left(\begin{array}{c|c} 0 & -R_h \\ \hline R_h & 0 \end{array}\right).$$

It is now natural to compare the Clifford system C_8 with the following notion, that was proposed by Th. Friedrich in [7].

DEFINITION 4.1. *A* Spin(9)-structure *on a 16-dimensional Riemannian manifold* (M,g) *is a rank* 9 *real vector bundle*

$$E^9 \subset \mathrm{End}(TM) \to M,$$

locally spanned by self-dual anti-commuting involutions $S_\alpha : TM \to TM$*:*

$$g(S_\alpha X, Y) = g(X, S_\alpha Y), \qquad \alpha = 1,\dots,9,$$
$$S_\alpha S_\beta = -S_\beta S_\alpha, \qquad \alpha \neq \beta,$$
$$S_\alpha^2 = \mathrm{Id}.$$

From this datum one gets on M local almost complex structures $S_{\alpha\beta} = S_\alpha S_\beta$, and the 9×9 skew-symmetric matrix of their Kähler 2-forms

$$\psi = (\psi_{\alpha\beta})$$

is naturally associated. The differential forms $\psi_{\alpha\beta}$, $\alpha < \beta$, are thus *a local system of Kähler* 2*-forms* on the Spin(9)-manifold (M^{16}, E^9).

On the model space $\mathbb{R}^{16}$, the standard Spin(9)-structure is defined by the generators $S_1,\dots,S_9$ of the Clifford algebra Cl_9, the endomorphisms' algebra of its 16-dimensional real representation $\Delta_9 = \mathbb{R}^{16} = \mathbb{O}^2$. Accordingly, unit vectors in $\mathbb{R}^9$ can be viewed as self-dual endomorphisms

$$w : \Delta_9 = \mathbb{O}^2 \to \Delta_9 = \mathbb{O}^2,$$

and the action of $w = u + r \in S^8$ ($u \in \mathbb{O}$, $r \in \mathbb{R}$, $u\overline{u} + r^2 = 1$), on pairs $(x,x') \in \mathbb{O}^2$ is given by

$$\begin{pmatrix} x \\ x' \end{pmatrix} \longrightarrow \begin{pmatrix} r & R_{\overline{u}} \\ R_u & -r \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}, \tag{4.1}$$

where $R_u, R_{\overline{u}}$ denote the right multiplications by $u, \overline{u}$, respectively (cf. [9, page 288]). The choices

$$w = (0,1),(0,i),(0,j),(0,k),(0,e),(0,f),(0,g),(0,h) \text{ and } (1,0) \in S^8 \subset \mathbb{O}\times\mathbb{R} = \mathbb{R}^9,$$

define the symmetric endomorphisms:

$$S_0, S_1, \dots, S_8$$

that constitute our Clifford system C_8.

The subgroup $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$ is characterized as preserving the vector subspace

$$E^9 =< S_0,\dots,S_8 >\subset \mathrm{End}(\mathbb{R}^{16}),$$

whereas it is easy to check that the only matrices in $\mathrm{SO}(16)$ that preserve all the individual involutions $S_0,\dots,S_8$ are just $\pm\mathrm{Id}$.

It is useful to have explicitly the following right multiplications on $\mathbb{O}$:

$$R_i = \left(\begin{array}{c|c} R^{\mathbb{H}}_i & 0 \\ \hline 0 & -R^{\mathbb{H}}_i \end{array}\right), \qquad R_j = \left(\begin{array}{c|c} R^{\mathbb{H}}_j & 0 \\ \hline 0 & -R^{\mathbb{H}}_j \end{array}\right), \qquad R_k = \left(\begin{array}{c|c} R^{\mathbb{H}}_k & 0 \\ \hline 0 & -R^{\mathbb{H}}_k \end{array}\right),$$

$$R_e = \left(\begin{array}{c|c} 0 & -\mathrm{Id} \\ \hline \mathrm{Id} & 0 \end{array}\right),$$

$$R_f = \left(\begin{array}{c|c} 0 & L^{\mathbb{H}}_i \\ \hline L^{\mathbb{H}}_i & 0 \end{array}\right), \qquad R_g = \left(\begin{array}{c|c} 0 & L^{\mathbb{H}}_j \\ \hline L^{\mathbb{H}}_j & 0 \end{array}\right), \qquad R_h = \left(\begin{array}{c|c} 0 & L^{\mathbb{H}}_k \\ \hline L^{\mathbb{H}}_k & 0 \end{array}\right).$$

