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HARMONICITY PROPERTIES OF PARACONTACT METRIC MANIFOLDS

Sommario. We shall describe some recent results concerning harmonicity properties of the Reeb vector field of a paracontact metric manifold, and their application to the study of paracontact Ricci solitons.

1. Introduction

Contact Riemannian structures are a natural odd-dimensional analogue to complex structures. Similarly, in pseudo-Riemannian settings, paracontact metric structures were introduced in [21] as an odd-dimensional counterpart to paraHermitian geometry.

Up to recent years, the study of paracontact metric manifolds essentially focused on the special case of paraSasakian manifolds. However, starting on 2009 with the work of Zamkovoy [29], a systematic study of paracontact metric structures began. Since then, paracontact metric manifolds have been studied under several different points of view, emphasizing similarities and differences with respect to the corresponding properties in the contact Riemannian case. Some recent results on paracontact and almost paracontact metric structures may be found in [4],[6]-[9],[15],[26] and references therein.

Harmonicity conditions of vector fields over pseudo-Riemannian manifolds have been intensively studied in recent years. We may refer to [3] and the monograph [18] and references therein for an overview on harmonicity properties of vector fields. Because of these results, it is a natural problem to investigate when the Reeb vector field of a paracontact metric manifold satisfies some harmonicity properties.

Given a (smooth, oriented, connected) semi-Riemannian manifold (M,g) and a unit vector field V on M, the *energy* of V is the energy of the corresponding smooth map $V:(M,g)\to(T_1M,g^s)$, where (T_1M,g^s) is the unit tangent bundle of (M,g), equipped with the Sasaki metric. V is said to be a *harmonic vector field* if $V:(M,g)\to(T_1M,g^s)$ is a critical point for the energy functional restricted to maps defined by unit vector fields.

The Reeb vector field ξ of a contact Riemannian manifold is harmonic if and only if ξ is a Ricci eigenvector [23]. This led to define *H-contact Riemannian manifolds* as contact metric manifolds, whose Reeb vector field is harmonic. Since their introduction, H-contact Riemannian manifolds have been intensively studied and their relations to other contact geometry properties are now well understood (see, for example, [2, Section 10.3.1], [16], [18, Chapter 4] and references therein). Correspondingly, *H-paracontact (metric) manifolds*, that is, paracontact metric manifolds whose Reeb

[1]Joint works with A. Perrone and D. Perrone.

vector field is harmonic, were introduced in [10]. It turns out that a paracontact metric manifold is H-paracontact if and only if the Reeb vector field is a Ricci eigenvector. Although formally similar to its contact Riemannian counterpart, this result needs a completely different approach, because of the deep differences arising between Riemannian and semi-Riemannian settings. In fact, in the Riemannian case, a self-adjoint operator admits an orthonormal basis of eigenvectors, but this property does not hold any more in pseudo-Riemannian settings.

Besides their intrinsic interest, the study of H-paracontact manifolds is also motivated by their relations with some other relevant geometric properties, like the Reeb vector field being an *infinitesimal harmonic transformation* or the existence of *paracontact Ricci solitons*. Under these points of view, a deep difference arises between the Riemannian case, where some strong rigidity results hold (see [24] and references therein), and the pseudo-Riemannian one, which allows several nontrivial interesting behaviours.

More precisely, the Reeb vector field of a contact metric manifold (M,φ,ξ,η,g) is an infinitesimal harmonic transformation (in particular, satisfies the Ricci soliton equation) if and only if the structure is both K-contact and Einstein. As such, it yields a trivial Ricci soliton, given by a Killing vector field together with an Einstein manifold. On the other hand, a positive answer was given in [8] to the open question about the existence of nontrivial paracontact Ricci solitons. A complete description of these objects can be achieved for the three-dimensional case, and their relationship with (κ,μ)-nullity condition arises.

The aim of the present paper is to illustrate these recent results, obtained in [7], [8] and [10], concerning harmonicity properties of the Reeb vector field of a paracontact metric manifold. The paper is organized in the following way. In Section 2 we report some basic information about paracontact metric manifolds and harmonicity properties of vector fields. The characterization of H-paracontact metric manifolds in terms of the Ricci operator is illustrated in Section 3. The relationship between H-paracontact metric manifolds and paracontact metric manifolds, whose Reeb vector field is 1-harmonic (equivalently, an infinitesimal harmonic transformation) is then discussed. In Section 4 we turn our attention to paracontact metric manifolds whose vector field determines a Ricci soliton, with particular regard to the study of nontrivial three-dimensional examples.

2. Preliminaries

2.1. Paracontact metric structures

An *almost paracontact structure* on a $(2n+1)$-dimensional (connected) smooth manifold M is a triple (φ,ξ,η), where φ is a $(1,1)$-tensor, ξ a global vector field and η a 1-form, such that

(1) $$\varphi(\xi)=0, \qquad \eta\circ\varphi=0, \qquad \eta(\xi)=1, \qquad \varphi^2=Id-\eta\otimes\xi$$

and the restriction J of φ on the horizontal distribution $\ker\eta$ is an almost paracomplex structure (that is, the eigensubbundles D^+, D^- corresponding to the eigenvalues $1, -1$ of J have equal dimension n).

