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A SPLITTING THEOREM FOR COMPACT VAISMAN MANIFOLDS

In memory of Sergio Console

Abstract. We extend to metric compact mapping tori a splitting result for coKähler manifolds. In particular, we prove that a compact Vaisman manifold is finitely covered by the product of a Sasakian manifold and a circle.

1. Introduction

It is quite often the case that two geometric structures are intimately related to one another. This is true, for instance, for Sasakian and Kähler structures. Indeed, assume that K is a Kähler manifold such that the Kähler class $[\omega]$ is integral. The Boothby-Wang construction ([5]) produces a principal bundle $S^1 \to S \to K$ with a connection whose curvature is ω; moreover, the total space S admits a Sasakian structure. This construction can be reversed if S is compact and the Sasakian structure is regular. Moreover, given a manifold S endowed with an almost contact metric structure, the product $S \times \mathbb{R}^{>0}$ with the cone metric is Kähler if and only if the structure is Sasakian.

But more is true. Sasakian structures are also related to Vaisman structures: if φ is an automorphism of a Sasakian manifold S, then the mapping torus S_φ has a natural Vaisman structure. Conversely, Ornea and Verbitsky showed in [15] that a compact Vaisman manifold is always diffeomorphic to the mapping torus of an automorphism of a Sasakian manifold (see Example 2).

In this short note we propose to explore further the relation between Vaisman and Sasakian structures. Recall that a compact Vaisman manifold V is a mapping torus S_φ of a Sasakian autotomorphism. Then the idea is to apply the techniques of [3] to show that the structure group of such mapping torus is finite. Hence, a compact Vaisman manifold is finitely covered by the product of a compact Sasakian manifold and a circle. From this, we obtain topological information about compact Vaisman manifolds.

2. Preliminaries

DEFINITION 1. *Let X be a topological space and let $\varphi\colon X \to X$ be a homeomorphism. The* mapping torus *or* suspension *of (X,φ), denoted X_φ, is the quotient space*

$$\frac{X \times [0,1]}{(x,0) \sim (\varphi(x),1)}.$$

The pair (X,φ) *is the* fundamental data *of* X_φ.

Notice that $\mathrm{pr}_2\colon X\times[0,1]\to[0,1]$ induces a projection $\pi\colon X_\varphi\to S^1$, where $S^1=\mathbb{R}/\mathbb{Z}$, whose fiber is X. Hence X_φ is a fiber bundle with base S^1 and fiber X. It can be shown (compare [3, Proposition 6.4]) that the structure group of the bundle $X\to X_\varphi\to S^1$ is the cyclic group $\langle\varphi\rangle\subset\mathrm{Homeo}(X)$. Moreover, X_φ is trivial as a bundle over S^1, i.e. $X_\varphi\cong X\times S^1$ over S^1, if and only if φ lies in the connected component of the identity of $\mathrm{Homeo}(X)$.

Here is an equivalent definition: given the fundamental data (X,φ), we consider the following $\mathbb{Z}$-action on the product $X\times\mathbb{R}$:

$$m\cdot(x,t)=(\varphi^m(x),t+m).$$

Notice that this action is free and properly discontinuous. The quotient space $(X\times\mathbb{R})/\mathbb{Z}$ is homeomorphic to X_φ. In particular, if M is a smooth manifold and $\varphi\colon M\to M$ is a diffeomorphism, we conclude that M_φ is a smooth manifold and $\pi\colon M_\varphi\to S^1$ is a smooth fiber bundle with fiber M. The length 1-form σ on S^1 pulls back under π to a closed 1-form $\theta\in\Omega^1(M_\varphi)$. Actually, since $[\sigma]\in H^1(S^1;\mathbb{Z})$, $[\theta]\in H^1(M_\varphi;\mathbb{Z})$ and $[\theta]$ itself gives the map π under the usual correspondence $H^1(M_\varphi,\mathbb{Z})\equiv[M_\varphi,S^1]$.

DEFINITION 2. *Let* (M,g) *be a Riemannian manifold and let* $\varphi\colon M\to M$ *be an isometry. We call* M_φ *the* metric mapping torus *of* (M,g,φ).