The space $\Lambda^2\mathbb{R}^{16}$ of 2-forms in $\mathbb{R}^{16}$ decomposes under $\mathrm{Spin}(9)$ as [7, page 146]:

$$\Lambda^2\mathbb{R}^{16} = \Lambda^2_{36} \oplus \Lambda^2_{84} \tag{4.2}$$

where $\Lambda^2_{36} \cong \mathfrak{spin}(9)$ and Λ^2_{84} is an orthogonal complement in $\Lambda^2 \cong \mathfrak{so}(16)$. Bases of the two subspaces are given respectively by

$$S_{\alpha\beta} = S_\alpha S_\beta \qquad \text{if } \alpha < \beta \qquad \text{and} \qquad S_{\alpha\beta\gamma} = S_\alpha S_\beta S_\gamma \qquad \text{if } \alpha < \beta < \gamma,$$

all complex structures on $\mathbb{R}^{16}$. We will need for later use the following ones:

$$S_{01} = \left(\begin{array}{c|c} R_i & 0 \\ \hline 0 & -R_i \end{array}\right), \; S_{02} = \left(\begin{array}{c|c} R_j & 0 \\ \hline 0 & -R_j \end{array}\right), \; S_{03} = \left(\begin{array}{c|c} R_k & 0 \\ \hline 0 & -R_k \end{array}\right), \; S_{04} = \left(\begin{array}{c|c} R_e & 0 \\ \hline 0 & -R_e \end{array}\right),$$

$$S_{05} = \left(\begin{array}{c|c} R_f & 0 \\ \hline 0 & -R_f \end{array}\right), \; S_{06} = \left(\begin{array}{c|c} R_g & 0 \\ \hline 0 & -R_g \end{array}\right), \; S_{07} = \left(\begin{array}{c|c} R_h & 0 \\ \hline 0 & -R_h \end{array}\right), \; S_{08} = \left(\begin{array}{c|c} 0 & -\mathrm{Id} \\ \hline \mathrm{Id} & 0 \end{array}\right).$$

Via invariant polynomials, one can then get global differential forms on manifolds M^{16}, and prove the following facts, completing some of the statements already proved in [15]:

THEOREM 4.1. *(i) The families of complex structures*

$$S^A = \{S_{\alpha\beta}\}_{1\leqslant\alpha<\beta\leqslant 7}, \qquad S^B = \{S_{\alpha\beta}\}_{0\leqslant\alpha<\beta\leqslant 7}, \qquad \textit{and} \;\; S^C = \{S_{\alpha\beta}\}_{0\leqslant\alpha<\beta\leqslant 8},$$

provide bases of Lie subalgebras $\mathfrak{spin}_\Delta(7)$, $\mathfrak{spin}(8)$, *and* $\mathfrak{spin}(9) \subset \mathfrak{so}(16)$, *respectively.*
(ii) Let

$$\psi^A = (\psi_{\alpha\beta})_{1\leqslant\alpha,\beta\leqslant 7}, \qquad \psi^B = (\psi_{\alpha\beta})_{0\leqslant\alpha,\beta\leqslant 7}, \qquad \psi^C = (\psi_{\alpha\beta})_{0\leqslant\alpha,\beta\leqslant 8}$$

be the skew-symmetric matrices of the Kähler 2-forms associated with the mentioned families of complex structures $S_{\alpha\beta}$. *If* τ_2 *and* τ_4 *are the second and fourth coefficient of the characteristic polynomial, then:*

$$\frac{1}{6}\tau_2(\psi^A) = \Phi_{\mathrm{Spin}_\Delta(7)}, \quad \frac{1}{4}\tau_2(\psi^B) = \Phi_{\mathrm{Spin}(8)}, \quad \tau_2(\psi^C) = 0, \quad \frac{1}{360}\tau_4(\psi^C) = \Phi_{\mathrm{Spin}(9)},$$

where $\Phi_{\mathrm{Spin}_\Delta(7)} \in \Lambda^4(\mathbb{R}^{16})$ *restricts on any summand of* $\mathbb{R}^{16} = \mathbb{R}^8 \oplus \mathbb{R}^8$ *to the usual* Spin(7) *4-form, and where* $\Phi_{\mathrm{Spin}(9)} \in \Lambda^8(\mathbb{R}^{16})$ *is the canonical form associated with the standard* Spin(9)*-structure in* $\mathbb{R}^{16}$.

The 8-form $\Phi_{\mathrm{Spin}(9)}$ was originally defined by M. Berger in 1972, cf. [2].

Proof. (i) The three families refer to Lie subalgebras of $\mathfrak{spin}(9)$. Now, the family $S^C = \{S_{\alpha\beta}\}_{0\leqslant\alpha<\beta\leqslant 8}$ is known to be a basis of $\mathfrak{spin}(9)$, cf. [7, 15]. Look at the construction of the $S_{\alpha\beta} = S_\alpha S_\beta$, following the approach to Spin(9) as generated by transformations of type (4.1) (cf. [9, pages 278–279]). In this construction, matrices in the subalgebra $\mathfrak{spin}(8) \subset \mathfrak{spin}(9)$ are characterized through the infinitesimal triality principle as:

