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EXAMPLES OF 3-QUASI-SASAKIAN MANIFOLDS

Dedicated to Prof. Anna Maria Pastore on the occasion of her 70th birthday

Abstract. We provide a general method to construct examples of quasi-Sasakian 3-structures on a $(4n+3)$-dimensional manifold. Moreover, among this class, we give the first explicit example of a compact 3-quasi-Sasakian manifold which is not the global product of a 3-Sasakian manifold and a hyper-Kähler manifold.

1. Introduction

The class of quasi-Sasakian manifolds was introduced by Blair in [1], and then studied by several authors (e.g. [14, 13, 10]) in order to unify the most important classes of almost contact metric manifolds, namely the Sasakian and coKähler ones, which are quasi-Sasakian manifolds of maximal and minimal rank, respectively. Moreover any quasi-Sasakian manifold is canonically endowed with a transversely Kähler foliation, so that they can be thought as an odd-dimensional analogue of Kähler manifolds.

When on a smooth manifold M there are defined three distinct quasi-Sasakian structures, with the same compatible metric, which are related to each other by certain relations similar to the quaternionic identities, one says that M is a 3-quasi-Sasakian manifold (see Section 2 for the precise definition). The class of 3-quasi-Sasakian manifolds was extensively studied a few years ago in [5] and [6], where several properties on 3-quasi-Sasakian manifolds, which do not hold for a general quasi-Sasakian structure, were proved. In particular, it was proved that the aforementioned quaternionic-like structure forces any 3-quasi-Sasakian manifold of non-maximal rank $4l+3$ to be the local Riemannian product of a 3-c-Sasakian manifold and a hyper-Kähler manifold. Therefore a natural question arises: are there examples of 3-quasi-Sasakian manifolds which are not the global product of a 3-c-Sasakian manifold and a hyper-Kähler manifold? In this article we give an affirmative answer to this problem. We present a general procedure to produce a large class of examples, and we prove that the 11-dimensional 3-quasi-Sasakian manifold in this class is not a global product of 3-Sasakian and hyper-Kähler manifolds.

All manifolds considered in this paper will be assumed to be smooth and connected. For wedge product, exterior derivative and interior product we use the conventions as in Goldberg's book [9].

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2. Preliminaries

We start with a few preliminaries on almost contact metric manifolds, referring the reader to the monographs [2, 4] and to the survey [8] for further details.

An *almost contact metric structure* on a $(2n+1)$-dimensional manifold M is the data of a $(1,1)$-tensor ϕ, a vector field ξ, called *Reeb vector field*, a 1-form η and a Riemannian metric g satisfying

$$\phi^2 = -I + \eta \otimes \xi, \ \ \eta(\xi) = 1, \ \ g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{1}$$

for all $X, Y \in \Gamma(TM)$, where I denotes the identity mapping on TM. From (1) it follows that $g(X, \phi Y) = -g(\phi X, Y)$, so that we can define the 2-form Φ on M by $\Phi(X,Y) = g(X, \phi Y)$, which is called the *fundamental* 2*-form* of the almost contact metric manifold (M, ϕ, ξ, η, g).

The manifold is said to be *normal* if the tensor field $N_\phi := [\phi, \phi]_{FN} + 2d\eta \otimes \xi$ vanishes identically. Normal almost contact metric manifolds such that both η and Φ are closed are called *coKähler manifolds* and those such that $d\eta = c\Phi$ are called *c-Sasakian manifolds*, where c is a non-zero real number (for $c = 2$ one obtains the well-known *Sasakian manifolds*).

The notion of quasi-Sasakian structure was introduced by Blair in his Ph.D. thesis in order to unify those of Sasakian and coKähler structures. A *quasi-Sasakian manifold* is defined as a normal almost contact metric manifold whose fundamental 2-form is closed. A quasi-Sasakian manifold M is said to be of *rank* $2p$ (for some $p \leq n$) if $(d\eta)^p \neq 0$ and $\eta \wedge (d\eta)^p = 0$ on M, and to be of *rank* $2p+1$ if $\eta \wedge (d\eta)^p \neq 0$ and $(d\eta)^{p+1} = 0$ on M (cf. [1, 14]). Blair proved that there are no quasi-Sasakian manifolds of even rank. Just like Blair and Tanno implicitly did, we will only consider quasi-Sasakian manifolds of constant (odd) rank. If the rank of M is $2p+1$, then the module $\Gamma(TM)$ of vector fields over M splits into two submodules as follows: $\Gamma(TM) = \mathcal{E}^{2p+1} \oplus \mathcal{E}^{2q}$, $p + q = n$, where

