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SUBMANIFOLDS AND HOLONOMY: A TRIBUTE TO SERGIO CONSOLE

Abstract. This is an expository article about our joint published research with Sergio Console.

1. Introduction.

In this expository article we will refer to our joint published research with Sergio Console and his related publications. We will put this in the perspective of future developments on the subjects covered: submanifold geometry and homogeneous Riemannian geometry. Most of our joint work was on the first subject and is related to the so-called normal holonomy. This is explained in our joint book, together with Jürgen Berndt, that inspired our title. In the last part of this survey we will explain in detail one of our last papers which is related to homogeneous Riemannian geometry. We have chosen this article because the subjects involved are very general, non-technical and would be of a wide interest for geometers.

Sergio has also done very interesting research in other areas on which I am not able to give an authoritative review (and I hope that other coauthors of his will comment on it).

2. The isoparametric rank rigidity

In this section we will refer to the article [CO1], where the so-called (local) *isoparametric rank*, $rank^i(M)$, of a submanifold M^n of the Euclidean space $\mathbb{R}^{n+k}$ is defined. Namely, $rank^i(M)$ is the maximal number of linearly independent, locally defined, isoparametric parallel normal vector fields. Let us say that a normal field ξ is called isoparametric if A_ξ has constant eigenvalues, where A is the shape operator of M. The main result is the following:

THEOREM 2.1. *Let $M^n \subset \mathbb{R}^{n+k}$ be a locally irreducible and full submanifold which is contained in a sphere. If $rank^i(M) \geqslant 2$ then M is a submanifold with constant principal curvatures.*

Observe, from the well-known theorem of Thorbergsson [Th] and the classification of Dadok [Da], that M is either an inhomogeneous isoparametric submanifold of the sphere or an orbit of an s-representation (i.e. the isotropy representation of a semisimple symmetric space).

The results in this paper, that uses normal holonomy of focal manifolds, were the starting point of some important results as the general higher rank rigidity theorem for submanifolds [DO] and the Berger-type theorem for complex projective submanifolds [CDO2] (that I will comment on later).

For "genuine" submanifolds of hyperbolic space the isoparametric rank is always trivial [OW].

3. Algebraically constant second fundamental form and isoparametric hypersurfaces

We will refer to the article [CO2]. In particular we give a conceptual proof of the classification of Cartan of isoparametric hypersurfaces of the sphere with three principal curvatures.

As a corollary of the paper [CO1] one has the following

COROLLARY 3.1. *Let M^n be a submanifold of the sphere S^N with a non-zero parallel mean curvature vector. Assume that the second fundamental form is algebraically constant. Then M is either an inhomogeneous isoparametric submanifold of the sphere or an orbit of an s-representation.*

Observe, since any compact Lie subgroup of $SO(N+1)$ has a minimal orbit, the above result is not true if the mean curvature is zero. The main result of [CO2] is to extend, of course under further assumptions, the above corollary and to apply it to isoparametric hypersurfaces of the sphere.

THEOREM 3.1. *Let M be a submanifold of a space of constant curvature X_κ which has a parallel (possibly zero) mean curvature vector. Assume that the second fundamental form is algebraically constant and it is the same as that of a submanifold N of X_κ with parallel second fundamental form. Then M has a parallel second fundamental form (and so M, up to an isometry of the ambient space, is an open part of N).*

The proof of this result depends on a Simons, Chern, doCarmo, Kobayashi type formula for the Laplacian of the length of the second fundamental form.

From the above result and the classification of Clifford systems we obtain a short alternative proof of the following result:

COROLLARY 3.2. *(Cartan). Let M be a compact isoparametric hypersurface of the sphere with three distinct principal curvatures. Then M is a tube around the Veronese embeddings of the real, complex, quaternionic projective spaces or the Cayley projective plane (in particular, M is homogeneous and so an orbit of a rank two irreducible s-representation).*

4. A Berger type theorem for the normal holonomy

In [CD] Sergio Console and Antonio J. Di Scala computed the normal holonomy of parallel complex submanifolds of the complex projective space (which correspond to the complex orbit, in the projectivized space, of a Hermitian s-representation). They made a conjecture that was proved in [CDO2], the so-called Berger type theorem for the normal holonomy of complex projective submanifolds. Namely,

THEOREM 4.1. *Let M be a full and complete complex projective submanifold of $\mathbb{C}P^n$. Then the following are equivalent:*

1) *The normal holonomy is not transitive on the unit sphere of the normal space.*
2) *M is the complex orbit, in the complex projective space, of the isotropy representation of a Hermitian symmetric space of rank greater than or equal to* 3.

