Rend. Sem. Mat. Univ. Pol. Torino
Vol. 73/2, 3–4 (2015), 253 – 259

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IDENTITIES FOR PRODUCTS AND SQUARES OF HORADAM AND (GENERAL) LUCAS NUMBERS

Abstract. For products and squares of Horadam and (general) Lucas numbers that generalize Fibonacci and Lucas numbers we describe several identities that are motivated by the Cusumano identity for Fibonacci numbers stated long ago as a problem in the journal The Fibonacci Quarterly.

1. Horadam and (general) Lucas numbers

The sequence F_0, F_1, F_2, F_3, $\ldots$ of Fibonacci numbers is defined recursively so that $F_0 = 0$ and $F_1 = 1$ and we require $F_{k+2} = F_{k+1} + F_k$ for every $k \geq 0$. Hence, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, $F_8 = 21$, $F_9 = 34$, etc.

Quite similar is also the sequence L_0, L_1, L_2, L_3, $\ldots$ of Lucas numbers when we take $L_0 = 2$ and $L_1 = 1$ while the recursion $L_{k+2} = L_{k+1} + L_k$ for every $k \geq 0$ remains the same. Hence, $L_2 = 3$, $L_3 = 4$, $L_4 = 7$, $L_5 = 11$, $L_6 = 18$, $L_7 = 29$, $L_8 = 47$, $L_9 = 76$, etc.

Let s, t, p and q be complex numbers such that $\Delta \neq 0$, where $\Delta = \sqrt{p^2 + 4q}$. Let $\alpha = \frac{p+\Delta}{2}$, $\beta = \frac{p-\Delta}{2}$, $a = t - s\beta$, $b = t - s\alpha$,

$$w_n = w_n(s,t;p,q) = \frac{a\alpha^n - b\beta^n}{\Delta},$$

$$x_n = x_n(s,t;\,p,\,q) = a\alpha^n + b\beta^n,$$

$$u_n = w_n(0,\,1;\,p,\,1), \qquad v_n = x_n(0,\,1;\,p,\,1),$$

$$g_n = w_n(0,\,1;\,p,\,-1), \qquad h_n = x_n(0,\,1;\,p,\,-1),$$

$$U_n = w_n(s,\,t;\,1,\,1), \qquad V_n = x_n(s,\,t;\,1,\,1),$$

$$G_n = w_n(s,\,t;\,-1,\,1), \qquad \text{and} \qquad H_n = x_n(s,\,t;\,-1,\,1)$$

for any integer n.

The numbers w_n are known as Horadam numbers (see [6]), the numbers x_n as the associated (general) Lucas numbers while the remaining are eight kinds of generalized Fibonacci and Lucas numbers. The first four have been used in [4] and they have 0 and 1 for initial values and arbitrary parameter p in the second order recurrence relation. Hence, these are polynomials in one variable p known as Fibonacci (u_n) and Lucas (v_n) polynomials. The polynomials g_n and h_n are the dual polynomials of u_n and v_n, respectively. The last four have arbitrary initial values s and t and the second order recurrence relation is either the sum or the difference of adjacent terms (see [6], [8, Chapter 7], and [11]). Hence, these numbers are linear sums $\mu s + \nu t$, where μ and ν

are adjacent Fibonacci and Lucas numbers (or their opposites). Again, G_n and H_n are considered as the dual numbers related to U_n and V_n, respectively. We call U_n the (s,t)-*Fibonacci numbers* and V_n the (s,t)-*Lucas numbers*. Let $\delta = -ab = qs^2 - t^2 + pst$.

In the paper [1] the authors prove several identities for the (s,t)-Fibonacci numbers U_n and the (s,t)-Lucas numbers V_n and their dual numbers G_n and H_n. Analogous identities for the polynomials u_n, v_n, g_n and h_n have been discovered earlier in [2].

The main goal of this paper is to present the versions of these identities for Horadam numbers w_n and (general) Lucas numbers x_n. Our results have corresponding identities both in [1] and [2] as special cases.

