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ORBIFOLDS, GEOMETRIC STRUCTURES AND FOLIATIONS. APPLICATIONS TO HARMONIC MAPS

Dedicated to Anna Maria Pastore on her 70th birthday

Abstract. In recent years a lot of attention has been paid to topological spaces which are a bit more general than smooth manifolds - orbifolds. Orbifolds are intuitively speaking manifolds with some singularities. The formal definition is also modelled on that of manifolds, an orbifold is a topological space which locally is homeomorphic to the orbit space of a finite group acting on R^n . Orbifolds were defined by Satake, as V-manifolds, cf. [25], then studied by W. Thurston, cf. [29], who introduced the term "orbifold". Due to their importance in physics, and in particular in the string theory, orbifolds have been drawing more and more attention. In this paper we propose to show that the classical theory of geometrical structures, cf. [27, 15, 22], easily translates itself to the context of orbifolds and is closely related to the theory of foliated geometrical structures, cf. [32]. Finally, we propose a foliated approach to the study of harmonic maps between Riemannian orbifolds based on our previous research into transversely harmonic maps.

1. Orbifolds and their smooth complete maps

Let X be a topological space, and fix $n \geq 0$. An n -dimensional orbifold chart on X is given by a connected open subset $\tilde{U} \subset R^n$, a finite group Γ of smooth diffeomorphisms of \tilde{U} , and a map $\phi: \tilde{U} \rightarrow X$ so that ϕ is Γ -invariant and induces a homeomorphism of \tilde{U}/Γ onto an open subset $U \subset X$.

An embedding $\lambda: (\tilde{U}, \Gamma, \phi) \rightarrow (\tilde{V}, \Delta, \psi)$ between two such charts is a smooth embedding $\lambda: \tilde{U} \rightarrow \tilde{V}$ with $\psi\lambda = \phi$. Then there exists a homomorphism $\alpha: \Gamma \rightarrow \Delta$ such that the mapping λ is α -equivariant.

An orbifold atlas on X is a family $\mathcal{U} = \{(\tilde{U}, \Gamma, \phi)\}$ of such charts, which cover X and are locally compatible:

given any two charts $(\tilde{U}, \Gamma, \phi)$ for $U = \phi(\tilde{U}) \subset X$ and $(\tilde{V}, \Delta, \psi)$ for $V = \psi(\tilde{V}) \subset X$ and a point $x \in U \cap V$, there exists an open neighborhood $W \subset U \cap V$ of x and a chart $(\tilde{W}, \Lambda, \tau)$ for $W = \tau(\tilde{W}) \subset X$ such that there are two embeddings $(\tilde{W}, \Lambda, \tau) \rightarrow (\tilde{U}, \Gamma, \phi)$ and $(\tilde{W}, \Lambda, \tau) \rightarrow (\tilde{V}, \Delta, \psi)$.

An atlas \mathcal{U} is said to refine another atlas \mathcal{V} if for every chart in \mathcal{U} there exists an embedding into some chart of \mathcal{V} . Two orbifold atlases are said to be equivalent if they have a common refinement.

With these notions defined above we can formulate the definition of an orbifold

DEFINITION 1. *An effective orbifold X of dimension n is a paracompact Hausdorff space X equipped with an equivalence class $[\mathcal{U}]$ of n -dimensional orbifold atlases.*

In this paper we will consider only effective orbifolds without further mentioning it.

For any effective orbifold we can find an atlas as in the following definition due to Borzellino and Brunsten, cf. [2]

DEFINITION 2. An n -dimensional smooth orbifold O consists of a paracompact, Hausdorff topological space X_O called the underlying space, with the following local structure. For each $x \in X_O$ and neighborhood U of x , there is a neighborhood $U_x \subset U$ of x , an open set \tilde{U}_x diffeomorphic to \mathbb{R}^n , a finite group Γ_x acting smoothly and effectively on \tilde{U}_x which fixes $0 \in \tilde{U}_x$, and a homeomorphism $\phi_x : \tilde{U}_x/\Gamma_x \rightarrow U_x$ with $\phi_x(0) = x$. These actions are subject to the condition that for a neighborhood $U_z \subset U_x$ with the corresponding $\tilde{U}_z \cong \mathbb{R}^n$, group Γ_z and homeomorphism $\phi_z : \tilde{U}_z/\Gamma_z \rightarrow U_z$ there is a smooth embedding $\tilde{\Psi}_{zx} : \tilde{U}_z \rightarrow \tilde{U}_x$ and an injective homomorphism $\theta_{zx} : \Gamma_z \rightarrow \Gamma_x$ so that $\tilde{\Psi}_{zx}$ is equivariant with respect to θ_{zx} . This means that for any $\gamma \in \Gamma_z; \tilde{\Psi}_{zx}(\gamma\tilde{y}) = \theta_{zx}(\gamma)\tilde{\Psi}_{zx}(\tilde{y})$ for all $\tilde{y} \in \tilde{U}_z$ and the following diagram commutes:

$$\begin{array}{ccc}
 \tilde{U}_z & \xrightarrow{\tilde{\Psi}_{zx}} & \tilde{U}_x \\
 \downarrow & & \downarrow \\
 \tilde{U}_z/\Gamma_z & \xrightarrow{\Psi_{zx}=\tilde{\Psi}_{zx}/\Gamma_z} & \tilde{U}_x/\theta_{zx}(\Gamma_z) \\
 \downarrow \phi_z & & \downarrow \phi_x \\
 U_z & \xrightarrow{\subset} & U_x
 \end{array}$$

For a given orbifold atlas \mathcal{U} , the underlying topological space X_O is homeomorphic to the topological space $X_{\mathcal{U}}$ defined as the quotient of the disjoint union of \tilde{U}_i by the equivalence relation induced by the action of the groups Γ_i and the compatibility condition. When there is no ambiguity the underlying space of the orbifold X will be denoted by the same letter X .