$$\mathcal{E}^{2q} = \{X \in \Gamma(TM) \mid i_X d\eta = 0 \text{ and } i_X \eta = 0\}$$

and $\mathcal{E}^{2p+1} = \mathcal{E}^{2p} \oplus \langle \xi \rangle$, $\mathcal{E}^{2p}$ being the orthogonal complement of $\mathcal{E}^{2q} \oplus \langle \xi \rangle$ in $\Gamma(TM)$. These modules satisfy $\phi \mathcal{E}^{2p} = \mathcal{E}^{2p}$ and $\phi \mathcal{E}^{2q} = \mathcal{E}^{2q}$ ([14]).

We now come to the main topic of our paper, i.e. 3-quasi-Sasakian geometry, which is framed into the more general setting of almost 3-contact geometry. An *almost contact metric* 3*-structure* on a smooth manifold M is the data of three almost contact structures (ϕ_1, ξ_1, η_1), (ϕ_2, ξ_2, η_2), (ϕ_3, ξ_3, η_3) satisfying the following relations, for any even permutation (α, β, γ) of $(1,2,3)$,

$$\begin{gathered} \phi_\gamma = \phi_\alpha \phi_\beta - \eta_\beta \otimes \xi_\alpha = -\phi_\beta \phi_\alpha + \eta_\alpha \otimes \xi_\beta, \\ \xi_\gamma = \phi_\alpha \xi_\beta = -\phi_\beta \xi_\alpha, \ \ \eta_\gamma = \eta_\alpha \circ \phi_\beta = -\eta_\beta \circ \phi_\alpha, \end{gathered} \tag{2}$$

and a Riemannian metric g compatible with each of them. This definition was introduced, independently, by Kuo ([12]) and Udriste ([15]). In particular, they proved that necessarily $\dim(M) = 4n+3$. It is well known that in any almost 3-contact metric manifold the Reeb vector fields ξ_1, ξ_2, ξ_3 are orthonormal with respect to the compatible metric g and that the structural group of the tangent bundle is reducible to $Sp(n) \times I_3$.

Moreover, by putting $\mathcal{H} = \bigcap_{\alpha=1}^{3} \ker(\eta_\alpha)$ one obtains a $4n$-dimensional *horizontal distribution* on M and the tangent bundle splits as the orthogonal sum $TM = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V} = \langle \xi_1, \xi_2, \xi_3 \rangle$ is the *vertical distribution*.

DEFINITION 1. *A* quasi-Sasakian 3-structure *is an almost contact metric 3-structure* $\{(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)\}_{\alpha \in \{1,2,3\}}$ *on a smooth manifold M such that each almost contact metric structure is quasi-Sasakian. The manifold M will be called a* 3-quasi-Sasakian manifold*.*

In particular, a quasi-Sasakian 3-structure such that each structure is Sasakian is called a *Sasakian 3-structure* and the manifold is said to be a 3-*Sasakian manifold*. A quasi-Sasakian 3-structure such that each structure is coKähler is called a *cosymplectic 3-structure* and the manifold is said to be a 3-*cosymplectic manifold*.

Let us collect some known results on 3-quasi-Sasakian manifolds. The following theorem combines the results obtained in Theorems 3.4 and 4.2 of [5], and Theorem 3.7 of [6].

THEOREM 1. *Let* $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ *be a 3-quasi-Sasakian manifold. Then the 3-dimensional distribution* $\mathcal{V}$ *generated by* ξ_1, ξ_2, ξ_3 *is integrable. Moreover,* $\mathcal{V}$ *defines a Riemannian foliation with totally geodesic leaves on M, and for any even permutation* (α, β, γ) *of* $(1,2,3)$ *and for some* $c \in \mathbb{R}$

$$[\xi_\alpha, \xi_\beta] = c\xi_\gamma.$$

Moreover, $c = 0$ *if and only if the structure is 3-cosymplectic.*

Using Theorem 1 we may divide 3-quasi-Sasakian manifolds in two classes according to the behaviour of the leaves of the foliation $\mathcal{V}$: those 3-quasi-Sasakian manifolds for which each leaf of $\mathcal{V}$ is locally $SO(3)$ (or $SU(2)$) (which corresponds to take in Theorem 1 the constant $c \neq 0$), and those for which each leaf of $\mathcal{V}$ is locally an abelian group (this corresponds to the case $c = 0$).