$$\begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix},$$

where $a_+, a_- \in \mathfrak{so}(8)$ are *triality companions*, i.e. for each $u \in \mathbb{O}$ there exists $v = a_0(u)$ such that $R_v = a_+ R_u a_-^t$. It is easily checked that all matrices $S_{\alpha\beta}$ with $0 \leqslant \alpha < \beta \leqslant 7$ satisfy this condition. Moreover, matrices in $\mathfrak{spin}_\Delta(7) \subset \mathfrak{spin}(8)$ are characterized as those with $a_+ = a_-$ (thus $a_0 = 1$) ([9, pages 278–279, 285]). Thus only the $S_{\alpha\beta}$ with $1 \leqslant \alpha < \beta \leqslant 7$ are in $\mathfrak{spin}_\Delta(7)$.
(ii) Here one can write explicit expressions of the $\psi_{\alpha\beta}$ in the coordinates of $\mathbb{R}^{16}$ (cf. [15, pages 334–335]). These formulas allow to compute the τ_2 and the τ_4 appearing in the statements. It is convenient to begin with the matrix ψ^B, by adding up squares of the 2-forms $\psi_{\alpha\beta}$ with $0 \leqslant \alpha < \beta \leqslant 7$:

$$\frac{1}{4}\tau_2(\psi^B) = \frac{1}{4}\sum_{0\leqslant\alpha<\beta\leqslant 7}\psi^2_{\alpha\beta} = 121'2' + 123'4' + 125'6' - 127'8' + 341'2' + 343'4' - 345'6'$$
$$+347'8' + 561'2' - 563'4' + 565'6' + 567'8' - 781'2' + 783'4' + 785'6' + 787'8' + 131'3' - 132'4'$$
$$+135'7' + 136'8' - 241'3' + 242'4' + 245'7' + 246'8' + 571'3' + 572'4' + 575'7' - 576'8' + 681'3'$$
$$+682'4' - 685'7' + 686'8' + 141'4' + 142'3' + 145'8' - 146'7' + 231'4' + 232'3' - 235'8' + 236'7'$$
$$+581'4' - 582'3' + 585'8' + 586'7' - 671'4' + 672'3' + 675'8' + 676'7' + 151'5' - 152'6' - 153'7'$$
$$-154'8' - 261'5' + 262'6' - 263'7' - 264'8' - 371'5' - 372'6' + 373'7' - 374'8' - 481'5' - 482'6'$$
$$-483'7' + 484'8' + 161'6' + 162'5' - 163'8' + 164'7' + 251'6' + 252'5' + 253'8' - 254'7' - 381'6'$$
$$+382'5' + 383'8' + 384'7' + 471'6' - 472'5' + 473'8' + 474'7' + 171'7' + 172'8' + 173'5' - 174'6'$$
$$+281'7' + 282'8' - 283'5' + 284'6' + 351'7' - 352'8' + 353'5' + 354'6' - 461'7' + 462'8' + 463'5'$$
$$+464'6' + 181'8' - 182'7' + 183'6' + 184'5' - 271'8' + 272'7' + 273'6' + 274'5' + 361'8' + 362'7'$$
$$+363'6' - 364'5' + 451'8' + 452'7' - 453'6' + 454'5',$$

and this can be defined as $\Phi_{\mathrm{Spin}(8)}$. By computing the sum of squares of the $\psi_{\alpha\beta}$ with $1 \leqslant \alpha < \beta \leqslant 7$ one gets instead:

$$\begin{aligned}\tau_2(\psi^A) = \sum_{1\leqslant\alpha<\beta\leqslant 7}\psi^2_{\alpha\beta} &= \frac{6}{4}\tau_2(\psi^B) + 6\big[1234 + 5678 + 1'2'3'4' + 5'6'7'8'\big]\\ &- 3\big[15 + 26 + 37 + 48\big]^2 - 3\big[1'5' + 2'6' + 3'7' + 4'8'\big]^2\\ &- 6\big[1278 - 1368 + 1467 + 2358 - 2457 + 3456 + 1'2'7'8' - 1'3'6'8' + 1'4'6'7' + 2'3'5'8'\\ &\qquad - 2'4'5'7' + 3'4'5'6'\big].\end{aligned}$$

Thus, by defining $\Phi_{\mathrm{Spin}_\Delta(7)} = \frac{1}{6}\tau_2(\psi^A)$ one has that its restriction to any of the two summands $\mathbb{R}^{16} = \mathbb{R}^8 \oplus \mathbb{R}^8$ is the usual Spin(7) form (cf. [15, pages 332–333]). Moreover, computations carried out in [15, pages 336 and 343] show that:

$$\tau_2(\psi^C) = \sum_{1\leqslant\alpha<\beta\leqslant 9}\psi^2_{\alpha\beta} = 0, \qquad \Phi_{\mathrm{Spin}(9)} = \frac{1}{360}\tau_4(\psi^C).$$

□

The coefficients in the above equalities are chosen in such a way that, when reading

$$\Phi_{\mathrm{Spin}_\Delta(7)}, \qquad \Phi_{\mathrm{Spin}(8)} \in \Lambda^4, \qquad \Phi_{\mathrm{Spin}(9)} \in \Lambda^8$$

in the coordinates of $\mathbb{R}^{16}$, the g.c.d. of coefficients be 1.

