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GENERALIZED DERIVATIONS ON SEMIGROUP IDEALS AND COMMUTATIVITY OF PRIME NEAR-RINGS

Abstract. A non empty subset U of a near-ring N is said to be a semigroup left (resp. right) ideal of N if $NU \subseteq U$ (resp. $UN \subseteq U$) and if U is both a semigroup left ideal and a semigroup right ideal, it is called a semigroup ideal. In the present paper, we investigate the commutativity of prime near-rings satisfying certain identities involving generalized derivations on semigroup ideals or ideals. Furthermore, we give examples to show that the restrictions imposed on the hypothesis of the various theorems are not superfluous.

1. INTRODUCTION

Throughout the paper, N will denote a zero symmetric left near-ring. N is called a prime near-ring if $xNy = \{0\}$ implies $x = 0$ or $y = 0$. Given an integer $n > 1$, near-ring N is said to be n-torsion free, if for $x \in N$, $nx = 0$ implies $x = 0$. A nonempty subset U of N is called semigroup left ideal (resp. semigroup right ideal)if $NU \subseteq U$ (resp. $UN \subseteq U$)and if U is both a semigroup left ideal and a semigroup right ideal, it will be called a semigroup ideal. The symbol Z will denote the multiplicative center of N, that is, $Z = \{x \in N \mid xy = yx \text{ for all } y \in N\}$. For any $x, y \in N$ the symbol $[x, y] = xy - yx$ stands for multiplicative commutator of x and y, while the symbol xoy will denote $xy + yx$. Finally the notation $\pm(xoy)$ represents either $+(xoy)$ i.e.; $xy + yx$ or $-(xoy)$ i.e.; $-(yx) - (xy)$.

An additive mapping d from N to N is called a derivation of N if $d(xy) = d(x)y + xd(y)$ (or equivalently $d(xy) = xd(y) + d(x)y$) holds for all $x, y \in N$. An additive mapping $F : N \longrightarrow N$ is called a right generalized derivation with associated derivation d if $F(xy) = F(x)y + xd(y)$, for all $x, y \in N$ and F is called a left generalized derivation with associated derivation d if $F(xy) = d(x)y + xF(y)$, for all $x, y \in N$. F is called a generalized derivation with associated derivation d if it is both a left as well as a right generalized derivation with associated derivation d. For terminologies concerning near-rings, we refer to G.Pilz [14].

The existing literature on prime near-rings contains a number of theorems concerning multiplicative commutativity of near-rings. H. E. Bell, G. Mason and A. Boua, L. Oukhtite have proved several results on commutativity of prime near-rings with derivations in [5, 6, 7] and [8] respectively. The notion of generalized derivation in rings was introduced by Matej Brešar [9] in the year 1991 and subsequently a number of authors have studied generalized derivation in the setting of prime and semiprime rings (for reference see [1],[2],[3],[4],[13],[15] where further references can be found). Motivated by the notion of generalized derivation in rings, Öznur Gölbasi introduced generalized derivation in near-rings. Several commutativity theorems of prime near-rings with generalized derivations have also been proved by Öznur Gölbasi [10, 11, 12]. It is natural to look for comparable results of prime near-rings having generalized derivations with semigroup ideals and ideals. Our aim in this paper is to study the commutativity of prime near-rings satisfying certain identities involving generalized derivations on ideals and semigroup ideals of a near-ring.

2. PRELIMINARY RESULTS

Now throughout onward in the paper the symbols N and U will denote a prime near-ring and a nonzero semigroup ideal of N respectively. Also we will use (F,d) for left generalized derivation F with associated nonzero derivation d of N unless stated otherwise. We begin by stating some well known results which will be used in throughout the text without any specific mention.

Lemma 3(iv) of [5] due to H.E.Bell states that if N is 2-torsion-free near-ring such that $d^2 = 0$, then $d = 0$. Later the same author obtained certain results in [6] which state as following: (i) If U is a nonzero semigroup right ideal or semigroup left ideal of N, then $d(U) \neq \{0\}$. (ii) If U is a nonzero semigroup right ideal of N and x is an element of N which centralizes U, then $x \in Z$. (iii) If $x, y \in N$ and $xUy = \{0\}$ then $x = 0$ or $y = 0$. (iv) Let U be nonzero semigroup right ideal or a nonzero semigroup left ideal of N. If N admits a nonzero derivation d for which $d(U) \subseteq Z$, then N is a commutative ring. (v) If d is a nonzero derivation on N such that $d^2(U) = 0$, then $d^2 = 0$. (vi) If Z contains a nonzero semigroup left ideal or semigroup right ideal, then N is a commutative ring. In a left near-ring right distributive property does not hold in general. But Gölbasi proved in Lemma 2.3(ii) of [10] that limited right distributive property holds in N and proved that if (F,d) is a left generalized derivation of N, then $(d(x)y + xF(y))z = d(x)yz + xF(y)z$, for all $x, y, z \in N$. Now we begin from the following lemmas:

