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SOME PARACONTACT METRIC STRUCTURES ON CONTACT METRIC MANIFOLDS[1]

Abstract. We consider contact metric manifolds such that the Jacobi operator anticommutes with the structure tensor field φ. These manifolds admit two paracontact metric structures compatible with the contact form η. We describe some geometric properties of these structures.

1. Introduction

Various results in the recent literature revealed remarkable interplays between contact metric and paracontact metric structures [3, 4, 5]. Such a relation has been particularly investigated on contact metric (κ,μ)-spaces, that is contact metric manifolds (M,φ,ξ,η,g) such that the Riemannian curvature satisfies

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \tag{1}$$

for every vector fields X,Y on M, and for some real constants κ,μ. Here $2h$ is the Lie derivative of the structure tensor φ in the direction of ξ. As shown in [3], a non-Sasakian contact metric (κ,μ)-space, for which $\kappa < 1$, admits two paracontact metric structures (φ_i,ξ,η,g_i), $i = 1,2$, where

$$\varphi_1 = \frac{1}{\sqrt{1-\kappa}}\varphi h, \quad \varphi_2 = \frac{1}{\sqrt{1-\kappa}}h, \quad g_i := d\eta(\,\cdot\,,\varphi_i\,\cdot\,) + \eta\otimes\eta.$$

The triplet $(\varphi_1,\varphi_2,\varphi)$ provides an almost bi-paracontact structure. In particular, in [3] the author proves that the curvature tensors of the semi-Riemannian metrics g_1 and g_2 satisfy nullity conditions formally similar to (1). The structure (φ_2,ξ,η,g_2) is deeply investigated in [4], where it is shown that this structure, called canonical, induces on the underlying contact manifold (M,η) a sequence of compatible contact and paracontact metric structures satisfying nullity conditions.

In this note we investigate the existence of paracontact metric structures on the class of contact metric manifolds such that the Jacobi operator $l := R(\,\cdot\,,\xi)\xi$ anticommutes with φ. This class includes contact metric manifolds with vanishing Jacobi operator, called M_l-manifolds, and more generally contact metric Jacobi $(0,\mu)$-spaces, for which $l = \mu h$, for some real constant μ. Notice that contact metric $(0,\mu)$-spaces are special Jacobi $(0,\mu)$-spaces (see Section 3 for more details).

We prove that any contact metric manifold (M,φ,ξ,η,g) such that $\varphi l + l\varphi = 0$ admits two paracontact metric structures given by (h,ξ,η,g_1) and $(\varphi h,\xi,\eta,g_2)$, where

$$g_1 := d\eta(\,\cdot\,,h\,\cdot\,) + \eta\otimes\eta, \qquad g_2 := d\eta(\,\cdot\,,\varphi h\,\cdot\,) + \eta\otimes\eta.$$

[1]This paper is dedicated to Professor Anna Maria Pastore with deep gratitude for her teachings.

We study the basic properties of these geometric structures, determining the Jacobi operators $l_1 := R_1(\cdot,\xi)\xi$ and $l_2 := R_2(\cdot,\xi)\xi$ defined by the curvature tensors R_1 and R_2 of the semi-Riemannian metrics g_1 and g_2 respectively. It is worth remarking that in both cases $l = 0$ and $l = 4h$, the structure (h,ξ,η,g_1) provides an example of paracontact metric structure such that the symmetric operator $h_1 := \frac{1}{2}\mathcal{L}_\xi h$ is not vanishing and satisfies $h_1^2 = 0$. These structures are of special interest in the context of paracontact metric geometry, since they have no counterpart in contact metric geometry [2, 10, 11].

2. Preliminaries

An *almost contact manifold* is a $(2n+1)$-dimensional smooth manifold M endowed with a structure (φ,ξ,η), where φ is a $(1,1)$-tensor field, ξ a vector field and η a 1-form, such that

$$\varphi^2 = -I + \eta\otimes\xi, \quad \eta(\xi) = 1,$$

implying that $\varphi\xi = 0$, $\eta\circ\varphi = 0$ and φ has rank $2n$. An almost contact manifold admits a compatible metric, that is a Riemannian metric g such that, for every $X,Y \in \Gamma(TM)$,

$$g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y).$$

Then $\eta = g(\cdot,\xi)$, and the tangent bundle of M splits into the orthogonal sum $TM = \mathcal{D}\oplus[\xi]$, where $\mathcal{D}$ is the $2n$-dimensional distribution defined as $\mathrm{Ker}(\eta)$ or, equivalently, as $\mathrm{Im}(\varphi)$. The manifold (M,φ,ξ,η,g) is called an *almost contact metric manifold*. It is said to be a *contact metric manifold* if $d\eta(X,Y) = g(X,\varphi Y)$ for every vector fields X,Y; in this case the 1-form η turns out to be a *contact form*, in the sense that $\eta\wedge(d\eta)^n \neq 0$ everywhere on M. A *Sasakian manifold* is defined as a contact metric manifold for which the tensor field $N_\varphi := [\varphi,\varphi] + 2d\eta\otimes\xi$ vanishes identically.