A metric mapping torus M_φ has a natural Riemannian metric, best described if one thinks of M_φ as a quotient of $M\times\mathbb{R}$. Indeed, consider the product metric $\tilde{h}=g+dt^2$ on $M\times\mathbb{R}$. Then, since $\mathbb{Z}$ acts by isometries on $(M\times\mathbb{R},\tilde{h})$, the metric $\tilde{h}$ descends to a metric h on the mapping torus M_φ. The vector field ∂_t on $M\times\mathbb{R}$ maps to the tangent vector field to S^1 under the map $M\times\mathbb{R}\to M_\varphi\xrightarrow{\pi}S^1$. Moreover, the 1-form θ is unitary (that is, it has length 1) and parallel. Thus, if we consider the standard Riemannian metric on S^1, $\pi\colon M_\varphi\to S^1$ is a Riemannian submersion with totally geodesic fibers.

DEFINITION 3. *We call* h *the* adapted metric *on the metric mapping torus* M_φ.

Let (g,J,ω) be a Hermitian structure on a manifold V of dimension $2n+2$, $n\geqslant 1$. Associated to it is the *Lee form*, defined by

$$\theta=-\frac{1}{n}\delta\omega\circ J;$$

here δ is the co-differential. The Hermitian structure (g,J,ω) is *Kähler* if ω is parallel. In particular, $\theta=0$ in this case. The structure (g,J,ω) is *Vaisman* if the Lee form is non-zero and parallel and $d\omega=\theta\wedge\omega$. In fact, we will assume, without the loss of generality, that θ is unitary. Thus, a Vaisman structure is a particular case of a *locally conformal Kähler* structure, where θ is only required to be closed with $d\omega=\theta\wedge\omega$. Note that, if $n\geqslant 2$, the last condition implies the closedness of the Lee form. Locally conformal Kähler geometry is a very active area of research (see [9, 13, 14]) and has recently attracted interest in Physics (see [17]). Locally conformal Kähler manifolds with parallel Lee 1-form were studied for the first time by Vaisman in [18]. In a subsequent paper (see [19]), Vaisman discussed these structures under the name of *generalized Hopf structures*. Indeed, the main example of a compact Vaisman manifold is the Hopf manifold $S^{2n+1} \times S^1$.

Let S be an odd-dimensional manifold. Consider an almost contact metric structure (ξ, η, g, ϕ) on S and let Ω be the Kähler form* (see [4] for an exposition on almost contact metric geometry). The structure is *coKähler* if $\nabla\Omega = 0$; it is possible to show that $\nabla\eta = 0$ in this case, hence, in particular, $d\eta = 0$.

EXAMPLE 1. If (g, J, ω) is a Kähler structure on a manifold K and $\varphi\colon K \to K$ is a holomorphic isometry, then K_φ is a so-called Kähler mapping torus, providing an example of a coKähler manifold. The metric on K_φ is the adapted one. Note that, in this case, the parallel 1-form giving the map $K_\varphi \to S^1$ is $[\eta] \in H^1(K_\varphi; \mathbb{Z})$. Conversely, a compact coKähler manifold is diffeomorphic to a Kähler mapping torus (see [11]).

An almost contact metric structure (ξ, η, g, ϕ) is *Sasakian* if $d\eta = \Omega$ and $N_\phi + 2d\eta \otimes \xi = 0$, where N_ϕ is the Nijenhuis torsion of ϕ. Let S be a manifold endowed with a Sasakian structure. A diffeomorphism $\varphi\colon S \to S$ is a *Sasakian automorphism* if $\varphi^*\eta = \eta$ and $\varphi^* g = g$.

A Sasakian manifold S is endowed with a 1-dimensional foliation $\mathcal{F}_\xi$, the *characteristic foliation*, whose tangent sheaf is generated by ξ. The foliation $\mathcal{F}_\xi$ is Riemannian and transversally Kähler, see [6, Section 7.2]. A p-form $\alpha \in \Omega^p(S)$ is *basic* if $\iota_\xi \alpha = 0$ and $\iota_\xi d\alpha = 0$. We denote basic forms by $\Omega^*(\mathcal{F}_\xi)$; this is a differential subalgebra of $\Omega^*(S)$. Its cohomology, called the *basic cohomology* of $\mathcal{F}_\xi$, is denoted $H^*(\mathcal{F}_\xi)$. We collect the most relevant features of the basic cohomology:

THEOREM 1 ([6, Proposition 7.2.3 and Theorem 7.2.9]). *Let S be a compact manifold of dimension $2n+1$ endowed with a Sasakian structure (ξ, η, g, ϕ). Then:*

