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REPRESENTATIONS AND INVARIANTS OF LINKS IN LENS SPACES

Abstract.
We analyze different representations of knots and links in lens spaces, as disk diagrams, grid diagrams, mixed diagrams; together with the associated moves describing the knot/link equivalence. Using such representations we study some invariants of these type of knots/links, as fundamental group of the complement, Alexander polynomial, twisted Alexander polynomial, HOMFLY-PT polynomial and lifting in the 3-sphere. Some of these results are previously unpublished.

1. Introduction

Knot theory is a widespread branch of geometric topology. Although the classical theory regards knots and links in the 3-sphere, the cases where the ambient is a more general (compact) 3-manifolds are recently widely investigated. In this work we focus on lens spaces, the simplest class of closed non-simply connected 3-manifolds: we review some results on the knot/link theory in these spaces and state a few open problems. The study of this kind of knots is also relevant for the Berge conjecture [24], in theoretical physics [38] and in biology [6].

Knots/links in lens spaces may be represented with different techniques, as disk diagrams [9], grid diagrams [1], band diagrams [27], mixed link diagrams [29] and other methods. In this work we describe some of these representations, together with the moves describing the equivalence of the associated knots/links, which generalize the Reidemester moves for the case of the 3-sphere.

Using these representations, "Wirtinger type" presentations for the fundamental group of the knot/link complements are given. We discuss whether or not a knot in a lens space is determined by its complement. Moreover, we describe some polynomial invariants such as the Alexander and twisted Alexander polynomials and the HOMFLY-PT polynomial.

Another point of view for the investigation of knots/links in lens spaces is to considering the lift of them in the 3-sphere, under the standard universal covering. In this setting, we present an algorithm producing a classical diagram for the lift, starting from a disk diagram of the link in the lens space. Using this construction we describe examples of different knots and links in $L(p,q)$ with the same lift, showing that the lift is not a complete invariant for knots/links in lens spaces.

The aforementioned results can be found in [9, 32, 31].

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2. Preliminaries

Let $S^3 = \{(z_1,z_2)\in \mathbf{C}^2 \mid |z_1|^2+|z_2|^2=1\}$ be the 3-sphere and let $p,q\in\mathbb{N}$, with $\gcd(p,q)=1$ and $p\neq 0$. The *lens space* $L(p,q)$ is the quotient of S^3 by the action of the cyclic group of order p generated by the homeomorphism $\theta_{p,q}: S^3\to S^3$,

$$\theta_{p,q}(z_1,z_2)=(z_1e^{\frac{2\pi i}{p}},z_2e^{\frac{2\pi iq}{p}}).$$

Clearly $L(1,0)=S^3$ and $L(2,1)=\mathbf{RP}^3$. Since we are interested in lens spaces different from S^3, we always assume $p>1$, and therefore $q>0$. Note that $L(p,q)$ inherit an orientation from the natural orientation of S^3 induced by $\mathbb{C}^2$,

There are a few of notions of "equivalence" for links in a compact 3-manifold M. We are interested in the following two:

- *diffeo-equivalence*: two links L and L' in M are *diffeo-equivalent* if there exists a diffeomorphism $h: M\to M$ such that $h(L)=L'$
- *isotopy equivalence*: two links L and L' in M are *isotopy equivalent* if there exists a continuous map $H\colon M\times[0,1]\to M$ such that $h_0=id_M$, $h_1(L)=L'$ and h_t is an diffeomorphism of M for each $t\in[0,1]$, where $h_t(x):=H(x,t)$.

Obviously isotopy equivalent links are also diffeo-equivalent. In S^3, there are only two isotopy classes of diffeomorphism: the orientation preserving one and the orientation reversing one. So if two links $L,L'\subset S^3$ are diffeo-equivalent, then L is isotopy equivalent to L' or to its mirror image. In $L(p,q)$ we have the following result.

Theorem 1 ([5], [26]). *Let $\mathcal{D}_{p,q}=\pi_0(\mathrm{Diffeo}(L(p,q)))$, (resp. $\mathcal{D}^+_{p,q}=\pi_0(\mathrm{Diffeo}^+(L(p,q)))$) be the group of diffeomorphism (resp. orientation preserving diffeomorphisms) of the lens space $L(p,q)$, up to isotopy. We have:*

- $\mathcal{D}_{p,q}=\mathcal{D}^+_{p,q}\cong Z_2\oplus Z_2$ *if* $q^2\equiv 1 \mod p$ *and* $q\not\equiv\pm1 \mod p$*;*
- $\mathcal{D}_{p,q}\cong Z_4$ *and* $\mathcal{D}^+_{p,q}\cong Z_2$ *if* $q^2\equiv -1 \mod p$ *and* $p\neq 2$*;*
- $\mathcal{D}_{p,q}\cong Z_2$ *and* $\mathcal{D}^+_{p,q}\cong 1$ *if* $p=2$*;*
- $\mathcal{D}_{p,q}=\mathcal{D}^+_{p,q}\cong Z_2$*, otherwise.*

Sometimes, we will need to deal with oriented links: we say that two oriented link L,L' are diffeo-equivalent if the diffeomorphism h and its restriction to L are orientation preserving; isotopy equivalent if the restriction of the diffeomorphism h_1 to L is orientation preserving.

3. Representations of links in lens spaces

There are many different ways to represent links in lens spaces, arising from the different models of the lens spaces. In this section we focus on three of them: disc diagrams, grid diagrams and mixed link diagrams. Moreover, for each diagrammatic representation, we describe a complete set of moves corresponding to isotopy equivalence on links, while a complete set of moves for diffeo-equivalence can be found in [8].

Nevertheless these are not the only ones: other different representations are, for example, band diagrams introduced in [27] or punctured disk diagrams, described in [18], both relying on the Dehn surgery representation of $L(p,q)$.

Having so many different representation, it is important to know how to pass from one to another: in [30] it is described how to pass from a disc diagram to both a grid and a band diagram.

3.1. Disk diagram representation and moves

The disc diagram representation, developed in [9], has been used fruitfully to describe and study the lift of a link under the universal covering map $\theta_{p,q} : \mathrm{S}^3 \to L(p,q)$ (see Section 5).

The model used for lens spaces, which is also called the "prism" model (see [40, §9B]), is the following. Let p and q be two coprime integers such that $0 \leqslant q < p$. Let $B^3 = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_1^2+x_2^2+x_3^2 \leqslant 1\}$ and denote with E_+ and E_- the upper and the lower closed hemisphere of ∂B^3, respectively. The intersection of the plane $x_3 = 0$ with B^3 is the equatorial disk B_0^2. Moreover we set $N = (0,0,1)$ and $S = (0,0,-1)$. Consider the rotation $g_{p,q}\colon E_+ \to E_+$ of $2\pi q/p$ radians around the x_3-axis and the reflection $f_3\colon E_+ \to E_-$ with respect to the plane $x_3 = 0$ (see Figure 1).

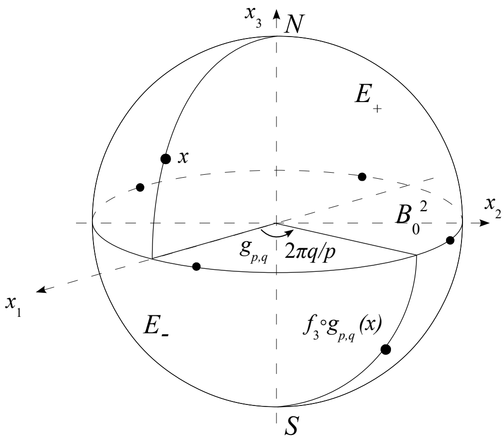


Figure 1: A model for $L(p,q)$.

The *lens space* $L(p,q)$ is the quotient of B^3 under the equivalence relation on ∂B^3 which identifies $x \in E_+$ with $f_3 \circ g_{p,q}(x) \in E_-$. Denote by $F\colon B^3 \to L(p,q) = B^3/\sim$

the quotient map. Note that on the equator $\partial B_0^2 = E_+ \cap E_-$ each equivalence class contains p points. The lens space $L(p,q)$ has a natural orientation induced by the standard orientation of $B^3 \subset \mathbb{R}^3$.

Let $L \subset L(p,q)$ be a link and consider $L' = F^{-1}(L)$. By moving L via a small isotopy in $L(p,q)$, we can suppose that L' is a disjoint union of closed curves embedded in $\mathrm{int}(B^3)$ and arcs properly embedded in B^3 not containing N and S. Denote with $\mathbf{p}\colon B^3 - \{N,S\} \to B_0^2$ the projection defined by $\mathbf{p}(x) = c(x) \cap B_0^2$, where $c(x)$ is the euclidean circle (possibly a line) through N, x and S. Project L' using $\mathbf{p}_{|_{L'}}: L' \to B_0^2$.

As in the classical case, we can assume, by moving L via a small isotopy, that the projection $\mathbf{p}_{|_{L'}}: L' \to B_0^2$ of L' is *regular*, namely:

- the projection of L' contains no cusps;
- all auto-intersections of $\mathbf{p}(L')$ are transversal;
- the set of multiple points is finite, and all of them are actually double points;
- no double point is on ∂B_0^2.

Moreover, in ∂B_0^2 each point is equivalent to other $p-1$ points, so we assume that L' is disjoint from ∂B_0^2 in order to have stability in the diagram under small isotopies of the link. As a consequence, boundary points in the diagram are coupled under the equivalence $\sim$.

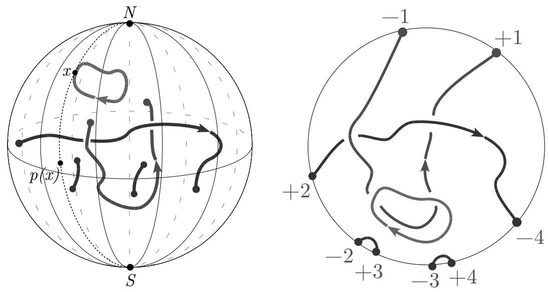


Figure 2: A link in $L(9,1)$ and its corresponding disk diagram.