On a contact metric manifold one can define the $(1,1)$-tensor field $h := \frac{1}{2}\mathcal{L}_\xi\varphi$. This operator satisfies $h\xi = 0$ and is symmetric with respect to g, so that $\eta\circ h = 0$. Furthermore, it anticommutes with φ and satisfies

$$\nabla_X\xi = -\varphi X - \varphi hX \tag{2}$$

where ∇ is the Levi-Civita connection of g. Denoting by R the Riemannian curvature tensor, we shall consider the *Jacobi operator* $l := R(\cdot,\xi)\xi$. It is known that l satisfies the following equations [1]:

$$\nabla_\xi h = \varphi - h^2\varphi - \varphi l, \tag{3}$$

$$\frac{1}{2}\big(-l + \varphi l\varphi\big) = h^2 + \varphi^2. \tag{4}$$

We recall now the basic notions on paracontact geometry [16]. An *almost paracontact structure* on a $(2n+1)$-dimensional manifold M is given by a $(1,1)$-tensor field $\tilde\varphi$, a vector field $\tilde\xi$ and a 1-form $\tilde\eta$, such that

(i) $\tilde\varphi^2 = I - \tilde\eta\otimes\tilde\xi$, $\tilde\eta(\tilde\xi) = 1$,

(ii) $\tilde{\varphi}$ induces an almost paracomplex structure on each fibre of the distribution $\mathcal{D} := \mathrm{Ker}(\tilde{\eta})$, i.e. the eigendistributions $\mathcal{D}^+_{\tilde{\varphi}}$ and $\mathcal{D}^-_{\tilde{\varphi}}$ corresponding to the eigenvalues $+1$ and -1 of $\tilde{\varphi}|_{\mathcal{D}}$, respectively, have dimension equal n.

From the definition it follows that $\tilde{\varphi}\tilde{\xi} = 0$, $\tilde{\eta} \circ \tilde{\varphi} = 0$ and $\tilde{\varphi}$ has constant rank $2n$. Any almost paracontact manifold M admits a semi-Riemannian metric $\tilde{g}$ satisfying

$$\tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) = -\tilde{g}(X,Y) + \tilde{\eta}(X)\tilde{\eta}(Y)$$

for all $X, Y \in \Gamma(TM)$, which necessarily has signature $(n+1, n)$. Then $(M, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is called an *almost paracontact metric manifold*. If furthermore $d\tilde{\eta}(X,Y) = \tilde{g}(X, \tilde{\varphi}Y)$ for all $X, Y \in \Gamma(TM)$, then $(M, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is called a *paracontact metric manifold*, in which case $\tilde{\eta}$ is a contact form. A *para-Sasakian manifold* is defined as a paracontact metric manifold such that the tensor field $N_{\tilde{\varphi}} := [\tilde{\varphi}, \tilde{\varphi}] - 2d\tilde{\eta} \otimes \tilde{\xi}$ vanishes identically.

On a paracontact metric manifold one can consider the $(1,1)$-tensor field $\tilde{h}$ defined by $\tilde{h} := \frac{1}{2}\mathcal{L}_{\tilde{\xi}}\tilde{\varphi}$, which satisfies $\tilde{h}\tilde{\xi} = 0$. Furthermore, $\tilde{h}$ is symmetric and anticommutes with $\tilde{\varphi}$. Denoting by $\tilde{\nabla}$ the Levi-Civita connection of $\tilde{g}$, one has

$$\tilde{\nabla}_X \tilde{\xi} = -\tilde{\varphi}X + \tilde{\varphi}\tilde{h}X. \tag{5}$$

If $\tilde{R}$ is the curvature tensor of $\tilde{g}$, the operator $\tilde{l} := \tilde{R}(\cdot, \tilde{\xi})\tilde{\xi}$ satisfies

$$\tilde{\nabla}_{\tilde{\xi}}\tilde{h} = -\tilde{\varphi} + \tilde{h}^2\tilde{\varphi} - \tilde{\varphi}\tilde{l}, \tag{6}$$

$$\frac{1}{2}\left(\tilde{l} + \tilde{\varphi}\tilde{l}\tilde{\varphi}\right) = \tilde{h}^2 - \tilde{\varphi}^2. \tag{7}$$

3. Contact metric structures with $\varphi l + l\varphi = 0$

THEOREM 1. *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric manifold. Then, the following conditions are equivalent:*

i) $\varphi l + l\varphi = 0$,

ii) (h, ξ, η) *is an almost paracontact structure,*

iii) $(\varphi h, \xi, \eta)$ *is an almost paracontact structure.*

If any of the above conditions holds, then M is endowed with two paracontact metric structures (h, ξ, η, g_1) and $(\varphi h, \xi, \eta, g_2)$, where for every $X, Y \in \Gamma(TM)$,

$$g_1(X,Y) = d\eta(X, hY) + \eta(X)\eta(Y), \tag{8}$$

$$g_2(X,Y) = d\eta(X, \varphi hY) + \eta(X)\eta(Y). \tag{9}$$

Proof. From (4) it follows that $\varphi l + l\varphi = 0$ if and only if $h^2 + \varphi^2 = 0$, or equivalently $h^2 = I - \eta \otimes \xi$. This is also equivalent to $(\varphi h)^2 = I - \eta \otimes \xi$. Indeed, since h anticommutes with φ, then $(\varphi h)^2 = -\varphi^2 h^2 = h^2$.