3. Basic properties of 3-quasi-Sasakian manifolds

For a 3-quasi-Sasakian manifold one can consider the ranks, a priori distinct, of the three quasi-Sasakian structures $(\phi_1, \xi_1, \eta_1, g)$, $(\phi_2, \xi_2, \eta_2, g)$, $(\phi_3, \xi_3, \eta_3, g)$. The following theorem assures that these three ranks coincide.

THEOREM 2 ([5, 6]). *Let* $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ *be a 3-quasi-Sasakian manifold of dimension* $4n+3$*. Then the 1-forms* η_1, η_2 *and* η_3 *have the same rank* $4l+3$*, for some integer* $l \leq n$*, or* 1 *according to* $[\xi_\alpha, \xi_\beta] = c\xi_\gamma$ *with* $c \neq 0$ *or* $c = 0$*, respectively.*

According to Theorem 2, the common rank of η_1, η_2, η_3 is called the *rank* of the 3-quasi-Sasakian manifold $(M,\phi_\alpha,\xi_\alpha,\eta_\alpha,g)$

Furthermore, for any 3-quasi-Sasakian manifold of rank $4l+3$ one can consider the following distribution

$$\mathcal{E}^{4m} := \left\{X \in \Gamma(\mathcal{H}) \mid i_X d\eta_\alpha = 0,\ \alpha = 1,2,3\right\} \quad (l+m=n)$$

and its orthogonal complement $\mathcal{E}^{4l+3} := (\mathcal{E}^{4m})^\perp$. In [6] it was proved the following remarkable property of 3-quasi-Sasakian manifolds, which in general does not hold for a general quasi-Sasakian structure.

THEOREM 3. *Let $(M,\phi_\alpha,\xi_\alpha,\eta_\alpha,g)$ be a 3-quasi-Sasakian manifold of rank $4l+3$. Then the distributions $\mathcal{E}^{4l+3}$ and $\mathcal{E}^{4m}$ are integrable and define Riemannian foliations with totally geodesic leaves.*

In particular it follows that $\nabla\mathcal{E}^{4l+3}\subset\mathcal{E}^{4l+3}$ and $\nabla\mathcal{E}^{4m}\subset\mathcal{E}^{4m}$. The leaves of such foliations are 3-c-Sasakian manifolds (i.e., for each $\alpha \in \{1,2,3\}$, $d\eta_\alpha = c\Phi_\alpha$) and hyper-Kähler manifolds, respectively (cf. Theorem 5.4 and Theorem 5.6 of [6]). Thus we can state the following corollary.

COROLLARY 1. *Any 3-quasi-Sasakian manifold of rank $4l+3$, with $1 \leq l < n$, is the local product of a 3-c-Sasakian manifold and of a hyperKähler manifold.*

Another strong consequence of Theorem 3 is the following

COROLLARY 2. *Any 3-quasi-Sasakian manifold of maximal rank $4n+3$ is necessarily 3-c-Sasakian.*

Thus in the two extremal cases — maximal and minimal rank — the geometry of a 3-quasi-Sasakian manifold is well known. In the rank 1 case, the structure turns out to be 3-cosymplectic and we can refer the reader to [7] for the main properties of these geometric structures and non-trivial examples. In the rank $(4n+3)$ case, by applying a certain homothety one can obtain a 3-Sasakian structure.

Thus we shall deal with the non-trivial cases $\mathrm{rank}(M) \neq 1$, $\mathrm{rank}(M) \neq \dim(M)$.

4. A general construction

Let $(M',\phi'_\alpha,\xi'_\alpha,g')$ and (M'',J''_α,g'') be a 3-Sasakian and a hyper-Kähler manifold, respectively. Set $\dim(M') = 4l+3$ and $\dim(M'') = 4m$. We define a canonical 3-quasi-Sasakian structure on the product manifold $M := M' \times M''$ in the following way.