We will refer to the neighborhood U_x or (\tilde{U}_x, Γ_x) or $(\tilde{U}_x, \Gamma_x, \rho_x, \phi_x)$ as an orbifold chart. In the 4-tuple notation, we are making explicit the representation $\rho_x : \Gamma_x \rightarrow \text{Diff}(\tilde{U}_x)$. The isotropy group of x is the group Γ_x . The definition of orbifold implies that the germ of the action of Γ_x in a neighborhood of the origin of \mathbb{R}^n is unique, so that by shrinking \tilde{U}_x if necessary, Γ_x is well-defined up to isomorphism. The singular set of O is the set of points $x \in X_O$ with $\Gamma_x \neq \{e\}$.

We present the definition of a complete orbifold map from [2].

DEFINITION 3. A C^∞ complete orbifold map $(f, \{\tilde{f}_x\}, \{\Theta_{f,x}\})$ between smooth orbifolds O and \mathcal{P} consists of the following:

- (1) a continuous map $f : X_O \rightarrow X_{\mathcal{P}}$ of the underlying topological spaces,
- (2) for each $y \in X_O$, there is a group homomorphism $\Theta_{f,y} : \Gamma_y \rightarrow \Gamma_{f(y)}$,
- (3) a smooth $\Theta_{f,y}$ -equivariant lift $\tilde{f}_y : \tilde{U}_y \rightarrow \tilde{V}_{f(y)}$ where (\tilde{U}_y, Γ_y) is an orbifold chart at y and $(\tilde{V}_{f(y)}, \Gamma_{f(y)})$ is an orbifold chart at $f(y)$. That is the following diagram commutes:

$$\begin{array}{ccc}
 \tilde{U}_y & \xrightarrow{\tilde{f}_y} & \tilde{V}_{f(y)} \\
 \downarrow & & \downarrow \\
 \tilde{U}_y/\Gamma_y & \xrightarrow{\tilde{f}_y/\Theta_{f,y}} & \tilde{V}_{f(y)}/\Theta_{f,y}(\Gamma_y) \\
 \downarrow & & \downarrow \\
 & & \tilde{V}_{f(y)}/\Gamma_{f(y)} \\
 \downarrow & & \downarrow \\
 U_y & \xrightarrow{f} & V_{f(y)}
 \end{array}$$

(4) (Equivalence) Two complete orbifold maps $(f, \{\tilde{f}_x\}, \{\Theta_{f,x}\})$ and $(g, \{\tilde{g}_x\}, \{\Theta_{g,x}\})$ are considered equivalent if for each $x \in X_O$ $\tilde{f}_x = \tilde{g}_x$ as germs at x and $\Theta_{f,x} = \Theta_{g,x}$. That is, there exists an orbifold chart (\tilde{U}_x, Γ_x) at x such that $\tilde{f}_x|_{\tilde{U}_x} = \tilde{g}_x|_{\tilde{U}_x}$ and $\Theta_{f,x} = \Theta_{g,x}$. Note that this implies that $f = g$.

The set of smooth complete orbifold maps from O to \mathcal{P} will be denoted by $C_{Orb}^\infty(O, \mathcal{P})$. In a very similar way to submanifolds one can define suborbifolds. The first definition of a suborbifold was given by W. Thurston in [29], but it seems to be too restrictive. We recall and use the definition formulated in [3].

DEFINITION 4. An (embedded) suborbifold P of an orbifold O consists of the following:

- i) A subspace $X_P \subset X_O$ equipped with the subspace topology,
- ii) For each $x \in X_P$ and neighbourhood W of x in X_O there is an orbifold chart $(\tilde{U}_x, \Gamma_x, \rho_x, \phi_x)$ about x in O with $U_x \subset W$, a subgroup $\Lambda_x \subset \Gamma_x$ of the isotropy group of x in O and a $\rho_x(\Lambda_x)$ -invariant submanifold $\tilde{V}_x \subset \tilde{U}_x \cong \mathbb{R}^n$, so that $(\tilde{V}_x, \Lambda_x/\Omega_x, \rho_x|_{\Lambda_x}, \Psi_x)$ is an orbifold chart for P where $\Omega_x = \{\gamma \in \Lambda_x : \rho_x(\gamma)|_{\tilde{V}_x} = id\}$. (In particular, the intrinsic isotropy subgroup at x),
- iii) $V_x = \Psi_x(\tilde{V}_x/\rho_x(\Lambda_x)) = U_x \cap X_P$ is an orbifold chart domain for $x \in P$.

Thurston's definition is a bit more restrictive:

DEFINITION 5. A $Y \subset X$ is called a full suborbifold of X if Y is a suborbifold with $\Lambda_x = \Gamma_x$ for all points x of Y .

For the differences in the definitions and the reasons for them see [3].

2. Fibre bundles over orbifolds

Let F be a smooth manifold.

