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SOME MYERS TYPE THEOREMS AND HITCHIN–THORPE INEQUALITIES FOR SHRINKING RICCI SOLITONS

Abstract. In this expository paper, we shall give a brief survey of recent development on some compactness criteria and diameter estimates for complete shrinking Ricci solitons. As an application, we shall provide some new sufficient conditions for four-dimensional compact shrinking Ricci solitons to satisfy the Hitchin–Thorpe inequality.

1. Ricci Solitons

In this expository paper, we shall give a brief survey of recent development on some compactness criteria and diameter estimates for complete shrinking Ricci solitons. Ricci solitons were introduced by R. Hamilton [20] and are natural generalizations of Einstein manifolds. They correspond to self-similar solutions to the Ricci flow and often arise as singularity models [6, 21]. The importance of Ricci solitons was demonstrated in a series of three papers by G. Perelman [33, 34, 35], where Ricci solitons played crucial roles in the affirmative resolution of the Poincaré conjecture. Besides their geometric importance, Ricci solitons are also of great interest in theoretical physics and have been studied actively in relation to string theory [8, 15].

DEFINITION 1. A complete Riemannian manifold (M,g) is called a *Ricci soliton* [20] if there exists a vector field $V \in \mathfrak{X}(M)$ satisfying the equation

$$\mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_V g = \lambda g \tag{A.1}$$

for some constant $\lambda \in \mathbb{R}$, where Ric_g denotes the Ricci curvature of (M,g) and $\mathcal{L}_V$ is the Lie derivative in the direction of V. We say that the soliton (M,g) is *shrinking*, *steady* and *expanding* described as $\lambda > 0, \lambda = 0$ and $\lambda < 0$, respectively. A typical example of Ricci solitons is an Einstein manifold, where V is given by a Killing vector field. In such a case, we say that the soliton is *trivial*. When V is replaced with the gradient vector field ∇f for some smooth function $f : M \to \mathbb{R}$, the soliton (M,g) is called a *gradient Ricci soliton*. We refer to f as a *potential function*. Then (A.1) becomes

$$\mathrm{Ric}_g + \mathrm{Hess}\, f = \lambda g, \tag{A.2}$$

where $\mathrm{Hess}\, f$ denotes the Hessian of the potential function f.

EXAMPLE 1 (Cigar solitons). A typical example of gradient Ricci solitons is a *cigar soliton* $(\mathbb{R}^2, \frac{dx^2+dy^2}{1+x^2+y^2})$ discovered by R. Hamilton [20], where its potential function is

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given by the function $f(x,y) = -\log(1+x^2+y^2)$. This is a complete non-compact steady Ricci soliton and is also known in the physics literature as *Witten's black hole*.

EXAMPLE 2 (Gaussian solitons). Another typical example of gradient Ricci solitons is a *Gaussian soliton* $(\mathbb{R}^n, g_0)$, where g_0 is the canonical flat metric on $\mathbb{R}^n$ and its potential function is given by the function $f(x) = \pm\frac{1}{4}|x|^2$. This is a complete non-compact shrinking (respectively, expanding) Ricci soliton.

Thanks to G. Perelman [33], any vector field on compact Ricci solitons appearing in (A.1) must be the sum of a gradient vector field and a Killing vector field. It is well-known now that compact steady and expanding Ricci solitons must be trivial, as well as compact shrinking Ricci solitons in dimension two and three [6]. Examples of compact non-trivial shrinking Kähler–Ricci solitons were constructed by N. Koiso [23], H.-D. Cao [5], X.-J. Wang and X. Zhu [44] and F. Podestà and A. Spiro [36]. Examples of non-compact non-trivial Kähler–Ricci solitons were given by M. Feldman, T. Ilmanen and D. Knopf [11] and A. S. Dancer and M. Y. Wang [10].

2. Some Myers type theorems

In this section, we shall introduce some Myers type theorems for complete shrinking Ricci solitons. Throughout this paper, we shall assume that all Riemannian manifolds are smooth, connected without boundary. To give nice compactness criteria for complete Riemannian manifolds is one of the most interesting problems in Riemannian geometry. The celebrated theorem of Myers guarantees the compactness of complete Riemannian manifolds under some positive lower bounds on the Ricci curvature.