It is also proved in this paper that the normal holonomy of a complex irreducible and full complete submanifold of $\mathbb{C}^n$ is always transitive. This is no longer true if we drop the completeness assumption.

Recently Antonio Di Scala and Francisco Vittone obtained a full characterization of the complex (non-complete) submanifolds of the projective space as Mok varieties.

The proof of the above theorem, though geometric, is very involved and uses most of the techniques that relate submanifolds and normal holonomy.

5. Killing fields as sections of the canonical bundle, Weyl scalar invariants and cohomogeneity

This section is based on [CO3, CO4]. We will explain the main techniques in the second reference which in particular imply also the results in the first reference.

Let M be a Riemannian manifold and let us consider the so-called *canonical bundle* $TM \oplus \Lambda^2(M)$ over M. We will always identify, by means of the Riemannian metric of M, $\Lambda^2(M)_p \simeq \mathfrak{so}(T_pM)$, for all $p \in M$. We endow E with the following connection $\tilde{\nabla}$, which depends on the Levi-Civita connection ∇ on M. If (v,B) is a section of E (i.e., v is a vector field and B is a skew-symmetric tensor field of type $(1,1)$ on M) then

$$\tilde{\nabla}_X(v,B) = (\nabla_X v - B.X, \nabla_X B - R_{X,v})$$

where R is the curvature tensor on M and, as usual, $(\nabla_X B).Y = \nabla_X(B.Y) - B.\nabla_X Y$.

The *canonical lift* of vector field Z on M is the section $\hat{Z}$ of E given by

$$\hat{Z}(p) = (Z(p), [(\nabla Z)_p]^{\mathrm{skew}})$$

where $[\ ]^{\mathrm{skew}}$ denotes the skew symmetric part.

The following result is well-known and elementary to show. Nevertheless, we include a proof of it (cf. also [BFP, Section 3.5.2]).

PROPOSITION 5.1. *The canonical lift gives an isomorphism between the set $\mathcal{K}(M)$ of Killing fields on M and the parallel sections of E with respect to $\widetilde{\nabla}$*

Proof. By the Killing equation, a vector field Z on M is a Killing field if and only if $(\nabla Z)_p$ is skew symmetric for all $p \in M$. Observe that in this case Z satisfies the affine Jacobi equation. Namely,

$$\nabla_X(\nabla Z) - R_{X,Z} = 0$$

for all X. This equation is derived from the fact that the flow associated to Z preserves the Levi-Civita connection (and using that ∇ is torsion free and the curvature tensor R satisfies the first Bianchi identity).

So, if Z is Killing then $\widehat{Z}$ is a parallel section of E. Conversely, if (v,B) is a parallel section of E, then the first component of $\widetilde{\nabla}(v,B) = 0$ implies that ∇v is skew-symmetric and hence v is a Killing field on M. □

A straightforward computation gives the curvature tensor $\widetilde{R}$ of E. The only things to use are the first Bianchi identity, for the first component of $\widetilde{R}$ and the second Bianchi identity for the second component. It yields

$$\widetilde{R}_{X,Y}(v,B) = (0, (\nabla_v R)_{X,Y} - (B.R)_{X,Y}) \qquad (*)$$

where B acts on R as a derivation.

The proof of the following result is straightforward and makes use of the so-called Ricci identity $\nabla^2_{X,Y}T - \nabla^2_{Y,X}T = R_{X,Y}.T$, where $R_{X,Y}$ acts as a derivation.