DEFINITION 6. An orbifold E is called an orbifold fibre bundle over the orbifold X with standard fibre F if

- i) there exists a smooth orbifold map $p: E \rightarrow X$,
- ii) there exists an orbifold atlas \mathcal{U} of X , i.e., for any $x \in X$ there exists an orbifold chart (U_i, Γ_i, ψ_i) of \mathcal{U} such that $x \in U_i$ and \tilde{U}_i an open subset of R^n , Γ_i is a finite group of diffeomorphisms of \tilde{U}_i , and \tilde{U}_i/Γ_i is homeomorphic to U_i , ψ_i being the homeomorphism,
- iii) let $V_i = p^{-1}(U_i)$ and $\tilde{V}_i = \tilde{U}_i \times F$, then there exist a group Λ_i of fibre preserving diffeomorphisms of \tilde{V}_i and a homeomorphism $\phi_i: \tilde{V}_i/\Lambda_i \rightarrow V_i$ such that $\{(\tilde{V}_i, \Lambda_i)\}$ form an atlas of the orbifold E ,
- iv) and the following diagram is commutative

$$\begin{array}{ccc}
 \tilde{U}_i \times F & \xrightarrow{\tilde{p}=p \times id} & \tilde{U}_i \\
 \downarrow & & \downarrow \\
 \tilde{V}_i/\Lambda_i & & \tilde{U}_i/\Gamma_i \\
 \downarrow \phi_i & & \downarrow \psi_i \\
 V_i & \xrightarrow{p} & U_i
 \end{array}$$

Remark We can assume that \tilde{p} is (Λ_i, Γ_i) -equivariant. Obviously, p is a C^∞ -complete orbifold map. In fact, we will consider only fibre bundles for which the groups Λ_i and Γ_i are isomorphic.

Examples

- a) The **tangent bundle** TX of an orbifold X , $F = R^n$

We construct the tangent bundle as follows. Take any orbifold atlas $\mathcal{U} = \{(\tilde{U}_i, \Gamma_i, \phi_i)\}$. Then take $\tilde{V}_i = T\tilde{U}_i = \tilde{U}_i \times R^n$, as the group Σ_i local transformations take $\Sigma_i = \{d\gamma: \gamma \in \Gamma_i\}$ and as ψ_i the quotient map $\tilde{V}_i \rightarrow \tilde{V}_i/\Sigma_i$. The condition of local compatibility of these charts is obviously satisfied with the required embeddings provided by the differentials of the embeddings of the compatibility condition of the atlas \mathcal{U} . The topological space TX is defined as the quotient of the disjoint union of $T\tilde{U}_i$ by the equivalence relation induced by the action of the groups Σ_i and the embeddings from the definition of an orbifold atlas. Equivalent atlases of the orbifold X define equivalent atlases of TX .

A smooth complete orbifold mapping $f: X \rightarrow Y$ defines the mapping $df: TX \rightarrow TY$ of the orbifold tangent bundles which is called the differential of f . If $(\tilde{U}, \Gamma, \phi)$ is a chart of X and $(\tilde{W}, \Delta, \rho)$ is a chart of Y such that $f(U) \subset W$ and $\tilde{f}: \tilde{U} \rightarrow \tilde{W}$ covers f then the differential $d\tilde{f}$ satisfies the condition of a complete orbifold mapping for the just defined atlases of the orbifold tangent bundles TX and TY . It is a simple exercise

to verify that df is a smooth complete map of the orbifolds TX and TY . Thus we have defined a functor from the category of (effective) orbifolds and their complete smooth mappings into the category of orbifold vector bundles and their complete smooth vector bundle mappings.

b) The **linear frame** bundle of an orbifold, $F = GL(n)$

The same construction as in the point a) using the frame bundle of open subsets of R^n instead of the tangent bundle defines the orbifold principal frame bundle of an orbifold. As the local groups are finite, and the fact that one can make them local isometries, the total space of the orbifold principal bundle is in fact a manifold. Any smooth complete mapping of orbifolds defines a smooth complete mapping of their frame bundles. The above correspondence is a functor from the category of (effective) orbifolds and their complete smooth mappings into the category of principal frame bundles and their smooth bundle mappings.

c) **Higher order frame and tangent** bundles

The same procedure can be applied to the functor of higher order frame and tangent bundles. In this way we define the functor from the category of smooth orbifolds and their smooth complete maps to the category of orbifold fibre bundles and their smooth complete fibre maps.

Similar constructions work well in the cases of fibre bundles listed below:

d) associated bundles to higher order frame bundles

e) natural bundles.

It is not difficult to verify that one can define the dual vector bundle E^* of any orbifold vector bundle E , and that E^* is an orbifold vector bundle. Moreover, any tensor product of orbifold vector bundles over a given orbifold is itself an orbifold vector bundle. Therefore we can define the orbifold tensor algebra

$$\bigotimes TX = \bigoplus_{p,q} \bigotimes_p^q TX = \bigoplus_{p,q} (\bigotimes^p TX) \otimes (\bigotimes^q TX^*).$$

Likewise for any finite number of orbifold vector bundles over a given orbifold X we can define their skewsymmetric product which itself is an orbifold vector bundle over X .

Another classical construction for fibre bundles over smooth manifolds works well within the framework of orbifolds, the pullback. Let $f: X \rightarrow Y$ be a smooth complete orbifold mapping between two orbifolds X and Y . Let $p: E \rightarrow Y$ be a smooth orbifold bundle, then the pullback bundle

$$f^{-1}E = \{(x, w) \in X \times E: f(x) = p(w)\}$$

is a well-defined orbifold bundle over X with the same standard fibre as the bundle E .

Remark The theory of natural bundles is well presented in [21], see also [28].