We suppose now that i) holds. Then, being $h^2 = I - \eta \otimes \xi$, the symmetric operator h admits non-zero eigenvalues $+1$ and -1 of multiplicity n. Indeed, since h anticommutes with φ, if X is an eigenvector with eigenvalue $+1$, then φX is an eigenvector with eigenvalue -1. Therefore, h induces an almost paracomplex structure on $\mathcal{D}$ and (h, ξ, η) is an almost paracontact structure. Now define g_1 as in (8) so that

$$g_1(X,Y) = g(X, \varphi hY) + \eta(X)\eta(Y). \tag{10}$$

Being φh symmetric with respect to g, g_1 is symmetric. Furthermore, it is compatible with h since, being $i_\xi d\eta = 0$, we have

$$g_1(hX,hY) = d\eta(hX, h^2Y) = -d\eta(Y,hX) = -g_1(X,Y) + \eta(X)\eta(Y).$$

Finally, $g_1(X,hY) = d\eta(X,h^2Y) = d\eta(X,Y)$, so that (h,ξ,η,g_1) is a paracontact metric structure. Analogously, if i) holds the tensor field φh induces an almost paracomplex structure on $\mathcal{D}$, since it satisfies $(\varphi h)^2 = I - \eta \otimes \xi$, it is symmetric and anticommutes with φ. Hence, $(\varphi h, \xi, \eta)$ is an almost paracontact structure. In this case, one can define a metric g_2 as in (9), so that

$$g_2(X,Y) = -g(X,hY) + \eta(X)\eta(Y). \tag{11}$$

One can easily see that g_2 is symmetric and compatible with φh. On the other hand, $g_2(X, \varphi hY) = d\eta(X,Y)$, so that $(\varphi h, \xi, \eta, g_2)$ is a paracontact metric structure. □

REMARK 1. Notice that if $(M, \varphi, \xi, \eta, g)$ is a contact metric manifold satisfying $\varphi l + l\varphi = 0$, then the triplet $(\varphi h, h, \varphi)$ is an almost bi-paracontact structure on M, in the sense of the definition given in [3]. In particular, one has

$$\mathcal{D}^{\pm}_{\varphi h} = \{X + \varphi X | X \in \mathcal{D}^{\pm}_{h}\}, \qquad \mathcal{D}^{\pm}_{h} = \{X + \varphi X | X \in \mathcal{D}^{\mp}_{\varphi h}\},$$

where $\mathcal{D}^{+}_{\varphi h}$ and $\mathcal{D}^{-}_{\varphi h}$ denote the eigendistributions of φh corresponding to the eigenvalues $+1$ and -1 respectively, while $\mathcal{D}^{+}_{h}$ and $\mathcal{D}^{-}_{h}$ are the eigendistributions of h corresponding to the eigenvalues $+1$ and -1. Furthermore, one can take a local orthonormal frame $\{\xi, e_i, \varphi e_i\}$, $i = 1, \ldots, n$, such that $he_i = e_i$ and consequently $h\varphi e_i = -\varphi e_i$. Then the orthogonal vector fields $e_i + \varphi e_i$, $i = 1, \ldots, n$, span the eigendistribution $\mathcal{D}^{+}_{\varphi h}$, while the orthogonal vector fields $e_i - \varphi e_i$, $i = 1, \ldots, n$, span the eigendistribution $\mathcal{D}^{-}_{\varphi h}$.

A first class of almost contact metric manifolds satisfying $\varphi l + l\varphi = 0$ is obviously given by contact metric manifolds with vanishing Jacobi operator, also known as M_l-manifolds in literature. This class is particularly large. For instance, the normal bundle of a Legendre submanifold of a Sasakian manifold admits a contact metric structure with $l = 0$, see [1, Theorem 9.16]. Contact metric 3-manifolds with $l = 0$ are studied in [13, 8, 9].

In [7] Ghosh and Sharma introduced a new class of contact metric manifolds. They define a Jacobi (κ, μ)-contact space as a contact metric manifold such that the Jacobi operator l satisfies

$$l = -\kappa\varphi^2 + \mu h, \tag{12}$$

for some constants κ and μ. In particular, this class includes contact metric (κ,μ)-spaces, for which the Riemannian curvature tensor satisfies (1). Now, if l is given by (12), then $\varphi l + l\varphi = 2\kappa\varphi$ and $\mathtt{trace}(l) = 2n\kappa$, so that

$$\varphi l + l\varphi = 0 \Leftrightarrow \kappa = 0 \Leftrightarrow \mathtt{trace}(l) = 0.$$