THEOREM 1 (S. B. Myers [32]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $\lambda > 0$ such that the Ricci curvature satisfies $\mathrm{Ric}_g \geqslant \lambda g$. Then (M,g) must be compact with finite fundamental group. Moreover,*

$$\mathrm{diam}(M,g) \leqslant \pi\sqrt{\frac{n-1}{\lambda}}.$$

The Myers theorem above has been widely generalized by many authors [1, 4, 18, 19, 31]. The first generalization was given by W. Ambrose [1], where the positive lower bound on the Ricci curvature was replaced with an integral condition of the Ricci curvature along some geodesics.

THEOREM 2 (W. Ambrose [1]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some point $p \in M$ for which every geodesic $\gamma : [0,+\infty) \to M$ emanating from p satisfies*

$$\int_0^{+\infty} \mathrm{Ric}_g(\dot\gamma(s), \dot\gamma(s))\,ds = +\infty.$$

Then (M,g) must be compact.

On the other hand, motivated by relativistic cosmology, G. J. Galloway [18] proved the following compactness theorem by perturbing the constant lower bound on the Ricci curvature by the derivative in the radial direction of some bounded function:

THEOREM 3 (G. J. Galloway [18]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exist some constants $\lambda > 0$ and $L \geqslant 0$ such that for every pair of points in M and minimal geodesic γ joining those points, the Ricci curvature satisfies*

$$\mathrm{Ric}_g(\dot\gamma,\dot\gamma)|_{\gamma(s)} \geqslant \lambda + \frac{d\phi}{ds}(s),$$

where ϕ is some smooth function of the arc length satisfying $|\phi| \leqslant L$ along γ. Then (M,g) must be compact. Moreover,

$$\mathrm{diam}(M,g) \leqslant \frac{\pi}{\lambda}\left(L + \sqrt{L^2+(n-1)\lambda}\right).$$

DEFINITION 2. A *smooth metric measure space* is a complete Riemannian manifold (M,g) with the weighted volume form $d\mu := e^{-f}d\mathrm{vol}_g$, where $f : M \to \mathbb{R}$ is a smooth function on M and vol_g is the Riemannian density with respect to the metric g. For a smooth metric measure space (M,g) and a positive constant $k \in (0,+\infty)$, we put

$$\mathrm{Ric}_f := \mathrm{Ric}_g + \mathrm{Hess}\, f \quad \text{and} \quad \mathrm{Ric}_f^k := \mathrm{Ric}_g + \mathrm{Hess}\, f - \frac{1}{k} df \otimes df \tag{A.3}$$

and call them a *Bakry–Émery Ricci curvature* and a *k-Bakry–Émery Ricci curvature*, respectively. We call f a *potential function*. More generally, for a smooth vector field $V \in \mathfrak{X}(M)$ and a positive constant $k \in (0,+\infty)$, we define

$$\mathrm{Ric}_V := \mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_V g \quad \text{and} \quad \mathrm{Ric}_V^k := \mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_V g - \frac{1}{k} V^* \otimes V^*,$$

where V^* is the metric dual of V with respect to g. We call them a *modified Ricci curvature* and a *k-modified Ricci curvature*, respectively. Then, we put

$$\Delta_f := \Delta_g - \nabla f \cdot \nabla \quad \text{and} \quad \Delta_V := \Delta_g - V \cdot \nabla \tag{A.4}$$

and call them a *Witten–Laplacian* and a *V–Laplacian*, respectively. Here, Δ_g is the Laplacian with respect to g.

Note that, if $f : M \to \mathbb{R}$ is constant in (A.3) and (A.4), then the Bakry–Émery Ricci curvature and the Witten–Laplacian are reduced to the Ricci curvature and the Laplacian, respectively. As with the ordinary case, for any smooth functions u, v on M with compact support, we have

$$\int_M g(\nabla u, \nabla v)d\mu = -\int_M (\Delta_f u)v d\mu = -\int_M u(\Delta_f v)d\mu.$$

Moreover, D. Bakry and M. Émery [3] proved that, for any smooth function u on M,

$$\frac{1}{2}\Delta_f|\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \mathrm{Ric}_f(\nabla u, \nabla u) + g(\nabla\Delta_f u, \nabla u), \tag{A.5}$$

which may be regarded as a natural extension of the Bochner–Weitzenböck formula.