2. Generalization of Cusumano identity

In [3] (see also [9]), Andrew Cusumano proposed the square of the following identity that holds for every integer $i \geq 0$.

$$F_{i+1}\,F_i - F_{i+1}^2 = -F_i^2 - (-1)^i. \tag{1}$$

It is written here on purpose in the most awkward form, with too many minuses, in order to easily recognize (2.1) as a special case of the following identity for Horadam numbers w_n

$$w_{i+j}\,w_i\,z_j - w_{i+j}^2 = q^j(-1)^j\,w_i^2 + q^i(-1)^i\,\delta\,y_j^2, \tag{2}$$

which is true for all integers i, j and where

$$y_n = y_n(p,q,n) = w_n(0,1,p,q,n)$$

and

$$z_n = z_n(p,q,n) = x_n(0,1,p,q,n).$$

Indeed, when $s = 0$, $t = 1$, $p = 1$, $q = 1$ and $j = 1$ since $w_n = F_n$, $y_1 = 1$, $(-1)^j = -1$ and $\delta = -1$, it follows that from (2.2) we get (2.1).

Moreover, when $q = 1$ and $p = 1$, then $w_n = U_n$ and $y_n = F_n$ so that (2.2) reduces to [1, (2.2)]. On the other hand, when $q = 1$ and $p = -1$, then $w_n = G_n$ so that (2.2) reduces to [1, (2.5)].

Similarly, when $q = 1$, $s = 0$ and $t = 1$, then $w_n = u_n$ so that (2.2) reduces to [2, (2.2)]. On the other hand, when $q = -1$, $s = 0$ and $t = 1$, then $w_n = g_n$ so that (2.2) reduces to [2, (2.5)].

In all other identities in this paper analogous reductions to results in [1] and [2] occur in the above described selections of parameters s, t, p and q. Hence, we shall not repeat these remarks in the future.

We end this section with a version of (2.2) where (general) Lucas numbers x_n play the main role.

$$x_{i+j}\,x_i\,z_j - x_{i+j}^2 = q^j(-1)^j\,x_i^2 - \Delta^2\,q^i(-1)^i\,\delta\,y_j^2. \tag{3}$$

3. Similar identities for products

In this section we show two identities involving the products $\mu_n = w_n x_n$ between Horadam numbers and (general) Lucas numbers instead of their squares w_n^2, x_n^2. For all integers i and j we have

$$(4) \qquad w_{i+j}\, x_i\, z_j - \mu_{i+j} = q^j (-1)^j \mu_i - q^i (-1)^i \delta y_{2j},$$

$$(5) \qquad x_{i+j}\, w_i\, z_j - \mu_{i+j} = q^j (-1)^j \mu_i + q^i (-1)^i \delta y_{2j}.$$

4. Simpler identities for squares

The formula (2.2) gives the square w_{i+j}^2 in terms of $w_{i+j}\, w_i\, z_j$ and of the squares w_i^2 and y_j^2. We now present simpler formula for the squares w_{i+j}^2, x_{i+j}^2 and the products μ_{i+j}

$$(6) \qquad w_{i+2j}\, w_i - w_{i+j}^2 = q^i (-1)^i \delta y_j^2,$$

$$(7) \qquad x_{i+j}^2 - x_{i+2j}\, x_i = \Delta^2 q^i (-1)^i \delta y_j^2,$$

$$(8) \qquad x_{i+2j}\, w_i - \mu_{i+j} = \mu_{i+j} - w_{i+2j}\, x_i = q^i (-1)^i \delta y_{2j}.$$

5. Identities for more indices

A natural question arises about the existence of similar identities for more than two integer indices i, j. The answer is of course affirmative because, for example, when we take three integers i, j and k, it is possible to consider the sum $i+j+k$ as $(i+j)+k$ and apply the formula (2.2) replacing the first index i with the sum $i+j$ and rename the index j as k which will give us one of such identities. Completely different identity emerges when the sum $i+j+k$ is represented as $i+(j+k)$ and we apply the formula (2.2) leaving the index i unchanged and replacing the index j with $j+k$.

We now propose identities for three integers i, j and k that could be described as beautiful.