LEMMA 1. *If (F,d) is a left generalized derivation of N such that $F(U)a = \{0\}$ where $a \in N$, then $a = 0$.*

Proof. Since $F(U)a = \{0\}$, we find that $F(ru)a = 0$ for all $u \in U, r \in N$. By [10, Lemma 2.3(ii)] we obtain $d(r)ua + rF(u)a = 0$ for all $u \in U, r \in N$. By hypothesis we get $d(r)ua = 0$, which shows that $d(r)Ua = \{0\}$. As $d \neq 0$, using Lemma 1.4(i) of [6] we conclude that $a = 0$. □

LEMMA 2. *If (F,d) is a left generalized derivation of N and U is a nonzero left semigroup ideal of N then $F(U) \neq \{0\}$.*

Proof. Suppose that F is a left generalized derivation of N and if possible let $F(U) = \{0\}$ i.e., $F(ru) = 0$ for all $u \in U, r \in N$. This shows that $d(r)u + rF(u) = 0$ and hence by hypothesis we obtain that $d(r)u = 0$ i.e.; $d(r)su = 0$ for all $u \in U, s, r \in N$. Finally we obtain that $d(r)Nu = \{0\}$, as $U \neq \{0\}$, primeness of N yields $d(r) = 0$ for all $r \in N$

i.e., $d = 0$ a contradiction. □

LEMMA 3. *If N admits left generalized derivations (F_1,d) and (F_2,d) such that $F_1(u) = F_2(u)$ for all $u \in U$, then $F_1 = F_2$.*

Proof. By hypothesis we have $F_1(ur) = F_2(ur)$ for all $u \in U, r \in N$ i.e., $d(u)r + uF_1(r) = d(u)r + uF_2(r)$, which implies that $u(F_1(r) - F_2(r)) = 0$. Previous relation gives us that $uN(F_1(r) - F_2(r)) = \{0\}$. Since $U \neq \{0\}$, primeness of N yields $F_1 = F_2$. □

LEMMA 4. *If N possesses a left generalized derivation (F,d) where d is arbitrary such that $F(u)v = uF(v)$ for all $u,v \in U$, then $d = 0$.*

Proof. Since $F(u)v = uF(v)$ for all $u,v \in U$. Now putting vw for v, where $w \in U$ in the previous relation we have $F(u)vw = uF(vw)$, which implies that $F(u)vw = ud(v)w + uvF(w)$. By hypothesis we obtain $F(u)vw = ud(v)w + uF(v)w$ which also gives us that $F(u)vw = ud(v)w + F(u)vw$ i.e., $ud(v)w = 0$ for all $u,v,w \in U$. Lastly putting ur where $r \in N$ for u in the previous relation we have $uNd(v)w = \{0\}$. Since $U \neq \{0\}$ and N is a prime near-ring, we conclude that $d(v)w = 0$ for all $v,w \in U$. Now putting sw where $s \in N$ for w in the relation $d(v)w = 0$ we have $d(v)Nw = \{0\}$. Since $U \neq \{0\}$ and N is a prime near-ring, we conclude that $d(v) = 0$ for all $v \in U$ i.e., $d(U) = \{0\}$. We claim that $d = 0$, for otherwise Lemma 1.3(ii) of [6] forces $d(U) \neq \{0\}$, leading to a contradiction. Hence our claim stands proved. □

LEMMA 5. *Let N be a 2-torsion free near-ring. If N admits a generalized derivation (F,d) where d is arbitrary such that $F^2(U) = \{0\}$, then $d = 0$.*

Proof. Since $F^2(U) = \{0\}$, we find that $F^2(F(u)v) = 0$ for all $u,v \in U$ i.e. $F(F^2(u)v + F(u)d(v)) = 0$. Using hypothesis we get $F(F(u)d(v)) = 0$ i.e., $F^2(u)d(v) + F(u)d^2(v) = 0$ for all $u,v \in U$. Hypothesis assures us $F(u)d^2(v) = 0$ for all $u,v \in U$ i.e.; $F(U)d^2(v) = \{0\}$. We claim that $d = 0$, for otherwise Lemma 1 gives $d^2(v) = 0$. Under this situation $d^2(U) = \{0\}$. By using Lemma 3.1 of [6] we have $d^2 = 0$ and by Lemma 3(iv) of [5] we conclude that $d = 0$, leading to a contradiction. □