PROPOSITION 5.2. *Let T be a given tensor on M and let (v,B) be a section of E that satisfies the equation*

$$\nabla_v T = B.T$$

Then $\widetilde{\nabla}_X(v,B)$ satisfies the above equation, for all vector fields X on M, if and only if (v,B) also satisfies the following equation

$$\nabla_v(\nabla T) = B.(\nabla T)$$

5.1. Invariant Weyl tensors

A tensorial *Weyl invariant* is a tensor T that can be naturally defined in any Riemannian manifold M^n by means of the metric tensor $\langle\ ,\ \rangle$, the curvature tensor R and all of its covariant derivatives $\nabla^k R$. Equivalently, any coefficient T^I_J, $I = (i_1,\ldots,i_l)$, $J = (j_1,\ldots,j_s)$, of this tensor with respect to any orthonormal basis $e_1,\ldots,e_n$ is a (fixed) polynomial in the components of the curvatures tensor and all of its derivatives up to a fixed order. In particular, the formal expression of T^I_J does not depend on the chosen orthonormal basis. If T is a function, i.e., a real-valued tensor of type $(0,0)$, then T is called a *scalar Weyl invariant*.

Examples of tensorial Weyl invariants are:

- $R_{X,Y}$, tensor of type $(0,2)$ with values in the skew-symmetric endomorphisms of the tangent space.
- $R_{X,Y}Z$, tensor of type $(0,3)$ with values in the tangent space
- $\langle R_{X,Y}Z, W\rangle$, tensor of type $(0,4)$ (with real values).
- $||R||^2 = \sum_{i,j,k,l}\langle R_{e_i.e_j}e_k, e_l\rangle^2$, a scalar Weyl invariant.
- The Ricci tensor $Ric(X) = \sum_i R(X,e_i)e_i$, tensor of type $(0,1)$ with values in the tangent space.
- $\langle(\nabla_{R_{X,Y}Z}R)_{Ric(U),V}W, H\rangle$, tensor of type $(0,7)$.
- The sum and the tensor product of any two tensorial Weyl invariants.

From the Weyl theory of invariants one has that the scalar Weyl invariants are obtained as a linear combination of complete traces, with respect of some paring of the indexes, of tensors of the form

$$\langle \nabla^{m_1}R, \ \rangle \dots \langle \nabla^{m_s}R, \ \rangle$$

where $m_1, \dots, m_s \geqslant 0$.

5.2. Extension of Killing vector fields

Let $\mathcal{V}_p$ be the subspace of T_pM spanned by the gradients of the Weyl scalar invariants. In a open and dense subset Ω of M^n the dimension of $\mathcal{V}$ is locally constant and so $\mathcal{V}$ defines a smooth distribution in any connected component of Ω. Since we will work locally we will assume that $M = \Omega$. Observe that the distribution $\mathcal{D} = \mathcal{V}^\perp$ is integrable: in fact, the integral manifolds of $\mathcal{D}$ are the (regular) level sets of the scalar Weyl invariants which foliate M.

Let $p \in M$ and let $f_1, \dots, f_r$ be scalar Weyl invariants such that $\mathrm{grad}(f_1)(p), \cdots$, $\mathrm{grad}(f_r)(p)$ is a basis of ν_p, $p \in M$. Since we work locally, we may assume that the vectorial Weyl invariants $\mathrm{grad}(f_1)(x), \cdots$, $\mathrm{grad}(f_r)(x)$ are linearly independent, for all $x \in M$. So the regular level sets of $f_1, \dots, f_r$ in M coincide with the (regular) level sets of the family of scalar Weyl invariants. Note that the gradient of any scalar Weyl scalar is a tensorial Weyl invariant of type $(1,0)$. Let now $S_1, \dots, S_s$ be tensorial Weyl invariants and consider $S = S_1 \otimes \cdots \otimes S_s$. Identify, for any $x \in M$, T_xM with $\mathbb{R}^n$ by means of a linear isometry h_x. Let $\mathbb{W}$ be the tensor algebra of $\mathbb{R}^n$ of the same type as S (so that $h_x(S_x) \in \mathbb{W}$). Let $p_1, \dots, p_r$ be $O(n)$-invariant polynomials in $\mathbb{W}$ that distinguish the orbits of the orthogonal group $O(n)$ in $\mathbb{W}$. Then $g_i(x) := p_i(h_x(S_x))$, $i = 1, \dots, r$, does not depend on the chosen isometries h_x, because of the $O(n)$-invariance, and so it defines a scalar Weyl invariant (see [PTV]). If F^{n-r} is a level set of $f_1, \dots, f_r$, then $g_1(x), \dots, g_r(x)$ are constant on F. This implies that for any $x \in F$ there exists a linear isometry $\ell_x : T_pM \to T_xM$ which maps S_p into S_x (cf. [CO3, Section 3]). It follows that given p, q in the same level set F, then there exists a linear isometry $h : T_pM \to T_qM$ which maps any covariant derivative $(\nabla^kR)_q$ to the same object at p, for any $k \geqslant 0$.