Let $\zeta = i+j+k$ (the sum of all), $\zeta_i = j+k$ (the sum of all except i), $\zeta_j = k+i$ and $\zeta_k = i+j$, we have

$$(9) \qquad z_{\zeta_k}\, w_{\zeta_j}\, w_{\zeta_i} - w_\zeta^2 = q^{\zeta_k} (-1)^{\zeta_k} w_k^2 - q^{\zeta_j} (-1)^{\zeta_j} w_j^2 + q^{\zeta_i} (-1)^{\zeta_i} \left[s\, z_{i-j}\, w_{\zeta_k} - w_i^2 \right].$$

For (general) Lucas numbers x_n we obtain a similar (nice) formula in the following two versions.

$$\text{(10)}\quad z_{\zeta_k} x_{\zeta_j} x_{\zeta_i} - x_\zeta^2 = \\ q^{\zeta_k}(-1)^{\zeta_k} x_k^2 - q^{\zeta_j}(-1)^{\zeta_j} x_j^2 + q^{\zeta_i}(-1)^{\zeta_i}\left[(2t-ps)z_{i-j} x_{\zeta_k} - x_i^2\right],$$

$$\text{(11)}\quad z_{\zeta_k} x_{\zeta_j} x_{\zeta_i} - x_\zeta^2 + 4q^\zeta(-1)^\zeta \delta = \\ \Delta^2\left(q^{\zeta_k}(-1)^{\zeta_k} w_k^2 + q^{\zeta_j}(-1)^{\zeta_j} w_j^2 - q^{\zeta_i}(-1)^{\zeta_i}\left[s z_{i-j} w_{\zeta_k} - w_i^2\right]\right).$$

The above three identities (5.1), (5.2) and (5.3) have respectively identities (2.2) and (2.3) as special cases. In fact we retrieve them when we consider $i = 0$ and rename index k as i, leaving index j unchanged.

For example, from (5.1) we obtain (2.2) using the identity

$$q^j(-1)^j s(z_{-j} w_j - s) - w_j^2 = \delta y_j^2.$$

Hence, beautiful identities for three integer indices contain (beautiful) identities for two integer indices as special cases.

We can also find the identities (3.1) and (3.2), for products of Horadam and (general) Lucas numbers, as special cases of the following generalizations to more indices.

$$\text{(12)}\quad \mu_\zeta - \Delta^2 w_{\zeta_i} w_{\zeta_j} y_{\zeta_k} = \\ q^j(-1)^{\zeta_j}\mu_j - q^k(-1)^{\zeta_k}\mu_k + q^i(-1)^{\zeta_i}\left[(2t-ps)z_{i-j} w_{\zeta_k} - \mu_i\right],$$

$$\text{(13)}\quad w_{\zeta_i} x_{\zeta_j} z_{\zeta_k} - \mu_\zeta = q^j(-1)^{\zeta_j}\mu_j + q^k(-1)^{\zeta_k}\mu_k + q^i(-1)^{\zeta_i}\left[s\Delta^2 y_{i-j} w_{\zeta_k} - \mu_i\right].$$

6. Proof of (2.2)

For the sake of simplicity, we pose $k = i + j$ and $b_n = (-1)^n$. Moreover recalling the definitions of w_n, y_n, z_n, if we set $A = \alpha^i$, $B = \beta^i$, $C = \alpha^j$, $D = \beta^j$ so that $z_j = C + D$, $y_j = \frac{C-D}{\Delta}$, $w_k = \frac{aAC-bBD}{\Delta}$, $w_i = \frac{aA-bB}{\Delta}$, we have

$$\Delta^2\left[w_k w_i z_j - w_k^2 - q^j b_j w_i^2 - q^i b_i \delta y_j^2\right] = abP - a^2Q - b^2R - S,$$

where

$$P = AB\left(2q^j b_j - C^2 - D^2\right), \qquad Q = A^2\left(q^j b_j - CD\right),$$
$$R = B^2\left(q^j b_j - CD\right), \qquad S = q^i b_i (C-D)^2\delta.$$

Since $CD = q^j b_j$, we have $Q = R = 0$. On the other hand, since $ab = -\delta$ and again $CD = q^j b_j$, it follows that $abP - S = 0$. Hence, $abP - a^2 Q - b^2 R - S = 0$. This concludes the proof of the formula (2.2).

7. Proof of (2.3)

With the same conventions used in the previous proof, recalling the definition of x_n, we get $x_k = aAC + bBD$, $x_i = aA + bB$ and we obtain

$$x_k x_i z_j - x_k^2 - q^j b_j x_i^2 + \Delta^2 b_i \delta y_j^2 = abP + a^2 Q + b^2 R + S,$$

where

$$P = AB\left(C^2 + D^2 - 2q^j b_j\right), \qquad Q = A^2\left(CD - q^j b_j\right),$$
$$R = B^2\left(CD - q^j b_j\right), \qquad S = q^i b_i (C - D)^2 \delta.$$

Since $CD = q^j b_j$, we have $Q = R = 0$. On the other hand, since $ab = -\delta$ and again $CD = q^j b_j$, it follows that $abP + S = 0$. Hence, $abP + a^2 Q + b^2 R + S = 0$ so that (2.3) holds.