Theorem 4.1 suggests to consider, besides the Clifford systems C_5 and C_8 on $\mathbb{R}^{16}$, the following intermediate Clifford systems:

$$C_6 = (S_1, \dots, S_7), \qquad C_7 = (S_0, \dots, S_7),$$

or, in accordance with Theorem 3.1, the equivalent Clifford systems

$$C'_6 = (S_0, S_1, \dots, S_5, S_8), \qquad C'_7 = (S_0, S_1, \dots, S_6, S_8).$$

REMARK 4.1. Similarly to what observed in Remark 2.1, a representative of the other equivalence class of Clifford systems with $m = 8$ can be constructed as $\widetilde{C}_8 = (S_0, \widetilde{S}_1, \dots, \widetilde{S}_7, S_8)$, where $\widetilde{S}_1, \dots, \widetilde{S}_7$ are defined like $S_1, \dots, S_7$ but using the left octonion multiplications $L_i, \dots, L_h$ instead of the right ones $R_i, \dots, R_h$.

REMARK 4.2. A family of calibrations $\phi_{4k}(\lambda) \in \Lambda^{4k}(\mathbb{R}^{16})$ has been constructed by J. Dadok and F. R. Harvey for any λ in the unit sphere $S^7 \subset \mathbb{R}^8$ by squaring positive spinors $S(\lambda) \in \mathbf{S}^+(16)$, through the following procedure, cf. [5]. Write $\mathbb{R}^{16}$ as $\mathbb{O}^+ \oplus \mathbb{O}^-$, and let $(s_1 = i, \dots, s_8 = h)$ be the standard basis of the octonions $\mathbb{O}^+, \mathbb{O}^-$. Then one looks at the model for the Clifford algebra

$$\Lambda(\mathbb{R}^{16}) \cong \mathrm{Cl}_{16} \cong \mathrm{Cl}_8 \otimes \mathrm{Cl}_8 \cong \mathrm{End}(\mathbb{O}^+ \oplus \mathbb{O}^-) \otimes \mathrm{End}(\mathbb{O}^+ \oplus \mathbb{O}^-) \cong$$
$$\mathrm{End}(\mathbb{O}^+ \otimes \mathbb{O}^+) \oplus \mathrm{End}(\mathbb{O}^- \otimes \mathbb{O}^-) \oplus \mathrm{End}(\mathbb{O}^+ \otimes \mathbb{O}^-) \oplus \mathrm{End}(\mathbb{O}^- \otimes \mathbb{O}^+),$$

and the spinors $S(\lambda)$ are in the diagonal $D \subset \mathbf{S}^+(16) = (\mathbb{O}^+ \otimes \mathbb{O}^+) \oplus (\mathbb{O}^- \otimes \mathbb{O}^-)$. Then it is proved in [5] that by squaring such spinors one gets inhomogeneous exterior forms in $\mathbb{R}^{16}$ as

$$256\, S(\lambda) \circ S(\lambda) = 1 + \phi_4 + \phi_8 + \phi_{12} + vol,$$

where the ϕ_{4k} are calibrations. In particular, calibrations corresponding to Spin(7) and Spin(8) geometries are determined and discussed in [5]. This construction can be related with the present point of view in terms of Clifford systems, as we plan to show in a forthcoming work.

5. Clifford systems C_m on Riemannian manifolds

The definition 4.1 of a Spin(9) structure on a Riemannian manifold M^{16}, using locally defined Clifford systems C_8 on its tangent bundle, and yielding a rank 9 vector subbundle of the endomorphism bundle, suggests to give the following more general definition.

DEFINITION 5.1. *A* Clifford system C_m on a Riemannian manifold (M^N, g) *is the datum of a rank* $m+1$ *vector subbundle* $E^{m+1} \subset \mathrm{End}(TM)$ *locally generated by Clifford systems that are related in the intersections of trivializing open sets by matrices in* $\mathrm{SO}(m+1)$.

Some of the former statements, like Propositions 2.1, 2.2, 2.3, 4.1, and similar properties discussed for C_6, C_7, C_8 in Section 4, can be interpreted on Riemannian manifolds. One can then recognize that the datum of a Clifford system C_m on a Riemannian manifold M^N, $N = 2\delta(m)$, is equivalent to the reduction of its structure group to the group G according to the following Table D.

Table D: Clifford systems C_m and G-structures on Riemannian manifolds M^N

m	1	2	3	4	5	6	7	8	9	10	11	12
N	2	4	8	8	16	16	16	16	32	64	128	128
G	U(1)	U(2)	$\mathrm{Sp}(1)^3$	Sp(2)Sp(1)	SU(4)Sp(1)	Spin(7)U(1)	Spin(8)	Spin(9)	Spin(10)	Spin(11)	Spin(12)	Spin(13)

REMARK 5.1. Although $\mathrm{Spin}(7)$ structures on 8-dimensional Riemannian manifolds cannot be described through a rank 7 vector bundle of symmetric endomorphisms of the tangent bundle (cf. [15, Corollary 9]), this can definitely be done for a $\mathrm{Spin}_\Delta(7)$ structure in dimension 16. The above discussion shows in fact that the 7 symmetric endomorphisms $S_1,\dots,S_7$ allow to deal with a $\mathrm{Spin}_\Delta(7)$ structure as a Clifford system on a 16-dimensional Riemannian manifold. Indeed, most of the known examples of $\mathrm{Spin}(9)$ manifolds carry the subordinated structure $\mathrm{Spin}_\Delta(7)$, cf. [7, 14].