- *the groups $H^p(\mathcal{F}_\xi)$ are finite dimensional, $H^{2n}(\mathcal{F}_\xi) \cong \mathbb{R}$ and $H^p(\mathcal{F}_\xi) = 0$ for $p > 2n$;*
- *$[d\eta]^p \in H^{2p}(\mathcal{F}_\xi)$ is non-trivial for $p = 1, \ldots, n$;*
- *the map $L^p\colon H^{n-p}(\mathcal{F}_\xi) \to H^{n+p}(\mathcal{F}_\xi)$, $[\alpha] \mapsto [(d\eta)^p \wedge \alpha]$ is an isomorphism for $0 \leqslant p \leqslant n$.*

Recall that a connected commutative differential graded algebra (A, d) is *cohomologically Kählerian* if its cohomological dimension is even (say $2n$), it satisfies Poincaré duality and there exists a 2-cocycle ω such that the map $H^{n-p}(A) \to H^{n+p}(A)$, $[\alpha] \mapsto [\omega^{n-p} \wedge \alpha]$, is an isomorphism for $0 \leqslant p \leqslant n$. As a consequence of Theorem 1, $H^*(\mathcal{F}_\xi)$, considered as a commutative differential graded algebra with trivial differential, is cohomologically Kählerian.

*We denote by ω the Kähler form of an almost Hermitian structure and by Ω the Kähler form of an almost contact metric structure.

EXAMPLE 2. Let (ξ, η, g, ϕ) be a Sasakian structure on a manifold S and let $\varphi\colon S \to S$ be a Sasakian automorphism. The mapping torus S_φ has a Vaisman structure with the adapted metric. In [13, Structure Theorem], Ornea and Verbitsky claimed that every compact manifold endowed with a Vaisman structure is the mapping torus of a Sasakian manifold and a Sasakian automorphism. In [15] they argued that this is actually imprecise, but provided a modified version of this statement. In [15, Corollary 3.5], they proved that if a compact manifold V admits a Vaisman structure, then V admits another Vaisman structure which arises as the mapping torus of a Sasakian manifold and a Sasakian automorphism. Thus, up to diffeomorphism, every compact Vaisman manifold is the mapping torus of a Sasakian manifold and a Sasakian automorphism.

3. Main result

Consider a metric mapping torus with adapted metric, (M_φ, h), and let $\theta \in \Omega^1(M_\varphi)$ be the closed 1-form described in Section 1. In Examples 1 and 2, θ is not only a closed form, but it is also unitary and *parallel* with respect to the Levi-Civita connection of the adapted metric. More generally, we can suppose we are given a mapping torus M_φ with a Riemannian metric $\bar{h}$ such that $\theta \in \Omega^1(M_\varphi)$ is a parallel 1-form. In such a case, M_φ is locally isometric to the product $M \times \mathbb{R}$ and it follows that θ is unitary and parallel. There is therefore no loss of generality in assuming that $\bar{h}$ is the adapted metric.

We prove the following result:

THEOREM 2. *Let (M, g) be a compact Riemannian manifold, let $\varphi\colon M \to M$ be an isometry and let (M_φ, h) be the mapping torus with the adapted metric. Let $\theta \in \Omega^1(M_\varphi)$ be the unitary and parallel 1-form. Then there is a finite cover $p\colon M \times S^1 \to M_\varphi$ whose deck group is isomorphic to a finite group $\mathbb{Z}_m$ for some $m > 0$ which acts diagonally and by translations on the second factor. We have a diagram of fiber bundle*

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\cong"'] & M \times S^1 \arrow[r] \arrow[d, "p"] & S^1 \arrow[d, "\cdot m"] \\ M \arrow[r] & M_\varphi \arrow[r] & S^1 \end{tikzcd}$$

and M_φ fibers over the circle $S^1/\mathbb{Z}_m$ with finite structure group $\mathbb{Z}_m$.