Since Ricci solitons are generalizations of Einstein manifolds, it is natural to ask whether classical theorems for Einstein manifolds with positive Ricci curvature remain valid in the case of Ricci solitons with positive modified Ricci curvature. However, a positive lower bound on the modified Ricci curvature does not imply the compactness of complete Ricci solitons. In fact, the shrinking Gaussian soliton is non-compact.

The compactness of complete shrinking Ricci solitons may be characterized by the boundedness of the norm of its vector fields.

THEOREM 4 (M. Fernández-López and E. García-Río [12]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $\lambda > 0$ such that the modified Ricci curvature satisfies* $\mathrm{Ric}_V \geqslant \lambda g$. *Then M is compact if and only if $|V|$ is bounded on M.*

Recently, the Bakry–Émery Ricci curvature and the Witten–Laplacian have received much attention in various areas of mathematics, since they are good substitutes for the Ricci curvature and the Laplacian respectively, allowing us to establish many interesting results in smooth metric measure spaces, such as eigenvalue estimates [16], Li–Yau Harnack inequalities [26] and comparison theorems [45]. In particular, G. Wei and W. Wylie [45] proved the following Myers type theorem via the Bakry–Émery Ricci curvature which extends Theorem 1 to the case of smooth metric measure spaces:

THEOREM 5 (G. Wei and W. Wylie [45]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $\lambda > 0$ such that the Bakry–Émery Ricci curvature satisfies* $\mathrm{Ric}_f \geqslant \lambda g$. *If the potential function satisfies $|f| \leqslant H$ for some non-negative constant $H \geqslant 0$, then (M,g) must be compact. Moreover,*

$$\mathrm{diam}(M,g) \leqslant \pi\sqrt{\frac{n-1}{\lambda}} + \frac{4H}{\sqrt{(n-1)\lambda}}. \tag{A.6}$$

REMARK 1. Under the same assumption as in Theorem 5, M. Limoncu [29] gave the upper diameter estimate

$$\mathrm{diam}(M,g) \leqslant \frac{\pi}{\sqrt{\lambda}}\sqrt{n-1+2\sqrt{2}H}, \tag{A.7}$$

which may be sharper than (A.6) if $H > \frac{(n-1)\pi}{8}(\sqrt{2}\pi - 4)$.

On the other hand, the author [39] gave the following diameter estimate under the same assumption as in Theorem 5:

THEOREM 6 ([39]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $\lambda > 0$ such that the Bakry–Émery*

Ricci curvature satisfies $\mathrm{Ric}_f \geqslant \lambda g$. *If the potential function satisfies* $|f| \leqslant H$ *for some non-negative constant* $H \geqslant 0$, *then* (M,g) *must be compact. Moreover,*

$$\mathrm{diam}(M,g) \leqslant \frac{\pi}{\sqrt{\lambda}}\sqrt{n-1+\frac{8H}{\pi}}. \tag{A.8}$$

REMARK 2. Since $\frac{8}{\pi} < 2\sqrt{2}$, the diameter estimate (A.8) above is sharper than (A.7) by M. Limoncu [29]. Moreover, we may easily see that (A.8) is also sharper than (A.6) by G. Wei and W. Wylie [45] without any assumptions on H.

Recall from Theorem 4 that the compactness of complete shrinking Ricci solitons may be characterized by the boundedness of the norm of its vector fields. However, no an upper diameter estimate was given in Theorem 4. By extending the proof of the Myers theorem, M. Limoncu [28] gave such a diameter estimate.

THEOREM 7 (M. Limoncu [28]). *Let* (M,g) *be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant* $\lambda > 0$ *such that the modified Ricci curvature satisfies* $\mathrm{Ric}_V \geqslant \lambda g$. *If the vector field satisfies* $|V| \leqslant K$ *for some non-negative constant* $K \geqslant 0$, *then* (M,g) *must be compact. Moreover,*

$$\mathrm{diam}(M,g) \leqslant \frac{\pi}{\lambda}\left(\frac{K}{\sqrt{2}}+\sqrt{\frac{K^2}{2}+(n-1)\lambda}\right). \tag{A.9}$$

On the other hand, the diameter estimate (A.9) above was improved by the author [40] under the same assumption as in Theorem 7.

