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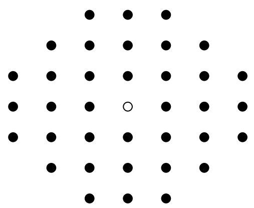
DE NUGIS GROEBNERIALIUM 3: SOLITAIRE, REIß, GRÖBNER*

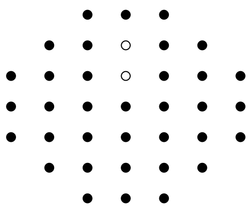
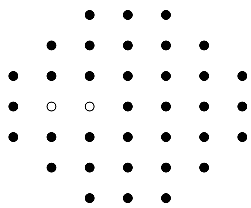
Abstract. The analysis of the Solitaire given by Reiß is here reformulated via groebner technology.

1. Solitaire

Solitaire † is a patience board game of which Leibniz was fond and whose first mention is to be found in a french book dating from 1668. The game is played on a board with thirtyseven holes in the following configuration



and thirtysix pieces, one in each hole except the central one.
The game entails removing one piece after another until a single piece is left; the only move allowed is like a capture in draughts move: one piece jumps over another into an empty hole, and the piece which was jumped over is removed. So, from the original position one could reach e.g. the configurations

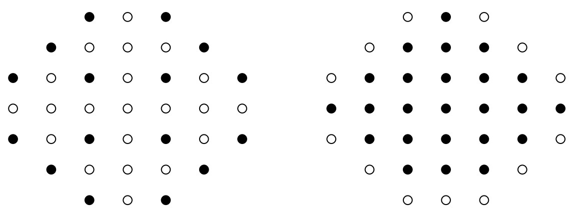
 

In the second half of the Nineteenth Century, it became popular a kind of puzzle whose

*This series [4, 5, 6] of notes, whose title mimics Walter Map's *De nugis curialium* [3] is devoted to trifles related to Buchberger Theory and its application.

†I learned all the historical informations on Solitaire, the quoted configurations and Reiß' solution from [1], which is the first in a series of books discussing arithmetical and combinatorial divertissements, some of which are now classical computer science problems like Hamiltonian paths, binary arithmetic, traversing labyrinths, the eight queens problem, etc.. The books are now available in a reprinted version [2].

aim was either to end with a specific configuration like



which are known as “The Four Evangelists and the Twelve Apostles” and “The Pentagon”, starting either by the original configuration or by any configuration using all the pieces (i.e. covering all the holes except one); or to begin with such configuration and remove all pieces except one[‡].

2. Reiß

This naturally raised the question of proving the impossibility of a given problem. A solution was proposed in Crelle by Reiß[§] [7], who extended the allowed configurations, allowing any number of pieces and that each hole could contain any number of them, and considered two configurations *congruent* if one of them is obtained by applying a move to the other one. Then he posed (and solved) the question of finding a set of canonical representatives for the congruence classes.

Following the presentation in [1] I assume that each hole could contain also a *negative* number of pieces and I remark that the result holds even in the case of an *infinite* board equivalent to (and indexed by) the lattice $\mathbb{Z}^2$ [¶].

Let us denote, $\forall (m,n) \in \mathbb{Z}^2, C(m,n)$ the configuration where there is a single piece in the hole indexed by (m,n) and no piece everywhere else, and by

$$\sum_{(m,n)\in\mathbb{Z}^2} \gamma(m,n)C(m,n)$$

the configuration which has exactly $\gamma(m,n)$ pieces in each hole indexed by (m,n).

In this setting it is easy to describe the results stated and proved by Reiß:

[‡] A less easy puzzle was already discussed by Leibniz, which, in a January 17, 1716 letter to Montmort, writes “I took it in the other direction, i.e. instead of undoing a piece composition according the rules of the game [...], I thought that it would be better to restore what has been destroyed [...]; in this way one could intend to achieve a specific configuration. [...] Why do so?, one could ask. I answer: to improve the art of inventing.”

[§] a student of Gauß which taught in Bruxelles and Frankfurt a. M.

[¶] In fact, Lucas embeds the board in the lattice $\mathbb{N}^2$ and describes his notation as a generalization of the one introduced for chess problems in [8] where “On trouve [...] une solution curieuse du problème du cavalier des échecs sur un échiquier cubique, et les premiers éléments scientifiques de la *géometrie des tissus à fils curvilignes*,” an argument in applied industrial mathematics in which Lucas worked a lot.

An embedding in $\mathbb{Z}^2$ is instead suggested by Reiß’s notation.