In [6] Cho and Inoguchi provide new examples of 3-dimensional Jacobi (κ,μ)-contact spaces which are neither contact (κ,μ)-spaces nor M_l-manifolds. They consider non-unimodular 3-dimensional Lie groups $G(\alpha,\gamma)$, α,γ real constants with $\alpha \neq 0$, equipped with a left invariant contact metric structure (φ,ξ,η,g), whose Lie algebra $\mathfrak{g}(\alpha,\gamma)$ is spanned by a basis $\{\xi,e_1,e_2\}$, with $e_2 = \varphi e_1$, satisfying the commutation relations

$$[\xi,e_1] = -\gamma e_2, \quad [\xi,e_2] = 0, \quad [e_1,e_2] = \alpha e_2 + 2\xi,$$

(see also [14]). Every Lie group $G(\alpha,\gamma)$ is a Jacobi (κ,μ)-contact space, with $\kappa = -\frac{1}{4}(\gamma^2 - 4)$ and $\mu = \gamma + 2$. Furthermore, except for the Sasakian case $G(\alpha,0)$, $G(\alpha,\gamma)$ is not a contact (κ,μ)-space. In particular the Lie groups $G(\alpha,2)$ are Jacobi $(0,4)$-contact spaces, for which l is not vanishing and anticommutes with φ.

In [6] the authors also provide an example of a non-homogeneous Jacobi (κ,μ)-contact space, which is not a (κ,μ)-space. This manifold M, described by Perrone in [15], is the open submanifold $\{(x,y,z) \in \mathbb{R}^3 | x \neq 0\}$ of $\mathbb{R}^3$, endowed with the contact metric structure (φ,ξ,η,g), where $\xi = \partial/\partial z$, $\eta = xydx + dz$, and φ is defined on the global frame fields

$$e_1 = -\frac{2}{x}\frac{\partial}{\partial y}, \quad e_2 = \frac{\partial}{\partial x} - \frac{4z}{x}\frac{\partial}{\partial y} - xy\frac{\partial}{\partial z}, \quad e_3 = \xi$$

by $\varphi\xi = 0$, $\varphi e_1 = e_2$, $\varphi e_2 = -e_1$. Finally g is the Riemannian metric with respect to which $\{e_1,e_2,e_3\}$ is orthonormal. In this case the Jacobi operator is given by $l = 4h$, so that M is a Jacobi $(0,4)$-contact space.

Now, given a contact metric manifold (M,φ,ξ,η,g) such that $\varphi l + l\varphi = 0$, with associated paracontact metric structures (h,ξ,η,g_1) and $(\varphi h,\xi,\eta,g_2)$, we denote by h_1 and h_2 the tensor fields defined by

$$h_1 := \frac{1}{2}\mathcal{L}_\xi h, \quad h_2 := \frac{1}{2}\mathcal{L}_\xi(\varphi h).$$

The operator h_1 is symmetric with respect to g_1 and anticommutes with h, while h_2 is symmetric with respect to g_2 and anticommutes with φh. We shall denote by ∇^1 and ∇^2 the Levi-Civita connections of g_1 and g_2, respectively, with curvature tensors R_1 and R_2. The associated Jacobi operators are defined by

$$l_1 := R_1(\cdot,\xi)\xi, \quad l_2 := R_2(\cdot,\xi)\xi.$$

LEMMA 1. *Let (M,φ,ξ,η,g) be a contact metric manifold such that $\varphi l + l\varphi = 0$. Then the following equations hold:*

$$\nabla_\xi h = -\varphi l, \tag{13}$$

$$lh - hl = 0. \tag{14}$$

Proof. Equation (13) immediately follows from (3), being $h^2 = I - \eta \otimes \xi$. Next, applying (13) and $\nabla_\xi(h^2) = 0$, we have

$$lh - hl = \varphi(\nabla_\xi h)h - h\varphi(\nabla_\xi h) = \varphi\big((\nabla_\xi h)h + h(\nabla_\xi h)\big) = \varphi\nabla_\xi(h^2) = 0.$$

□

PROPOSITION 1. *Let (M,φ,ξ,η,g) be a contact metric manifold with $\varphi l + l\varphi = 0$, and associated paracontact metric structures (h,ξ,η,g_1) and $(\varphi h,\xi,\eta,g_2)$. Then*

$$h_1 = \varphi + \varphi h - \frac{1}{2}\varphi l, \qquad h_2 = -h + \frac{1}{2}l. \tag{15}$$

Proof. Applying (13) and (2), we have

$$\begin{aligned} 2h_1X &= [\xi,hX] - h[\xi,X] \\ &= (\nabla_\xi h)X - \nabla_{hX}\xi + h(\nabla_X\xi) \\ &= -\varphi lX + \varphi hX + \varphi h^2X - h\varphi X - h\varphi hX \\ &= 2\varphi X + 2\varphi hX - \varphi lX. \end{aligned}$$

As regards h_2, we have

$$h_2 = \frac{1}{2}\big((\mathcal{L}_\xi\varphi)h + \varphi(\mathcal{L}_\xi h)\big) = h^2 + \varphi h_1 = h^2 + \varphi^2 + \varphi^2 h - \frac{1}{2}\varphi^2 l = -h + \frac{1}{2}l.$$

□

4. The paracontact metric structure (h,ξ,η,g_1).

LEMMA 2. *Let (M,φ,ξ,η,g) be a contact metric manifold such that $\varphi l + l\varphi = 0$. Then the operators h_1 and l_1 associated to the structure (h,ξ,η,g_1) satisfy the following identities:*

(a) $lh_1 + h_1 l = 0$,

(b) $h_1\varphi + \varphi h_1 = 2\varphi^2$,

(c) $h_1\varphi - \varphi h_1 = 2h - l$,

(d) $h_1^2 = \frac{1}{4}l^2 - hl$,

(e) $l_1 h + hl_1 = -2h - 2l + \frac{1}{2}l^2 h$.