On the other hand, a $\mathrm{Spin}(7)$ structure on a 8-dimensional Riemannian manifold is an example of even Clifford structure, as defined in the Introduction. Here the defining vector bundle E has rank 7 and one can choose $E \subset \mathrm{End}(TM)^-$ locally spanned as

$$< \mathcal{I},\mathcal{J},\mathcal{K},\mathcal{E},\mathcal{F},\mathcal{G},\mathcal{H} >,$$

i.e. multiplying by elements in the canonical basis of octonions. As already mentioned, in situations like this, we call *essential* the even Clifford structure.

6. The Clifford system C_9 and the essential Clifford structure on EIII

Our recipe for producing Clifford systems, according to Theorem 3.1, gives on $\mathbb{R}^{32}$ the following Clifford system $C_9 = (T_0, T_1, \dots, T_9)$:

$$T_0 = \left(\begin{array}{c|c} 0 & \mathrm{Id} \\ \hline \mathrm{Id} & 0 \end{array}\right), \dots, T_\alpha = \left(\begin{array}{c|c} 0 & -S_{0\alpha} \\ \hline S_{0\alpha} & 0 \end{array}\right), \dots, T_9 = \left(\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline 0 & -\mathrm{Id} \end{array}\right).$$

Here $\alpha = 1,\dots,8$ and any block in the matrices is now of order 16.

In [17] we showed that the group of orthogonal transformations preserving the vector subspace $E^{10} =< C_9 >\subset \mathrm{End}(\mathbb{R}^{32})$ is the image of $\mathrm{Spin}(10)$ under a real representation in SO(32). Indeed, one can also look at the half-Spin representations of $\mathrm{Spin}(10)$ into $\mathrm{SU}(16) \subset \mathrm{SO}(32)$, that are related with the notion of *even Clifford structure*, as defined in the Introduction.

Note that, according to the definitions, any Clifford system C_m on a Riemannian manifold M^N gives rise to an even Clifford structure of rank $m+1$: this is for example the case of C_4 on 8-dimensional quaternion Hermitian manifolds, or of C_8

on 16-dimensional Spin(9) manifolds. Indeed, one has also a notion of *parallel* even Clifford structure, requiring the existence of a metric connection ∇ on (E,h) such that φ is connection preserving. Thus for example parallel even Clifford structures with $m = 4,8$ correspond to a quaternion Kähler structure in dimension 8 and to holonomy Spin(9) in dimension 16. In [13, page 955], a classification is given of complete simply connected Riemannian manifolds with a parallel non-flat even Clifford structure.

This classification statement includes one single example for each value of the rank $m+1 = 9,10,12,16$ (and no other examples when $m+1 > 8$). These examples are the ones in the last row (or column) of Table A, namely the projective planes over the four algebras $\mathbb{O}$, $\mathbb{C}\otimes\mathbb{O}$, $\mathbb{H}\otimes\mathbb{O}$, $\mathbb{O}\otimes\mathbb{O}$.

In the Cartan labelling they are the symmetric spaces:

$$\mathrm{FII} \;= \mathrm{F}_4/\mathrm{Spin}(9), \qquad \mathrm{EIII} = \mathrm{E}_6/\mathrm{Spin}(10)\cdot\mathrm{U}(1),$$
$$\mathrm{EVI} \;= \mathrm{E}_7/\mathrm{Spin}(12)\cdot\mathrm{Sp}(1), \quad \mathrm{EVIII} = \mathrm{E}_8/\mathrm{Spin}(16)^+.$$

In this respect we propose the following

DEFINITION 6.1. *Let M be a Riemannian manifold. An even Clifford structure (E,h), with rank $m+1$ and defining map $\varphi:\,\mathrm{Cl}_0(E)\to\mathrm{End}(TM)$, is said to be* essential *if it is not a Clifford system, i.e. if it is not locally spanned by anti-commuting self-dual involutions.*

We have already seen that Spin(7) structures in dimension 8 are examples of essential even Clifford structures, cf. Remark 4.1. As mentioned, both quaternion Hermitian structures in dimension 8 and Spin(9) structures in dimension 16 are instead non-essential. For example, on the Cayley plane FII, local Clifford systems on its three coordinate open affine planes $\mathbb{O}^2$ fit together to define the Spin(9) structure and hence the even Clifford structure. This property has no analogue for the other three projective planes EIII, EVI, EVIII. As a matter of fact it has been proved in [11] that the projective plane EIII over complex octonions cannot be covered by three coordinate open affine planes $\mathbb{C}\otimes\mathbb{O}^2$. We have:

THEOREM 6.1. *The parallel even Clifford structure on* EIII *is essential.*

Proof. Note first that the statement cannot follow from Proposition 3.1. However, as observed in Table D, the structure group of a 32-dimensional manifold carrying a Clifford system C_9 reduces to $\mathrm{Spin}(10)\subset\mathrm{SU}(16)$. This would be the case of the holonomy group, assuming that such a Clifford system induces the parallel even Clifford structure of EIII. Thus, EIII would have a trivial canonical bundle, in contradiction with the positive Ricci curvature property of Hermitian symmetric spaces of compact type. □

As showed in [17], the vector bundle defining the even Clifford structure is the E^{10} locally spanned as $< I > \oplus < S_0,\dots,S_8 >$. Here I is the global complex structure of the Hermitian symmetric space EIII, and $S_0,\dots,S_8$ (matrices in $\mathrm{SO}(16)\subset\mathrm{SU}(16)$)

define, together with I, the $\mathrm{Spin}(10)\cdot\mathrm{U}(1)\subset\mathrm{U}(16)$ structure given by its holonomy. The construction of the even Clifford structure follows of course the alternating composition, so that $I\wedge S_\alpha = -S_\alpha\wedge I$, allowing to get a skew symmetric matrix $\psi^D = (\psi_{\alpha\beta})$ of Kähler forms associated with all compositions of two generators (cf. Theorem 4.1).

REMARK 6.1. As proved in [17], the cohomology classes of the Kähler form and of the global differential form $\tau_4(\psi^D)$ generate the cohomology of EIII.

7. Clifford systems C_{10} up to C_{16}

For the next step, we need now the following order 32 matrices:

$$T_{01} = \left(\begin{array}{c|c|c|c} R_i & 0 & 0 & 0 \\ \hline 0 & -R_i & 0 & 0 \\ \hline 0 & 0 & -R_i & 0 \\ \hline 0 & 0 & 0 & R_i \end{array}\right), \dots, T_{07} = \left(\begin{array}{c|c|c|c} R_h & 0 & 0 & 0 \\ \hline 0 & -R_h & 0 & 0 \\ \hline 0 & 0 & -R_h & 0 \\ \hline 0 & 0 & 0 & R_h \end{array}\right),$$

$$T_{08} = \left(\begin{array}{c|c|c|c} 0 & -\mathrm{Id} & 0 & 0 \\ \hline \mathrm{Id} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \mathrm{Id} \\ \hline 0 & 0 & -\mathrm{Id} & 0 \end{array}\right), \quad T_{09} = \left(\begin{array}{c|c|c|c} 0 & 0 & -\mathrm{Id} & 0 \\ \hline 0 & 0 & 0 & -\mathrm{Id} \\ \hline \mathrm{Id} & 0 & 0 & 0 \\ \hline 0 & \mathrm{Id} & 0 & 0 \end{array}\right),$$

and now we can write on $\mathbb{R}^{64}$ the following matrices of the Clifford system C_{10}:

$$U_0 = \left(\begin{array}{c|c} 0 & \mathrm{Id} \\ \hline \mathrm{Id} & 0 \end{array}\right), \dots, U_\alpha = \left(\begin{array}{c|c} 0 & -T_{0\alpha} \\ \hline T_{0\alpha} & 0 \end{array}\right), \dots, U_{10} = \left(\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline 0 & -\mathrm{Id} \end{array}\right).$$

Here $\alpha = 1,\dots,9$ and any block in the matrices is of order 32.

The subgroup of $\mathrm{SO}(64)$ preserving the subbundle $E^{11} =< C_{10} >\subset \mathrm{End}(\mathbb{R}^{64})$ is now $\mathrm{Spin}(11)$, a subgroup of $\mathrm{Sp}(16)\subset\mathrm{SO}(64)$ under its Spin representation.

Still another step, through

$$U_{01} = \left(\begin{array}{c|c} T_{01} & 0 \\ \hline 0 & -T_{01} \end{array}\right), \dots, U_{09} = \left(\begin{array}{c|c} T_{09} & 0 \\ \hline 0 & -T_{09} \end{array}\right), U_{0,10} = \left(\begin{array}{c|c} 0 & -\mathrm{Id} \\ \hline \mathrm{Id} & 0 \end{array}\right),$$

we go to the first Clifford system C_{11} in $\mathbb{R}^{128}$. Its matrices are:

$$V'_0 = \left(\begin{array}{c|c} 0 & \mathrm{Id} \\ \hline \mathrm{Id} & 0 \end{array}\right), \dots, V'_\alpha = \left(\begin{array}{c|c} 0 & -U_{0\alpha} \\ \hline U_{0\alpha} & 0 \end{array}\right), \dots, V'_{11} = \left(\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline 0 & -\mathrm{Id} \end{array}\right),$$

now with $\alpha = 1, \dots, 10$ and any block of order 64.