THEOREM 8 ([40]). *Let* (M,g) *be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant* $\lambda > 0$ *such that the modified Ricci curvature satisfies* $\mathrm{Ric}_V \geqslant \lambda g$. *If the vector field satisfies* $|V| \leqslant K$ *for some non-negative constant* $K \geqslant 0$, *then* (M,g) *must be compact. Moreover,*

$$\mathrm{diam}(M,g) \leqslant \frac{1}{\lambda}\left(2K+\sqrt{4K^2+(n-1)\lambda\pi^2}\right).$$

An interesting problem in smooth metric measure spaces is to establish Ambrose and Galloway types theorems via the Bakry–Émery Ricci curvature. An Ambrose type theorem via the Bakry–Émery Ricci curvature was first established by S. Zhang [46] under the assumption that the potential function of the Bakry–Émery Ricci curvature has at most linear growth.

THEOREM 9 (S. Zhang [46]). *Let* (M,g) *be an n-dimensional complete Riemannian manifold. Suppose that there exists some point* $p \in M$ *for which every geodesic* $\gamma : [0,+\infty) \to M$ *emanating from p satisfies*

$$\int_0^{+\infty} \mathrm{Ric}_f(\dot\gamma(s),\dot\gamma(s))ds = +\infty$$

and the potential function satisfies $f(x) \leqslant \gamma(d(x,p)+\alpha)$ *for some constants* γ *and* α*, where* $d(x,p)$ *is the distance between* x *and* p*, then* (M,g) *must be compact.*

More generally, we may prove the following Ambrose type theorem via the modified Ricci curvature which may be considered as a generalization of Theorem 4:

THEOREM 10 ([41]). *Let* (M,g) *be an* n*-dimensional complete Riemannian manifold. Suppose that there exists some point* $p \in M$ *for which every geodesic* $\gamma : [0,+\infty) \to M$ *emanating from* p *satisfies*

$$\int_0^{+\infty} \mathrm{Ric}_V(\dot{\gamma}(s),\dot{\gamma}(s))ds = +\infty$$

and the vector field satisfies $|V| \leqslant K$ *for some non-negative constant* $K \geqslant 0$*, then* (M,g) *must be compact.*

M. P. Cavalcante, J. Q. Oliveira and M. S. Santos [7] established the following Galloway type theorem via the Bakry–Émery Ricci curvature:

THEOREM 11 (M. P. Cavalcante, J. Q. Oliveira and M. S. Santos [7]). *Let* (M,g) *be an* n*-dimensional complete Riemannian manifold. Suppose that there exist some constants* $\lambda > 0$ *and* $L \geqslant 0$ *such that for every pair of points in* M *and minimal geodesic* γ *joining those points, the Bakry–Émery Ricci curvature satisfies*

$$\mathrm{Ric}_f(\dot{\gamma},\dot{\gamma})|_{\gamma(s)} \geqslant \lambda + \frac{d\phi}{ds}(s),$$

where ϕ *is some smooth function of the arc length satisfying* $|\phi| \leqslant L$ *along* γ*. If the potential function satisfies* $|f| \leqslant H$ *for some non-negative constant* $H \geqslant 0$*, then* (M,g) *must be compact. Moreover,*

$$\text{(A.10)} \qquad \mathrm{diam}(M,g) \leqslant \frac{\pi}{\lambda}\left(L + \sqrt{L^2 + \{(n-1)+2\sqrt{2}H\}\lambda}\right).$$

REMARK 3. Under the same assumption as in Theorem 11, the author [41] gave the upper diameter estimate

$$\text{(A.11)} \qquad \mathrm{diam}(M,g) \leqslant \frac{1}{\lambda}\left(2L + \sqrt{4L^2 + \{(n-1)\pi + 8H\}\lambda\pi}\right),$$

which is slightly sharper than (A.10). Moreover, by taking $L = 0$, (A.11) is reduced to (A.8).