A.1. Sections of fibre bundles over an orbifold

We shall consider sections of orbifold fibre bundles which are smooth complete orbifold mappings.

If $p: E \rightarrow X$ is a smooth orbifold bundle, $Sect_{comp}(X, E)$ denotes the space of smooth complete sections of E , i.e., the set of all smooth complete orbifold mapping $s: X \rightarrow E$ such that $ps = id_X$.

Let \mathcal{U} be an orbifold atlas of the orbifold X . The existence of a section s of the fibre bundle $s: X \rightarrow E$ is equivalent to the existence on each U_i of a section s_i of $\tilde{U}_i \times F$ which is (Γ_i, Λ_i) -equivariant and satisfy the compatibility condition.

Riemannian metrics

A Riemannian metric on an orbifold X is given by a family of Riemannian metrics g_i on \tilde{U}_i which are Γ_i invariant, i.e. elements of Γ_i are isometries of the Riemannian metrics g_i .

PROPOSITION 1. *On any orbifold X there exists a Riemannian metric g_X .*

As the metrics g_i on \tilde{U}_i are compatible, so are the associated Levi-Civita connections ∇^i . The induced object on the orbifold X we call the Levi-Civita connection of the Riemannian metric g . The operator ∇ can be characterized as a two-linear mapping

$$\mathcal{X}_{comp}(X) \times \mathcal{X}_{comp}(X) \rightarrow \mathcal{X}_{comp}(X)$$

satisfying the standard conditions for connections. The operator ∇ can be extended to the orbifold tensor algebra $\otimes TX$ in the same way as in the case of manifolds. For a given complete vector field Z on the orbifold X and a complete (p, q) tensor field T , $\nabla_Z T$ is also a complete (p, q) tensor field, or ∇T a complete $(p, q + 1)$ tensor field.

A complete section of the orbifold tensor bundle $\otimes_q^p TX$ is called a (p, q) -orbi-tensor field on the orbifold X . Let $T_0 \in (\otimes^p R^n) \otimes (\otimes^q R^{n*})$ be a (p, q) -tensor of R^n . A tensor field T is called 0-deformable if for any point $x \in X$ there exists a frame p at that point such that $p^* T_x = T_0$ for some tensor T_0 .

A.2. Geometrical structures on orbifolds

In the case of manifolds a very general definition defines a geometrical structure as a submanifold or a subbundle of a natural bundle over the chosen manifold. The same approach can be extended to the case of orbifolds.

A suborbifold H of the total space of an orbifold natural bundle K which is at the same time an orbifold natural bundle is called a orbifold (fibre) subbundle.

DEFINITION 7. A geometrical structure on an orbifold X is an orbifold subbundle of a natural orbifold bundle H over X .

In particular, one can develop the theory of classical orbifold G -structures for any classical Lie group $G \subset GL(n)$. For a fixed Lie group G orbifold G -structures on an orbifold of dimension n is an orbifold subbundle $B(X, G)$ of the linear frame bundle $L(X)$. As in the case of manifolds any 0-deformable orbi-tensor field T determines an orbifold G_T -structure. In this case the corresponding G -structure is defined as, for any point $x \in X$,

$$B(T)_x = \{p \in L(X)_x : p^*T_x = T_0\}$$

and the structure group is the Lie subgroup of $GL(n)$:

$$G(T_0) = \{A \in GL(n) : A^*T_0 = T_0\}.$$

Remark Many geometrical objects or procedures associated to G -structures over manifolds have their orbifold counterparts. It is not difficult to see that for an orbifold G -structure we can define the structure tensor in the same way as for manifolds, cf. [15, 27]. The structure tensor c is a smooth function with values in

$$Hom(\mathbb{R}^n \wedge \mathbb{R}^n, \mathbb{R}^n) / \partial Hom(\mathbb{R}^n, Lie(G)).$$

The vanishing of the structure tensor of an orbifold G -structure is equivalent to the existence of a torsionless orbifold connection in this orbifold G -structure. Likewise, one can prove that the 1st prolongation of an orbifold G -structure is an orbifold bundle. So the theory of prolongations of G -structures works well for orbifolds, cf. [15, 27].

Like in the classical case of manifolds we can define many geometrical structures in the dual way, via orbi-tensor fields or via orbifold reductions of the orbifold frame bundle.

Examples

a) **Riemannian structure (Riemannian metric)** In the previous subsection we have defined an orbifold Riemannian metric as a section of the orbifold bundle $\wedge^2 T^*X$. It is equivalent to a choice of an orbifold $O(n)$ reduction of the orbifold linear frame bundle LX .

b) **Symplectic structure** A symplectic structure on the orbifold X can be introduced by choosing a symplectic form on X , or equivalently an orbifold $Sp(2m)$ -reduction of the orbifold linear frame bundle $L(X)$, cf. [31, 1].

c) **Kähler structure** A $U(n)$ -reduction of the orbifold linear frame bundle LX of the orbifold X would give us an almost Kähler structure on X . The vanishing of its structure tensor is equivalent to the existence of a $U(n)$ torsionless connection, thus to the integrability of the almost-complex structure.

d) In a similar way we can define hyperKähler or quaternionic structures on orbifolds, cf. [4].

Remark Having said that we realize that with no difficulty we can define on orbifolds contact structures, K-contact and Sasakian structures as well as 3-Sasakian or K-structures. The only difference is that all the objects considered have to be orbifold tensor fields.