A pseudo-Riemannian metric g on M is *compatible* with the almost paracontact structure (φ, ξ, η) when

$$g(\varphi X, \varphi Y) = -g(X,Y) + \eta(X)\eta(Y). \tag{2}$$

In such a case, (φ, ξ, η, g) is said to be an *almost paracontact metric structure*. We can observe that by (1) and (A.3), $\eta(X) = g(\xi, X)$ for any compatible metric.

Any almost paracontact structure admits compatible metrics, which, by (A.3), have signature $(n+1, n)$. The *fundamental* 2*-form* Φ of an almost paracontact metric structure (φ, ξ, η, g) is defined by $\Phi(X,Y) = g(X, \varphi Y)$, for all tangent vector fields X, Y. If $\Phi = d\eta$, then the manifold (M, η, g) (or $(M, \varphi, \xi, \eta, g)$) is called a *paracontact metric manifold* and g the *associated metric*.

Throughout the paper, we shall denote with ∇ the Levi-Civita connection and by R the curvature tensor of g, taken with the sign convention

$$R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y].$$

An almost paracontact metric structure (φ, ξ, η, g) is said to be *normal* if

$$[\varphi, \varphi] - 2d\eta \otimes \xi = 0, \tag{3}$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion tensor of φ. A *paraSasakian manifold* is a normal paracontact metric manifold.

We recall that by definition ([15]), a *paracontact* (κ, μ)*-space* is a paracontact metric manifold $(M, \varphi, \xi, \eta, g)$, satisfying the curvature condition

$$R(X,Y)\xi = \kappa(\eta(X)Y - \eta(Y)X) + \mu(\eta(X)hY - \eta(Y)hX), \tag{4}$$

for all vector fields X, Y on M, where κ and μ are smooth functions, and $h := \frac{1}{2}\mathcal{L}_\xi \varphi$ is a $(1,1)$-tensor which plays an important role in the study of paracontact metric geometry. These manifolds generalize the paraSasakian ones, for which $\kappa = -1$ and μ is undetermined.

We also recall that any almost paracontact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ admits (at least, locally) a φ*-basis* [29], that is, a pseudo-orthonormal basis of vector fields of the form $\{\xi, E_1, .., E_n, \varphi E_1, .., \varphi E_n\}$, where $\xi, E_1, .., E_n$ are space-like vector fields and so, by (A.3), vector fields $\varphi E_1, .., \varphi E_n$ are time-like.

Observe that if $(M^3, \varphi, \xi, \eta, g)$ is a three-dimensional almost paracontact metric manifold, then any (local) pseudo-orthonormal basis of $\ker\eta$ determines a φ-basis, up to sign. In fact, if $\{e_2, e_3\}$ is a (local) pseudo-orthonormal basis of $\ker\eta$, with e_3, time-like, then (A.3) yields that $\varphi e_2 \in \ker\eta$ is time-like and orthogonal to e_2, so that $\varphi e_2 = \pm e_3$. Hence, $\{\xi, e_2, \pm e_3\}$ is a φ-basis.

We can provide a local description of all paracontact metric three-manifolds. Indeed, a local description was obtained in [7] for the much larger class of three-dimensional *natural* almost paracontact metric structures, for which one only requires the much weaker condition $\xi \in \ker d\eta$.

Let (φ,ξ,η,g) be a three-dimensional natural almost paracontact metric structure on M. Then,

$$2hX = \varphi(\nabla_X\xi) - \nabla_{\varphi X}\xi. \tag{5}$$

Let now $\{\xi,e,\varphi e\}$ denote a (local) φ-basis on M, with φe time-like. Then,

$$he = a_1 e + a_2\varphi e, \qquad h\varphi e = (-\varphi he) = -a_2 e - a_1\varphi e, \tag{6}$$

for some smooth functions a_1,a_2. Consequently,

$$\|h\|^2 = \mathrm{tr}h^2 = 2(a_1^2 - a_2^2). \tag{7}$$

In particular, by (6) and (7) we have that the following conditions are equivalent:

(i) $h^2 = 0$, that is, h is two-step nilpotent;

(ii) $\mathrm{tr}h^2 = 0$;

(iii) $a_2 = \varepsilon a_1 = \pm a_1$.