On the other hand, a Galloway type theorem via the modified Ricci curvature was first established by the author [41].

THEOREM 12 ([41]). *Let* (M,g) *be an* n*-dimensional complete Riemannian manifold. Suppose that there exist some constants* $\lambda > 0$ *and* $L \geqslant 0$ *such that for every pair of* *points in M and minimal geodesic γ joining those points, the modified Ricci curvature satisfies*

$$\mathrm{Ric}_V(\dot\gamma,\dot\gamma)|_{\gamma(s)} \geqslant \lambda + \frac{d\phi}{ds}(s),$$

where ϕ is some smooth function of the arc length satisfying $|\phi| \leqslant L$ along γ. If the vector field satisfies $|V| \leqslant K$ for some non-negative constant $K \geqslant 0$, then (M,g) must be compact. Moreover,

$$\mathrm{diam}(M,g) \leqslant \frac{1}{\lambda}\left(2(L+K)+\sqrt{4(L+K)^2+(n-1)\lambda\pi^2}\right).$$

REMARK 4. By taking $L = 0$, Theorem 12 above is reduced to Theorem 8.

M. Limoncu [28] established the following Myers type theorem via the k-modified Ricci curvature without making any assumptions on its vector field:

THEOREM 13 (M. Limoncu [28]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $\lambda > 0$ such that the k-modified Ricci curvature satisfies $\mathrm{Ric}_V^k \geqslant \lambda g$, where $k \in (0,+\infty)$. Then (M,g) must be compact. Moreover,*

$$\mathrm{diam}(M,g) \leqslant \frac{\pi}{\sqrt{\lambda}}\sqrt{n+k-1}. \tag{A.12}$$

REMARK 5. In the case that the vector field V is replaced with the gradient of some smooth function $f : M \to \mathbb{R}$ as $V = \nabla f$, Theorem 13 was already proved by Z. Qian [37].

On the other hand, M. P. Cavalcante, J. Q. Oliveira and M. S. Santos [7] proved the following Ambrose type theorem via the k-Bakry–Émery Ricci curvature:

THEOREM 14 (M. P. Cavalcante, J. Q. Oliveira and M. S. Santos [7]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some point $p \in M$ for which every geodesic $\gamma : [0,+\infty) \to M$ emanating from p satisfies*

$$\int_0^{+\infty} \mathrm{Ric}_f^k(\dot\gamma(s),\dot\gamma(s))\,ds = +\infty,$$

where $k \in (0,+\infty)$. Then (M,g) must be compact.

The key ingredient in the proof of Theorem 14 above is the Riccati inequality

$$\mathrm{Ric}_f^k(\partial_r,\partial_r) \leqslant -\dot m_f - \frac{(m_f)^2}{n+k-1},$$

which may be derived by applying the Bochner–Weitzenböck formula (A.5) to the distance function $r = r(x)$. Here $m_f := \Delta_f r$.

The Bochner–Weitzenböck formula (A.5) may be extended as follows:

LEMMA 1 (Y. Li [27]). *Let (M,g) be a Riemannian manifold. For any smooth vector field $V \in \mathfrak{X}(M)$ and smooth function $u : M \to \mathbb{R}$, we have*

$$\frac{1}{2}\Delta_V|\nabla u|^2 = |\operatorname{Hess} u|^2 + \operatorname{Ric}_V(\nabla u, \nabla u) + g(\nabla \Delta_V u, \nabla u). \tag{A.13}$$

REMARK 6. In Lemma 1 above, if the vector field V is replaced with the gradient of some smooth function $f : M \to \mathbb{R}$ as $V = \nabla f$, then (A.13) is reduced to the Bochner–Weitzenböck formula (A.5) via the Bakry–Émery Ricci curvature.

By applying the Bochner–Weitzenböck formula (A.13) to the distance function $r = r(x)$, we may obtain the Riccati inequality for the k-modified Ricci curvature

$$\operatorname{Ric}_V^k(\partial_r, \partial_r) \leqslant -\dot{m}_V - \frac{(m_V)^2}{n+k-1},$$

where $m_V := \Delta_V r$. By using this Riccati inequality, we may prove the following Ambrose type theorem via the k-modified Ricci curvature:

THEOREM 15 ([41]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some point $p \in M$ for which every geodesic $\gamma : [0,+\infty) \to M$ emanating from p satisfies*

$$\int_0^{+\infty} \operatorname{Ric}_V^k(\dot{\gamma}(s), \dot{\gamma}(s))ds = +\infty,$$

where $k \in (0,+\infty)$. Then (M,g) must be compact.