A.3. Harmonic mappings

Any smooth complete orbifold mapping $f: X \rightarrow Y$ defines a smooth complete orbifold mapping $df: TX \rightarrow TY$ called the differential of f . The differential df is a smooth complete section of the bundle

$$TX \otimes f^{-1}TY.$$

If the orbifolds X and Y are Riemannian, then the associated Levi-Civita connections ∇^X and ∇^Y , respectively, define a connection D in the vector bundle $TX \otimes f^{-1}TY$. Therefore it makes sense to consider

$$Ddf$$

and its trace

$$\tau(f)$$

$\tau(f)$ is a complete section of the bundle $f^{-1}TY \rightarrow X$. We call it the tension field of the complete mapping f between the Riemannian orbifolds (X, g_X) and (Y, g_Y) .

DEFINITION 8. *A complete orbifold mapping $f: X \rightarrow Y$ between two Riemannian orbifolds (X, g_X) and (Y, g_Y) is harmonic if its tension field vanishes.*

Remark A different approach to the study of harmonic maps between orbifolds was proposed by Y.-J. Chiang in [7]. The same approach can be applied to the study of bi-harmonic, f-harmonic, etc. mappings as well as harmonic morphisms between orbifolds using the theory developed by Y.-J. Chiang and the author, cf. [8, 9, 10, 11, 12].

3. Orbifolds and foliations

At the beginning of the seventies foliations with all leaves compact were an object of very interesting studies. In the case of compact manifolds it was proved that the local boundedness of the volume of leaves is equivalent to the leaf space being Hausdorff, or to the fact that the holonomy group of any leaf is finite. That in turn means that the leaf space of such a foliation is an orbifold, cf. [14, 22, 13]. In this case the

foliation is Riemannian and taut, cf. [23]. A few years later A. Haefliger et al. cf. [16], remarked that any orbifold can be realized as the leaf space of a Riemannian foliation with compact leaves. It is a simple remark, having noticed that an orbifold admits a Riemannian metric, they take the associated orthonormal frame bundle, which in this case is a smooth manifold. If we take a compact orbifold, its orthonormal frame bundle is a compact manifold foliated by a foliation with all leaves compact, the fibres of this bundle. Its leaf space can be identified with the initial orbifold. Therefore Riemannian foliations with all leaves compact, or more general Riemannian foliations with leaves of finite holonomy, can be considered as natural desingularization objects of orbifolds. For a given orbifold X we will denote this foliated manifold by (O_X, \mathcal{F}_X) .

In the study of the geometry of a single orbifold this duality between the orbifold and its orthogonal frame bundle foliated by the compact fibres, between orbifold objects and foliated objects, works well, cf. e.g. [1].

The theory of foliated G-structures was developed mainly by Pierre Molino, cf. [23] and his numerous papers. The most general theory was proposed by the author in [32]. It is based on the theory of natural bundles.

Let (M, g, \mathcal{F}) be a compact Riemannian foliated manifold with compact leaves. In fact, we can assume less: it is sufficient to have a complete bundle-like Riemannian metric and leaves of finite holonomy. Then according to the Reeb stability theorem, cf. [24, 5], each leaf L admits a foliated (saturated by leaves) tubular neighbourhood U . Any fibre of such a tubular neighbourhood is a transverse submanifold intersecting all leaves contained in the neighbourhood. The holonomy of the zero section, a leaf of \mathcal{F} , defines a finite subgroup Γ of diffeomorphisms of the fibre D . The space U/\mathcal{F} of leaves contained in the tubular neighbourhood U is identified with the orbit space D/Γ . Let U and V be two such tubular neighbourhoods of leaves L_1 and L_2 , respectively. If the intersection is not empty, it is an open saturated subset. For any leaf L in $U \cap V$ we can find an open saturated tubular neighbourhood $W \subset U \cap V$. The choice of a fibre of W defines two embeddings of W/\mathcal{F} into U/\mathcal{F} and V/\mathcal{F} , respectively. Thus the orbifold charts of the leaf space M/\mathcal{F} defined using these tubular neighbourhoods are compatible, hence we have an orbifold atlas on M/\mathcal{F} . Different choices of tubular neighbourhoods and of different fibres in them produce different but equivalent orbifold atlases. Thus we have defined an orbifold structure on M/\mathcal{F} .

In the language of foliations, the orbifold associated to a foliations with compact leaves and finite holonomy contains the following data: the underlying topological space is the space of leaves of the foliation, the orbifold atlas corresponds to the choice of a complete transverse manifold, and the local finite groups are the elements of the holonomy pseudogroup induced on this transverse manifold. The embeddings from the definition of the orbifold atlas are also induced by elements of the holonomy pseudogroup, those which map open subsets of one connected component of the transverse manifold into another. The choices described in the previous paragraph lead to different but equivalent pseudogroups, cf. [17].

In [32], the author demonstrated that foliated geometrical structures are in one-to-one correspondence with holonomy invariant geometrical structures on the transverse manifold. In the case of foliated manifolds with foliations with compact leaves and finite

holonomy, this result can be reformulated saying that foliated geometrical structures are in one-to-one correspondence with orbifold geometrical structures on the leaf space equipped with the induced orbifold structure. In particular, any bundle-like Riemannian metric of the foliated manifold (M, \mathcal{F}) induces an orbifold Riemannian metric on the leaf space M/\mathcal{F} and vice versa.