We define as Reeb vector fields $\xi_\alpha := \xi'_\alpha$, for each $\alpha \in \{1,2,3\}$. Next, let ϕ_α be the $(1,1)$-tensor field determined by

$$\phi_\alpha X := \begin{cases} \phi'_\alpha X, & \text{if } X \in \Gamma(TM') \\ J''_\alpha X, & \text{if } X \in \Gamma(TM''). \end{cases}$$

Finally, we consider the product metric $g := g' + g''$ and we define three 1-forms η_1, η_2, η_3 by $\eta_\alpha := g(\cdot, \xi_\alpha)$. From the definition it follows that the horizontal distribution $\mathcal{H} := \bigcap_{\alpha=1}^{3} \ker(\eta_\alpha)$ coincides with $\mathcal{H}' \oplus TM''$, where $\mathcal{H}'$ is the horizontal distribution of the 3-Sasakian manifold M'. Then on $\mathcal{H}$ the triple (ϕ_1, ϕ_2, ϕ_3) satisfies the quaternionic relations

$$\phi_\alpha \phi_\beta = -\phi_\beta \phi_\alpha = \phi_\gamma$$

for a cyclic permutation (α, β, γ) of $\{1,2,3\}$. On the other hand, $\phi_\alpha \xi_\beta = \phi'_\alpha \xi'_\beta = \xi'_\gamma = \xi_\gamma = -\phi_\beta \xi_\alpha$. Hence

$$\begin{aligned} \phi_\gamma &= \phi_\alpha \phi_\beta - \eta_\beta \otimes \xi_\alpha = -\phi_\beta \phi_\alpha + \eta_\alpha \otimes \xi_\beta, \\ \xi_\gamma &= \phi_\alpha \xi_\beta = -\phi_\beta \xi_\alpha, \;\; \eta_\gamma = \eta_\alpha \circ \phi_\beta = -\eta_\beta \circ \phi_\alpha \end{aligned}$$

and we conclude that $\{(\phi_\alpha, \xi_\alpha, \eta_\alpha)\}_{\alpha \in \{1,2,3\}}$ is an almost contact 3-structure on M. By the very definition of g and ϕ_α then we have that g is a compatible metric.

Let us show that $\{(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)\}_{\alpha \in \{1,2,3\}}$ is a 3-quasi-Sasakian structure on M. Notice that each fundamental 2-form $\Phi_\alpha := g(\cdot, \phi_\alpha \cdot)$ is given by

$$\Phi_\alpha(X,Y) := \begin{cases} \Phi'_\alpha(X,Y), & \text{if } X,Y \in \Gamma(TM') \\ 0, & \text{if } X \in \Gamma(TM'), \text{ if } Y \in \Gamma(TM'') \\ \Omega''_\alpha(X,Y), & \text{if } X,Y \in \Gamma(TM'') \end{cases}$$

where Φ'_α and Ω''_α denote the fundamental 2-forms of $(M', \phi'_\alpha, \xi'_\alpha, g')$ and (M'', J''_α, g''), respectively. By using the well-known formula

$$\begin{aligned} d\Phi_\alpha(X,Y,Z) = {} & X(\Phi_\alpha(Y,Z)) + Y(\Phi_\alpha(Z,X)) + Z(\Phi_\alpha(X,Y)) \\ & - \Phi_\alpha([X,Y],Z) - \Phi_\alpha([Y,Z],X) - \Phi_\alpha([Z,X],Y) \end{aligned}$$

we see that

$$d\Phi_\alpha(X,Y,Z) = \begin{cases} d\Phi'_\alpha(X,Y,Z), & \text{if } X,Y,Z \in \Gamma(TM') \\ 0, & \text{if } X,Y \in \Gamma(TM'),\, Z \in \Gamma(TM'') \\ 0, & \text{if } X \in \Gamma(TM'),\, Y,Z \in \Gamma(TM'') \\ d\Omega''_\alpha(X,Y,Z), & \text{if } X,Y,Z \in \Gamma(TM''). \end{cases}$$

Since Φ'_α and Ω''_α are closed, we conclude that also each Φ_α is closed. Moreover, in order to prove the normality of the 3-structure $\{(\phi_\alpha, \xi_\alpha, \eta_\alpha)\}_{\alpha \in \{1,2,3\}}$, it is enough to check the vanishing of N_{ϕ_α} on the couples of vector fields of this type:

$$N_{\phi_\alpha}(X', Y'), \quad N_{\phi_\alpha}(X', Y''), \quad N_{\phi_\alpha}(Y', Y''),$$

where X', Y' are vector fields on M' and X'', Y'' are vector fields on M''. Since $d\eta_\alpha = 0$ on TM'', using the definitions of ϕ_α and N_{ϕ_α},

$$\begin{aligned} N_{\phi_\alpha}(X', Y') &= N_{\phi'_\alpha}(X', Y') = 0, \\ N_{\phi_\alpha}(X'', Y'') &= N_{J''_\alpha}(X'', Y'') = 0, \end{aligned}$$

because M' is 3-Sasakian and M'' hyper-Kähler, and

$$N_{\phi_\alpha}(X',Y'') = \phi_\alpha^2[X',Y''] + [\phi_\alpha X', \phi_\alpha Y''] - \phi_\alpha[\phi_\alpha X', Y''] - \phi_\alpha[X', \phi_\alpha Y''] = 0$$

since each summand in the last equation is zero.