3. COMMUTATIVITY THEOREMS

Recently Öznur Gölbasi [11, Theorems 3.1&3.2] proved that if (F,d) is a left generalized derivation of N and satisfying either of the following identities (i) $F([x,y]) = 0$ for all $x,y \in N$ or (ii) $F([x,y]) = \pm[x,y]$ for all $x,y \in N$, then N is a commutative ring. We have shown that these results are still true if both identities hold on some nonzero semigroup ideal U of N. In fact, we proved the following.

THEOREM 1. *If N admits a left generalized derivation (F,d) satisfying either of the following identities (i) $F([x,y]) = 0$, for all $x,y \in U$ or (ii) $F([x,y]) = \pm[x,y]$ for all $x,y \in U$, then N is a commutative ring.*

Proof. (i) Assume that $F([x,y]) = 0$ for all $x,y \in U$. Putting xy in place of y, obtaining

$F([x,xy]) = F(x[x,y]) = d(x)[x,y] + xF([x,y]) = 0$. Since the second term is zero, it is clear that

$$d(x)xy = d(x)yx \tag{3.1}$$

for all $x,y \in U$. Replacing y by yz where $z \in N$ in (3.1) and using this relation again, we get $d(x)U[x,z] = \{0\}$ for all $x \in U, z \in N$. Hence by Lemma 1.4(i) of [6] for each $x \in U$ either $d(x) = 0$ or $x \in Z$. If $x \in Z$ then $xr = rx$ for all $r \in N$, which gives us $d(x)r + xd(r) = rd(x) + d(r)x$ for all $r \in N$. Previous relation implies that $d(x) \in Z$. Hence we conclude that $d(U) \subseteq Z$. Now by Theorem 2.1 of [6] we infer that N is a commutative ring.

(ii) Assume that $F([x,y]) = \pm[x,y]$ for all $x,y \in U$. Putting xy in place of y, obtaining $F([x,xy]) = F(x[x,y]) = d(x)[x,y] + xF([x,y]) = \pm x[x,y]$. Using our hypothesis we get $d(x)xy = d(x)yx$ for all $x,y \in U$ which is identical with the relation (3.1) of Theorem 1(i). Now arguing in the similar way as in the Theorem 1(i) we conclude that N is a commutative ring. □

The conclusion of Theorem 1 remains valid if we replace the product $[x,y]$ by xoy and nonzero semigroup ideals by nonzero ideals respectively. In fact, we obtain the following results:

THEOREM 2. *If N admits a left generalized derivation (F,d) satisfying either of the following identities* (i) $F(xoy) = 0$, *for all $x,y \in I$ or* (ii) $F(xoy) = \pm(xoy)$ *for all $x,y \in I$, where I is a nonzero ideal of N, then N is a commutative ring.*

Proof. (i) Suppose that $F(xoy) = 0$ for all $x,y \in I$. Putting xy in place of y, obtaining $F(xoxy) = F(x(xoy)) = d(x)(xoy) + xF(xoy) = 0$. Since the second term is zero, it is clear that

$$d(x)xy = -d(x)yx \tag{3.2}$$

for all $x,y \in I$. Replacing y by yz where $z \in N$ in (3.2) and using this relation again, we get $d(x)y(-x)z + d(x)yzx = 0$ i.e., $d(x)I((-x)z + zx) = \{0\}$ for all $x,y \in I, z \in N$. Since$(I,+)$ is a normal subgroup of $(N,+)$, therefore $x \in I$ implies that $-x \in I$. Now replacing x by $-x$ in the preceding relation we get $d(-x)I[x,z] = \{0\}$ for all $x \in I, z \in N$. As I is an ideal of N, I will be also a semigroup ideal of N. Hence by Lemma 1.4(i) of [6] for each $x \in I$ either $d(-x) = 0$ i.e., $d(x) = 0$ or $x \in Z$. If $x \in Z$ then $xr = rx$ for all $r \in N$, which gives us $d(x)r + xd(r) = rd(x) + d(r)x$ for all $r \in N$. Previous relation implies that $d(x) \in Z$. Hence we conclude that $d(I) \subseteq Z$. Now by Theorem 2.1 of [6] we infer that N is a commutative ring.