Since $\nabla_e\xi$ is orthogonal to ξ, there exist two smooth functions b_1,b_2, such that $\nabla_e\xi = b_1e + b_2\varphi e$. So, (A.7) yields $\nabla_{\varphi e}\xi = (b_2-2a_1)e + (b_1-2a_2)\varphi e$. Moreover,

$$\nabla_\xi e = g(\nabla_\xi e,\xi)\xi + g(\nabla_\xi e,e)e - g(\nabla_\xi e,\varphi e)\varphi e = -g(\nabla_\xi e,\varphi e)\varphi e = a_3\varphi e,$$

where we put $a_3 := g(\nabla_\xi\varphi e,e)$. By similar computations and taking into account the compatibility of g, we obtain

$$\begin{cases} \nabla_e\xi = b_1e + b_2\varphi e, & \nabla_{\varphi e}\xi = (b_2-2a_1)e + (b_1-2a_2)\varphi e,\\ \nabla_\xi e = a_3\varphi e, & \nabla_\xi\varphi e = a_3 e,\\ \nabla_e e = -b_1\xi + a_4\varphi e, & \nabla_{\varphi e}\varphi e = (b_1-2a_2)\xi + a_5 e,\\ \nabla_e\varphi e = b_2\xi + a_4 e, & \nabla_{\varphi e}e = (2a_1-b_2)\xi + a_5\varphi e, \end{cases}$$

for some real smooth functions a_i,b_j. Equivalently, the Lie brackets of $\xi,e,\varphi e$ are described by

$$\begin{cases} [\xi,e] = -b_1e + (a_3-b_2)\varphi e,\\ [\xi,\varphi e] = (a_3+2a_1-b_2)e + (2a_2-b_1)\varphi e,\\ [e,\varphi e] = 2(b_2-a_1)\xi + a_4e - a_5\varphi e. \end{cases}$$

and must satisfy the Jacoby identity, which, by standard calculations, is proved to be equivalent to the following system of equations:

$$\begin{cases} \xi(b_2-a_1) - 2(a_2-b_1)(b_2-a_1) = 0,\\ \xi(a_4) - e(a_3+2a_1-b_2) - \varphi e(b_1) - a_4(2a_2-b_1) - a_5(a_3+2a_1-b_2) = 0,\\ \xi(a_5) + e(2a_2-b_1) + \varphi e(b_2-a_3) + a_4(b_2-a_3) + a_5b_1 = 0. \end{cases} \tag{8}$$

The above local description holds for any three-dimensional *natural* almost paracontact metric structure. In particular, the case of a paracontact metric is characterized by Equation $a_1-b_2=1$, which, by the first Equation in (8), also yields $b_1=a_2$. Therefore, we have the following result.

PROPOSITION 1. *Any three-dimensional paracontact metric manifold (M,φ,ξ,η,g) is locally described by*

$$\left\{\begin{array}{l}[\xi,e]=-a_2e+(a_3-a_1+1)\varphi e,\\ [\xi,\varphi e]=(a_3+a_1+1)e+a_2\varphi e,\\ [e,\varphi e]=-2\xi+a_4e-a_5\varphi e\end{array}\right. \tag{9}$$

with respect to a local φ-basis $\{\xi,e,\varphi e\}$, for some smooth functions $a_1,\dots,a_5$, satisfying

$$\left\{\begin{array}{l}\xi(a_4)-e(a_3+a_1)-\varphi e(a_2)-a_4a_2-a_5(a_3+a_1+1)=0,\\ \xi(a_5)+e(a_2)+\varphi e(a_1-a_3)+a_4(a_1-a_3-1)+a_5a_2=0.\end{array}\right. \tag{10}$$

2.2. Harmonicity properties of vector fields

We now report some basic information on harmonic vector fields over a pseudo-Riemannian manifold, referring to [3] and [18, Chapter 8] for more details.

Let (M,g) be an m-dimensional semi-Riemannian manifold, ∇ its Levi-Civita connection and V a smooth vector field on M. The *energy* of V is, by definition, the energy of the corresponding smooth map $V:(M,g)\to(TM,g^s)$, where g^s is the *Sasaki metric* (also referred to as the *Kaluza-Klein metric* in Mathematical Physics) on the tangent bundle TM of M. If M is compact, then

$$E(V)=\frac{1}{2}\int_M(\mathrm{tr}V^*g^s)dv=\frac{m}{2}\mathrm{vol}(M,g)+\frac{1}{2}\int_M g(V,V)dv,$$

(in the non-compact case, one works over relatively compact domains). Note that the energy of a vector field V, up to a constant, also corresponds to the *total bending* of V [28]. The Euler-Lagrange equation yields that a vector field V defines a harmonic map from (M,g) to (TM,g^s) if and only if its *tension field* $\tau(V)=\mathrm{tr}(\nabla dV)$ vanishes, that is, when

$$\mathrm{tr}[R(\nabla.V,V)\cdot]=0 \qquad \text{and} \qquad \bar{\Delta}V=0.$$

Here, $\bar{\Delta}V:=-\mathrm{tr}\nabla^2V$ is the so called *rough Laplacian* of V. With respect to any local pseudo-orthonormal frame field $\{E_1,..,E_m\}$ on (M,g), with $\varepsilon_i=g(E_i,E_i)=\pm1$ for all indices $i=1,\dots,m$, it is given by

$$\bar{\Delta}V=\sum_i\varepsilon_i\left(\nabla_{\nabla_{E_i}E_i}V-\nabla_{E_i}\nabla_{E_i}V\right).$$