Therefore $(M,\phi_\alpha,\xi_\alpha,\eta_\alpha,g)$ is a 3-quasi-Sasakian manifold with rank $4l+3 = \dim(M')$.

We say that $f\colon M \to M$ is a *3-quasi-Sasakian isometry* if it is an isometry of the Riemannian manifold (M,g) preserving each quasi-Sasakian structure, namely

$$f_* \circ \phi_\alpha = \phi_\alpha \circ f_*, \quad f_*\xi_\alpha = \xi_\alpha \tag{3}$$

for each $\alpha \in \{1,2,3\}$. Notice that from (3) it follows that

$$f^*\eta_\alpha = \eta_\alpha. \tag{4}$$

Indeed for any $X \in \Gamma(TM)$

$$f^*\eta_\alpha(X) = \eta_\alpha(f_*X) = g(f_*X,\xi_\alpha) = g(f_*X, f_*\xi_\alpha) = g(X,\xi_\alpha) = \eta_\alpha(X).$$

Given a free and properly discontinuous action of a discrete group (in particular, a free action of a finite group) G on a 3-quasi-Sasakian manifold M by 3-quasi-Sasakian isometries, the quotient M/G is a smooth manifold of the same dimension as M and inherits a 3-quasi-Sasakian structure from M.

Recall that $f\colon M'' \to M''$ is a hyper-Kähler isometry if f is an isometry of the Riemannian manifold (M'',g'') and

$$f_* \circ J''_\alpha = J''_\alpha \circ f_* \tag{5}$$

for each $\alpha \in \{1,2,3\}$. From (5) it follows that

$$f^*\Omega''_\alpha = \Omega''_\alpha.$$

Suppose G is a finite group that acts on M' by 3-Sasakian isometries and on M'' by hyper-Kähler isometries. Then G also acts on the product manifold $M' \times M''$ by $g \cdot (p',p'') = (g \cdot p', g \cdot p'')$, $g \in G$. It is easy to check that G preserves the 3-quasi-Sasakian structure on $M' \times M''$ defined above. If the action of G on $M' \times M''$ is free then the quotient $(M' \times M'')/G$ is a 3-quasi-Sasakian manifold.

As an application, we consider the 3-Sasakian manifold S^{4l+3}. We recall how the standard 3-Sasakian structure $(\phi'_\alpha, \xi'_\alpha, \eta'_\alpha, g')$ of the sphere is defined. Let us consider the sphere S^{4l+3} as an hypersurface in $\mathbb{H}^{l+1}$. Let (J_1,J_2,J_3) be the standard quaternionic structure of $\mathbb{H}^{l+1}$ that is upon identification of $T_x\mathbb{H}^{l+1}$ with $\mathbb{H}^{l+1}$ the operators J_1, J_2, J_3 act by multiplication with **i**, **j**, **k** on the left.

Let N be the outer vector field normal to the sphere. Then one can prove that the vector fields

$$\xi'_\alpha := -J_\alpha N \tag{6}$$

are tangent to the sphere. Moreover, for any $X \in \Gamma(TS^{4l+3})$, we decompose $J_\alpha X$ in their components tangent and normal to the sphere,

$$J_\alpha X = \phi'_\alpha X + \eta'_\alpha(X)N, \tag{7}$$

so obtaining, for each $\alpha \in \{1,2,3\}$, a tensor field ϕ'_α and a 1-form η'_α on S^{4l+3}. Then one can check that the geometric structure $\{(\phi'_\alpha, \xi'_\alpha, \eta'_\alpha, g')\}_{\alpha\in\{1,2,3\}}$ is a 3-Sasakian structure on S^{4l+3}, being g' the Riemannian metric induced by the Riemannian metric g of $\mathbb{H}^{l+1} \cong \mathbb{R}^{4l+4}$.