Finally, double points are endowed with underpasses and overpasses instructions as in the diagrams of links in S^3. A *disk diagram* of a link L in $L(p,q)$ is a regular projection of $L' = F^{-1}(L)$ on the equatorial disk B_0^2, with specified overpasses and underpasses (see Figure 2). If the link L is oriented, then any diagram of L inherits an orientation. In order to make the disk diagram more comprehensible, the boundary points of the projection of L' are indexed as follows: first, assume that the equator ∂B_0^2 is oriented counterclockwise if we look at it from N and fix a point P_0 on it (for example $(1,0,0)$); then, according to this orientation and starting from P_0, label with $+1,\dots,+t$

the endpoints of the projection of the link coming from the upper hemisphere, and with $-1,\dots,-t$ the endpoints coming from the lower hemisphere, respecting the rule $+i \sim -i$ (see an example in Figure 2). Note that t is exactly the number of arcs in the disc diagram, and that the central angle subtended by the points labelled with $+i$ and $-i$ is $2\pi q/p$ and so identifies the lens space.

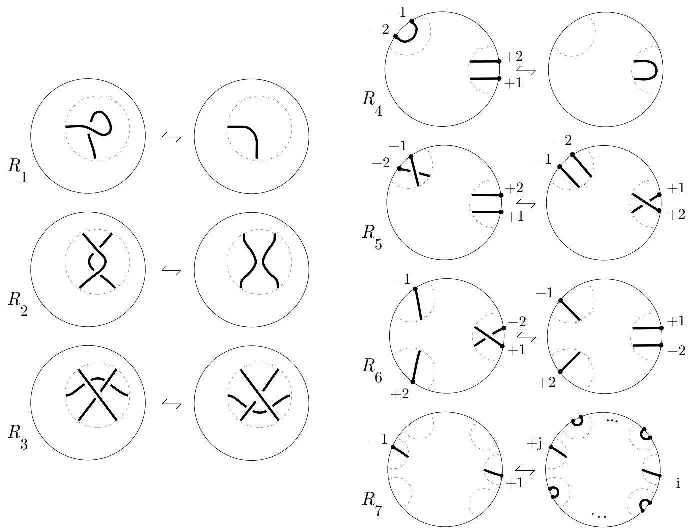


Figure 3: Equivalence moves on a disk diagram.

PROPOSITION 3.1 ([9]). *Two disk diagrams of links in $L(p,q)$ represent isotopy equivalent links, if and only if they are connected by a finite sequence of the seven Reidemeister type moves depicted in Figure 3.*

Note that the move R_7 is due to the assumption that $L' \cap \partial B_0^2 = \varnothing$, as depicted in Figure 4.

3.2. Grid diagrams and moves

We describe the representation developed [1] and the corresponding equivalence moves. This representation relies on the construction of lens spaces via Heegaard splittings and has been used to generalized to $L(p,q)$ invariants of skein type as the HOMFLY-PT polynomial (see Subsection 7.2) and of homology type, as the Link Floer Homology (see [1]).

Let $V = S^1 \times D^2$ be a solid torus and set $T^2 = \partial V = S^1 \times S^1$. We can obtain $L(p,q)$ by gluing together the boundaries of two copies V_1, V_2 of V, as follows. On

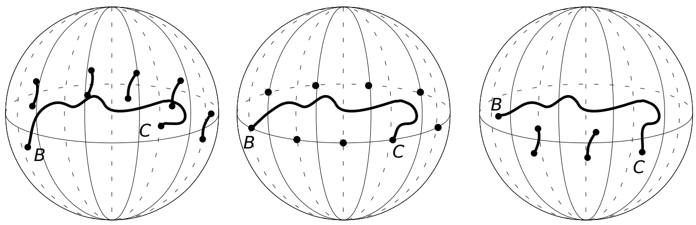


Figure 4: Avoiding ∂B_0^2 in $L(9,1)$.

$\partial V_1 = \mathrm{S}^1 \times \mathrm{S}^1$ consider the two (oriented*) curves $\alpha = \mathrm{S}^1 \times Q$ and $\beta = P \times \mathrm{S}^1$, where P, Q are fixed points on S^1. The isotopy type of an homeomorphism f on the torus depends only on the homological class of $f(\beta)$: if we glue V_1 and V_2 by an orientation reversing homeomorphism $\varphi_{p,q} \colon \partial V_2 \to \partial V_1$ sending the curve β to the curve $q\beta + p\alpha$, we obtain the lens space $L(p,q)$ where the orientation is induced by a fixed orientation of V. In Figure 5 the case $L(5,2)$ is illustrated.

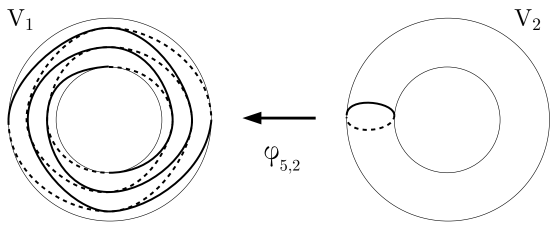


Figure 5: Heegaard splitting of $L(5,2)$.

In order to introduce the notion of *(toroidal) grid diagram*, we consider T^2 as the quotient $\mathbb{R}^2/\mathbb{Z}^2$, where $\mathbb{Z}^2$ is the lattice generated by the vectors $(1,0)$ and $(0,1)$. A *(toroidal) grid diagram* G in $L(p,q)$, with grid number n, is a 5-tuple $(T^2, \alpha, \beta, \mathbb{O}, \mathbb{X})$ satisfying the following conditions (see Figure 6 for an example in $L(4,1)$ with grid number 3):

- $\alpha = \{\alpha_0, \ldots, \alpha_{n-1}\}$ are the images in T^2 of the n lines in $\mathbb{R}^2$ described by the equations $y = i/n$, for $i = 0, \ldots, n-1$; the complement $T^2 - (\alpha_0 \cup \ldots \cup \alpha_{n-1})$ has n connected open annular components, called the *rows* of the grid diagram;

- $\beta = \{\beta_0, \ldots, \beta_{n-1}\}$ are the images in T^2 of the n lines in $\mathbb{R}^2$ described by the equations $y = -\frac{p}{q}(x - \frac{i}{pn})$, for $i = 0, \ldots, n-1$; the complement $T^2 - (\beta_0 \cup \ldots \cup$

* We assume that α and β are oriented such that the algebraic intersection between α and β is one.

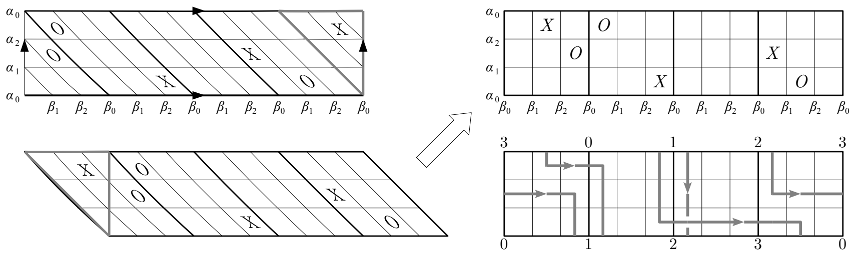


Figure 6: From a grid diagram with grid number 3 to its corresponding link in $L(4,1)$.

β_{n-1}) has n connected open annular components, called the *columns* of the grid diagram;

- $\mathbb{O} = \{O_0,\dots,O_{n-1}\}$ (resp. $\mathbb{X} = \{X_0,\dots,X_{n-1}\}$) are n points in $T^2 - (\alpha \cup \beta)$ called *markings*, such that any two points in $\mathbb{O}$ (resp. $\mathbb{X}$) lie in different rows and in different columns.

In order to make the identifications of the diagram boundary easier to understand, it is possible to perform the "shift" depicted in Figure 6. Notice that, if we forget about identifications, the curve β_0 divides the rectangle of a grid diagram into p adjacent squares, called *boxes* of the diagram.

A grid diagram G represents an oriented link $L \subset L(p,q)$ obtained as follows. First, denote with V_α and V_β two solid tori having α and β as meridians, respectively. Clearly $V_\alpha \cup_{T^2} V_\beta$ is a genus one Heegaard splitting representing $L(p,q)$. Then connect

(1) each X_i to the unique O_j lying in the same row with an arc embedded in the row and disjoint from the curves of α,

(2) each O_j to the unique X_l lying in the same column by an arc embedded in the column and disjoint from the curves of β,

obtaining in this way a union of curves immersed in T^2. Finally remove the double points, pushing the lines of (1) into V_α and the lines of (2) into V_β. The orientation on L is obtained by orienting any horizontal arc connecting the markings from X to O. Note that a pair of markings X and O in the same position corresponds to a trivial component of the link (see Figure 22).

By [1, Theorem 4.3], each link $L \subset L(p,q)$ can be represented by a grid diagram. The idea of the proof is a PL-approximation with orthogonal lines of the link projection on the torus.

In order to relate isotopy equivalent links we introduce some moves on grid diagram. A $(X : NW)$-*grid stabilization* is the move depicted in Figure 7. It increases the grid number by one. Similarly we can define stabilization with respect to a O

marking and/or with NE, SW and SE arrangements. The opposite move is called *grid destabilization*.

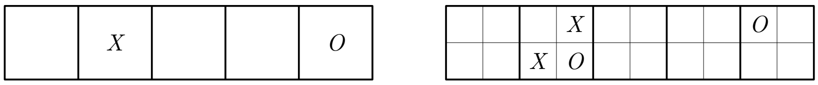


Figure 7: An example of (de)stabilization.

A *grid commutation* is a move that interchanges either two adjacent columns or two adjacent rows. Let A be the union of the closed vertical (resp. horizontal) annuli containing the two considered columns (resp. rows) c_1 and c_2. The rows (resp. columns) divide A into pn parts: denote with s_1 and s_2 the two parts containing the markings of c_1. A commutation is called *interleaving* if the markings of c_2 are in different components of $A-(s_1 \cup s_2)$, and *non-interleaving* otherwise (see Figure 8).

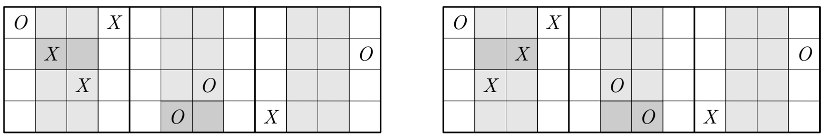


Figure 8: An example of non-interleaving commutation.