Let $f: (M_1, \mathcal{F}_1) \rightarrow (M_2, \mathcal{F}_2)$ be a foliated smooth mapping between two foliated manifolds. We can choose two open coverings \mathcal{U}_1 and \mathcal{U}_2 by tubular neighbourhoods of leaves of (M_1, \mathcal{F}_1) and (M_2, \mathcal{F}_2) , respectively, such that for any $U \in \mathcal{U}_1$ there exists $V \in \mathcal{U}_2$ such that $f(U) \subset V$. This choice ensures that the foliated mapping f induces a holonomy invariant mapping \bar{f} of the associated transverse manifolds, cf. [18]. Therefore the induced mapping $\bar{f}: M_1/\mathcal{F}_1 \rightarrow M_2/\mathcal{F}_2$ between the leaf spaces, considered as orbifolds, is a complete orbifold mapping. The remark can be understood as an orbifold version of the considerations of Section 3.2 of [18] about the mapping induced on the transverse manifolds by a foliated mapping.

Let us recall the precise definition of the mapping \bar{f} presented in [18].

Let $f: (M_1, \mathcal{F}_1) \rightarrow (M_2, \mathcal{F}_2)$ be a foliated smooth mapping between two foliated manifolds. We suppose that $\mathcal{U} = \{(U_i, \phi_i, g_{ij})\}_I$ is a cocycle defining the foliation \mathcal{F}_1 , and denote by $\mathcal{V} = \{(V_\alpha, \psi_\alpha, h_{\alpha\beta})\}_A$ a cocycle defining the foliation \mathcal{F}_2 such that, for any $i \in I$, there exists $\alpha(i) \in A$ such that $f(U_i) \subset V_{\alpha(i)}$. Let $\bar{U}_i = \phi_i(U_i)$ and $\bar{V}_\alpha = \psi_\alpha(V_\alpha)$. Then the manifold $N_1 = \coprod \bar{U}_i$ is a transverse manifold of the foliation \mathcal{F}_1 , and $N_2 = \coprod \bar{V}_\alpha$ is a transverse manifold of the foliation \mathcal{F}_2 . The transformations g_{ij} generate a pseudogroup \mathcal{H}_1 , which is called the holonomy pseudogroup of \mathcal{F}_1 associated with the cocycle \mathcal{U} , and the transformations $h_{\alpha\beta}$ generate a pseudogroup \mathcal{H}_2 , which is called the holonomy pseudogroup of \mathcal{F}_2 associated with the cocycle \mathcal{V} . On the level of transverse manifolds, the map f induces a smooth map \bar{f} , as, for any $i \in I$, the following diagram is commutative:

$$\begin{array}{ccc} U_i & \xrightarrow{f|_{U_i}} & \tilde{U}_x \\ \downarrow \tilde{\phi}_i & & \downarrow \tilde{\psi}_{\alpha(i)} \\ \bar{U}_i & \xrightarrow{\bar{f}_{\alpha(i)i}} & \bar{V}_{\alpha(i)} \end{array}$$

The map $\bar{f}: N_1 \rightarrow N_2$ is defined as follows:

$$\bar{f}|_{\bar{U}_i} = \bar{f}_{\alpha(i)i}.$$

Taken open subsets U_i, U_j with $U_i \cap U_j \neq \emptyset$, then $f(U_i \cap U_j) \subset V_{\alpha(i)} \cap V_{\alpha(j)}$. This intersection covers the open subsets $\bar{U}_{ji} \subset \bar{U}_i$ and $\bar{U}_{ij} \subset \bar{U}_j$. Likewise $V_{\alpha(i)} \cap V_{\alpha(j)}$ covers $\bar{V}_{\alpha(j)\alpha(i)} \subset \bar{V}_{\alpha(i)}$ and $\bar{V}_{\alpha(i)\alpha(j)} \subset \bar{V}_{\alpha(j)}$. Moreover, the map $g_{ji}: \bar{U}_{ji} \rightarrow \bar{U}_{ij}$ is a diffeomorphism as well as the map $h_{\alpha(j)\alpha(i)}: \bar{V}_{\alpha(j)\alpha(i)} \rightarrow \bar{V}_{\alpha(i)\alpha(j)}$. Then

$$h_{\alpha(j)\alpha(i)} \bar{f}_{\alpha(i)i} |_{\bar{U}_{ji}} = \bar{f}_{\alpha(j)\alpha(i)} |_{\bar{U}_{ji}} \circ g_{ji} |_{\bar{U}_{ji}}.$$

Let X and Y be two orbifolds, and $O(X)$ and $O(Y)$ their respective foliated desingularizations, i.e., foliated manifolds whose leaf space is X and Y , respectively. It would have been very nice if any smooth complete orbifold mapping $f: X \rightarrow Y$ could be lifted to a smooth foliated mapping $O(f): O(X) \rightarrow O(Y)$. In general, this is not the case as the differential df does not send orthonormal frames into orthonormal frames. It is the case only if f is an isometry of (X, g_X) into (Y, g_Y) . To rectify this we can consider linear frame bundles $L(X)$ and $L(Y)$ of the orbifolds X and Y , respectively. Any smooth complete orbifold local diffeomorphism $f: X \rightarrow Y$ lifts to a smooth mapping $L(f): L(X) \rightarrow L(Y)$ which is a foliated mapping for the natural foliations \mathcal{F}_X and \mathcal{F}_Y by fibres of these two orbifold fibre bundles $L(X)$ and $L(Y)$, respectively. These foliations are Riemannian for the natural liftings of the Riemannian metrics of orbifolds. The holonomy groups of their leaves are finite.