Now we consider the isometry f of $\mathbb{H}^{l+1}$ given by the multiplication with $\mathbf{i}$ on the right. Notice that $f(S^{4l+3}) = S^{4l+3}$, because for any $x \in S^{4l+3}$ one has $\|f(x)\| = \|x\mathbf{i}\| = \|x\| = 1$. Hence f induces an isometry on (S^{4l+3}, g'), again denoted by f. Notice that the associativity of the product in $\mathbb{H}$ implies

$$f_* \circ J_\alpha = J_\alpha \circ f_*.$$

Thus f is a hyper-Kähler isometry. Moreover, for every $X \in \Gamma(TS^{4l+3})$, one has $g(f_*N, f_*X) = g(N,X) = 0$, so that $f_*N \in (TS^{4l+3})^\perp = \langle N\rangle$. Since $\|N\| = 1$ and f is an isometry, it follows that

$$f_*N = N.$$

Then by (6) and (7) we get

$$f_*\xi'_\alpha = -f_*J_\alpha N = -J_\alpha f_* N = -J_\alpha N = \xi'_\alpha,$$

and, for all $X \in \Gamma(TS^{4l+3})$,

$$\begin{aligned} f_*(\phi'_\alpha X) + \eta'_\alpha(X)N &= f_*(\phi'_\alpha X) + \eta'_\alpha(X) f_* N \\ &= f_* J_\alpha X \\ &= J_\alpha f_* X \\ &= \phi'_\alpha(f_*X) + \eta'_\alpha(f_*X)N, \end{aligned}$$

from which, taking the tangential and the normal components to the sphere, it follows that $f_* \circ \phi'_\alpha = \phi'_\alpha \circ f_*$ and $f^*\eta'_\alpha = \eta'_\alpha$. Thus f is a 3-Sasakian isometry of S^{4l+3}. Moreover, f^4 is the identity operator. Thus we get an action of $\mathbb{Z}_4$ on S^{4l+3} by 3-Sasakian isometries.

Let m be a positive integer. We denote the hyper-Kähler isometry of $\mathbb{H}^m$, $(q_1, \ldots, q_m) \mapsto (q_1\mathbf{i}, \ldots, q_m\mathbf{i})$, by h. The map h induces a hyper-Kähler isometry on the torus $\mathbb{T}^{4m} = \mathbb{H}^m/\mathbb{Z}^{4m}$. Thus h generates an action of $\mathbb{Z}_4$ on $\mathbb{T}^{4m}$ by hyper-Kähler isometries. Note, that $\mathbb{Z}_4$ acts freely on S^{4l+3}, but has a fixed point $[0]$ in $\mathbb{T}^{4m}$. Nevertheless, the resulting action of $\mathbb{Z}_4$ on $S^{4l+3} \times \mathbb{T}^{4m}$ is free. We will denote the 3-quasi-Sasakian manifold $(S^{4l+3} \times \mathbb{T}^{4m})/\mathbb{Z}_4$ by $M_{l,m}$.

Concerning this example, in view of Corollary 1, an interesting question is the following: is $M_{l,m}$ the global product of a 3-Sasakian manifold of dimension $4l+3$ and a hyperKähler manifold of dimension $4m$?

In the next Section we shall show that the answer is negative, at least in the case $l=1$ and $m=1$. Namely we will prove that the 3-quasi-Sasakian manifold

$$M_{1,1} := (S^7 \times \mathbb{T}^4)/\mathbb{Z}_4$$

is not topologically equivalent to the product of a 7-dimensional compact 3-Sasakian manifold and a 4-dimensional compact hyper-Kähler manifold.

5. The manifold $M_{1,1} = (S^7 \times \mathbb{T}^4)/\mathbb{Z}_4$

Let M be a compact Riemannian manifold and G a finite group freely acting on M. Denote by ρ_M the corresponding group homomorphism from G to $\mathrm{Aut}(M)$. Then from the Hodge theory we obtain the isomorphism

$$H^*(M/G) \cong H^*(M)^G := \{x \in H^*(M) \,|\, \rho(a)^* x = x, \text{ for all } a \in G\}. \tag{8}$$

Indeed, every harmonic form on M/G lifts to a G-periodic harmonic form on M and every G-periodic form on M defines a periodic form on M/G. Here it is important that the projection $M \to M/G$ is a local diffeomorphism and the Laplacian $\triangle$ is defined locally.