PROPOSITION 3.2 ([1]). *Two grid diagrams in $L(p,q)$ represent the same link, up to isotopy, if and only if there exists a finite sequence of (de)stabilizations and non-interleaving commutations connecting them.*

In [2, 11] an algebraic representation of links in 3-manifolds is developed via braid group of punctured surfaces, generalizing the plat representation of links in S^3 (see [3]). Such representation relies on Heegaard decompositions of the 3-manifolds, as well as grid presentation, in the case of $L(p,q)$. In this setting an open problem is the following one.

Question 1. Is it possible to interpret the moves on a grid diagrams in terms of braids of punctured surfaces, under the generalized plat decomposition?

3.3. Mixed link diagrams representation and moves

In this subsection we analyze a representation of links in lens spaces arising from the Dehn surgery model of lens spaces: mixed link diagrams (for detailed reference see [29]). One of the main features of this representation is that from the diagram it is possible to find a presentation for the *fundamental augmented rack* invariant: this is a classifying invariant for irreducible links having the form of an algebraic structure that, roughly speaking, encodes the equivalence due to Reidemeister-type moves (see [16]). Moreover, mixed link representation is strictly connected to punctured disc diagram representation introduced in [18] and used for finding a tabulation for knots and links in lens spaces.

We start by recalling the general definition of Dehn surgery in a general 3-manifold, since we will need it in Section 4. Let K be a knot in a orientable 3-manifold M. Denote with $N(K)$ a tubular neighborhood of K in M and fix a simple cosed curve γ on $\partial N(K)$. The *Dehn surgery on K along* γ is the manifold $M' = M(K;\gamma)$ obtained by gluing a solid torus $S^1 \times D^2$ with $M - \mathrm{int}(N(K))$ along their boundary, via a homeomorphism that identifies γ with the meridian of the solid torus. The curve γ is called *slope* of the surgery. Now let $M = S^3$. The homeomorphism type of the resulting manifold depends only on the homology class of γ in $\partial N(K) \cong S^1 \times S^1$, up to orientation change. We fix a base (m,l) for $H_1(\partial N(K))$, such that m is a meridian of K and l has algebraic intersection 1 with m. Moreover, if K is oriented we take l homologous to it in $N(K)$. If the homology class of γ is $pm+ql$, with $p,q \in \mathbb{Z}$, we call $M(K;\gamma)$ *rational* Dehn surgery on K with *framing index* p/q. This operation can be generalized to links and a theorem of Lickorish and Wallace states that every closed, connected, orientable 3-manifold can be obtained by surgery on some link in S^3. Therefore, a link L in a 3-manifold M can be represented by a diagram of a link $L' \cup J$ in S^3 such that: each connected component $J_1, \dots, J_\mu$ of J is equipped with a rational number, the surgery along J gives M and L is the image of L' under the surgery operation. This representation is called *mixed link diagram* and the link J is called the *surgery link*. In order to simplify notation we use the same symbol to denote both L' and L.

The surgery description of lens spaces can be done by a rational surgery over the trivial knot U in S^3 with framing index $-p/q$, so a link L in $L(p,q)$ can be described by a mixed link diagram of the link $L \cup U$. An example is depicted in Figure 9. Any other surgery description of lens spaces can be reduced to this one by Kirby moves.

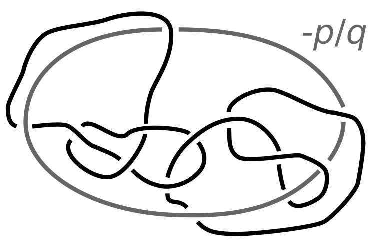


Figure 9: Example of a mixed link diagram of a link in $L(p,q)$.

Two mixed link diagrams of this kind represent the same link, up to isotopy, if they are connected by a finite sequence of Redeimeister moves that keep U fixed and a non-local move called a *band move* or *SL-move*. Informally, this move consists in exchanging the overpass with the underpass in a double point involving both L and U, and summing a (p,q)-torus knot to L. A precise statement can be found in [29, Theorem 5.8].

4. Knot complement and cosmetic pairs

In this section we collect some results on the generalization of the Gordon-Luecke Theorem for knot complements in lens spaces and cosmetic pair problem in Dehn surgery. So we discuss whether a knot is determined or not by its complement.

Let K be a non-trivial knot in a 3-manifold M. If we study manifolds via Dehn surgery, a natural question to ask is if there exists a non-trivial Dehn surgery on K which yields a manifold homeomorphic to M. In this case the pair (M,K) is called *cosmetic* as well as the surgery. This terminology was first used in [4], where a cosmetic pair is constructed in $L(49,18)$, and the interest is mainly for hyperbolic knots. On the contrary, Gabai in [17] and Gordon-Luecke in [22] proved the non existence of cosmetic pairs for $S^2 \times S^1$ and S^3, respectively. An important consequence of these two works is that two knots in $S^2 \times S^1$ or S^3 are diffeo-equivalent if and only if their complements are diffeomorphic. The example of [4] proves that the same result does not hold in lens spaces: after that, many authors tried to characterize the knots giving rise to cosmetic pairs.

The following results investigates the problem applying the result of [22] via the covering $\theta_{p,q} : S^3 \to L(p,q)$.

THEOREM 4.1 ([14]). *Let K be a knot in a lens space and let $\widetilde{K}$ denote its lifting to S^3. If $\widetilde{K}$ is a non-trivial knot then $L(p,q)(K;\gamma) \not\cong L(p,q)$, for all curves $\gamma \neq \mu$, where μ denotes a meridian of K.*

So, the problem remains open if $\widetilde{K}$ is the trivial knot or if it has more then one component. The following result deals with the first case, since an example of knots covered by the trivial knot is given by the axes of the Heegaard decomposition.

PROPOSITION 4.1 ([34]). *Let $L(p,q) = V \cup_\varphi V'$ and consider the axes of the Heegaard splitting, that is the knots K and K' such that $V = N(K)$ and $V' = N(K')$. Then K and K' are diffeo-equivalent if and only if $q^2 \equiv \pm 1 \mod p$.*

Since the complement of both K and K' is homeomorphic to $S^1 \times D^2$, this proposition produces a counter-example to a Gordon-Luecke result in lens spaces $L(p,q)$ with $q^2 \not\equiv \pm 1 \mod p$. The following result characterizes the cosmetic pairs $(L(p,q),K))$, when K is non-hyperbolic, non-local and not an axis.

THEOREM 4.2 ([35]). *There exists two infinite families $\mathcal{F}_{\mathrm{Sft}}$ and $\mathcal{F}_{\mathrm{Sat}}$ of cosmetic pairs (M,K) which satisfy all the following properties:*

1) M is a lens space;

2) *if* $(M,K) \in \mathcal{F}_{\mathrm{Sft}}$ *then* $M_K = M - N(K)$ *is a Seifert fibered manifold;*

3) *if* $(M,K) \in \mathcal{F}_{\mathrm{Sat}}$ *then* $M_K = M - N(K)$ *contains an essential 2-torus;*

4) *if* (M,K) *is a cosmetic pair such that* M *is a lens space and* K *is a non-hyperbolic knot, which is neither in a 3-ball nor an axis of* M*, then* $(M,K) \in \mathcal{F}_{\mathrm{Sft}} \cup \mathcal{F}_{\mathrm{Sat}}$*, up to diffeomorphism;*

5) K *admits admits a single non-trivial slope* γ *such that* $M(K;\gamma) \cong M$*;*

6) *there exists a homeomorphism on* M_K *which sends a meridian* μ_K *of* K *to* γ*;*

7) *if* (M,K) *and* (M,K') *both lie in* $\mathcal{F}_{\mathrm{Sft}}$ *and* $\mathcal{F}_{\mathrm{Sat}}$ *and* $M_K \cong M_{K'}$ *then* K' *is isotopy equivalent to* K*; .*

8) *all the diffeomorphisms between* M *and* $M(K;\gamma)$ *are orientation reversing.*

If we focus on non-hyperbolic knots, the case of Proposition 4.1 is the only counter-example of a Gordon-Luecke result.

PROPOSITION 4.2 ([35, 39]). *Non-hyperbolic knots in lens spaces are determined by their complement, except the axes in* $L(p,q)$ *when* $q^2 \equiv \pm 1 \bmod p$.

On the contrary, the example of [4] is hyperbolic and the surgery is *exotic*, that is, it does not satisfy the requirement 6) of Theorem 4.2. The following result deals with a family of knots whose covering in S^3 is a p-component link: that is nullhomologous knots (i.e., knots whose homology class in $H_1(L(p,q))$ is zero).

THEOREM 4.3 ([21]). *Nullhomologous knots in lens spaces are determined by their complements.*

Next theorem instead of characterizing the knots, deals with the coefficients of the lens spaces.

THEOREM 4.4 ([21]). *If* p *is square-free, then all knots (except eventually the axes) in* $L(p,q)$ *are determined by their complement.*

As shown by previous results, for non-hyperbolic knots the cosmetic surgery and complement problems are well understood. On the contrary, the hyperbolic case is still open, and there is also a lack of examples.

Question 2. Is it possible to characterize non-diffeomorphic hyperbolic knots having diffeomorphic complements? Is it possible to find new examples?

5. Link lifting in the 3-sphere

In this section we deal with the lifting of a link $L \subset L(p,q)$ under the universal covering map, describing how to find a diagram for the lifted link starting from a disc diagram representation.

Let L be a link in the lens space $L(p,q)$; we denote by $\widetilde{L} = P^{-1}(L)$ the *lift* of L in S^3 under the universal covering map $P\colon S^3 \to L(p,q)$. If L has ν components, then, from the monodromy map $\omega_P : H_1(L(p,q)) \to \mathbb{Z}_p$, the lift of the i-th component L_i has $\gcd(\delta_i, p)$ components, where δ_i is the homology class of L_i, arbitrary oriented[†], in $H_1(L(p,q)) \cong \mathbb{Z}_p$. As a consequence, the number of components of $\widetilde{L}$ is

$$\sum_{i=1}^{\nu} \gcd(\delta_i, p).$$

The construction of the lift via grid presentation can be found in [23]. In the following we explain the construction of a diagram for $\widetilde{L} \subset S^3$, starting from a disk diagram of $L \subset L(p,q)$.