Next, for any real constant $r\neq0$, let $\mathfrak{X}^r(M)=\{V\in\mathfrak{X}(M):g(V,V)=r\}$ denote the set of tangent vector fields of constant length r. A vector field $V\in\mathfrak{X}^r(M)$ is called *harmonic* if it is a critical point for the energy functional $E|_{\mathfrak{X}^r(M)}$, restricted to vector fields of the same length. The Euler-Lagrange equation of this variational condition yields that V is harmonic if and only if

$$\bar{\Delta}V \quad \text{is collinear to} \quad V. \tag{11}$$

This characterization, obtained in the Riemannian case by G. Wiegmink [28] and C.M. Wood [27], was successively generalized in pseudo-Riemannian settings, to vector fields of constant length, if not light-like [3].

Let T_1M denote the *unit tangent sphere bundle* over M, and g^s the metric induced on T_1M by the Sasaki metric of TM. Then, the map $V : (M,g) \to (T_1M, g^s)$ is harmonic if V is a harmonic vector field and the additional condition

$$\mathrm{tr}[R(\nabla_{\cdot}V,V)\cdot] = 0 \tag{12}$$

holds. In analogy with the contact metric case [23], we now introduce the following.

DEFINITION 1. A paracontact metric manifold (M,φ,ξ,η,g) is said to be *H-paracontact* if its Reeb vector field ξ is a harmonic vector field.

Let again (M^m,g) denote a pseudo-Riemannian manifold and $f : x \mapsto x'$ a point transformation in (M,g). If $\nabla(x)$ denotes the Levi-Civita connection at x and $\nabla'(x) := f^{-1}(\nabla(x'))$ [25], the *Lie difference* at x is defined as $\nabla'(x)-\nabla(x)$. The map f is said to be *harmonic* if $\mathrm{tr}(\nabla'(x)-\nabla(x)) = 0$.

Consider now a vector field V on M and the local one-parameter group of infinitesimal point transformations f_t generated by V. In this case, $(L_V\nabla)(x) = \nabla'(x) - \nabla(x)$ and so, V generates a group of harmonic transformations if and only if

$$\mathrm{tr}(L_V\nabla) = 0.$$

In this case, V is said to be an *infinitesimal harmonic transformation* [22].

Infinitesimal harmonic transformations are also critical points for a suitable energy functional. In fact, if g^c denotes the *complete lift metric* of g to TM, which is of neutral signature (n,n), a vector field V on M defines a harmonic section $V : (M,g) \to (TM, g^c)$ if and only if V is an infinitesimal harmonic transformation [22].

Consequently, infinitesimal harmonic transformations are also called 1*-harmonic vector fields*, because this harmonicity property is equivalent to the vanishing of the linear part of the tension field of the local one-parameter group of infinitesimal point transformations [17]. A vector field V is an infinitesimal harmonic transformation if and only if $\bar{\Delta}V = QV$, where Q denotes the Ricci operator (see for example [10],[13]).

3. H-paracontact metric manifolds

We start with the following result, obtained in [10].

THEOREM 1. [10] *Let (M,φ,ξ,η,g) be a $(2n+1)$-dimensional paracontact metric manifold. Then,*

$$\bar{\Delta}\xi = -4n\xi - Q\xi = \parallel \nabla\xi \parallel^2 \xi - \mathrm{pr}_{|\ker\eta} Q\xi, \tag{13}$$

where $\parallel \nabla\xi \parallel^2 = -(2n+\mathrm{tr}h^2)$ *and* $\mathrm{pr}_{|\ker\eta}$ *denotes the projection on* $\ker\eta$.

Differently from its analogue proved for the contact Riemannian case in [23], the above result could not be proved using the existence of an orthonormal basis of eigenvectors for the tensor h and so, it required a completely new and *ad hoc* argument.

Since ξ is harmonic if and only if $\bar{\nabla}\xi$ is collinear to ξ, as a direct consequence of the above result we get at once the following characterization.

THEOREM 2. *A paracontact metric manifold is H-paracontact if and only if the Reeb vector field ξ is an eigenvector of the Ricci operator.*

The characterization given in Theorem 2 implies that the class of H-paracontact manifolds is indeed very large. In fact, it is easy to check that paraSasakian and K-paracontact manifolds, paracontact (κ,μ)-spaces, three-dimensional homogeneous paracontact metric manifolds, η-Einstein paracontact manifolds all are examples of H-paracontact spaces. Moreover, it again follows from Theorem 2 that any paracontact metric structure, obtained applying a *$\mathcal{D}$-homothetic deformation* to an H-paracontact structure, is again paracontact. Thus, the property that ξ is harmonic is invariant under $\mathcal{D}$-homothetic deformations. We may refer to [10] for more details.

We now turn our attention to the case when ξ is an infinitesimal harmonic deformation. Observe that in general, a harmonic vector field needs not be 1-harmonic, nor conversely. This statement may be easily proved, for example, by comparing the classifications of harmonic and 1-harmonic left-invariant vector fields over three-dimensional Lorentzian Lie algebras, given respectively in [3] and [13].