8. Proof of (3.1)

Using the same notations introduced in the previous proofs, recalling that $\mu_n = w_n x_n$, and pointing out that $y_{2j} = \frac{C^2 - D^2}{\Delta}$, we find

$$\Delta\left[w_k x_i z_j - \mu_k - q^j b_j \mu_i^2 + q^i b_i \delta y_{2j}\right] = abP + a^2 Q - b^2 R - S,$$

where

$$P = AB\left(C^2 - D^2\right), \qquad Q = A^2\left(CD - q^j b_j\right),$$
$$R = B^2\left(CD - q^j b_j\right), \qquad S = q^i b_i (C^2 - D^2)\delta.$$

Since $CD = q^j b_j$, we have $Q = R = 0$. On the other hand, since $ab = -\delta$ and $AB = q^i b_i$, it follows that $abP - S = 0$. Hence,

$$abP + a^2 Q - b^2 R - S = 0.$$

This concludes the proof of the formula (3.1).

The identity (3.2) is proved similarly.

9. Proof of (4.1)

If we pose $m = i + 2j$ and, with the previous conventions, we observe that

$$w_m = \frac{aAC^2 - bBD^2}{\Delta},$$

then we find

$$w_m\, w_i - w_k^2 - q^i\, b_i\, \delta\, y_j^2 = -a\, b A B (C-D)^2 - q^i\, b_i (C-D)^2\, \delta.$$

Since $a\, b = -\delta$ and $A\, B = q^i\, b_i$, it follows that

$$-a\, b A B (C-D)^2 - q^i\, b_i (C-D)^2\, \delta = 0.$$

Hence, $w_m\, w_i - w_k^2 - q^i\, b_i\, \delta\, y_j^2 = 0$ so that (4.1) is true.

The other identities in section 4 are proved similarly.

10. Proof of (5.1)

Keeping unchanged our previous notations, we also pose $E = \alpha^k$, $F = \beta^k$. Letting

$$L = z_{\zeta_k}\, w_{\zeta_j}\, w_{\zeta_i} - w_{\zeta}^2,$$

$$R = q^{\zeta_k}\, b_{\zeta_k}\, w_k^2 - q^{\zeta_j}\, b_{\zeta_j}\, w_j^2 + q^{\zeta_i}\, b_{\zeta_i} (s\, z_{i-j}\, w_{\zeta_k} - w_i^2),$$

thanks to the definitions of w_n, z_n, we can easily evaluate w_{ζ_i}, w_{ζ_j}, w_{ζ_k}, z_{ζ_k}, z_{i-j} and we have

$$C D \Delta^2 (L-R) = m_1\, a^2 - m_2\, a\, b + m_3\, b^2 - m_4\, a + m_5\, b,$$

where

$$m_1 = b_{j+k}\, q^{j+k} (b_j\, q^j A^2 + b_i\, q^i\, C^2),$$

$$m_2 = b_{j+k}\, q^{j+k} (C B + D A)(A C + B D),$$

$$m_3 = b_{j+k}\, q^{j+k} (b_j\, q^j\, B^2 + b_i\, q^i\, D^2),$$

$$m_4 = s \Delta b_{k+j}\, q^{k+j} A C (A D + B C),$$

$$m_5 = s \Delta b_{k+j}\, q^{k+j} B D (A D + B C).$$

When we replace B, D and F with $\frac{b_i q^i}{A}$, $\frac{b_j q^j}{C}$ and $\frac{b_k q^k}{E}$, respectively, it follows that

$$C D \Delta^2 (L-R) = \frac{b_{j+k}\, q^{j+k}\, (a - b - s\Delta) M}{A^2 C^2},$$

where

$$M = \left(b_j\, q^j A^2 + b_i\, q^i\, C^2\right) \left(a A^2 C^2 - b\, b_{i+j}\, q^{i+j}\right).$$

Now, since $a - b - s\Delta = 0$, we get $L = R$ and this is the identity (5.1).

The other identities in section 5 are proved similarly.

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AMS Subject Classification: 11B39, 11Y55, 05A19

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Lavoro arrivato in redazione il 9-2-16.