Let B_t be the braid group on t strands and let $\sigma_1, \dots, \sigma_{t-1}$ be the Artin generators of B_t. Consider the inverse of the Garside braid Δ_t^{-1} on t strands illustrated in Figure 10 and defined by $(\sigma_{t-1}\sigma_{t-2}\cdots\sigma_1)(\sigma_{t-1}\sigma_{t-2}\cdots\sigma_2)\cdots(\sigma_{t-1})$. This braid can be seen also as a positive half-twist of all the strands and it belongs to the center of the braid group. Moreover Δ_t^{-1} can be represented by $(\sigma_{t-1}^{-1}\sigma_{t-2}^{-1}\cdots\sigma_1^{-1})(\sigma_{t-1}^{-1}\sigma_{t-2}^{-1}\cdots\sigma_2^{-1})\cdots(\sigma_{t-1}^{-1})$.

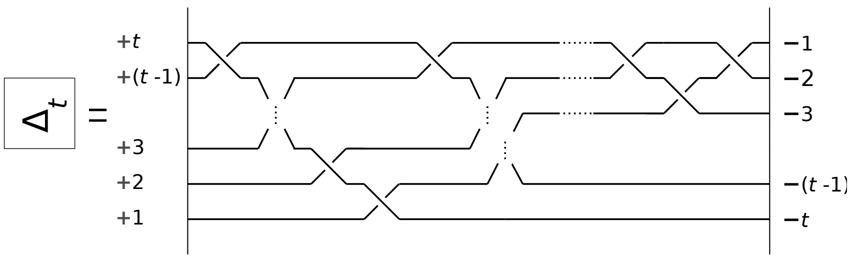


Figure 10: The Garside braid Δ_t.

In order to construct a diagram for $\widetilde{L}$, we introduce the notion of standard diagram: a disc diagram is called *standard* if the labels on its boundary points, read according to the orientation on ∂B_0^2, are $+1, \dots, +t, -1, \dots, -t$. Any link admits a representation by a standard diagram (see [31]). An example of a standard diagram is depicted in Figure 11.

THEOREM 5.1 ([31]). *Let L be a link in the lens space $L(p,q)$ and D be a standard disk diagram for L; then a diagram for the lift $\widetilde{L} \subset S^3$ can be found as follows (refer to Figure 12):*

[†]Note that even if δ_i depends on the orientation chosen for L_i, the number of the components does not depend on it.

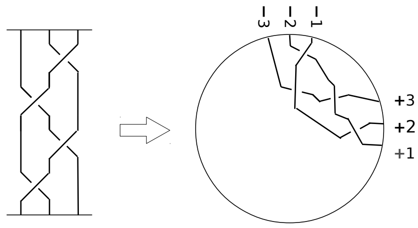


Figure 11: The braid $B = \sigma_2\sigma_1\sigma_2\sigma_1$ becomes a standard disk diagram.

- *consider p copies $D_1, \ldots, D_p$ of the standard disk diagram D;*
- *for each $i = 1, \ldots, p-1$, connect the diagram D_{i+1} with the diagram D_i via the braid Δ_t, joining the boundary point $-j$ of D_{i+1} with the boundary point $+j$ of D_i;*
- *connect D_1 with D_p via the braid Δ_t^{1-2q}, where the boundary points are connected as before.*

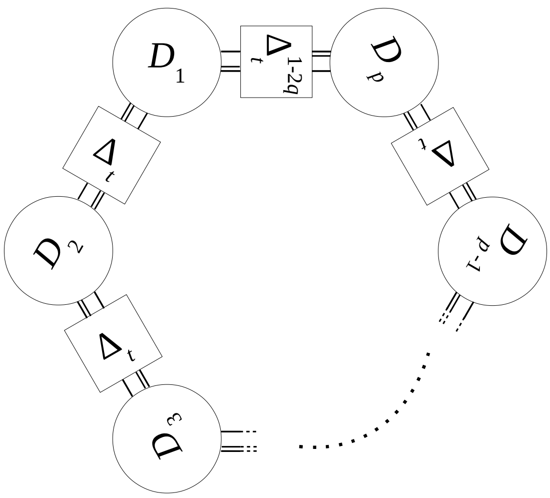


Figure 12: Diagram of the lift in S^3 of a link in $L(p,q)$.

Another planar diagram of the lift, with smaller number of crossing points, is described in the following result. Let $\overline{D}$ be the diagram obtained by mirroring D across a line and then exchanging all overpasses/underpasses.

PROPOSITION 5.1 ([31]). *Let L be a link in the lens space $L(p,q)$ and let D be a standard disk diagram for L; then a diagram for the lift $\widetilde{L} \subset S^3$ can be found as follows (refer to Figure 13):*

- *consider p copies $D_1, \dots, D_p$ of the standard disk diagram D and set $F_i = D_i$ if i is odd, and $F_i = \overline{D}_i$ otherwise;*
- *for each $i = 1, \dots, p-1$, connect the diagram F_{i+1} with the diagram F_i via a trivial braid, joining the boundary point $-j$ of F_{i+1} with the boundary point $+j$ of F_i;*
- *connect F_1 with F_p via the braid Δ_t^{p-2q}, where the boundary points are connected as before.*

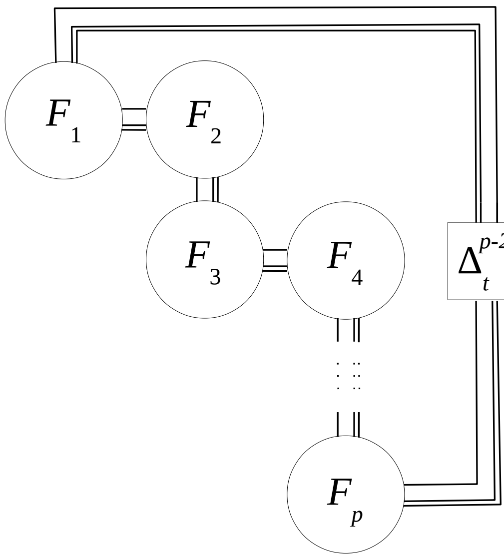


Figure 13: Another diagram of the lift in S^3 of a link in $L(p,q)$.

We can construct a link $L \subset L(p,q)$ starting from a braid B on t strands by considering the standard disk diagram where the braid B has the two ends of its strands on the boundary, indexed respectively by the points $(+1, \dots, +t)$ and $(-1, \dots, -t)$. See Figure 11 for an example. In this case, we say that B represents L.

PROPOSITION 5.2 ([31]). *If $L \subset L(p,q)$ is a link represented by the braid B on t strands, then the lift $\widetilde{L}$ is the link obtained by the closure in S^3 of the braid $(B\Delta_t)^p\Delta_t^{-2q}$.*

The braid construction of the lift can be used in order to find different links in lens spaces with isotopic lifts. This proves that the lift is not a complete invariant for link in lens spaces.

A first example of different knots with the same lift, is given by the axes of the solid tori of the Heegaard decomposition of $L(p,q)$. They both lift to the trivial knot[‡] in S^3, but by Proposition 4.1, they are diffeo-equivalent if and only if $q^2 \equiv \pm 1$ mod p. An example is depicted in Figure 14, for $L(p,2)$ or $L(p,p-2)$: in this case the two knots are isotopic for $p=3$, diffeo-equivalent but not isotopic for $p=5$ and not diffeo-equivalent if $p>5$.

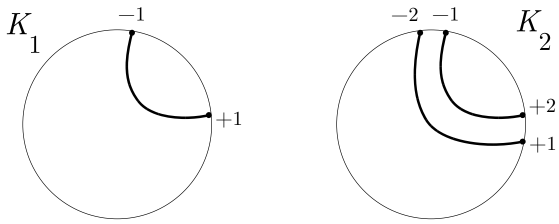


Figure 14: Two non-isotopic knots with trivial lift in $L(p,2)$ or $L(p,p-2)$, with $p>3$.

Another example is given by the links L_A and L_B in $L(4,1)$ depicted in Figure 15. They are non-diffeo-equivalent because they have a different number of components, but they have the same lift, the Hopf link.

The previous two examples are not completely satisfying, because it is easy to distinguish the two pairs of links (different number of components or different homology classes). In [31] an infinite family of non isotopic links with the same number of components, same homology class and same lifting is shown, using satellite links of the previous example. A couple of these links is depicted in Figure 25. In Section 7.2 we will show that these two links are non-isotopic since they have different HOMFLY-PT polynomial.

Question 3. Is it possible to find hyperbolic knots/links not diffeo-equivalent, but having diffeomorphic lifts? In [8] the behavior of the lift with respect to the diffeo-equivalence is investigated.

As we see in the previous section the lifting can be used to study the cosmetic surgery and complement problems.

Question 4. Is it possible to characterize the knots in $L(p,q)$ lifting to the trivial knot in S^3? In [14] an answer is given in the case of the projective space $L(2,1)$.

[‡]Note that the trivial knot in S^3 has multiple invariant presentations under the action of the symmetry $\theta_{p,q}$, distinguished by the restriction of the action to the trivial knot.