Now, let M and N be two compact manifolds with G-action given by $\rho_M \colon G \to \mathrm{Aut}(M)$ and $\rho_N \colon G \to \mathrm{Aut}(N)$. We will write $\rho \colon G \to \mathrm{Aut}(M \times N)$ for the corresponding action on the product $M \times N$. If ω is a q-form on M and σ is a p-form on N, then $\mathrm{pr}_M^* \omega \wedge \mathrm{pr}_N^* \sigma$ is a $(p+q)$-form on $M \times N$. Moreover,

$$\rho(a)^* \left(\mathrm{pr}_M^* \omega \wedge \mathrm{pr}_N^* \sigma\right) = \mathrm{pr}_M^* \rho_M(a)^* \omega \wedge \mathrm{pr}_M^* \rho_N(a)^* \sigma$$

for $a \in G$. By Künneth theorem we have

$$H^k(M \times N) = \bigoplus_{p+q=k} H^q(M) \otimes H^p(N).$$

From the above we see that $H^q(M) \otimes H^p(N)$ is a G-invariant subspace of $H^k(M \times N)$. Therefore

$$H^k(M \times N)^G = \bigoplus_{q+p=k} \left(H^q(M) \otimes H^p(N)\right)^G. \tag{9}$$

Let us now specialize to the case of $M = S^7$ and $N = \mathbb{T}^4$ with the action of $\mathbb{Z}_4$ on S^7 and $\mathbb{T}^4$ defined in the previous section. Note that since the isometry $f \colon S^7 \to S^7$ was orientation preserving, the induced action of $\mathbb{Z}_4$ on $H^7(S^7) \cong \mathbb{R}$ is trivial. It is also clear that $\mathbb{Z}_4$ acts trivially on $H^0(S^7) \cong \mathbb{R}$. Thus for any $0 \le k \le 4$

$$\left(H^0(S^7) \otimes H^k(\mathbb{T}^4)\right)^{\mathbb{Z}_4} \cong H^k(\mathbb{T}^4)^{\mathbb{Z}_4}, \qquad \left(H^7(S^7) \otimes H^k(\mathbb{T}^4)\right)^{\mathbb{Z}_4} \cong H^k(\mathbb{T}^4)^{\mathbb{Z}_4}. \tag{10}$$

Let us denote the Betti numbers of $M_{1,1}$ by b_k and we write $\tilde{b}_k$ for $\dim H^k(\mathbb{T}_4)^{\mathbb{Z}_4}$. Then, from (8), (9), and (10) it follows that

$$\begin{array}{llllll} b_0 = \tilde{b}_0 = 1, & b_1 = \tilde{b}_1, & b_2 = \tilde{b}_2, & b_3 = \tilde{b}_3, & b_4 = \tilde{b}_4, & b_5 = b_6 = 0, \\ b_7 = \tilde{b}_0 = 1, & b_8 = \tilde{b}_1, & b_9 = \tilde{b}_2, & b_{10} = \tilde{b}_3, & b_{11} = \tilde{b}_4 = 1. & \end{array} \tag{11}$$

Now we compute $\tilde{b}_1$, $\tilde{b}_2$, and $\tilde{b}_3$. Note that from the above equations and Poincaré duality for $M_{1,1}$, we get $\tilde{b}_1 = b_1 = b_{10} = \tilde{b}_3$. Thus it is enough to compute $\tilde{b}_1$ and $\tilde{b}_2$. The cup product on $H^*(\mathbb{T}^4)$ induces the $\mathbb{Z}_4$-invariant isomorphism

$$\begin{aligned}\Lambda^* H^1(\mathbb{T}^4) &\longrightarrow H^*(\mathbb{T}^4)\\ [\alpha_1]\wedge\cdots\wedge[\alpha_k] &\longmapsto [\alpha_1\wedge\cdots\wedge\alpha_k],\end{aligned}$$

where $\Lambda^* V$ stands for the exterior algebra of a vector space V. Thus

$$\tilde{b}_k = \dim(\Lambda^k H^1(\mathbb{T}^4))^{\mathbb{Z}_4} = \dim(\Lambda^k H^1(\mathbb{T}^4))^{h^*},$$

where $h\colon \mathbb{T}^4 \to \mathbb{T}^4$ was defined in the previous section. Let $x_{\mathbf{1}}$, $x_{\mathbf{i}}$, $x_{\mathbf{j}}$, $x_{\mathbf{k}}$ be the coordinate functions on $\mathbb{T}^4$ induced from $\mathbb{H}$. Let $\theta_{\mathbf{1}}$, $\theta_{\mathbf{i}}$, $\theta_{\mathbf{j}}$, and $\theta_{\mathbf{k}}$ be the dual 1-forms. Then the classes $[\theta_{\mathbf{1}}]$, $[\theta_{\mathbf{i}}]$, $[\theta_{\mathbf{j}}]$, and $[\theta_{\mathbf{k}}]$ give a basis of $H^1(\mathbb{T}^4)$. The matrix of h^* in this basis is