However, in the case of the Reeb vector field of a paracontact metric manifold (M,φ,ξ,η,g), by Theorem 1 and equation $\rho(\xi,\xi) = -2n + \mathrm{tr}h^2$ for the Ricci curvature in the direction of ξ (see [29]), we get at once that

$$\bar{\Delta}\xi = Q\xi \quad \Longleftrightarrow \quad Q\xi = -2n\xi \quad \Longleftrightarrow \quad \mathrm{tr}h^2 = 0 \quad \text{and} \quad Q\xi \text{ is collinear to } \xi.$$

Therefore, taking into account Theorem 2, we have the following result.

THEOREM 3. *Let (M,φ,ξ,η,g) be a paracontact metric manifold. Then, the following properties are equivalent:*

1) $Q\xi = -2n\xi$*;*

2) ξ *is an infinitesimal harmonic transformation (equivalently,* 1*-harmonic);*

3) M *is H-paracontact and* $\mathrm{tr}h^2 = 0$.

By the above Theorem 3, if the Reeb vector field of a paracontact metric manifold is 1-harmonic, then it is harmonic. However, the converse does not hold, because of the additional condition $\mathrm{tr}h^2 = 0$. Several explicit examples of H-paracontact manifolds with $\mathrm{tr}h^2 \neq 0$ may be found in [10]. In particular, all paracontact (κ,μ)-spaces are H-paracontact; however, their Reeb vector field is 1-harmonic (that is, $\mathrm{tr}h^2 = 0$) only if $\kappa = -1$.

REMARK 1. Observe that the Reeb vector field of a contact *Riemannian* manifold is an infinitesimal harmonic transformation if and only if it is a Killing vector field [24], that is, when the contact Riemannian structure is K-contact.

Such a rigidity result does not hold for paracontact spaces. In fact, there exist paracontact metric manifolds for which ξ is 1-harmonic (for example, paracontact $(-1,\mu)$-spaces), which are not K-paracontact. Some explicit examples, related to the issue of the existence of nontrivial paracontact Ricci solitons, will be presented in the next Section.

4. Paracontact Ricci solitons

A *Ricci soliton* is a pseudo-Riemannian manifold (M,g), admitting a smooth vector field X, such that

$$\mathcal{L}_X g + \rho = \lambda g, \tag{14}$$

where $\mathcal{L}_X$, ρ and λ denote the Lie derivative in the direction of X, the Ricci tensor and a real number, respectively. A Ricci soliton is said to be *shrinking*, *steady* or *expanding*, according to whether $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively.

It is obvious that Einstein manifolds (together with $X = 0$ or, more generally, a Killing vector field X) satisfy the above equation. So, they are considered trivial Ricci solitons.

Ricci solitons were introduced in Riemannian Geometry [20] as the self-similar solutions of the *Ricci flow*, and play an important role in understanding its singularities. A wide survey on Riemannian Ricci solitons may be found in [14].

Recently, Ricci solitons have also been extensively studied in pseudo-Riemannian settings. For some recent results and further references on pseudo-Riemannian Ricci solitons, we may refer to [1],[5],[12] and references therein.

Given a class of pseudo-Riemannian manifolds (M,g), it is then a natural problem to solve Equation (14), especially when it holds for a smooth vector field playing a special role in the geometry of these manifolds. Under this point of view, the Reeb vector field ξ of a contact metric manifold would be a natural candidate.

However, in these settings, a strong rigidity result holds: the Reeb vector field of a contact Riemannian (or Lorentzian) manifold (M,φ,ξ,η,g) satisfies (14) if and only if (M,φ,ξ,η,g) is K-contact Einstein [11]. Thus, contact Riemannian or Lorentzian Ricci solitons are necessarily trivial.

As proved in [25], the study of Ricci solitons is closely related to the one of infinitesimal harmonic transformations. In fact, a vector field X determining a Ricci soliton (that is, satisfying (14)) is necessarily an infinitesimal harmonic transformation. The same argument, initially obtained in [25] for the Riemannian case, also applies to pseudo-Riemannian manifolds. We now introduce the following.

DEFINITION 2. A *paracontact Ricci soliton* is a paracontact metric manifold (M,φ,ξ,η,g), such that Equation (14) holds with $X=\xi$.

Consider now a paracontact Ricci soliton (M,φ,ξ,η,g). Then, in particular ξ is an infinitesimal harmonic transformation. Hence, Theorem 3 yields that M is H-paracontact and $Q\xi=-2n\xi$. Comparing with Equation (14), we then necessarily have that $\lambda=-2n$ and so, we have the following result.

THEOREM 4. *A paracontact Ricci soliton is H-paracontact, and is necessarily expanding.*

As we already recalled, in the contact Riemannian case, if ξ is an infinitesimal harmonic transformation, then ξ is Killing. Consequently, there do no exist nontrivial *contact* Ricci solitons.