Proof. Identity (a) follows from (15), taking into account the fact that l anticommutes with φ and commutes with h. Identities (b), (c) and (d) also follow from (15). Finally, from (7) we get $l_1 + hl_1h = -2h^2 + 2h_1^2$, and using (d) we get (e). □

PROPOSITION 2. *Let (M,φ,ξ,η,g) be a contact metric manifold such that $l = 0$. Then (h,ξ,η,g_1) is a paracontact metric structure such that $h_1 \neq 0$ and $h_1^2 = 0$.*

Proof. From (15) we have $h_1 = \varphi + \varphi h$, which is not vanishing since φ is skew-symmetric with respect to g, while φh is symmetric. On the other hand, from (d) of Lemma 2, $h_1^2 = 0$. □

PROPOSITION 3. *Let (M,φ,ξ,η,g) be a Jacobi $(0,4)$-contact space. Then (h,ξ,η,g_1) is a paracontact metric structure such that $h_1 \neq 0$ and $h_1^2 = 0$.*

Proof. The Jacobi operator of the contact metric structure (φ,ξ,η,g) is $l = 4h$. From (15) we have $h_1 = \varphi - \varphi h \neq 0$, while (d) of Lemma 2 implies $h_1^2 = 0$. □

REMARK 2. In both cases $l = 0$ and $l = 4h$, from (e) of Lemma 2, one can easily verify that $l_1 h + h l_1 = -2h \neq 0$, and in particular $l_1 \neq 0$.

In the following, we shall obtain an explicit expression for the Jacobi operator l_1. First, we determine the relation between the Levi-Civita connections of the Riemannian metrics g and g_1.

THEOREM 2. *Let (M,φ,ξ,η,g) be a contact metric manifold such that $\varphi l + l\varphi = 0$. Then for every $X,Y \in \Gamma(TM)$, the Levi-Civita connections ∇^1 and ∇ satisfy:*

$$\nabla^1_X Y = \nabla_X Y - \eta(Y)hX - \eta(X)hY - \eta(X)\eta(Y)\xi - \frac{1}{2}g(lX,Y)\xi + \frac{1}{2}\varphi h\big((\nabla_X\varphi)Y + (\nabla_X\varphi h)Y - R(\xi,X)Y\big) + g(X + hX - \varphi hX, Y)\xi. \tag{16}$$

Proof. First, we prove that for every $X,Y \in \Gamma(TM)$ and $U \in \Gamma(\mathcal{D})$, we have

$$\begin{aligned} g(\nabla^1_X Y, U) &= g(\nabla_X Y, U) - \eta(Y)g(hX,U) - \eta(X)g(hY,U) \\ &\quad + \frac{1}{2}g\big(\varphi h((\nabla_X\varphi)Y + (\nabla_X\varphi h)Y - R(\xi,X)Y), U\big). \end{aligned} \tag{17}$$

Let us consider $X,Y \in \Gamma(TM)$ and $Z \in \Gamma(\mathcal{D})$. Applying the Koszul formula, (10) and the fact that φh is symmetric, we have

$$\begin{aligned} 2g_1(\nabla^1_X Y, Z) &= X(g_1(Y,Z)) + Y(g_1(Z,X)) - Z(g_1(X,Y)) \\ &\quad + g_1([X,Y],Z) + g_1([Z,X],Y) - g_1([Y,Z],X) \\ &= X(g(Y,\varphi hZ)) + Y(g(X,\varphi hZ)) - Z(g(X,\varphi hY)) \\ &\quad - Z(\eta(X)\eta(Y)) + g([X,Y],\varphi hZ) + g([Z,X],\varphi hY) \\ &\quad + \eta([Z,X])\eta(Y) - g([Y,Z],\varphi hX) - \eta([Y,Z])\eta(X) \\ &= 2g(\nabla_X Y,\varphi hZ) + g(Y,\nabla_X(\varphi hZ)) + g(X,\nabla_Y(\varphi hZ)) \\ &\quad - g(X,\nabla_Z(\varphi hY)) - g(\nabla_X Z,\varphi hY) - g(\nabla_Y Z,\varphi hX) \\ &\quad + g(\nabla_Z Y,\varphi hX) - 2d\eta(Z,X)\eta(Y) - 2d\eta(Z,Y)\eta(X) \\ &= 2g(\nabla_X Y,\varphi hZ) - 2g(Z,\varphi X)\eta(Y) - 2g(Z,\varphi Y)\eta(X) \\ &\quad + g((\nabla_Y\varphi h)Z,X) + g((\nabla_X\varphi h)Z,Y) - g((\nabla_Z\varphi h)Y,X). \end{aligned}$$