Let $c(t)$ be a curve in F with $c(0) = p$ and let $\tau_t : T_pM \to T_{c(t)}M$ be the parallel transport along $c(t)$. The parallel transport is a linear isometry (from the corresponding tangent spaces) and by the previous paragraph one has that $\tau_t^{-1}(\nabla^kR)_{c(t)}$ lies in the same $O(T_pM)$ orbit of $(\nabla^k R)_p$, for all t. Differentiating this condition at $t = 0$ yields that there exists $B \in \mathfrak{so}(T_pM)$ such that

$$(\nabla_v(\nabla^k R))_p = B.(\nabla^k R)_p$$

where $v = c'(0) \in T_pF$. Observe that v is arbitrary in T_pF, since $c(t)$ is an arbitrary curve in F.

Let, for $q \in M$, E^k_q be the subspace of E_q which consists of all the pairs (v, B) such that

$$(\nabla_v(\nabla^i R))_q - B.(\nabla^i R)_q = 0$$

for all $0 \leqslant i \leqslant k$. Notice that the projection to the first component maps E^k_q onto $T_qF(q)$, where $F(q)$ is the level set of the scalar Weyl invariants by q. So, dim $E^k_q \geqslant r$, where r is the dimension of $F(q)$. It is clear that there exists $j(q) \leqslant \dim E_q = n + \frac{1}{2}n(n-1)$ such that $\dim E^{j(q)} = \dim E^{j(q)+1}_q$.

By making, if necessary, M smaller we may assume that $\dim E^{j(q)}_q$ does not depend on q. This gives rise to a subbundle $\bar{E}$ of E whose fibers are $E^{j(q)}_q$. By Proposition 5.2 we have that $\bar{E}$ is a parallel subbundle of E which must be flat by (*). Therefore any $(v, B) \in \bar{E}_p$, p fixed in M, gives rise to a parallel section $(\tilde{v}, \widetilde{B})$ of $\bar{E}$ and so to a parallel section of E (we may assume that M is simply connected). By Proposition 1, this parallel section corresponds to a Killing field on M, whose value at p is v, which is arbitrary in $T_pF(p)$. This Killing field must be always tangent to any level set, because the scalar Weyl invariants are preserved by isometries. This proves:

THEOREM 5.1 ([CO3, CO4]). *The cohomogeneity of a Riemannian manifold M (with respect to the full isometry group) coincides locally with the codimension of the foliation by regular level sets of the scalar Weyl invariants.*

COROLLARY 5.1 (Prüfer, Tricerri and Vanhecke [PTV]). *Let M be an n-dimensional Riemannian manifold. Then M is locally homogeneous if and only if all scalar Weyl invariants of order s with* $s \leqslant \frac{n(n-1)}{2}$ *are constant.*

From the proof the following well-known result of Singer [Si, NT], used in [PTV], follows.

COROLLARY 5.2 (Singer [Si]). *Let M^n be a Riemannian manifold. Then M is locally homogeneous if and only if for any $p, q \in M$ there is a linear isometry $h : T_pM \to T_qM$ such that $h^*(\nabla^s R_q) = \nabla^s R_p$, for any $s \leqslant \frac{1}{2}n(n+1)$.*

REMARK 5.1 (On the pseudo-Riemannian case). Singer's Theorem was generalized to the pseudo Riemannian case by F. Podestà and A. Spiro, [PS]. Our proof can also be applied to this setting and extends to any affine connection without torsion.

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