On the other hand, the above Theorem 4 does not exclude the existence of nontrivial *paracontact* Ricci solitons. A careful analysis of the three-dimensional case shows that nontrivial paracontact Ricci solitons do exist.

As showed in Proposition 1, a three-dimensional paracontact metric manifold (M,φ,ξ,η,g) is completely described, with respect to a local φ-basis $\{\xi,e,\varphi e\}$, by Equations (9) and (10). Using these Equations, we can then obtain a complete description of the Levi-Civita connection and the curvature of such a manifold. First of all, the Levi-Civita connection is described by

$$\begin{array}{llll} & \nabla_\xi\xi=0 & \nabla_e\xi=a_2e+(a_1-1)\varphi e, & \nabla_{\varphi e}\xi=-(a_1+1)e-a_2\varphi e, \\ (15) & \nabla_\xi e=a_3\varphi e, & \nabla_e e=-a_2\xi+a_4\varphi e, & \nabla_{\varphi e}e=(a_1+1)\xi+a_5\varphi e \\ & \nabla_\xi\varphi e=a_3e, & \nabla_e\varphi e=(a_1-1)\xi+a_4e, & \nabla_{\varphi e}\varphi e=-a_2\xi+a_5e, \end{array}$$

Consequently, we find

$$\begin{aligned} \operatorname{tr}(\mathcal{L}_\xi\nabla)=&(\mathcal{L}_\xi\nabla)(\xi,\xi)+(\mathcal{L}_\xi\nabla)(e,e)-(\mathcal{L}_\xi\nabla)(\varphi e,\varphi e) \\ =&4\big(a_1^2-a_2^2\big)\xi \\ &+\big(e(a_2)+\varphi e(a_3+a_1)-\xi(a_5)+2a_1a_4+3a_2a_5+a_4(a_3+a_1+1)\big)e \\ &+\big(e(a_1-a_3)+\varphi e(a_2)+\xi(a_4)+2a_1a_5+3a_2a_4-a_5(a_3-a_1+1)\big)\varphi e. \end{aligned}$$

In particular, substituting $\xi(a_4)$ and $\xi(a_5)$ from (10), we conclude that $\operatorname{tr}(\mathcal{L}_\xi\nabla)=0$ if and only if $a_2=\varepsilon a_1=\pm a_1$ and $(e+\varepsilon\varphi e)(a_1)+2\varepsilon a_1a_4+2a_1a_5=0$. So, we proved the following result.

PROPOSITION 2. [8] *Let (M,φ,ξ,η,g) be a three-dimensional paracontact metric manifold. Then, the Reeb vector field ξ is an infinitesimal harmonic transformation if and only if the manifold is locally described by*

$$(16)\qquad \begin{cases} [\xi,e]=-\varepsilon a_1e+(a_3-a_1+1)\varphi e, \\ [\xi,\varphi e]=(a_3+a_1+1)e+\varepsilon a_1\varphi e, \\ [e,\varphi e]=-2\xi+a_4e-a_5\varphi e, \end{cases}$$

with respect to a local φ*-basis* $\{\xi, e, \varphi e\}$*, for some smooth functions* a_1, a_3, a_4, a_5*, satisfying*

$$
\left\{
\begin{array}{l}
\xi(a_4) - e(a_3) + \varepsilon a_1 a_4 - a_5(a_3 - a_1 + 1) = 0, \\
\xi(a_5) - \varphi e(a_3) - \varepsilon a_1 a_5 - a_4(a_3 + a_1 + 1) = 0, \\
(e + \varepsilon\varphi e)(a_1) + 2\varepsilon a_1 a_4 + 2a_1 a_5 = 0.
\end{array}
\right.
\tag{17}
$$

Observe that if ξ is an infinitesimal harmonic transformation, then $a_2 = \varepsilon a_1$ and so, $\mathrm{tr}h^2 = 0$, compatibly with the result of Theorem 3.

We now determine the Ricci tensor of any paracontact metric three-manifold whose Reeb vector field is an infinitesimal harmonic transformation. Using (15) with $a_2 = \varepsilon a_1$ and taking into account (17), standard calculations yield

$$
\left\{
\begin{array}{l}
R(\xi, e)\xi = -\big(\varepsilon\xi(a_1) + 2a_1 a_3 + 1\big)e - \big(\xi(a_1) + 2\varepsilon a_1 a_3\big)\varphi e, \\
R(\xi, \varphi e)\xi = \big(\xi(a_1) + 2\varepsilon a_1 a_3\big)e + \big(\varepsilon\xi(a_1) + 2a_1 a_3 - 1\big)\varphi e, \\
R(e, \varphi e)\xi = 0, \\
R(e, \varphi e)e = \big(\varphi e(a_4) - e(a_5) + 1 - 2a_3 + a_4^2 - a_5^2\big)\varphi e,
\end{array}
\right.
\tag{18}
$$

which easily imply that, with respect to $\{\xi, e, \varphi e\}$, the Ricci tensor ρ is completely described by the matrix

$$
\rho = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -B - \varepsilon A & A \\ 0 & A & B - \varepsilon A \end{pmatrix}, \qquad
\begin{array}{l} A := \xi(a_1) + 2\varepsilon a_1 a_3, \\ B := e(a_5) - \varphi e(a_4) + 2a_3 - a_4^2 + a_5^2. \end{array}
\tag{19}
$$

Observe that by (19), we see that the Ricci operator Q satisfies $Q\xi = -2\xi$, compatibly with the characterization proved in Theorem 3.