Proof. The proof is basically a reproduction of the argument used in [3] for coKähler manifolds. We recall it briefly. The first step is to notice that $\upsilon \in \mathfrak{X}(M_\varphi)$, the metric dual of the 1-form θ, is a unitary and parallel vector field and, in particular, Killing. If the metric h happens to be adapted, then υ is the image under the derivative of the projection $M \times \mathbb{R} \to M_\varphi$ of the vector field ∂_t. By the Myers-Steenrod theorem (see [12]), $\mathrm{Isom}(M_\varphi, h)$ is compact, so the closure of the flow of υ in $\mathrm{Isom}(M_\varphi, h)$ is a torus T. This gives a free T-action on M_φ. Choose a vector field $\hat{\upsilon}$ in the Lie algebra of T, close enough to υ, and such that $\hat{\upsilon}$ generates a circle action on M_φ. At some point $x_0 \in M_\varphi$ we surely have $\theta(\hat{\upsilon})(x_0) \neq 0$, since $\theta(\upsilon)(x_0) \neq 0$. But being θ harmonic and $\hat{\upsilon}$

Killing, this implies that $\theta(\hat{\upsilon}) \neq 0$. We assume henceforth that $\theta(\hat{\upsilon}) > 0$ and denote by $\tau \subset M_\varphi$ an orbit of this S^1-action. Then

$$\int_\tau \theta = \int_0^1 \theta\left(\frac{d\tau}{dt}\right) dt = \int_0^1 \theta(\hat{\upsilon}) dt > 0. \tag{1}$$

Consider the orbit map $\alpha\colon S^1 \to M_\varphi$, $g \mapsto g \cdot x_0$ and the composition

$$H_1(S^1;\mathbb{Z}) \xrightarrow{\alpha_*} H_1(M_\varphi;\mathbb{Z}) \xrightarrow{\pi_*} H_1(S^1;\mathbb{Z}).$$

We remarked above that, under the correspondence $H^1(M_\varphi;\mathbb{Z}) \equiv [M_\varphi, S^1]$, π is given by θ; thus (1) tells us that π_* is non-zero when evaluated on an element of $H_1(M_\varphi;\mathbb{Z})$ coming from the orbit map. Since $H_1(S^1;\mathbb{Z}) = \mathbb{Z}$, this means that α_* is injective. We conclude that the S^1-action is homologically injective.

The second step consists in relating the homological injectivity of this S^1-action with the reduction of the structure group of the bundle $M_\varphi \to S^1$ to a finite group and the existence of a finite cover of M_φ with the desired properties. This uses the more general notion of transversal equivariance of a fibration over a torus with respect to a smooth torus action, developed by Sadowski in [16]. These ideas were developed first in the topological context by Conner and Raymond, see [8]. We refer to [3] for a detailed explanation of the result. □

REMARK 1. Prof. Dieter Kotschick has suggested to us that Theorem 2 can be proven in an easier way, without appealing to the results of Conner - Raymond and Sadowski. Indeed, by the Myers-Steenrod theorem, the isometry group $\mathrm{Isom}(M,g)$ of a compact Riemannian manifold (M,g) is a compact Lie group; in particular, it has a finite number of connected components. This implies that if φ is an isometry of (M,g), there exists an integer $m > 0$ such that φ^m belongs to the connected component of the identity $\mathrm{Isom}_0(M,g)$ of $\mathrm{Isom}(M,g)$. Indeed, if $\varphi^n \notin \mathrm{Isom}_0(M,g)$, for every integer $n > 0$, then $[\varphi^n] \neq [\varphi^m]$ in the quotient group $\mathrm{Isom}(M,g)/\mathrm{Isom}_0(M,g)$, for $n \neq m$. However, this is not possible, since $\mathrm{Isom}(M,g)/\mathrm{Isom}_0(M,g)$ is just the finite group of connected components of $\mathrm{Isom}(M,g)$. Consider now the map $\gamma_m\colon S^1 \to S^1$ given by $\gamma_m(z) = z^m$ and use it to pull back to the first S^1 the fiber bundle $M \to M_\varphi \to S^1$. It is clear that the structure group of $\gamma_m^* M_\varphi$ is generated by φ^m, hence $\gamma_m^* M_\varphi \cong M \times S^1$. This gives the splitting up to final cover. One also obtains an action of the finite group $\mathbb{Z}_m$, generated by the isotopy class of φ, on the product $M \times S^1$, which is diagonal and by translations on S^1. In [3] we overlooked this simple approach and appealed rather to the techniques of Conner - Raymond and Sadowski since our main goal was to investigate the rational-homotopic properties of compact coKähler manifolds. Among the outcomes of this research, we quote the proof of the toral rank conjecture for compact coKähler manifolds (see [2]).