To recognize the next Clifford system, C_{12} and again in $\mathbb{R}^{128}$, introduce the following matrices, of order 32:

$$\mathrm{Block}_{R_i^{\mathbb{H}}} = \left(\begin{array}{c|c|c|c} 0 & -\mathrm{Id} & 0 & 0 \\ \hline \mathrm{Id} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \mathrm{Id} \\ \hline 0 & 0 & -\mathrm{Id} & 0 \end{array}\right), \ \mathrm{Block}_{R_j^{\mathbb{H}}} = \left(\begin{array}{c|c|c|c} 0 & 0 & -\mathrm{Id} & 0 \\ \hline 0 & 0 & 0 & -\mathrm{Id} \\ \hline \mathrm{Id} & 0 & 0 & 0 \\ \hline 0 & \mathrm{Id} & 0 & 0 \end{array}\right), \ \mathrm{Block}_{L_k^{\mathbb{H}}} = \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & -\mathrm{Id} \\ \hline 0 & 0 & -\mathrm{Id} & 0 \\ \hline 0 & \mathrm{Id} & 0 & 0 \\ \hline \mathrm{Id} & 0 & 0 & 0 \end{array}\right),$$

block-wise extensions of matrices $R_i^{\mathbb{H}}, R_j^{\mathbb{H}}, L_k^{\mathbb{H}}$ considered in Section 2. We need also the further matrices, of order 64, block-wise extension of R_i, R_j, R_e, R_h:

$$\mathrm{Block}_{R_i} = \left(\begin{array}{c|c} \mathrm{Block}_{R_i^{\mathbb{H}}} & 0 \\ \hline 0 & -\mathrm{Block}_{R_i^{\mathbb{H}}} \end{array}\right), \qquad \mathrm{Block}_{R_j} = \left(\begin{array}{c|c} \mathrm{Block}_{R_j^{\mathbb{H}}} & 0 \\ \hline 0 & -\mathrm{Block}_{R_j^{\mathbb{H}}} \end{array}\right),$$

$$\mathrm{Block}_{R_e} = \left(\begin{array}{c|c} 0 & -\mathrm{Id} \\ \hline \mathrm{Id} & 0 \end{array}\right), \qquad \mathrm{Block}_{R_h} = \left(\begin{array}{c|c} 0 & \mathrm{Block}_{L_k^{\mathbb{H}}} \\ \hline \mathrm{Block}_{L_k^{\mathbb{H}}} & 0 \end{array}\right).$$

Then one easily writes the last matrices in the Clifford system C_{11} as:

$$V'_8 = \left(\begin{array}{c|c} 0 & -\mathrm{Block}_{R_i} \\ \hline \mathrm{Block}_{R_i} & 0 \end{array}\right), \ V'_9 = \left(\begin{array}{c|c} 0 & -\mathrm{Block}_{R_j} \\ \hline \mathrm{Block}_{R_j} & 0 \end{array}\right), \ V'_{10} = \left(\begin{array}{c|c} 0 & -\mathrm{Block}_{R_e} \\ \hline \mathrm{Block}_{R_e} & 0 \end{array}\right).$$

One gets:

PROPOSITION 7.1. *The matrices*

$$V_0 = V'_0, \dots, V_{10} = V'_{10}, V_{11} = \left(\begin{array}{c|c} 0 & -\mathrm{Block}_{R_h} \\ \hline \mathrm{Block}_{R_h} & 0 \end{array}\right), V_{12} = V'_{11}$$

give rise to the Clifford system C_{12} in $\mathbb{R}^{128}$.

Proof. The only point to check is that V_{11} anti-commutes with all the other matrices. This is a straightforward computation. □

The orthogonal transformations preserving C_{12} correspond to a real representation of Spin(12) in SO(128).

As a further step, we construct C_{13}, the first Clifford system in $\mathbb{R}^{256}$, whose involutions are:

$$W'_0 = \left(\begin{array}{c|c} 0 & \mathrm{Id} \\ \hline \mathrm{Id} & 0 \end{array}\right), \dots, W'_\alpha = \left(\begin{array}{c|c} 0 & -V_{0\alpha} \\ \hline V_{0\alpha} & 0 \end{array}\right), \dots, W'_{13} = \left(\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline 0 & -\mathrm{Id} \end{array}\right),$$

now with $\alpha = 1,\dots,12$. In particular

$$W'_{12} = \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & \mathrm{Id} \\ \hline 0 & 0 & -\mathrm{Id} & 0 \\ \hline 0 & -\mathrm{Id} & 0 & 0 \\ \hline \mathrm{Id} & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} 0 & -\mathrm{Block}^{128}_{R_e} \\ \hline \mathrm{Block}^{128}_{R_e} & 0 \end{array}\right),$$

when now the block matrix is or order 128. Then one recognizes that one can add three similar block matrices with $\mathrm{Block}^{128}_{R_i}, \mathrm{Block}^{128}_{R_j}, \mathrm{Block}^{128}_{L_h}$, extending C_{13} up to C_{16}, still on $\mathbb{R}^{256}$, and with intermediate Clifford systems C_{14} and C_{15}.