Next, from (18) we have that $R(\xi, e, \xi, e) = -\varepsilon A - 1$. On the other hand, it is well known that in dimension three, the curvature tensor R satisfies

$$
\begin{aligned}
R(X,Y,Z,V) = {} & g(X,Z)\rho(Y,V) - g(Y,Z)\rho(X,V) + g(Y,V)\rho(X,Z) \\
& - g(X,V)\rho(Y,Z) - \frac{r}{2}\big(g(X,Z)g(Y,V) - g(Y,Z)g(X,V)\big),
\end{aligned}
\tag{20}
$$

where r denotes the scalar curvature. In particular, for $X = Z = \xi$ and $Y = V = e$, we then get $R(\xi, e, \xi, e) = \rho(e,e) - 2 - \frac{r}{2}$ and so, $\rho(e,e) = -\varepsilon A + \frac{r}{2} + 1$. Comparing with (19), we then find $B = -\frac{r}{2} - 1$. Consequently, (19) becomes

$$
\rho = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -\varepsilon A + \frac{r}{2} + 1 & A \\ 0 & A & -\varepsilon A - \frac{r}{2} - 1 \end{pmatrix},
\tag{21}
$$

Next, for any paracontact metric three-manifold $(M, \varphi, \xi, \eta, g)$, if $h^2 = 0$, then applying (15) with $a_2 = \varepsilon a_1$, we easily find that, with respect to $\{\xi, e, \varphi e\}$,

$$
\mathcal{L}_\xi g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\varepsilon a_1 & -2a_1 \\ 0 & -2a_1 & 2\varepsilon a_1 \end{pmatrix}.
\tag{22}
$$

Therefore, ξ satisfies equation (14) if and only if

$$\lambda = -2, \qquad A = \xi(a_1) + 2\varepsilon a_1 a_3 = 2a_1 \qquad \text{and} \qquad r = -6. \tag{23}$$

Thus, we proved the following result.

THEOREM 5. *A three-dimensional paracontact metric manifold* $(M, \varphi, \xi, \eta, g)$ *is a paracontact Ricci soliton if and only if the manifold is locally described by equations* (16),(17) *and* (23)*, with respect to a local* φ*-basis* $\{\xi, e, \varphi e\}$*, for some smooth functions* a_1, a_3, a_4, a_5*. (In particular, the Ricci soliton is necessarily expanding.)*

Before giving some explicit examples, we now clarify the relationship between three-dimensional paracontact Ricci solitons and (κ, μ)-spaces. Checking Equation (1) for an arbitrary three-dimensional paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ with ξ an infinitesimal harmonic transformation (that is, by Proposition 2, locally described by (16) and (17)), by standard calculations (see also [8]) we get the following.

THEOREM 6. *A three-dimensional paracontact metric manifold* $(M, \varphi, \xi, \eta, g)$ *is a nontrivial paracontact Ricci soliton if and only if it is a* (κ, μ)*-space, with* $\kappa = -1$ *and* $\mu = -2\varepsilon$*, of scalar curvature* $r = -6$*.*

4.1. Homogeneous 3D nontrivial paracontact Ricci solitons

As proved in [4], a (simply connected, complete) homogeneous paracontact metric three-manifold is isometric to some Lie group G equipped with a left-invariant paracontact metric structure (φ, ξ, η, g). Then, denoting by $\mathfrak{g}$ the Lie algebra of G, we have that $\xi \in \mathfrak{g}$, η is a 1-form over $\mathfrak{g}$ and $\ker(\eta) \subset \mathfrak{g}$. Moreover, starting from a φ-basis of tangent vectors at the base point of G, by left translations one builds a φ-basis $\{\xi, e, \varphi e\}$ of the Lie algebra $\mathfrak{g}$.

Suppose now that the left-invariant paracontact metric structure (φ, ξ, η, g) is a nontrivial Ricci soliton. Then, with respect to the φ-basis $\{\xi, e, \varphi e\}$ of the Lie algebra $\mathfrak{g}$, standard calculations yield that necessarily

$$[\xi, e] = -a_1 e + (2 - a_1)\,\varphi e, \quad [\xi, \varphi e] = (2 + a_1)\,e + a_1 \varphi e, \quad [e, \varphi e] = -2\xi, \tag{24}$$

and the Reeb vector field of this paracontact metric structure satisfies (14).

The Lie algebra described in (24) is not solvable, as $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Indeed, comparing (24) with the classification of left-invariant paracontact metric structures obtained in [4] (see also [7]), we conclude that (24) corresponds to the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ of the universal covering of $SL(2, \mathbb{R})$. Thus, we have the following result.