On the other hand, a Galloway type theorem via the k-modified Ricci curvature was first established by M. P. Cavalcante, J. Q. Oliveira and M. S. Santos [7].

THEOREM 16 (M. P. Cavalcante, J. Q. Oliveira and M. S. Santos [7]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exist some constants $\lambda > 0$ and $L \geqslant 0$ such that for every pair of points in M and minimal geodesic γ joining those points, the k-modified Ricci curvature satisfies*

$$\operatorname{Ric}_V^k(\dot{\gamma}, \dot{\gamma})|_{\gamma(s)} \geqslant \lambda + \frac{d\phi}{ds}(s),$$

where ϕ is some smooth function of the arc length satisfying $|\phi| \leqslant L$ along γ and $k \in (0,+\infty)$. Then (M,g) must be compact. Moreover,

$$\operatorname{diam}(M,g) \leqslant \frac{\pi}{\lambda}\left(L + \sqrt{L^2 + (n+k-1)\lambda}\right). \tag{A.14}$$

REMARK 7. Under the same assumption as in Theorem 16, the author [41] gave the upper diameter estimate

$$\operatorname{diam}(M,g) \leqslant \frac{1}{\lambda}\left(2L + \sqrt{4L^2 + (n+k-1)\lambda\pi^2}\right), \tag{A.15}$$

which is slightly sharper than (A.14). Moreover, by taking $L = 0$, (A.15) is reduced to (A.12).

REMARK 8. In the case that the vector field V is replaced with the gradient of some smooth function $f : M \to \mathbb{R}$ as $V = \nabla f$, Theorem 16 was already proved by M. Rimoldi [38].

3. Diameter bounds for compact Ricci solitons

In this section, we shall introduce some recent development on lower and upper diameter bounds for compact shrinking Ricci solitons.

A.1. Lower diameter bounds

Diameter bounds for compact shrinking Ricci solitons have been recently investigated by many authors [2, 9, 13, 16, 17, 39, 40]. In particular, a lower diameter bound for compact non-trivial shrinking Ricci solitons was first investigated by M. Fernández-López and E. García-Río [13] in terms of the Ricci curvature and the range of the potential function.

THEOREM 17 (M. Fernández-López and E. García-Río [13]). *Let (M,g) be an n-dimensional compact non-trivial shrinking Ricci soliton satisfying* (A.2). *Then*

(A.16)
$$\operatorname{diam}(M,g) \geqslant \max\left\{\sqrt{\frac{2(f_{\max}-f_{\min})}{C-\lambda}}, \sqrt{\frac{2(f_{\max}-f_{\min})}{\lambda-c}}, 2\sqrt{\frac{2(f_{\max}-f_{\min})}{C-c}}\right\},$$

where $f_{\max}$ and $f_{\min}$ respectively denote the maximum and the minimum values of the potential function on the soliton.

In Theorem 17 above and throughout this paper, the numbers

$$C := \max_{v\in TM}\{\operatorname{Ric}_g(v,v) : |v| = 1\} \quad \text{and} \quad c := \min_{v\in TM}\{\operatorname{Ric}_g(v,v) : |v| = 1\}$$

respectively denote the maximum and the minimum values of the Ricci curvature on the unit sphere bundle over (M,g). Note that $cg \leqslant \operatorname{Ric} \leqslant Cg$.

When the soliton has positive Ricci curvature, the diameter bound (A.16) above may be written in terms of the range of the scalar curvature.

COROLLARY 1 (M. Fernández-López and E. García-Río [13]). *Let (M,g) be an n-dimensional compact non-trivial shrinking Ricci soliton satisfying* (A.2). *Suppose that the soliton has positive Ricci curvature. Then*

$$\operatorname{diam}(M,g) \geqslant \max\left\{\sqrt{\frac{R_{\max}-R_{\min}}{\lambda(C-\lambda)}}, \sqrt{\frac{R_{\max}-R_{\min}}{\lambda(\lambda-c)}}, 2\sqrt{\frac{R_{\max}-R_{\min}}{\lambda(C-c)}}\right\},$$

where $R_{\max}$ and $R_{\min}$ respectively denote the maximum and the minimum values of the scalar curvature on the soliton.