8. The symmetric spaces EVI and EVIII

As a consequence of Proposition 3.1 we have:

THEOREM 8.1. *The parallel even Clifford structures on* EVI *and on* EVIII *are essential.*

These even Clifford structures can in fact be defined by vector subbundles E^{12} and E^{16} of the endomorphism bundle, locally generated as follows:

(8.1) $$E^{12} :< \mathcal{I},\mathcal{J},\mathcal{K} > \oplus < S_0,\dots,S_9 > \longrightarrow \text{EVI},$$

(8.2) $$E^{16} :< \mathcal{I},\mathcal{J},\mathcal{K},\mathcal{E},\mathcal{F},\mathcal{G},\mathcal{H} > \oplus < S_0,\dots,S_9 > \longrightarrow \text{EVIII},$$

where $S_0,\dots,S_9$ are the involutions in SO(16) defining the Spin(9) structures.

Note that the quaternionic structure of EVI, one of the quaternion Kähler Wolf spaces, appears as part of its even Clifford structure.

As already mentioned concerning EIII, also on EVI and EVIII the compositions of generators of the even Clifford structure follows the alternating property e.g. $\mathcal{I} \wedge S_\alpha = -S_\alpha \wedge \mathcal{I}$. In this way one still has skew-symmetric matrices of Kähler forms associated with compositions of two generators (cf. Theorem 4.1 and Theorem 6.1).

We denote these skew-symmetric matrices by ψ^E for EVI and ψ^F for EVIII. One can look at the following sequence of the matrices we introduced:

$$\psi^A \subset \psi^B \subset \psi^C \subset \psi^D \subset \psi^E \subset \psi^F,$$

all producing, via invariant polynomials, global differential forms associated with the associated structure groups

$$\mathrm{Spin}(7)_\Delta,\ \mathrm{Spin}(8),\ \mathrm{Spin}(9),\ \mathrm{Spin}(10)\cdot \mathrm{U}(1),\ \mathrm{Spin}(12)\cdot \mathrm{Sp}(1),\ \mathrm{Spin}(16)^+.$$

(cf. also Remark 6.1). We can mention here that the (rational) cohomology of EVI is generated, besides by the class of the quaternion Kähler 4-form, by a 8-dimensional class and by a 12-dimensional class. It is thus tempting to represent these classes by $\tau_4(\psi^E)$ and by $\tau_6(\psi^E)$. As for EVIII, it is known that its rational cohomology is generated by classes of dimension $8, 12, 16, 20$. One can also observe, in this last situation of EVIII, that the local Kähler forms $\psi_{\alpha\beta}$ associated with the group $\mathrm{Spin}(16)^+$ can be seen for $\alpha < \beta$ in correspondence with a basis of its Lie algebra $\mathfrak{so}(16)$. As such, they exhaust both families of $36+84 = 120$ exterior 2-forms appearing in decomposition 4.2.

We conclude with two remarks relating the discussed subjects with some of our previous work.

REMARK 8.1. In [16] we described a procedure to construct maximal orthonormal systems of tangent vector fields on spheres. For that we essentially used, besides multiplication in $\mathbb{C}, \mathbb{H}, \mathbb{O}$, the Spin(9) structure of $\mathbb{R}^{16}$, applied also block-wise in higher dimension. Remind that the maximal number $\sigma(N)$ of linear independent vector field on an odd-dimensional sphere S^{N-1}, with $N = (2k+1)2^p 16^q$ and $0 \leqslant p \leqslant 3$, is given by the Hurwitz-Radon formula

$$\sigma(N) = 2^p + 8q - 1.$$

Thus, it does not appear easy to read this number out of Table B, even considering also reducible Clifford systems.

On the other hand, one can recognize from the construction of [16] that there is instead a simple relation with even Clifford structures, and that for example the construction of a maximal system of tangent vector fields on spheres S^{31}, S^{63}, S^{127} can be rephrased using the essential even Clifford structures of rank $10, 12, 16$ on $\mathbb{R}^{32}, \mathbb{R}^{64}, \mathbb{R}^{128}$. Such even Clifford structures exist and are parallel non-flat on the symmetric spaces EIII, EVI, EVIII (cf. proof of Theorem 6.1, and equations (8.1), (8.2)). Following [16], this point of view can be suitably applied to spheres of any odd dimension.

REMARK 8.2. In [14] we studied the structure of compact locally conformally parallel Spin(9) manifolds. They are of course examples, together with their Kähler, quaternion Kähler, and Spin(7) counterparts, of manifolds equipped with a *locally conformally parallel even Clifford structure*. We can here observe that the following Hopf manifolds

$$S^{31} \times S^1,\ S^{63} \times S^1,\ S^{127} \times S^1$$

are further examples of them, with the locally conformally flat metric coming from their universal covering. One can also describe some finite subgroups of $\mathrm{Spin}(10)$, $\mathrm{Spin}(12)$, $\mathrm{Spin}(16)^+$ acting freely on S^{31}, S^{63}, S^{127}, respectively, and the list of groups K mentioned in Example 6.6 of [14] certainly applies to these three cases. Accordingly, finite quotients like $(S^{N-1}/K) \times S^1$, with $N = 32, 64, 128$, still carry a locally conformally parallel even Clifford structure. Note however that the structure Theorem C proved in [14] cannot be reproduced for these higher rank locally conformally parallel even Clifford structures.

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