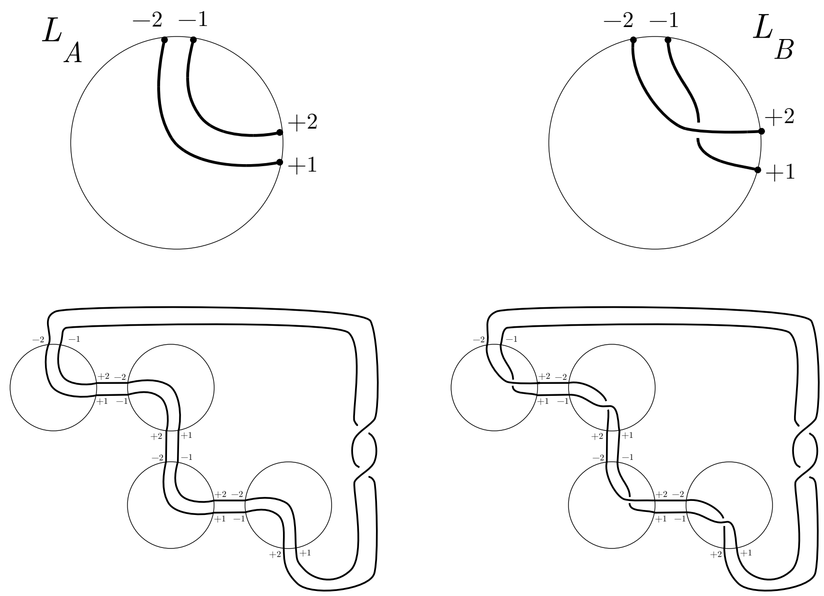


Figure 15: Non-diffeo-equivalent links with Hopf link lift in $L(4,1)$.

6. Fundamental group of link complements

One of the most important invariants of links in S^3 is the link group, i.e., the fundamental group of the link complement, which is, of course, an invariant of the link complement. As discussed in Section 4, the knot complement in lens spaces is not as strong as in S^3 in distinguishing knots. In the same way some results regarding the knot group in S^3 are no longer true in lens spaces. For example, a theorem of Norwood [37] states that every non-trivial knot in the 3-sphere admitting a 2-generator presentation for its group is prime: this result does not hold in lens spaces (see [30]). Another result states that the knot group classifies prime knot in S^3 (see for example [28, Theorem 6.1.12]). This is no longer true in lens spaces: the knots K_1 and K_2 of Figure 14 are non-isotopy equivalent, but an easy computation using Theorem 6.1 shows that they have both knot group isomorphic to $\mathbb{Z}$.

Given $L \subset L(p,q)$, the universal covering map $\theta_{p,q}$ induce, by restriction, a covering map $S^3 - N(\widetilde{L}) \to L(p,q) - N(L)$. As a consequence the fundamental group of the complement of L admits an index p subgroup isomorphic to the link group of a link in S^3. Particularly, if L is a *local* link (i.e., a link contained in a ball embedded in $L(p,q)$) the link group has the form $G * \mathbb{Z}_p$, where G is isomorphic to the link group of a link in S^3.

Question 5. Suppose that the link group of a link L in $L(p,q)$ can be decomposed as $G * \mathbb{Z}_p$: is it true that L is a local link?

In the next subsections we describe how to find a presentation for the link group, according to the different representations described in Section 3. As we will see, in all presentations generators correspond to "overpasses" of the diagram plus an element generating $\pi_1(L(p,q),*)$. On the contrary relations, beside those of Wirtinger type, depend on the chosen representation. Having different presentation for the link group, allows us to choose the one that works better with respect to the example or the problem we are dealing with.

6.1. Via disk diagrams

Let L be a link in $L(p,q)$, and consider a disk diagram of L. Fix an orientation for L, which induces an orientation on both L' and $\mathbf{p}(L')$ (see notation of Section 3.1). Perform an R_1 move on each overpass of the diagram having both endpoints on the boundary of the disk; in this way every overpass has at most one boundary point. Then label the overpasses as follows: $A_1,\dots,A_t$ are the ones having the boundary point labelled $+1,\dots,+t$, while $A_{t+1},\dots,A_{2t}$ are the overpasses having the boundary point labelled $-1,\dots,-t$. The remaining overpasses are labelled by $A_{2t+1},\dots,A_r$. For each $i=1\dots,t$, let $\varepsilon_i=+1$ if, according to the link orientation, the overpass A_i starts from the point $+i$; otherwise, if A_i ends in the point $+i$, let $\varepsilon_i=-1$.

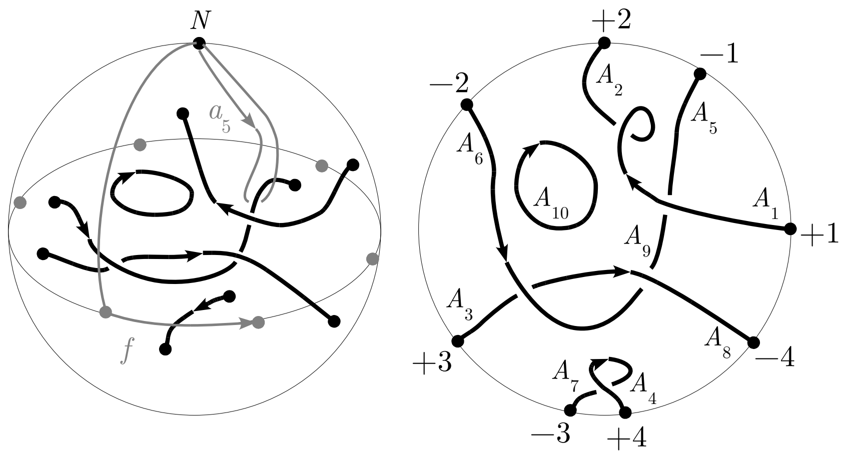


Figure 16: Example of overpasses labelling for a link in $L(6,1)$.

Associate to each overpass A_i a generator a_i, which is a loop around the overpass as in the classical Wirtinger theorem, oriented following the left hand rule and based on

N. Moreover let f be the generator of the fundamental group of the lens space depicted in Figure 16. The relations are the following:

W: $w_1, \ldots, w_s$ are the classical Wirtinger relations for each crossing, that is to say $a_i a_j a_i^{-1} a_k^{-1} = 1$ or $a_i a_j^{-1} a_i^{-1} a_k = 1$, according to Figure 17;

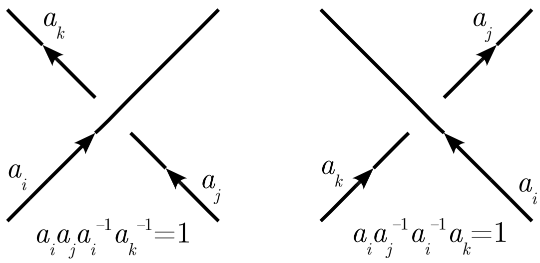


Figure 17: Wirtinger relations.

L: l is the lens relation $a_1^{\varepsilon_1} \cdots a_t^{\varepsilon_t} = f^p$;

M: $m_1, \ldots, m_t$ are relations (of conjugation) between loops corresponding to overpasses with identified endpoints on the boundary. If $t = 1$ the relation is $a_2^{\varepsilon_1} = a_1^{-\varepsilon_1} f^q a_1^{\varepsilon_1} f^{-q} a_1^{\varepsilon_1}$. Otherwise, consider the point $-i$ and, according to equator orientation, let $+j$ and $+j+1$ (mod t) be the type $+$ points next to it on each side. We distinguish two cases:

- if $-i$ lies on the diagram between -1 and $+1$ (including the case $-i = -1$), then the relation m_i is

$$a_{t+i}^{\varepsilon_i} = \Big(\prod_{k=1}^{j} a_k^{\varepsilon_k}\Big)^{-1} f^q \Big(\prod_{k=1}^{i-1} a_k^{\varepsilon_k}\Big) \quad a_i^{\varepsilon_i} \quad \Big(\prod_{k=1}^{i-1} a_k^{\varepsilon_k}\Big)^{-1} f^{-q} \Big(\prod_{k=1}^{j} a_k^{\varepsilon_k}\Big);$$

- otherwise, the relation m_i is

$$a_{t+i}^{\varepsilon_i} = \Big(\prod_{k=1}^{j} a_k^{\varepsilon_k}\Big)^{-1} f^{q-p} \Big(\prod_{k=1}^{i-1} a_k^{\varepsilon_k}\Big) \quad a_i^{\varepsilon_i} \quad \Big(\prod_{k=1}^{i-1} a_k^{\varepsilon_k}\Big)^{-1} f^{p-q} \Big(\prod_{k=1}^{j} a_k^{\varepsilon_k}\Big).$$

THEOREM 6.1 ([9]). *Let L be a link in $L(p,q)$, then*

$$\pi_1(L(p,q) - L, *) = \langle a_1, \ldots, a_r, f \mid w_1, \ldots, w_s, l, m_1, \ldots, m_t \rangle.$$

COROLLARY 6.1. *Let L be a link in $L(p,q)$, with components $L_1, \ldots L_\nu$. Choose an arbitrary orientation on each component of L and, for each $j = 1, \ldots, \nu$, let $\delta_j = [L_j] \in \mathbb{Z}_p = H_1(L(p,q))$. Then*

$$H_1(L(p,q) - L) \cong \mathbb{Z}^\nu \oplus \mathbb{Z}_d,$$

where $d = \gcd(\delta_1, \ldots, \delta_\nu, p)$.

6.2. Via grid diagrams

Let L be a link in $L(p,q)$ and let $G = (T, \alpha, \beta, \mathbb{O}, \mathbb{X})$ be a diagram of grid number n representing L. Up to stabilizations, we can suppose that the grid diagram does not contain a pair of X and O markings in the same position (as the one depicted Figure 22). Using notation of Section 3.2 and referring to Figure 18, denote with $B_1, \ldots, B_p$ (numbered left to right) the boxes of the diagram. Each box is divided by the β curves into n bands: denote with B_{ij} the i-th band (numbered left to right) in the j-th box for $i = 1, \ldots, n$ and $j = 1, \ldots, p$. Notice that the columns of the diagram are $B_{i1} \cup \cdots \cup B_{ip}$, for $i = 1, \ldots, n$.

Depict the projection of the link L into the diagram, by connecting the O markings with the X ones, according to conventions, and fix a point P inside the solid torus of the Heegaard splitting having β curves as meridians. For each $i = 1, \ldots, n$, let j be the first index such that B_{ij} contains a vertical strand of the link. Denote this vertical segment with V_i and associate to it a generator v_i starting from P, going around V_i once and coming back to P; we require that v_i intersects the diagram only in B_{ij} and we choose its orientation such that it enters to the left of V_i (with respect to the orientation of α_0). Moreover we associate a generator f to β_0: it starts from N goes once around the arc of β_0 separating B_1 and B_2; we orient it such that it enters into the diagram in B_{p1} and comes out in B_{12}.