THEOREM 7. *A homogeneous paracontact metric three-manifold* $(M, \varphi, \xi, \eta, g)$ *is a nontrivial paracontact Ricci soliton if and only if M is locally isometric to* $SL(2, \mathbb{R})$*, equipped with the left-invariant paracontact metric structure described in* (24)*.*

4.2. Inhomogeneous 3D nontrivial paracontact Ricci solitons

Using Darboux coordinates, the following local description can be given for any three-dimensional paracontact metric structure.

PROPOSITION 3. [8] *Any three-dimensional paracontact metric structure* (φ,ξ,η,g)*, in terms of local Darboux cordinates* (x,y,z)*, is explicitly described by*

$$\xi = 2\partial_z, \quad \eta = \frac{1}{2}(dz - ydx),$$

$$g = \frac{1}{4}\begin{pmatrix} a & b & -y \\ b & c & 0 \\ -y & 0 & 1 \end{pmatrix}, \qquad \varphi = \begin{pmatrix} -b & -c & 0 \\ (a-y^2) & b & 0 \\ -by & -cy & 0 \end{pmatrix}$$

for some smooth functions a,b,c*, satisfying* $ac - b^2 - cy^2 = -1$*. In particular,*

i) the structure is paraSasakian if and only if the functions a,b,c *do not depend on z,*

ii) $h^2 = 0$ *(equivalently,* $\mathrm{tr}h^2 = 0$*) if and only if* $b_z^2 - a_z c_z = 0$*.*

The above result emphasizes the fact that differently from the contact metric case, for paracontact metric structures the condition $h^2 = 0$ does not imply $h = 0$.

We now consider $M = \mathbb{R}^3(x,y,z)$, equipped with the paracontact metric structure (φ,ξ,η,g) defined by

$$\xi = 2\partial_z, \quad \eta = \frac{1}{2}(dz - ydx), \tag{25}$$

$$g = \frac{1}{4}\begin{pmatrix} F & 1 & -y \\ 1 & 0 & 0 \\ -y & 0 & 1 \end{pmatrix}, \qquad \varphi = \begin{pmatrix} -1 & 0 & 0 \\ F-y^2 & 1 & 0 \\ -y & 0 & 0 \end{pmatrix}, \tag{26}$$

where

$$F = F(x,y,z) = f(x) + \alpha e^{2z} + \beta y + \gamma,$$

for a smooth function $f(x)$ and some real constant $\alpha \neq 0, \beta, \gamma$. We have the following result.

THEOREM 8. *Let* (φ,ξ,η,g) *be the paracontact metric structure described by* (25) *and* (26)*. Then,* $(\mathbb{R}^3,\varphi,\xi,\eta,g)$ *is a paracontact Ricci soliton. Moreover, for any* $\beta \neq 0$*, such a paracontact metric structure is not locally homogeneous.*

Dimostrazione. The paracontact metric structure defined by (25) and (26) is of the type described in Proposition 3, with $a = F$, $b = 1$ and $c = 0$. This structure is not paraSasakian, because $a_z = F_z = 2\alpha e^{2z} \neq 0$. On the other hand, since b,c are constant, one concludes at once that $b_z^2 - a_z c_z = 0$. Therefore, $h^2 = 0 \neq h$.

We now determine a global φ- basis $(\xi, E, \varphi E)$ on M, taking

$$E := \frac{1}{\sqrt{2}}\big(4\partial_x + (2y^2 - 2F + 1)\partial_y + 4y\partial_z\big)$$

and so,

$$\varphi E = \frac{1}{\sqrt{2}}\big(-4\partial_x + (1 - 2y^2 + 2F)\partial_y - 4y\partial_z\big).$$

By a standard calculation, we then get

$$\begin{cases} [\xi, E] = -4\alpha e^{2z}(E + \varphi E), \\ [\xi, \varphi E] = 4\alpha e^{2z}(E + \varphi E), \\ [E, \varphi E] = -2\xi + \sqrt{2}(\beta - 2y)(E + \varphi E). \end{cases} \tag{27}$$

We can now compare (27) with (9), obtaining $a_1 = a_2 = 4\alpha e^{2z}$, $a_3 = -1$ and $a_4 = -a_5 = \sqrt{2}(\beta - 2y)$. It is then easy to check that the conditions in (17) and (23) are satisfied. Hence, by Theorem 5, we conclude that $(M, \varphi, \xi, \eta, g)$ is a paracontact Ricci soliton.

Finally, $(\mathbb{R}^3, \varphi, \xi, \eta, g)$ is a locally homogeneous paracontact metric manifold if and only if $\beta = 0$, in which case we get again the situation already described in Theorem 7, namely, it is locally isometric to the Lie group $SL(2, \mathbb{R})$ of. On the other hand, whenever $\beta \neq 0$, we described a paracontact Ricci soliton which is not locally homogeneous.

$\square$

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