Now, recall that in a contact metric manifold we have ([12])

$$g(R(\xi,X)Y,Z) = g((\nabla_X\varphi)Y,Z) - g(X,(\nabla_Y\varphi h)Z) + g(X,(\nabla_Z\varphi h)Y).$$

Therefore, applying again (10), the symmetry of φh and $(\varphi h)^2(Z) = Z$, we have

$$\begin{aligned}
2g(\nabla^1_X Y,\varphi hZ) &= 2g(\nabla_X Y,\varphi hZ) - 2g(Z,\varphi X)\eta(Y) - 2g(Z,\varphi Y)\eta(X)\\
&\quad - g(R(\xi,X)Y,Z) + g((\nabla_X\varphi)Y,Z) + g((\nabla_X\varphi h)Y,Z)\\
&= 2g(\nabla_X Y,\varphi hZ) - 2g(hX,\varphi hZ)\eta(Y) - 2g(hY,\varphi hZ)\eta(X)\\
&\quad + g\big(\varphi h(-R(\xi,X)Y + (\nabla_X\varphi)Y + (\nabla_X\varphi h)Y),\varphi hZ\big)
\end{aligned}$$

which implies (17). Now, we prove that for every $X,Y \in \Gamma(TM)$

$$g(\nabla^1_X Y,\xi) = g(\nabla_X Y,\xi) - \eta(X)\eta(Y) + g\big(X + hX - \varphi hX - \frac{1}{2}lX,Y\big). \tag{18}$$

Indeed, using (5) and the first identity of (15), we get

$$\nabla^1_X\xi = -hX + hh_1X = -hX - \varphi X + h\varphi X - \frac{1}{2}h\varphi lX.$$

Hence, applying also (2), we get

$$\begin{aligned}
g(\nabla^1_X Y,\xi) &= g_1(\nabla^1_X Y,\xi) = X(g_1(Y,\xi)) - g_1(Y,\nabla^1_X\xi)\\
&= X(g(Y,\xi)) - g(\varphi hY,\nabla^1_X\xi) - \eta(Y)\eta(\nabla^1_X\xi)\\
&= g(\nabla_X Y,\xi) + g(Y,-\varphi X - \varphi hX) + g\big(hX + \varphi X + \varphi hX - \frac{1}{2}\varphi hlX,\varphi hY\big)\\
&= g(\nabla_X Y,\xi) - \eta(X)\eta(Y) + g\big(X + hX - \varphi hX - \frac{1}{2}lX,Y\big).
\end{aligned}$$

From (17) and (18) we get the result. □

PROPOSITION 4. *Let (M,φ,ξ,η,g) be a contact metric manifold such that $\varphi l + l\varphi = 0$. The Jacobi operator l_1 is given by:*

$$l_1 = \varphi^2 + 2\varphi + 2\varphi h - lh + l\varphi - \frac{1}{4}l^2 + \frac{1}{2}h\varphi\nabla_\xi l. \tag{19}$$

If M is a Jacobi $(0,\mu)$-contact space, then

$$l_1 = \Big(1 + \mu - \frac{1}{4}\mu^2\Big)\varphi^2 + 2\varphi + (2-\mu)\varphi h. \tag{20}$$

In particular, if $l = 0$, then $l_1 = \varphi^2 + 2\varphi + 2\varphi h$.

Proof. Recall that for a contact metric structure $\nabla_\xi\varphi = 0$. Hence, from (16) and using also (13), we have

$$\nabla^1_\xi = \nabla_\xi - h + \frac{1}{2}\varphi hl. \tag{21}$$

Now, from (21) and the first identity in (15), applying also $\nabla_\xi\varphi = 0$, (13), (14), $hh_1 + h_1h = 0$, and identities (a), (c) in Lemma 2, we get

$$\begin{aligned}
\nabla^1_\xi h_1 &= \nabla_\xi h_1 - hh_1 + h_1 h + \frac{1}{2}(\varphi hlh_1 - h_1\varphi hl) \\
&= \varphi\nabla_\xi h - \frac{1}{2}\varphi\nabla_\xi l - 2hh_1 + \frac{1}{2}(\varphi h_1 - h_1\varphi)hl \\
&= l - \frac{1}{2}\varphi\nabla_\xi l - 2h\Big(\varphi + \varphi h - \frac{1}{2}\varphi l\Big) - l + \frac{1}{2}hl^2 \\
&= -\frac{1}{2}\varphi\nabla_\xi l + 2\varphi h + 2\varphi - \varphi hl + \frac{1}{2}hl^2.
\end{aligned}$$

On the other hand, from (6) we know that $\nabla^1_\xi h_1 = -h + h_1^2 h - hl_1$, and using (d) of Lemma (2), we have

$$2\varphi h + 2\varphi - \varphi hl + \frac{1}{2}hl^2 - \frac{1}{2}\varphi\nabla_\xi l = -h + \Big(\frac{1}{4}l^2 - hl\Big)h - hl_1.$$