There are n relators, one for each α curve (the dotted horizontal ones in figure) which are $r_h = \Pi_{j=1}^{p} v_1^{\varepsilon_{1j}} \cdots v_n^{\varepsilon_{nj}} f$ where $\varepsilon_{ij} = 1$ if α_h intersects the arc of L contained in B_{ij} and zero otherwise, for $h = 0, \ldots, n-1$.

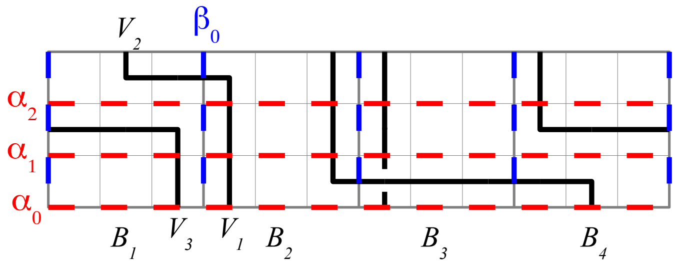


Figure 18: An example with grid number 3 in $L(4,1)$.

THEOREM 6.2. *Let L be a link in $L(p,q)$, and let $G = (T, \alpha, \beta, \mathbb{O}, \mathbb{X})$ be a grid* *diagram of grid number n representing L. Then*

$$\pi_1(L(p,q)-L,P) = \langle v_1,\ldots,v_n,f \mid r_0,\ldots,r_{n-1}\rangle,$$

with $r_h = \Pi_{j=1}^{p} v_1^{\varepsilon_{1j}} \cdots v_n^{\varepsilon_{nj}} f$ *where* $\varepsilon_{ij} = 1$ *if* α_h *intersects the arc of* L *contained in* B_{ij} *and zero otherwise, for* $h = 0,\ldots,n-1$.

Proof. This result, as well as the proof, generalizes the presentation for the link group described in [33] and [36] for links in S^3.

The link L is obtained from the union of curves immersed in T^2 by pushing the n horizontal arcs into V_α and the n vertical ones into V_β. We have that $\pi_1(V_\beta - L,P)$ is the free group generated by $v_1,\ldots v_n, f$. Moreover, $L(p,q)-L$ is homotopic to the space obtained by gluing a disc along each α_h and so a presentation for $\pi_1(L(p,q)-L,P)$ is obtained by adding to $\pi_1(V_\beta - L,P)$ the corresponding relators which are exactly r_h, for $h = 0,\ldots,n-1$. □

6.3. Via mixed link diagrams

Let L be a link in $L(p,q)$. Consider $L(p,q)$ as the result of rational Dehn surgery along the trivial knot U with framing index $-p/q$, and consider a mixed link diagram D for L.

Fix an orientation on L and on U, and consider the induced orientation on D. In order to find a presentation for $\pi_1(L(p,q)-L)$, we describe the generators referring to Figure 19. Call $D_1,\ldots,D_k$ the overpasses of L and $B_1,\ldots,B_h$ the overpasses of U. Fix a point C on U and let B_1 be the overpass containing C. Consider a loop based on N for every overpass, oriented following the left hand rule with respect to the orientation of the link, and call them $d_1,\ldots,d_k$ and $b_1,\ldots,b_h$ respectively. Let l be a loop which is nullhomologous in $S^3 - U$, starting from N, going to C avoiding all overpasses, following U along its orientation and then returning to N.

The relations are the following:

W: $w_1,\ldots w_s$ are the classical Wirtinger relations concerning both D and B type of overpasses, that is that is to say $a_i a_j a_i^{-1} a_k^{-1} = 1$ and or $a_i a_j^{-1} a_i^{-1} a_k = 1$, according to Figure 17, where a_i can take the value d_i or b_i;

X: $l = x$, where x is the product of the type d generators corresponding to the D overpasses crossing U starting from C and moving along U according to its orientation;

K: $b_1^p l^{-q} = 1$.

THEOREM 6.3. *Let L be a link in* $L(p,q)$*, then*

$$\pi_1(L(p,q)-L,N) = \langle d_1,\ldots,d_k,b_1,\ldots,b_h,l \mid w_1,\ldots,w_s,\ \ l = x,\ \ b_1^p l^{-q} = 1\rangle.$$

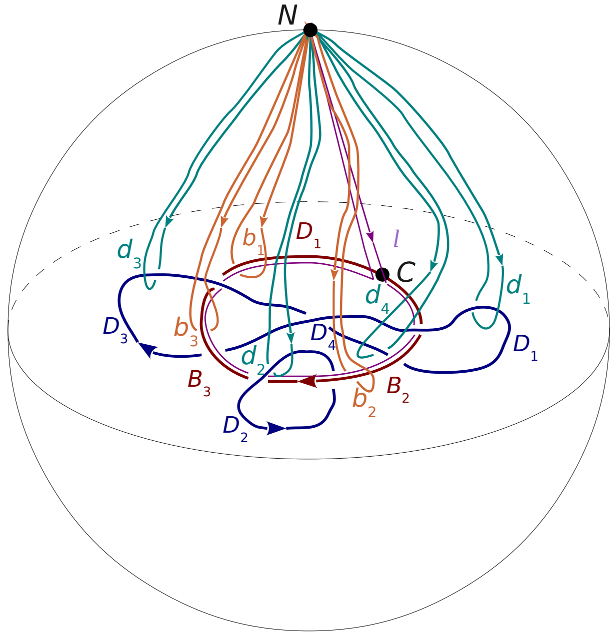


Figure 19: Generators of $\pi_1(L(p,q)-L)$.

Proof. According to the notation of Section 3.3, we have $L(p,q)-N(L) = (S^3 - N(U \cup L)) \cup_{\psi_{p,q}} S^1 \times D^2$, where the map induced by $\psi_{p,q} : \partial(S^1 \times D^2) \to \partial(S^3 - N(U))$ on the first homology groups satisfies

$$\begin{cases} \psi_{p,q}(\beta) = pm - ql \\ \psi_{p,q}(\alpha) = rm - sl, \end{cases}$$

for some $r, s \in \mathbb{Z}$ such that $|ps - qr| = 1$, where β and α are, respectively, a standard meridian and a parallel of the solid torus. We have $\pi_1(S^1 \times D^2) = \langle \alpha, \beta \mid \beta = 1 \rangle$ and $\pi_1(S^3 - N(U \cup L)) = \langle d_1, \ldots, d_k, b_1, \ldots, b_h \mid w_1, \ldots, w_s \rangle$. In $\pi_1(S^3 - N(U \cup L))$, we have $ml = lm$, since they belong to an embedded torus, so $\psi_{p,q}(\beta) = m^p l^{-q}$ and $\psi_{p,q}(\alpha) = m^r l^s$. Moreover, we can choose m such that $m = b_1$ in $\pi_1(L(p,q)-L)$ and so $x = l$. Now the result follows by the Seifert-Van Kampen Theorem. $\square$

7. Polynomial invariants

In this section we deal with polynomial invariants of links in the lens space. The first subsection is devoted to twisted Alexander polynomials: these invariants are rather invariants of the link group, and so are of "diffeo-type" (i.e., they don't detect non-isotopic diffeo-equivalent links). Moreover, even if one of these polynomials (the one corresponding to the trivial twisting) generalizes the classical Alexander polynomial in S^3, the possibility of having "different" Alexander polynomials, according to different twisting, is peculiar of lens spaces, since it depends on the torsion part of the homology of the link complement. In the second subsection we investigate a "skein type" invariant: the HOMFLY-PT polynomial which, on the contrary, is an invariant of isotopy type.

7.1. Twisted Alexander polynomial

In this section we collect some general results on twisted Alexander polynomials of links in lens spaces. We briefly recall the definition (for further reference see [41, §II.5]).

Given a finitely generated group π, denote with $H = \pi/\pi'$ its abelianization and let $G = H/\mathrm{Tors}(H)$. Take a presentation $\pi = \langle x_1,\dots,x_m \mid r_1 \dots, r_n\rangle$ and consider the Alexander-Fox matrix A associated to the presentation, that is $A_{ij} = \mathrm{pr}(\frac{\partial r_i}{\partial x_j})$, where pr is the natural projection $\mathbb{Z}[F(x_1,\dots,x_m)] \to \mathbb{Z}[\pi] \to \mathbb{Z}[H]$ and $\frac{\partial r_i}{\partial x_j}$ is the Fox derivative of r_i. Moreover let $E(\pi)$ be the first elementary ideal of π, which is the ideal of $\mathbb{Z}[H]$ generated by the $(m-1)$-minors of A. For each homomorphism $\sigma : \mathrm{Tors}(H) \to \mathbb{C}^* = \mathbb{C} - \{0\}$ we can define a twisted Alexander polynomial $\Delta^\sigma(\pi)$ of π as follows: fix a splitting $H = \mathrm{Tors}(H) \times G$ and consider the ring homomorphism that we still denote with $\sigma : \mathbb{Z}[H] \to \mathbb{C}[G]$ sending (f, g), with $f \in \mathrm{Tors}(H)$ and $g \in G$, to $\sigma(f)g$, where $\sigma(f) \in \mathbb{C}^*$. Since the ring $\mathbb{C}[G]$ is a unique factorization domain, we can define $\Delta^\sigma(\pi)$ as $\gcd(\sigma(E(\pi))$. This is an element of $\mathbb{C}[G]$ defined up to multiplication by elements of G and non-zero complex numbers. If $\Delta(\pi)$ denote the classic multivariable Alexander polynomial, we have $\Delta^1(\pi) = \alpha\Delta(\pi)$, with $\alpha \in \mathbb{C}^*$.

If $L \subset L(p,q)$ is a link in a lens space then the σ-*twisted Alexander polynomial* of L is $\Delta^\sigma_L = \Delta^\sigma(\pi_1(L(p,q) - L, *))$. Since in this case $\mathrm{Tors}(H) = \mathbb{Z}_d$ then $\sigma(\mathrm{Tors}(H))$ is contained in the cyclic group generated by ζ, where ζ is a d-th primitive root of the unity. When $\mathbb{Z}[\zeta]$ is a principal ideal domain, in order to define Δ^σ_L we can consider the restriction $\sigma : \mathbb{Z}[H] \to \mathbb{Z}[\zeta][G]$. Note that $\Delta^\sigma_L \in \mathbb{Z}[\zeta][G]$ is defined up to multiplication by $\zeta^h g$, with $g \in G$.