4. Transversely harmonic maps

Let \mathcal{F} be a foliation on a Riemannian n -manifold (M, g) . Then \mathcal{F} is defined by a cocycle $\mathcal{U} = \{U_i, f_i, g_{ij}\}_{i,j \in I}$ modeled on a q -manifold N_0 such that

- (1) $\{U_i\}_{i \in I}$ is an open covering of M ,
- (2) $f_i: U_i \rightarrow N_0$ are submersions with connected fibres,
- (3) $g_{ij}: N_0 \rightarrow N_0$ are local diffeomorphisms of N_0 with $f_i = g_{ij}f_j$ on $U_i \cap U_j$.

The connected components of the trace of any leaf of \mathcal{F} on U_i consist of the fibres of f_i . The open subsets $N_i = f_i(U_i) \subset N_0$ form a q -manifold $N = \coprod N_i$, which can be considered as a transverse manifold of the foliation \mathcal{F} . The pseudogroup \mathcal{H}_N of local diffeomorphisms of N generated by g_{ij} is called the holonomy pseudogroup of the foliated manifold (M, \mathcal{F}) defined by the cocycle \mathcal{U} . If the foliation \mathcal{F} is Riemannian for the Riemannian metric g , then it induces a Riemannian metric \bar{g} on N such that the submersions f_i are Riemannian submersions and the elements of the holonomy group are local isometries.

Let $\phi: U \rightarrow \mathbb{R}^p \times \mathbb{R}^q$, $\phi = (\phi^1, \phi^2) = (x_1, \dots, x_p, y_1, \dots, y_q)$ be an adapted chart on a foliated manifold (M, \mathcal{F}) . Then on U the vector fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_p}$ span the bundle $T\mathcal{F}$ tangent to the leaves of the foliation \mathcal{F} , the equivalence classes denoted by $\bar{\frac{\partial}{\partial y_1}}, \dots, \bar{\frac{\partial}{\partial y_q}}$ of $\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_q}$ span the normal bundle $N(M, \mathcal{F}) = TM/T\mathcal{F}$ which is isomorphic to the subbundle $T\mathcal{F}^\perp$.

Suppose that (M, \mathcal{F}, g) is a Riemannian foliation. The sheaf $\Gamma_b(T\mathcal{F}^\perp)$ of foliated sections of the vector bundle $T\mathcal{F}^\perp \rightarrow M$ may be described as follows: If U is an open subset of M , then $X \in \Gamma_b(U, T\mathcal{F}^\perp)$ if and only if for each local Riemannian submersion $\phi: U \rightarrow \bar{U}$ defining \mathcal{F} , the restriction of X to U is projectable via the map ϕ on a vector field \bar{X} on \bar{U} .

The vector bundle $N(M, \mathcal{F})$ is a foliated vector bundle over (M, \mathcal{F}) , i.e., it admits a foliation \mathcal{F}_N of the same dimension as \mathcal{F} whose leaves are covering spaces of leaves of \mathcal{F} . The vector bundle \mathcal{F}^\perp is also foliated as it is isomorphic to the normal bundle. As \mathcal{F}^\perp is foliated, so is its dual bundle $\mathcal{F}^{\perp*}$ and any tensor product of these vector

bundles as well as any pull-back of a foliated vector bundle by a foliated mapping.

DEFINITION 9. A basic partial connection (M, \mathcal{F}, g) is a sheaf operator

$D : \Gamma_b(U, T\mathcal{F}^\perp) \times \Gamma_b(U, T\mathcal{F}^\perp) \rightarrow \Gamma_b(U, T\mathcal{F}^\perp)$ such that

1. $D_{fX+hY}Z = fD_XY + hD_YZ$,

2. D_X is R -linear;

3. $D_XfY = X(f)Y + fD_XY$ (the transversal Leibniz rule),

for any $X, Y, Z \in \Gamma_b(U, T\mathcal{F}^\perp)$ and any basic functions $f, h \in C_b^\infty(U)$, where U is any open subset of M .

Let ∇ be the Levi-Civita connection of g , then for any open subset U of M and $X, Y \in \Gamma_b(U, T\mathcal{F}^\perp)$ we define D as

$$D_XY = (\nabla_XY)^\perp$$

where $(\nabla_XY)^\perp$ is the horizontal component of ∇_XY . It is easy to check that D is a basic partial connection on (M, \mathcal{F}, g) . Let $\phi : U \rightarrow \bar{U}$ be a Riemannian submersion defining the foliation \mathcal{F} on an open set U . Let us assume that $X, Y \in \Gamma_b(U, T\mathcal{F}^\perp)$, and \bar{X}, \bar{Y} be the push forward vector fields via the map ϕ . Then

$$d\phi(D_XY) = \nabla_{\bar{X}}^{\bar{g}}\bar{Y}$$

where $\nabla^{\bar{g}}$ is the Levi-Civita connection of the metric \bar{g} on the transverse manifold.

Let $f : (M_1, \mathcal{F}_1, g) \rightarrow (M_2, \mathcal{F}_2, h)$ be a foliated map between two foliated Riemannian manifolds. The covariant derivative $D(\Pi_2dfi_1)$ is a global foliated section of the bundle

$$(T\mathcal{F}_1^\perp)^* \otimes (T\mathcal{F}_1^\perp)^* \otimes f^{-1}T\mathcal{F}_2^\perp \rightarrow M_1$$

where i_1 is the inclusion of $(T\mathcal{F}_1^\perp)^*$ into TM_1 and Π_2 is the orthogonal projection of TM_2 onto $T\mathcal{F}_2^\perp$. Its trace

$$\tau_b(f)$$

is a foliated section of the bundle $f^{-1}T\mathcal{F}_2^\perp \rightarrow M_1$. We call it the tension field of the foliated mapping f .