(ii) Suppose that $F(xoy) = \pm(xoy)$ for all $x,y \in I$. Putting xy in place of y, obtaining $F(xoxy) = F(x(xoy)) = d(x)(xoy) + xF(xoy) = \pm(xoy)$. Using our hypothesis we get $d(x)xy = -d(x)yx$ for all $x,y \in I$. Which is identical with the relation (3.2) of Theorem 2(i). Now arguing in the similar way as in the Theorem 2(i), we conclude that N is a commutative ring. □

THEOREM 3. *If N admits a left generalized derivation (F,d) such that $F([x,y]) = \pm(xoy)$ for all $x,y \in U$, then N is a commutative ring.*

Proof. Assume that $F([x,y]) = \pm(xoy)$ for all $x,y \in U$. Putting xy in place of y, obtaining $F([x,xy]) = F(x[x,y]) = d(x)[x,y] + xF([x,y]) = \pm x(xoy)$. Using our hypothesis we get $d(x)xy = d(x)yx$ for all $x,y \in U$ which is identical with the relation (3.1) of Theorem $1(i)$. Now arguing in the similar way as in the Theorem $1(i)$, we conclude that N is a commutative ring. □

THEOREM 4. *If N admits a left generalized derivation (F,d) such that $F(xoy) = \pm[x,y]$ for all $x,y \in I$, where I is a nonzero ideal of N, then N is a commutative ring.*

Proof. Suppose that $F(xoy) = \pm[x,y]$ for all $x,y \in I$. Putting xy in place of y, obtaining $F(xoxy) = F(x(xoy)) = d(x)(xoy) + xF(xoy) = \pm[x,y]$. Using our hypothesis we get $d(x)xy = -d(x)yx$ for all $x,y \in I$. Which is identical with the relation (3.2) of Theorem $2(i)$. Now arguing in the similar way as in the Theorem $2(i)$, we conclude that N is a commutative ring. □

Recently Öznur Gölbasi [10, Theorem 2.6.] proved that if N possesses a generalized derivation (F,d) and $F(N) \subseteq Z$ then $(N,+)$ is an abelian. Moreover if N is 2-torsion free, then N is a commutative ring. We have generalized this result for semigroup ideals. The following result shows that "2-torsion free restriction" in the above result used by Öznur Gölbasi is superfluous. In fact, we have obtained the following:

THEOREM 5. *If N admits a left generalized derivation (F,d) such that $F(U) \subseteq Z$, then N is a commutative ring.*

Proof. For all $u_1, u_1^{'} \in U$, we have $F(u_1u_1^{'}) = d(u_1)u_1^{'} + u_1F(u_1^{'}) \in Z$. Hence $u_1\{d(u_1)u_1^{'} + u_1F(u_1^{'})\} = \{d(u_1)u_1^{'} + u_1F(u_1^{'})\}u_1$. Using the hypothesis and Lemma 2.3(ii) of [10] we get $u_1d(u_1)u_1{'} = d(u_1)u_1{'}u_1$. Now replacing $u_1{'}$ by $u_1{'}r$ where $r \in N$ in the preceding identity and using it again we have $d(u_1)u_1{'}[u_1,r] = 0$ i.e., $d(u_1)U[u_1,r] = \{0\}$. By Lemma 1.4(i) of [6] we infer that for each fixed $u_1 \in U$ either $d(u_1) = 0$ or $u_1 \in Z$. If second condition holds then $u_1r = ru_1$ for all $r \in N$, which gives us $d(u_1)r + u_1d(r) = rd(u_1) + d(r)u_1$ for all $r \in N$. Previous relation implies that $d(u_1) \in Z$. Lastly we conclude that $d(U) \subseteq Z$ therefore by Theorem 2.1 of [6], N is a commutative ring. □

The following example shows that the restriction of primeness imposed on the hypothesis of the above theorems is not superfluous.

EXAMPLE 1. Consider S to be a noncommutative zero symmetric left near-ring. Let

$$N = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a,b,0 \in S \right\}.$$

Then N is a near-ring and $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b,0 \in S \right\}$ is both a nonzero ideal and a nonzero semigroup ideal of N. Define $F : N \longrightarrow N$ by $F\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Then it is easy to see that (F,d) is a left generalized derivation of N, where d is defined by $d\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. If we set $p = \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix}$ with $0 \neq s$, then $pNp = \{0\}$ proving that N is not a semi prime near-ring. It can be easily shown that N satisfies the following properties: (i) $F([x,y]) = 0$, (ii) $F([x,y]) = \pm[x,y]$, (iii) $F(xoy) = 0$, (iv) $F(xoy) = \pm(xoy)$, (v) $F([x,y]) = \pm(xoy)$, (vi) $F(xoy) = \pm[x,y]$ for all $x,y \in I$ and (vii) $F(I) \subseteq Z$. However, N is not a commutative ring.