THEOREM 1. *The following holds:*

1. *It is possible to remove one piece from each of three contiguous holes in a row (column);*
2. *it is possible to remove two pieces from a hole;*
3. *a piece can jump two holes in both directions along a row (column);*
4. *the two configurations $C(m,n)$ and $C(m',n')$ are congruent iff $m \equiv m' \pmod 3$, and $n \equiv n' \pmod 3$;*
5. *each configuration C is equivalent to one of the following sixteen configurations:*

$$\gamma(0,0)C(0,0)+\gamma(1,0)C(1,0)+\gamma(0,1)C(0,1)+\gamma(1,1)C(1,1), \gamma(i,j)\in\mathbb{Z}_2.$$

3. Gröbner

When, in the Eighties, I read this result, which is one of the oldest non trivial application of the notion of “canonical form" I have ever seen, it seemed to me that the Gröbner technology would give an immediate proof of Reiß' result. In fact, it was sufficient

- to describe in terms of Laurent polynomials in $\mathbb{Z}[[X,Y]]$ the configurations associating $\sum_{(m,n)\in\mathbb{Z}^2}\gamma(m,n)C(m,n)$ to $\sum_{(m,n)\in\mathbb{Z}^2}\gamma(m,n)X^mY^n$
- and each Laurent polynomial

$$\sum_{(m,n)\in\mathbb{Z}^2}\gamma(m,n)X^mY^n\in\mathbb{Z}[[X,Y]]$$

to its canonical form in $\mathbb{Z}[X,T,Y,Z]$ modulo the ideal $(XT-1,YZ-1)$;

- to realize that, in this setting, the elementary moves are described by the polynomials

$$P := \left\{X^if_1, X^if_2, T^if_1, T^if_2, Y^if_3, Y^if_4, Z^if_3, Z^if_4, i\in\mathbb{N}\right\}$$

where

$$f_1 := T+1-X, f_2 := T-1-X, f_3 := Z+1-Y, f_3 := Z-1-Y.$$

and that, therefore, the problem posed by Reiß could be translated to

PROBLEM 1. *Describe the set of the canonical forms of the polynomials in $\mathbb{Z}[X,T,Y,Z]$ modulo the ideal $I\subset\mathbb{Z}[X,T,Y,Z]$ generated by*

$$P\cup\{XT-1,YZ-1\};$$

- to remark that

$$(1) \qquad 2 = (T+1-X) - (T-1-X), \quad 2 = (Z+1-Y) - (Z-1-Y),$$

- so that, as a consequence,

$$G := \{X^2+X-1, T-X-1, 2, Z-Y-1, Y^2+Y-1, XT-1, YZ-1\}$$

is a basis of I;

- to solve therefore the problem by computing the Gröbner basis of the ideal generated by G, using any degree compatible ordering $<$ s.t. $X < T < Y < Z$, which requires only

- to perform the Buchberger reductions

$$\begin{aligned} XT-1 &= X(T-X-1)+(X^2+X-1), \\ YZ-1 &= Y(Z-Y-1)+(Y^2+Y-1), \end{aligned}$$

in order

- to conclude that the required Gröbner basis is

$$\{X^2+X-1, T-X-1, 2, Z-Y-1, Y^2+Y-1\}$$

so that

$$\mathbb{Z}[X,T,Y,Z] \setminus I \cong \mathbb{Z}_2[X,Y] \setminus (X^2+X+1, Y^2+Y+1),$$

from which all the statements of Reiß's theorem follow immediately.

REMARK 1. Of course, there is no need to deal with Laurent polynomials, if, following Vandermonde and Lucas approach, we embed the board in $\mathbb{N}^2$ and we represent the configurations by polynomials in $\mathbb{Z}[X,Y]$; in this setting the elementary moves are the polynomials

$$P := \left\{X^i(X^2 \pm X-1), Y^i(Y^2 \pm Y-1), i \in \mathbb{N}\right\}$$

generating the ideal $I \subset \mathbb{Z}[X,Y]$ w.r.t. which we need to describe the set of the canonical forms. Once it is proved that $2 \in I$, it is evident that I is generated by the Gröbner basis $\{X^2+X-1, 2, Y^2+Y-1\}$.
However, the euclidean derivation

$$2 = (X-1)(X^2+X-1) - (X+1)(X^2-X-1) \in I$$

seems to me less elegant than (1), which moreover represents two effective moves.

REMARK 2. Solitaire could be directly interpreted as a game on a board of Boolean values, each move amounting to simultaneously negate three consecutive values in a line. Under this interpretation the configurations are represented by polynomials in $\mathbb{Z}_2[X,Y]$, the elementary moves are the polynomials

$$P := \left\{X^i(X^2+X+1), Y^i(Y^2+Y+1), i \in \mathbb{N}\right\},$$

the ideal $I \subset \mathbb{Z}_2[X,Y]$ w.r.t. which we need to describe the set of the canonical forms, is generated by the basis $\{X^2+X+1, Y^2+Y+1\}$, which is obviously Gröbner. In this way Reiß's theorem can be derived by a completely elementary and elegant argument.

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