Applying h on both sides of the last identity we get (19). Finally, if $l = \mu h$, applying (13) in (19), we obtain (20). □

If (M,φ,ξ,η,g) is a contact metric manifold such that $\varphi l + l\varphi = 0$, consider local orthonormal frames of type $\{\xi, e_i, \varphi e_i\}$, $i = 1,\dots,n$, such that $he_i = e_i$. Then, setting $u_i := e_i + \varphi e_i$ and $v_i := e_i - \varphi e_i$, one easily verifies that $\{\xi, u_i, v_i\}$, $i = 1,\dots,n$, is a local orthogonal frame with respect to g_1, such that each u_i is space-like and each v_i is time-like. Indeed, being $\varphi hu_i = u_i$, we have $g_1(u_i,u_i) = g(u_i,\varphi hu_i) = g(u_i,u_i) = 2$. Analogously, $\varphi hv_i = -v_i$ implies that $g_1(v_i,v_i) = -2$.

If M is a Jacobi $(0,\mu)$-contact space, being $l = \mu h$ one has the following ξ-sectional curvatures with respect to the Riemannian metric g:

$$K(\xi,e_i) = \mu, \qquad K(\xi,\varphi e_i) = -\mu, \qquad K(\xi,u_i) = K(\xi,v_i) = 0.$$

We compute now the sectional curvatures with respect to the semi-Riemannian metric g_1 of the nondegenerate 2-planes spanned by ξ and the vector fields u_i or v_i.

PROPOSITION 5. *Let (M,φ,ξ,η,g) be a Jacobi $(0,\mu)$-contact space. Consider a local orthonormal frame $\{\xi, e_i, \varphi e_i\}$, $i = 1,\dots,n$, such that $he_i = e_i$. Then the semi-Riemannian metric g_1 has ξ-sectional curvatures*

$$K_1(\xi,u_i) = 1 - 2\mu + \frac{1}{4}\mu^2, \qquad K_1(\xi,v_i) = -3 + \frac{1}{4}\mu^2.$$

In particular, if $l = 0$ then $K_1(\xi,u_i) = 1$ and $K_1(\xi,v_i) = -3$.

Proof. The sectional curvature of a nondegenerate 2-plane spanned by ξ and a vector field X is $K_1(\xi,X) = \frac{g_1(l_1X,X)}{g_1(X,X)}$. Applying (20), we have

$$l_1u_i = \Big(1 - 2\mu + \frac{1}{4}\mu^2\Big)u_i + 2\varphi u_i, \qquad l_1v_i = \Big(-3 + \frac{1}{4}\mu^2\Big)v_i + 2\varphi v_i,$$

which get the result. □

5. The paracontact metric structure $(\varphi h, \xi, \eta, g_2)$

THEOREM 3. *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric manifold such that $\varphi l + l\varphi = 0$. Then for every $X, Y, Z \in \Gamma(TM)$ the Levi-Civita connections ∇^2 and ∇ satisfy:*

$$g(\nabla^2_X Y, Z) = g(\nabla_X Y, Z) - \eta(X)g(\varphi hY, Z) - \eta(Y)g(\varphi hX, Z) - \frac{1}{2}g((\nabla_{hZ}h)Y, X) + \frac{1}{2}g\big((\nabla_X h)Y + (\nabla_Y h)X, hZ\big) - \frac{1}{2}\eta(Z)g(\varphi lX, Y). \tag{22}$$

Proof. First, from (5) and the second identity in (15), we get

$$\nabla^2_X \xi = -\varphi hX + \varphi hh_2 X = -\varphi hX - \varphi X + \frac{1}{2}\varphi hlX. \tag{23}$$

Hence, applying (11), (23) and (2), we have

$$\begin{aligned} g(\nabla^2_X Y, \xi) &= g_2(\nabla^2_X Y, \xi) = X(g_2(Y, \xi)) - g_2(Y, \nabla^2_X \xi) \\ &= X(g(Y, \xi)) + g(\nabla^2_X \xi, hY) \\ &= g(\nabla_X Y, \xi) - g(Y, \varphi X + \varphi hX) - g\big(\varphi hX + \varphi X - \frac{1}{2}\varphi hlX, hY\big) \\ &= g(\nabla_X Y, \xi) - \frac{1}{2}g(\varphi lX, Y). \end{aligned}$$

From (11), $g_2(\nabla^2_X Y, hZ) = -g(\nabla^2_X Y, h^2 Z)$ for every $X, Y, Z \in \Gamma(TM)$, and thus

$$g_2(\nabla^2_X Y, hZ) = -g(\nabla^2_X Y, Z) + \eta(Z)\Big(g(\nabla_X Y, \xi) - \frac{1}{2}g(\varphi lX, Y)\Big). \tag{24}$$