COROLLARY 1. *Let (M,g) be a compact Riemannian manifold, let $\varphi\colon M \to M$ be an isometry and let (M_φ, h) be the mapping torus with the adapted metric. Then*

$$H^*(M_\varphi;\mathbb{R}) \cong H^*(M;\mathbb{R})^G \otimes H^*(S^1;\mathbb{R}),$$

where $G \cong \mathbb{Z}_m$ and m is the smallest positive integer such that $\varphi^m \in \mathrm{Isom}_0(M,g)$.

Proof. For a finite G-cover $\tilde{X} \to X$, one has $H^*(X;\mathbb{R}) \cong H^*(\tilde{X};\mathbb{R})^G$. □

COROLLARY 2. *Let V be a compact Vaisman manifold. Then there exists a finite cover $p\colon S \times S^1 \to V$, where S is a compact Sasakian manifold, the deck group is isomorphic to $\mathbb{Z}_m$, for some $m > 0$, acts diagonally and by translations on the second factor. We have a diagram of fiber bundles*

$$\begin{tikzcd} S \arrow[r] \arrow[d, "\cong"'] & S \times S^1 \arrow[r] \arrow[d, "p"] & S^1 \arrow[d, "\cdot m"] \\ S \arrow[r] & V \arrow[r] & S^1 \end{tikzcd}$$

and V fibers over the circle $S^1/\mathbb{Z}_m$ with finite structure group $\mathbb{Z}_m$.

Proof. By the aforementioned result of Ornea and Verbitsky (see [15, Corollary 3.5]), V is diffeomorphic to a mapping torus S_φ where S is a compact Sasakian manifold and $\varphi\colon S \to S$ is a Sasakian automorphism. Under this identification, the Lee form θ, which is parallel by definition on a Vaisman manifold, gives the projection $V \cong S_\varphi \to S^1$. It is now enough to apply Theorem 2. □

In the next corollary, we show how to apply our splitting theorem to obtain well-known results on the topology of compact Vaisman manifolds. Compare [19].

COROLLARY 3. *Let V be a compact connected Vaisman manifold of dimension $2n+2$ and let $b_r(V)$ be the r^{th} Betti number of V. Then $b_p(V) - b_{p-1}(V)$ is even for p odd and $1 \leqslant p \leqslant n$. In particular, $b_1(V)$ is odd.*

Proof. By [15, Corollary 3.5], V is diffeomorphic to a mapping torus S_φ, where φ is an automorphism of the compact Sasakian manifold S. Let (ξ, η, g, ϕ) be the Sasakian structure of S and let $\mathcal{F}_\xi$ denote the characteristic foliation; then $(H^*(\mathcal{F}_\xi), 0)$ is a cohomologically Kählerian algebra with Kähler class $[d\eta] \in H^2(\mathcal{F}_\xi)$. If $G \cong \mathbb{Z}_m$ is the finite structure group of the finite cover $p\colon S \times S^1 \to S_\varphi$, then the G-action preserves $\mathcal{F}_\xi$, since φ, which generates the structure group of the mapping torus, is a Sasakian automorphism. Hence we can consider the invariant basic cohomology $H^*(\mathcal{F}_\xi)^G$. For the basic Kähler class, it holds $[d\eta] \in H^2(\mathcal{F}_\xi)^G$. There is an exact sequence (see [6, page 215])

$$\cdots \to H^p(S;\mathbb{R}) \to H^{p-1}(\mathcal{F}_\xi) \xrightarrow{e} H^{p+1}(\mathcal{F}_\xi) \to H^{p+1}(S;\mathbb{R}) \to \cdots \tag{2}$$

where e is the multiplication by the basic Kähler class $[d\eta]$. Since $\dim V = 2n+2$, the cohomological dimension of $H^*(\mathcal{F}_\xi)$ is $2n$. This implies that the map $e\colon H^{p-1}(\mathcal{F}_\xi) \to H^{p+1}(\mathcal{F}_\xi)$ is injective for $0 \leqslant p \leqslant n$, hence (2) splits and gives short exact sequences

$$0 \to H^{p-1}(\mathcal{F}_\xi) \xrightarrow{e} H^{p+1}(\mathcal{F}_\xi) \to H^{p+1}(S;\mathbb{R}) \to 0, \quad 0 \leqslant p \leqslant n. \tag{3}$$