DEFINITION 10. A foliated map $f : (M_1, \mathcal{F}_1, g) \rightarrow (M_2, \mathcal{F}_2, h)$ between two foliated Riemannian manifolds is called transversally harmonic if $\tau_b(f) = 0$.

As we have shown any foliated map $f : (M_1, \mathcal{F}_1, g) \rightarrow (M_2, \mathcal{F}_2, h)$ induces a map \bar{f} of the corresponding transverse manifolds. Since the tension tensor of a foliated map is itself a foliated section, it induces a holonomy invariant object on the transverse

manifold. In fact, if p_i is a local submersion defining the foliation \mathcal{F}_i for $i = 1, 2$, then we have the following relation between the tension tensors of the mappings f and \bar{f} :

$$dp_2\tau_b(f)_x = \tau(\bar{f})_{p_1(x)}.$$

Let $f : (M_1, \mathcal{F}_1, g) \rightarrow (M_2, \mathcal{F}_2, h)$ be a foliated mapping. The restriction of f to any leaf L of \mathcal{F}_1 is a smooth mapping of the manifold L into a leaf of the foliation \mathcal{F}_2 . Both submanifolds can be provided with the restrictions of the Riemannian metrics g and h , respectively. If for each leaf L of the foliation \mathcal{F}_1 the mappings just described are harmonic, we say that the mapping f is *leaf-wise harmonic*.

Taking into account our previous considerations we can formulate the following theorems, whose proofs can be found in [18, 19]:

THEOREM 1. *Let $f : (M_1, \mathcal{F}_1, g) \rightarrow (M_2, \mathcal{F}_2, h)$ be a smooth foliated mapping between regular foliated Riemannian manifolds. Let \mathcal{U} and \mathcal{V} be two cocycles defining the foliations \mathcal{F}_1 and \mathcal{F}_2 , respectively. Then the map f is transversally harmonic if and only if the induced map $\bar{f} : N_{\mathcal{U}} \rightarrow N_{\mathcal{V}}$ between the associated transverse manifolds is harmonic.*

THEOREM 2. *Let $f : (M_1, \mathcal{F}_1, g) \rightarrow (M_2, \mathcal{F}_2, h)$ be a foliated harmonic map between two manifolds with Riemannian foliations. Moreover, assume that all leaves of the foliation \mathcal{F}_1 are minimal, the foliation \mathcal{F}_2 is totally geodesic, and f is horizontal, that is $df(T\mathcal{F}_1^\perp) \subset T\mathcal{F}_2^\perp$, then the map f is transversally harmonic.*

COROLLARY 1. *If the foliation \mathcal{F}_1 is minimal, and \mathcal{F}_2 totally geodesic, then the map f is harmonic if and only if f is transversally harmonic and leaf-wise harmonic.*

THEOREM 3. *Let $f : (M_1, \mathcal{F}_1, g) \rightarrow (M_2, \mathcal{F}_2, h)$ be a smooth foliated mapping between foliated Riemannian manifolds. If the foliations \mathcal{F}_1 and \mathcal{F}_2 are totally geodesic and the horizontal distribution of \mathcal{F}_2 is integrable, then the map f is transversally harmonic if and only if the horizontal part of the tension tensor $\tau(f)$ is zero. In particular, the harmonicity of f implies its transverse harmonicity.*

The above results are "foliated" versions of Theorem 1.3.5 and its corollary of Smith's Ph.D. thesis, cf. [26], pp. 17 and 18, and the results of Xin, Theorem 6.4 and its corollary of [34], see also [33].

Finally, let us formulate the following two theorems for maps between Riemannian orbifolds which can serve as the starting point for the study of harmonic maps between orbifolds. The proofs are simple consequence of our previous considerations.

THEOREM 4. *Let $f : (M_1, \mathcal{F}_1, g) \rightarrow (M_2, \mathcal{F}_2, h)$ be a foliated map between two foliated Riemannian manifolds with compact leaves. Let \bar{f} be the induced map between the leaf spaces with induced orbifold Riemannian metrics, $\bar{f} : M_1/\mathcal{F}_1 \rightarrow M_2/\mathcal{F}_2$. The map f is transversely harmonic iff the map \bar{f} is harmonic.*

THEOREM 5. *Let $f: X \rightarrow Y$ be a smooth complete embedding of a Riemannian orbifold (X, g) into another Riemannian orbifold (Y, h) . Then f is harmonic iff the induced (foliated) mapping $L(f): (L(X), \mathcal{F}_X, g_L) \rightarrow (L(Y), \mathcal{F}_Y, h_L)$ is transversely harmonic.*

Remark The proposed approach to the study of harmonic maps between orbifolds has some deficiencies. We can not lift any smooth map to a foliated map of its foliated desingularizations. However, if we just want to find harmonic maps, then the proposed methods can offer some solutions. The foliations \mathcal{F}_X and \mathcal{F}_Y are taut, so harmonic foliated maps between O_X and O_Y are not far from being transversally harmonic, and thus from inducing harmonic maps between the orbifolds. On the other hand, any compact Sasakian manifold fibers over a Kählerian orbifold, cf. [4], so the study of geometrical and cohomological properties of orbifolds can lead to the development of new obstructions to the existence of such structures.

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