$$A := \begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{pmatrix}.$$

The eigenvalues of A over $\mathbb{C}$ are i and $-i$. Since 1 is not among the eigenvalues there is no element in $H^1(\mathbb{T}^4)$ which is h^*-invariant. Thus $\tilde{b}_1 = 0$. The matrix of h^* in the basis

$$[\theta_{\mathbf{1}}]\wedge[\theta_{\mathbf{i}}],\quad [\theta_{\mathbf{1}}]\wedge[\theta_{\mathbf{j}}]+[\theta_{\mathbf{i}}]\wedge[\theta_{\mathbf{k}}],\quad [\theta_{\mathbf{1}}]\wedge[\theta_{\mathbf{j}}]-[\theta_{\mathbf{i}}]\wedge[\theta_{\mathbf{k}}]$$

$$[\theta_{\mathbf{1}}]\wedge[\theta_{\mathbf{k}}]+[\theta_{\mathbf{i}}]\wedge[\theta_{\mathbf{j}}],\quad [\theta_{\mathbf{1}}]\wedge[\theta_{\mathbf{k}}]-[\theta_{\mathbf{i}}]\wedge[\theta_{\mathbf{j}}],\quad [\theta_{\mathbf{j}}]\wedge[\theta_{\mathbf{k}}]$$

of $\Lambda^2 H^1(\mathbb{T}^4)$ is $\mathrm{diag}(1,-1,1,1,-1,1)$. Thus $\tilde{b}_2 = \dim(\Lambda^2 H^1(\mathbb{T}^4))^{h^*} = 4$. Using that $\tilde{b}_1 = 0 = \tilde{b}_3$, $\tilde{b}_2 = 4$, we get from (11)

$$b_0 = b_4 = b_7 = b_{11} = 1,\quad b_1 = b_3 = b_5 = b_6 = b_8 = b_{10} = 0,\quad b_2 = b_9 = 4.$$

Thus the Poincaré polynomial of $M_{1,1}$ is

$$P(t) := 1 + 4t^2 + t^4 + t^7 + 4t^9 + t^{11} = (1+t^7)(1+4t^2+t^4). \tag{12}$$

Suppose $M_{1,1} \cong M' \times M''$, where M' is a 7-dimensional 3-Sasakian manifold and M'' a 4-dimensional hyper-Kähler manifold. Denote by P' and P'' the Poincaré polynomial of M', respectively of M''. Then by Künneth theorem

$$P(t) = P'(t)P''(t). \tag{13}$$

We will write $p_1 \le p_2$ for two polynomials with non-negative coefficients if all the coefficients of $p_2 - p_1$ are non-negative. We also write $p_1 < p_2$ if $p_1 \le p_2$ and $p_1 \ne p_2$. It is obvious that if $p_1 \le p_2$ then $p_1 p \le p_2 p$, and if $p_1 < p_2$ then $p_1 p < p_2 p$ for any non-zero polynomial p with non-negative coefficients.

With this notation we have $P''(t) \ge 1 + t^7$, since M'' is a compact orientable 7-dimensional manifold.

Let us recall the following well-known result that follows from Enriques-Kodaira classification [11] of compact closed surfaces (see e.g. [3]).

THEOREM 4. *If M^4 is a compact four-dimensional hyper-Kähler manifold, then M^4 is either a K3 surface or a four dimensional torus.*

If $M'' \cong \mathbb{T}^4$, then $P''(t) = 1+4t+6t^2+4t^3+t^4$. If M'' is a K3 surface then $P''(t) = 1+22t^2+t^4$. Thus in both cases $P''(t) > 1+4t^2+t^4$. Therefore

$$P(t) = P'(t)P''(t) > (1+t^7)(1+4t^2+t^7) = P(t),$$

which gives a contradiction to our assumption $M_{1,1} \cong M' \times M''$. So we finally proved

THEOREM 5. *There exists an 11-dimensional compact 3-quasi-Sasakian manifold of rank 7 which is not a global product of a 7-dimensional 3-Sasakian manifold and a 4-dimensional hyper-Kähler manifold.*

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