THEOREM 6. *Let (F,d) be a left generalized derivation of N such that $d(Z) \neq \{0\}$ and $F([x,y]) \in Z$ for all $x,y \in U$, then N is a commutative ring.*

Proof. We are given that for all $x,y \in U$, $F([x,y]) \in Z$.
CaseI: If $Z = \{0\}$, it follows that $F([x,y]) = 0$ for all $x,y \in U$. This is identical with Theorem $1(i)$. Hence for this case the proof of the Theorem $1(i)$ shows that N is a commutative ring.
CaseII: If $Z \neq \{0\}$, replacing y by yz where $z \in Z$ in our hypothesis, we get $d(z)[x,y] + zF[x,y] \in Z$ for all $x,y \in U, z \in Z$. Using our hypothesis again together with Lemma 2.3(ii) of [10] previous relation forces $d(z)[x,y] \in Z$ for all $x,y \in U, z \in Z$. Since $z \in Z$ hence $zr = rz$ for all $r \in N$, which gives us $d(z)r + zd(r) = rd(z) + d(r)z$ for all $r \in N$. Previous relation implies that $d(z) \in Z$. Therefore we find that $[d(z)[x,y],t] = d(z)[[x,y],t] = 0$ for all $t \in N$ and thus $d(z)N[[x,y],t] = \{0\}$ for all $x,y \in U, t \in N, z \in Z$. Now primeness of N yields $d(Z) = \{0\}$ or $[[x,y],t] = 0$ for all $x,y \in U$ and $t \in N$. By our hypothesis $d(Z) \neq \{0\}$ therefore $[[x,y],t] = 0$ for all $x,y \in U, t \in N$. Substituting xy for y in preceding relation we get $[x[x,y],t] = 0$ but $[x,y] \in Z$ and therefore $[x,y][x,t] = 0$ for all $x,y \in U, t \in N$. As $[x,y] \in Z$, $[x,y]N[x,y] = \{0\}$ for all $x,y \in U$. In the light of primeness of N, we obtain that $[x,y] = 0$ for all $x,y \in U$ i.e., $xy = yx$ for all $x,y \in U$. Putting yr for y where $r \in N$ in the previous relation and using the same again we get $y[x,r] = 0$. Now again replacing y by ys, where $s \in N$ in the relation $y[x,r] = 0$, we have $yN[x,r] = \{0\}$. Since N is a prime near-ring and $U \neq \{0\}$, we conclude that $U \subseteq Z$. Now by Lemma 1.5 of [6], N is a commutative ring. □

The following example demonstrates that the restriction of $d(Z) \neq \{0\}$ imposed on the hypothesis of the Theorem 6 is not superfluous even in the case of arbitrary rings.

EXAMPLE 2. Let

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z} \right\}$$

be the ring of 2×2 matrices over $\mathbb{Z}$, the ring of integers. It is easy to see that the ring R is prime with the center $Z = \left\{ \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \mid s,0 \in \mathbb{Z} \right\}$. Also it can be verified that $U = \left\{ \begin{pmatrix} m & n \\ p & q \end{pmatrix} \mid m,n,p,q \in 2\mathbb{Z} \right\}$, where $2\mathbb{Z}$ denotes the set of even integers, is a nonzero semigroup ideal of R. Assume $\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R$. Define $d, F : R \longrightarrow R$ as following, $d(x) = \alpha x - x\alpha$ and $F(x) = \alpha x + x\alpha$. It can be easily proved that F is a left generalized derivation of R with associated nonzero derivation d, satisfying the conditions, $d(Z) = \{0\}$ and $F[x,y] \in Z$ for all $x, y \in U$. However R is not a commutative ring.