On the other hand, using the Jacobi identity and (11)

$$\begin{aligned} 2g_2(\nabla^2_X Y, hZ) &= X(g_2(Y, hZ)) + Y(g_2(hZ, X)) - (hZ)(g_2(X, Y)) \\ &\quad + g_2([X, Y], hZ) - g_2([Y, hZ], X) + g_2([hZ, X], Y) \\ &= -X(g(Y, h^2 Z)) - Y(g(hZ, hX)) + (hZ)(g(X, hY)) \\ &\quad - (hZ)(\eta(X)\eta(Y)) - g([X, Y], h^2 Z) + g([Y, hZ], hX) \\ &\quad - \eta([Y, hZ])\eta(X) - g([hZ, X], hY) + \eta([hZ, X])\eta(Y) \\ &= -2g(\nabla_X Y, h^2 Z) - g(Y, \nabla_X(h^2 Z)) - g(X, \nabla_Y(h^2 Z)) \\ &\quad + g(X, \nabla_{hZ}(hY)) + g(\nabla_X(hZ), hY) + g(\nabla_Y(hZ), hX) \\ &\quad - g(\nabla_{hZ} Y, hX) + 2d\eta(Y, hZ)\eta(X) - 2d\eta(hZ, X)\eta(Y) \\ &= -2g(\nabla_X Y, Z) + 2\eta(Z)g(\nabla_X Y, \xi) \\ &\quad + 2g(Y, \varphi hZ)\eta(X) - 2g(hZ, \varphi X)\eta(Y) \\ &\quad - g(Y, (\nabla_X h)hZ) - g(X, (\nabla_Y h)hZ) + g(X, (\nabla_{hZ} h)Y). \end{aligned}$$

Comparing the above equation with (24), we obtain (22). □

PROPOSITION 6. *Let (M,φ,ξ,η,g) be a contact metric manifold such that $\varphi l + l\varphi = 0$. The Jacobi operator l_2 is given by:*

$$l_2 = -lh - \frac{1}{4}l^2 + \frac{1}{2}h\varphi\nabla_\xi l. \tag{25}$$

If M is a Jacobi $(0,\mu)$-contact space, then

$$l_2 = \mu\Big(1 - \frac{1}{4}\mu\Big)\varphi^2. \tag{26}$$

In particular, if $l = 0$, then $l_2 = 0$.

Proof. First we prove that

$$\nabla^2_\xi = \nabla_\xi + \frac{1}{2}\varphi hl. \tag{27}$$

Indeed, applying (22), (13) and (2), for every $Y,Z \in \Gamma(TM)$ we have

$$\begin{aligned} g(\nabla^2_\xi Y,Z) &= g(\nabla_\xi Y - \varphi hY, Z) - \frac{1}{2}g(\nabla_{hZ}(hY),\xi) + \frac{1}{2}g(-\varphi lY + h(\varphi Y + \varphi hY), hZ) \\ &= g(\nabla_\xi Y - \varphi hY, Z) + \frac{1}{2}g(hY, -\varphi hZ - \varphi Z) + \frac{1}{2}g(\varphi hlY + \varphi Y + \varphi hY, Z) \\ &= g(\nabla_\xi Y, Z) + \frac{1}{2}g(\varphi hlY, Z). \end{aligned}$$

From (27) we have $\nabla^2_\xi h_2 = \nabla_\xi h_2 + \frac{1}{2}(\varphi hlh_2 - h_2\varphi hl)$, where the operator $h_2 = -h + \frac{1}{2}l$ commutes with l and h, and anticommutes with φ. Therefore,

$$\nabla^2_\xi h_2 = -\nabla_\xi h + \frac{1}{2}\nabla_\xi l - h_2\varphi hl = \varphi l + \frac{1}{2}\nabla_\xi l - \Big(\frac{1}{2}l - h\Big)\varphi hl = \frac{1}{2}\left(\nabla_\xi l + \varphi hl^2\right).$$

On the other hand, from (6) we have $\nabla^2_\xi h_2 = -\varphi h + h_2^2\varphi h - \varphi hl_2$, and thus

$$\nabla^2_\xi h_2 = -\varphi h + \Big(h^2 - lh + \frac{1}{4}l^2\Big)\varphi h - \varphi hl_2 = \varphi h\Big(-lh + \frac{1}{4}l^2 - l_2\Big).$$

Comparing the two expressions of $\nabla^2_\xi h_2$, we have

$$\varphi h\Big(-lh + \frac{1}{4}l^2 - l_2\Big) = \frac{1}{2}\left(\nabla_\xi l + \varphi hl^2\right),$$

and applying φh to both sides in the above equation, we get (25). Finally, if $l = \mu h$, we easily obtain (26). □

Finally, let us consider a Jacobi $(0,\mu)$-contact space (M,φ,ξ,η,g), and fix a local orthonormal frame $\{\xi, e_i, \varphi e_i\}$, $i = 1,\ldots,n$, such that $he_i = e_i$. Notice that $g_2(e_i,e_i) = -1$ and $g_2(\varphi e_i,\varphi e_i) = 1$, and thus each e_i is time-like, while each φe_i is space-like with respect to g_2. Using (26), one can compute the sectional curvatures for g_2 of the non-degenerate 2-planes spanned by ξ and the vector fields e_i or φe_i, showing that

$$K_2(\xi,e_i) = K_2(\xi,\varphi e_i) = -\mu\Big(1 - \frac{1}{4}\mu\Big).$$

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