PROPOSITION 7.1 ([9]). *Let L be a local link in $L(p,q)$. Then $\Delta^\sigma_L = 0$ if $\sigma \neq 1$, and $\Delta_L = p \cdot \Delta_{\bar{L}}$ otherwise, where $\bar{L}$ is the link L considered as a link in S^3.*

As a consequence, a knot with a non trivial twisted Alexander polynomial cannot be local.

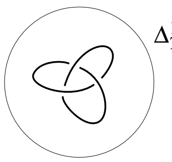
PROPOSITION 7.2 ([41]). *Let L be a knot in a lens space then:*

- $\Delta^\sigma_L(t) = \Delta^\sigma_L(t^{-1})$ *(i.e., the twisted Alexander polynomial is symmetric);*
- $\Delta(1) = |\mathrm{Tors}(H_1(L(p,q) - L))|$.

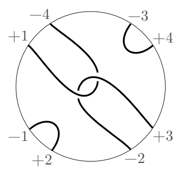
We say that a link $L \subset L(p,q)$ is *nontorsion* if $\mathrm{Tors}(H_1(L(p,q)-L)) = 0$, otherwise we say that L is *torsion*. Note that a local link L in a lens space is obviously torsion.

THEOREM 7.1 ([9]). *Let L be a link in $L(p,q)$ and let τ^σ be the σ-twisted Reidemeister torsion of L. If L is a nontorsion knot and t is a generator of its first homology group, then $\Delta_L^\sigma = \tau_L^\sigma \cdot (t-1)$. Otherwise $\Delta_L^\sigma = \tau_L^\sigma$.*

Figure 20 shows the twisted Alexander polynomials of a local trefoil knot in $L(4,1)$ and proves that twisted Alexander polynomial may distinguish knots with the same Alexander polynomial.

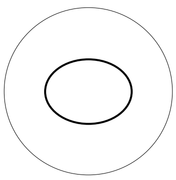


$\Delta_T^1 = 4(t^2-t+1)$
$\Delta_T^{-1} = 0$
$\Delta_T^i = 0$
$\Delta_T^{-i} = 0$

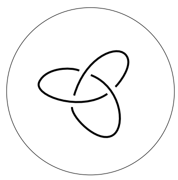


$\Delta_K^1 = 4(t^2-t+1)$
$\Delta_K^{-1} = 0$
$\Delta_K^i = 2(t-1)$
$\Delta_K^{-i} = 2(t-1)$

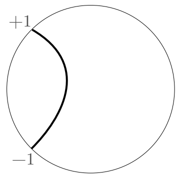
Figure 20: Twisted Alexander polynomials of two knots in $L(4,1)$.



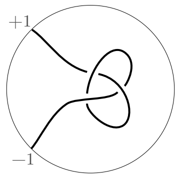
$\Delta_{K_0}^1 = 4$
$\Delta_{K_0}^{-1} = 0$
$\Delta_{K_0}^i = 0$
$\Delta_{K_0}^{-i} = 0$



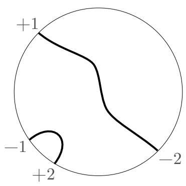
$\Delta_{K_0\sharp\overline{T}}^1 = 4(t^2-t+1)$
$\Delta_{K_0\sharp\overline{T}}^{-1} = 0$
$\Delta_{K_0\sharp\overline{T}}^i = 0$
$\Delta_{K_0\sharp\overline{T}}^{-i} = 0$



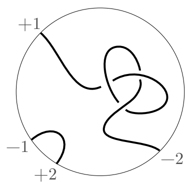
$\Delta_{K_1}^1 = 1$



$\Delta_{K_1\sharp\overline{T}}^1 = t^2-t+1$



$\Delta_{K_2}^1 = t+1$
$\Delta_{K_2}^{-1} = 1$



$\Delta_{K_2\sharp\overline{T}}^1 = (t+1)(t^4-t^2+1)$
$\Delta_{K_2\sharp\overline{T}}^{-1} = t^4+t^2+1$

Figure 21: Twisted Alexander polynomials of three knots in $L(4,1)$.

In Figure 21 we list the twisted Alexander polynomials of the connected sum of a trefoil knot $\overline{T}$ in S^3 with the three knots $K_0, K_1, K_2 \subset L(4,1)$ depicted in the left part of the figure, respectively. Note that for the case of $K_2 \sharp \overline{T}$, the map σ_2, that is the restriction of σ to $\mathbb{Z}[j_2(H_1(S^3 - \overline{T}))]$, sends the generator $g \in \mathbb{Z}[H_1(S^3 - \overline{T})]$ in $t^2 \in \mathbb{Z}[H_1(L(p,q) - K_2 \sharp \overline{T})]$ (resp. in $-t^2$) if $\sigma = 1$ (resp. if $\sigma = -1$), instead of t as in the case of classical Alexander polynomial.

An interesting question regards the connection, if any, between the twisted Alexander polynomials of a link L in a lens space and the Alexander polynomial of the lift $\widetilde{L}$ in S^3. In [25] there is a formula connecting them if both L and $\widetilde{L}$ are knots.

Proposition 7.2 ([25])**.** *Let K be a knot in $L(p,q)$ such that $\widetilde{K} \subset S^3$ is a knot. Denote with ζ a primitive p-root of unity. If the map $\sigma : \pi_1(L(p,q) - K, *) \to \mathbb{Z}[\zeta][t^{\pm 1}]$ is a representation for the knot group and the map $\tilde{\sigma} : \pi_1(S^3 - \widetilde{K}, *) \to \mathbb{Z}[t^{\pm 1}]$ is the lift of this representation, then*

$$\Delta_{\widetilde{K}}^{\tilde{\sigma}}(t^p) = \prod_{i=0}^{p-1} \Delta_K^{\sigma}(\zeta^i t).$$

In [13] some interesting characterizations for multi-variable Alexander of the lift of links in lens spaces are found.

In S^3 there are many equivalent ways to define the Alexander polynomial: via skein relations (as a specialization of the HOMFLY-PT polynomial), using Seifert surfaces, as the Euler characteric with respect to the Alexander grading of the Knot Floer homology, via braid group representations and so on.

Question 6. Do the different ways to define the Alexander polynomial generalize to lens spaces? Are they equivalent?

A partial answer is given in [19] where it is proved that the Alexander polynomial satisfies a skein relation. From this result another question arises.

Question 7. Do the twisted Alexander polynomials of links lens spaces satisfy a skein relation?

7.2. HOMFLY-PT polynomial

In this section we deal with the HOMFLY-PT polynomial developed in [15]. Throughout all the section we consider oriented link. We start by recalling its definition.

We say that a link in $L(p,q)$ is *pseudo-trivial* if it can be represented by a grid diagram satisfying the following conditions

- the markings in each box lie only on the principal diagonal (the one going from NW-corner to the SE-corner);
- all the O-markings are contained in the the first box (from the left);

- the X-markings in the same box are contiguous, and if the first box contains X-markings, one of them lies in the SE-corner;
- for each X-marking, all the other X-markings lying in a different box and in a row below, must lie in a column on the left.

A pseudo-trivial link will be denoted as $U_{i_0,i_1,\dots,i_{p-1}}$ where $i_j \in \mathbb{N}$ is the number of components of the link belonging to the j-th homology class. In Figure 22 is depicted the trivial link $U_{1,0,1,2} \subset L(4,1)$ having one 0-homologous component, zero 1-homologous components, one 2-homologous component and two 3-homologous components.

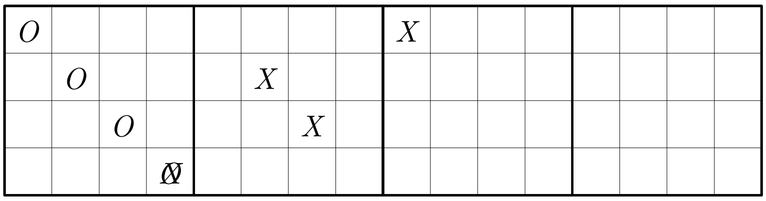


Figure 22: Grid diagram for the pseudo-trivial link $U_{1,0,1,2}$ in $L(4,1)$.

THEOREM 7.3 ([15]). *Let $\mathcal{L}$ be the set of isotopy classes of links in $L(p,q)$ and let $\mathcal{TL} \subset \mathcal{L}$ denote the set of isotopy classes of pseudo-trivial links. Define $\mathcal{TL}^* \subset \mathcal{TL}$ to be those pseudo-trivial links with no nullhomologous components. Let U be the isotopy class of the trivial knot (the one bounding an embedded disc). Given a value $J_{p,q}(T) \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ for every $T \in \mathcal{TL}^*$, there is a unique map $J_{p,q} \colon \mathcal{L} \to \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ such that*

- *$J_{p,q}$ satisfies the skein relation $a^{-p} J_{p,q}(L_+) - a^p J_{p,q}(L_-) = z J_{p,q}(L_0)$.*
- *$J_{p,q}(U) = \left(\frac{a^{-1}-a}{z}\right)^{p-1}$*
- *$J_{p,q}(U \sqcup L) = \left(\frac{a^{-p}-a^p}{z}\right) J_{p,q}(L)$*

As usual, the links L_+,L_-, and L_0 differ only in a small neighborhood of a double point: Figure 23 shows how this difference appears on grid diagrams. The HOMFLY-PT invariant produced by Theorem 7.3 is not yet a polynomial, in [15] the author suggests to produce a polynomial in two variables by defining $J_{p,q}$ on the pseudo-trivial links as the classic HOMFLY-PT polynomial of their lift in the 3-sphere: a natural question arises.

Question 8. Which is the connection between the HOMFLY-PT polynomial of a link in $L(p,q)$ and those of its lift in S^3?

In [12], some characterization of the HOMFLY-PT polynomial of link in S^3 that are lifts of link in lens spaces is given.