THEOREM 7. *Let (F,d) be a left generalized derivation of N such that $[F(x),y] \in Z$ for all $x,y \in U$, then N is a commutative ring.*

Proof. Assume that $[F(x),y] \in Z$ for all $x,y \in U$. Hence $[[F(x),y],t] = 0$ for all $x,y \in U, t \in N$. Replacing y by $F(x)y$ in the previous relation we find that $[F(x)[F(x),y],t] = 0$ for all $x,y \in U, t \in N$. In view of hypothesis, we get $[F(x),y][F(x),t] = 0$ i.e., $[F(x),y]$ $N[F(x),y] = \{0\}$ for all $x,y \in U$. Primeness of N yields $[F(x),y] = 0$ i.e., $F(x)y = yF(x)$ for all $x,y \in U$. Putting yr for y where $r \in N$ in the preceding relation and using the same again we arrive at $y[F(x),r] = 0$ for all $x,y \in U, r \in N$. Now substituting ys for y where $s \in N$ we get $yN[F(x),r] = \{0\}$. Since $U \neq \{0\}$, primeness of N yields $F(U) \subseteq Z$. By application of Theorem 5, we conclude that N is a commutative ring. □

The following example shows that the restriction of primeness imposed on the hypothesis of the above Theorems 6 and 7 is not superfluous.

EXAMPLE 3. Let S be a noncommutative zero symmetric left near-ring. Let

$$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} \mid x,y,z,0 \in S \right\}.$$

Then N is a near-ring and $U = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x,0 \in S \right\}$ is a nonzero semigroup ideal of N. Define $F : N \longrightarrow N$ by $F\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix}$. Then it is easy to see that (F,d) is a left generalized derivation of N where d is defined by $d\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. If we set $p = \begin{pmatrix} 0 & s & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ with $0 \neq s$, then $pNp = \{0\}$ proving that N is not a semi prime near-ring. It can be easily shown that (i) $F([x,y] \in Z$ and $d(Z) \neq \{0\}$ (ii) $[F(x),y] \in Z$ for all $x,y \in U$. However N is not a commutative ring.

The following example shows that the restriction of primeness imposed on the hypothesis of the above Theorems $1(i)$, $2(i)$, 5, 6 & 7 is not superfluous even in the case of arbitrary rings.

EXAMPLE 4. Consider the polynomial ring $Q[x]$ where Q is the ring of real quaternions. Suppose $R = Q[x] \times Q[x]$. It can be easily verified that R is semiprime ring which is not a prime ring. Let $I = Q[x] \times \{0\}$. It can be easily proved that I is a nonzero ideal as well as a semigroup ideal of R. Define $d, f : Q[x] \longrightarrow Q[x]$ such that $d(p(x)) = \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 + \cdots + t\alpha_t x^{t-1}$, where $p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \cdots + \alpha_t x^t$ and $f(q(x)) = q(x)a_0(x) + d(q(x))$, where $a_0(x)$ is a nonzero given element of $Q[x]$. One can easily prove that f is a left generalized derivation of $Q[x]$ with associated nonzero derivation d. Next define $D, F : R \longrightarrow R$ such that $D(p(x), q(x)) = (0, d(q(x)))$ and $F(p(x), q(x)) = (0, f(q(x)))$. Finally it can be easily shown that F is a left generalized derivation of R with associated nonzero derivation D satisfying the conditions (i) $F([u,v]) = 0$ (ii) $F(uov) = 0$ (iii) $F(I) \subseteq Z$ (iv) $F([u,v]) \in Z$ and $D(Z) \neq \{0\}$ and (v) $[F(u), v] \in Z$, for all $u, v \in I$, where Z is the center of R. However R is not a commutative ring.

THEOREM 8. *Let N be without nonzero divisors of zero, and U a nonzero semigroup right ideal of N. If N admits a left generalized derivation (F,d) such that $F([x,y]) = 0$ for all $x, y \in U$, then N is a commutative ring.*

Proof. Assume that $F([x,y]) = 0$ for all $x, y \in U$. Putting xy in place of y, obtaining $F([x,xy]) = F(x[x,y]) = d(x)[x,y] + xF([x,y]) = 0$. Since the second term is zero, it is clear that $d(x)[x,y] = 0$ for all $x, y \in U$. Thus by hypothesis for each $x \in U$, either $d(x) = 0$ or x centralizes U. Applying Lemma 1.3(iii) of [6], we see that either $d(x) = 0$ or $x \in Z$. If $x \in Z$ then $xr = rx$ for all $r \in N$, which gives us $d(x)r + xd(r) = rd(x) + d(r)x$ for all $r \in N$. Previous relation implies that $d(x) \in Z$. Therefore we conclude that $d(U) \subseteq Z$. Lastly result follows by Theorem 2.1 of [6]. □

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