Notice that (3) is a short exact sequence of G-modules. Since we are working with real coefficients, every term in (3) is a real vector space. By taking invariants, we obtain short exact sequences

(4) $$0 \to H^{p-1}(\mathcal{F}_\xi)^G \to H^{p+1}(\mathcal{F}_\xi)^G \to H^{p+1}(S;\mathbb{R})^G \to 0, \quad 0 \leqslant p \leqslant n;$$

the surjectivity of $H^{p+1}(\mathcal{F}_\xi)^G \to H^{p+1}(S;\mathbb{R})^G$ follows by averaging over G. By [2, Proposition 2.3], $H^*(\mathcal{F}_\xi)^G$ is also cohomologically Kählerian. We now set $\bar{b}_p(S) := \dim H^p(S;\mathbb{R})^G$ and $\bar{b}_p(\mathcal{F}_\xi) := \dim H^p(\mathcal{F}_\xi)^G$; then $\bar{b}_p(\mathcal{F}_\xi)$ is even for p odd. In view of Corollary 1, we have, for $1 \leqslant p \leqslant n$,

$$b_p(V) = \bar{b}_p(S) + \bar{b}_{p-1}(S) = \bar{b}_p(\mathcal{F}_\xi) - \bar{b}_{p-2}(\mathcal{F}_\xi) + \bar{b}_{p-1}(\mathcal{F}_\xi) - \bar{b}_{p-3}(\mathcal{F}_\xi),$$

hence $b_p(V) - b_{p-1}(V) = \bar{b}_p(\mathcal{F}_\xi) - 2\bar{b}_{p-2}(\mathcal{F}_\xi) + \bar{b}_{p-4}(\mathcal{F}_\xi)$. If p is odd, then $b_p(V) - b_{p-1}(V)$ is even. □

According to [14], the fundamental group of a compact Vaisman manifold V sits in an exact sequence

$$0 \to G \to \pi_1(V) \to \pi_1(X) \to 0$$

where $\pi_1(X)$ is the fundamental group of a Kähler orbifold and G is a quotient of $\mathbb{Z}^2$ by a subgroup of rank $\leqslant 1$. We give a different characterization of the fundamental group of a Vaisman manifold:

COROLLARY 4. *Let V be a compact Vaisman manifold. Then $\pi_1(V)$ has a subgroup of finite index of the form $\Gamma \times \mathbb{Z}$, where Γ is the fundamental group of a compact Sasaki manifold.*

Proof. It is enough to consider the finite cover $S \times S^1 \to S_\varphi$. □

Sasaki groups, and their relation with Kähler groups, have been investigated for instance in [7, 10].

Another application of the splitting theorem is to the group of automorphisms of a compact Sasakian manifold. Let S be a compact manifold endowed with a Sasakian structure (ξ, η, g, ϕ), let $\mathrm{Aut}(\xi, \eta, g, \phi)$ be the group of Sasakian automorphisms and let $\varphi \in \mathrm{Aut}(\xi, \eta, g, \phi)$. Form the mapping torus S_φ. By Corollary 2, we find $m > 0$ such that $\varphi^m \in \mathrm{Aut}_0(\xi, \eta, g, \phi)$, the identity component of $\mathrm{Aut}(\xi, \eta, g, \phi)$.

COROLLARY 5. *If S is a compact manifold endowed with a Sasakian structure (ξ, η, g, ϕ), then every element of the group $\mathrm{Aut}(\xi, \eta, g, \phi)/\mathrm{Aut}_0(\xi, \eta, g, \phi)$ has finite order.*

For a careful analysis of the automorphism group of a Sasakian manifold we refer to [6].

REMARK 2. In [1], the notion of a K-cosymplectic structure was introduced. This is an almost contact metric structure (ξ, η, g, ϕ) with $d\eta = 0$, $d\Omega = 0$ and $L_\xi g = 0$. One can prove that, in this case, the 1-form η is parallel. Examples of K-cosymplectic manifolds are given by mapping tori of almost Kähler manifolds (K, g, J, ω) with a diffeomorphism $\varphi\colon K \to K$ such that $\varphi^* g = g$ and $\varphi^*\omega = \omega$; hence they are metric mapping tori. If K is compact, so is K_φ. Hence we can apply Theorem 2 and obtain a finite cover $K \times S^1 \to K_\varphi$.

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