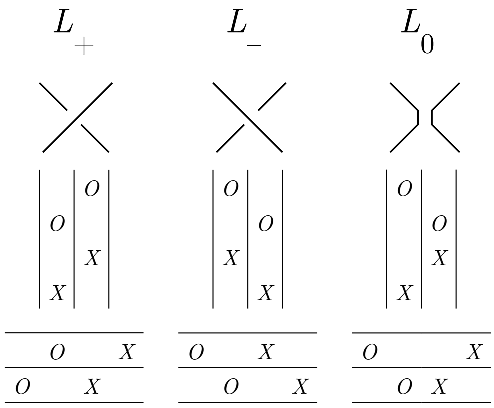


Figure 23: Grid skein relation.

PROPOSITION 7.3 ([10]). *Let L be an oriented link in $L(p,q)$ and denote with $-L$ the link obtained by reversing the orientation of all components. If the HOMFLY-PT invariant of L can be written as $J_{p,q}(L) = \sum a^k z^h J_{p,q}(U_{i_0,i_{p-1},i_{p-2},\dots,i_1})$, then $J_{p,q}(-L) = \sum a^k z^h J_{p,q}(U_{i_0,i_1,\dots,i_{p-2},i_{p-1}})$.*

Usually, in $L(p,q)$, the links L and $-L$ are not isotopy equivalent (since they are possibly homologically different). So, the last proposition suggests a way to construct examples of non-equivalent oriented links with the same lift in S^3 which are distinguished by the HOMFLY-PT invariant. Indeed, it is enough to find a link L which lifts to an invertible link and such that L is non-isotopic to $-L$. For example, the knots K_1 and $-K_1$ in $L(3,1)$ in Figure 24 are non-isotopic equivalent as oriented knots since the first one is 1-homologous whereas the second one is 2-homologous, but they both lift to the trivial knot in S^3 (note that K_1 is the knot of Figure 14, where the arc is oriented in such a way that the initial point is the one labelled with $+1$).

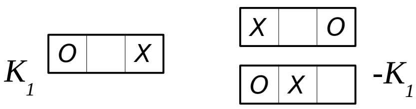


Figure 24: Grid diagrams for K_1 and $-K_1$ (two equivalent ones) in $L(3,1)$ both lifting to the trivial knot in S^3.

More generally, referring to Figures 14 and 15 and orienting all the arcs such that the $+$ point is the initial one, both the couples K_1, K_2 and L_A, L_B are pseudo-trivial links (see [10]) with isotopy equivalent oriented lifts. So, if we define $J_{p,q}$ on the pseudo-trivial links as the classic HOMFLY-PT polynomial of their lift in the 3-sphere, then the polynomial is not able to distinguish them.

Obviously this is not always the case. For example, the links $A_{2,2}$ and $B_{2,2}$ in $L(4,1)$ depicted in Figure 25 both lift to the Hopf link in S^3 (see [31]). Their HOMFLY-PT invariant is

$$\begin{aligned}
J_{4,1}(A_{2,2}) &= (a^{24} + 3a^{24}z^2 + a^{24}z^4)J_{4,1}(U_{0,0,2,0}) + \\
&\quad +(3a^{28}z + 4a^{28}z^3 + a^{28}z^5)J_{4,1}(U_{1,0,0,0}) + \\
&\quad +(3a^{24}z^2 + 4a^{24}z^4 + a^{24}z^6)J_{4,1}(U_{0,1,0,1}) \\
J_{4,1}(B_{2,2}) &= (a^{24} + 2a^{24}z^2 + a^{24}z^4)J_{4,1}(U_{0,0,2,0}) + \\
&\quad +(a^{28}z + 2a^{28}z^3 + a^{28}z^5)J_{4,1}(U_{1,0,0,0}) + \\
&\quad +(a^{24}z^2 + 2a^{24}z^4 + a^{24}z^6)J_{4,1}(U_{0,1,0,1}) + \\
&\quad +(a^2z + a^{20}z^3)J_{4,1}(U_{0,2,1,0}) + \\
&\quad +(a^2z + a^{20}z^3)J_{4,1}(U_{0,0,1,2}) + a^{24}z^2J_{4,1}(U_{0,2,0,2}).
\end{aligned}$$

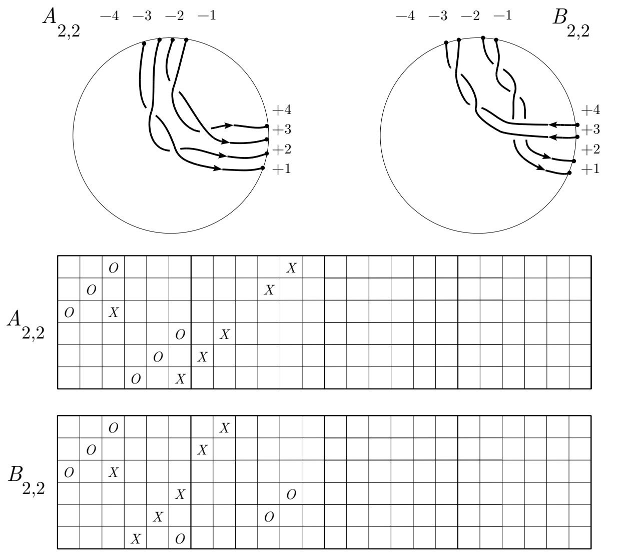


Figure 25: Grid diagrams for non-isotopic links in $L(4,1)$ with the same lift.

The lift of $U_{1,0,0,0}$ is the trivial link with four components, the lift of $U_{0,1,0,1}$ is the Hopf link (the closure of Δ_2^2), and $U_{0,2,1,0}$, $U_{0,2,0,2}$, $U_{0,0,1,2}$, $U_{0,0,2,0}$ lift to the closure of the braid Δ_4^2 (see [30]). Moreover, all the lifts have the orientation induced by orienting Δ_t from left to right (see Figure 10). So, if we set $J_{4,1}(L) := J_{1,0}(\widetilde{L})$ on pseudo-trivial links, we get the following different HOMFLY-PT polynomials

$$\begin{aligned}
J_{4,1}(A_{2,2}) \quad = \quad & a^9z^{-3} - 3a^{11}z^{-3} + 3a^{13}z^{-3} - a^{15}z^{-3} + 3a^{25}z^{-2} - 9a^{27}z^{-2} + \\
& +9a^{29}z^{-2} - 3a^{31}z^{-2} + 3a^9z^{-1} - 15a^{11}z^{-1} + 21a^{13}z^{-1} + \\
& -9a^{15}z^{-1} + 4a^{25} - 12a^{27} + 12a^{29} - 4a^{31} + a^9z - 25a^{11}z + \\
& +62a^{13}z - 38a^{15}z + 3a^{25}z - 3a^{27}z + a^{25}z^2 - 3a^{27}z^2 + \\
& +3a^{29}z^2 - a^{31}z^2 - 19a^{11}z^3 + 102a^{13}z^3 - 99a^{15}z^3 + 7a^{25}z^3 + \\
& -4a^{27}z^3 - 7a^{11}z^5 + 94a^{13}z^5 - 155a^{15}z^5 + 5a^{25}z^5 - a^{27}z^5 + \\
& -a^{11}z^7 + 46a^{13}z^7 - 129a^{15}z^7 + a^{25}z^7 + 11a^{13}z^9 - 56a^{15}z^9 + \\
& +a^{13}z^{11} - 12a^{15}z^{11} - a^{15}z^{13}
\end{aligned}$$

$$\begin{aligned}
J_{4,1}(B_{2,2}) \quad = \quad & a^9z^{-3} - 3a^{11}z^{-3} + 3a^{13}z^{-3} - a^{15}z^{-3} + 2a^5z^{-2} - 6a^7z^{-2} + \\
& +6a^9z^{-2} - 2a^{11}z^{-2} + a^{25}z^{-2} - 3a^{27}z^{-2} + 3a^{29}z^{-2} - a^{31}z^{-2} + \\
& +3a^9z{-1} - 15a^{11}z^{-1} + 21a^{13}z^{-1} - 9a^{15}z^{-1} + 2a^5 - 18a^7 + 30a^9 + \\
& -14a^{11} + 2a^{25} - 6a^{27} + 6a^{29} - 2a^{31} + a^9z - 25a^{11}z + \\
& +62a^{13}z - 38a^{15}z + a^{25}z - a^{27}z - 20a^7z^2 + 70a^9z^2 + \\
& -50a^{11}z^2 + a^{25}z^2 - 3a^{27}z^2 + 3a^{29}z^2 - a^{31}z^2 - 19a^{11}z^3 + \\
& +102a^{13}z^3 - 99a^{15}z^3 + 3a^{25}z^3 - 2a^{27}z^3 - 10a^7z^4 + 88a^9z^4 + \\
& -110a^{11}z^4 - 7a^{11}z^5 + 94a^{13}z^5 - 155a^{15}z^5 + 3a^{25}z^5 - a^{27}z^5 + \\
& -2a^7z^6 + 58a^9z^6 - 128a^{11}z^6 - a^{11}z^7 + 46a^{13}z^7 - 129a^{15}z^7 + \\
& +a^{25}z^7 + 18a^9z^8 - 74a^{11}z^8 + 11a^{13}z^9 - 56a^{15}z^9 + 2a^9z^{10} + \\
& -20a^{11}z^{10} + a^{13}z^{11} - 12a^{15}z^{11} - 2a^{11}z^{12} - a^{15}z^{13}.
\end{aligned}$$

As observed in the previous subsection, it is possible to obtain the Alexander polynomial of a link in S^3 by specializing the HOMFLY-PT polynomial: namely by imposing $a = 1$ and $z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$. The same substitutions do not work in lens spaces.

Question 9. Is it possible to obtain the Alexander polynomial as a specialization of the HOMFLY-PT invariant $J_{p,q}$? See also Question 6.

In [20] a presentation for the HOMFLY-PT skein module is computed for $L(p,1)$, while the problem of finding a presentation is still open when $q > 1$.

Question 10. How is the HOMFLY-PT invariant $J_{p,q}$ connected with the presentation of the HOMFLY-PT skein module found in [20]? Are the pseudo-trivial links free generators for the HOMFLY-PT skein module in $L(p,q)$ with $q > 1$?

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