On the other hand, a universal lower diameter bound for compact non-trivial shrinking Ricci solitons was first given by A. Futaki and Y. Sano [17] in relation to study of the first non-zero eigenvalue of the Witten–Laplacian.

THEOREM 18 (A. Futaki and Y. Sano [17]). *Let (M,g) be an n-dimensional compact non-trivial shrinking Ricci soliton satisfying* (A.2)*. Then*

$$\operatorname{diam}(M,g) \geqslant \frac{10\pi}{13\sqrt{\lambda}}. \tag{A.17}$$

The lower diameter bound (A.17) above was improved by many authors [2, 9, 16]. In particular, the following sharper diameter bound was given independently by Y. Chu and Z. Hu [9] and A. Futaki, H. Li and X.-D. Li [16]:

THEOREM 19 (Y. Chu and Z. Hu [9] and A. Futaki, H. Li and X.-D. Li [16]). *Let (M,g) be an n-dimensional compact non-trivial shrinking Ricci soliton satisfying* (A.2)*. Then*

$$\operatorname{diam}(M,g) \geqslant \frac{2(\sqrt{2}-1)\pi}{\sqrt{\lambda}}. \tag{A.18}$$

REMARK 9. Theorem 19 above says that, if a compact shrinking Ricci soliton (M,g) does not satisfy the inequality (A.18), then the soliton (M,g) must be trivial. Hence, this theorem gives us a gap phenomenon between Einstein manifolds and non-trivial Ricci solitons. See [14, 25] for other gap theorems for gradient Ricci solitons and Kähler–Ricci solitons, respectively.

A.2. Upper diameter bounds

Myers type theorems via the Bakry–Émery and modified Ricci curvatures in the previous section are closely related to an upper diameter bound for compact shrinking Ricci solitons. We recall the following useful propositions:

PROPOSITION 1 (R. Hamilton [21]). *Let (M,g) be an n-dimensional gradient Ricci soliton satisfying* (A.2)*. Then*

$$R + |\nabla f|^2 - 2\lambda f = C_0 \tag{A.19}$$

for some real constant C_0, where R denotes the scalar curvature on the soliton.

PROPOSITION 2 (M. Fernández-López and E. García-Río [13, 14]). *Let (M,g) be an n-dimensional compact shrinking Ricci soliton satisfying* (A.2)*. Then*

$$|\nabla f|^2 \leqslant R_{\max} - R. \tag{A.20}$$

Moreover, if the soliton has positive Ricci curvature, then

$$2\lambda f_{\max} - 2\lambda f_{\min} = R_{\max} - R_{\min}. \tag{A.21}$$

Recall from Theorem 17 and Corollary 1 that a lower diameter bound for compact non-trivial shrinking Ricci solitons was given in terms of the range of the potential function, as well as in terms of the range of the scalar curvature. M. Fernández-López and E. García-Río [13] conjectured that an upper diameter bound for compact shrinking Ricci solitons would also be given in terms of the range of the potential function, as well as in terms of the range of the scalar curvature. By combining Theorem 8 and the gradient estimate (A.20), we may give the following upper diameter bound for compact shrinking Ricci solitons which may be considered as an answer to the conjecture by M. Fernández-López and E. García-Río:

THEOREM 20 ([40]). *Let (M,g) be an n-dimensional compact shrinking Ricci soliton satisfying* (A.2)*. Then*

$$\operatorname{diam}(M,g) \leqslant \frac{1}{\lambda}\left(2\sqrt{R_{\max}-R_{\min}}+\sqrt{4(R_{\max}-R_{\min})+(n-1)\lambda\pi^2}\right). \tag{A.22}$$

REMARK 10. In Theorem 20 above, if the soliton has constant scalar curvature, then the soliton appears as an Einstein manifold and the diameter bound (A.22) above is reduced to the Myers diameter bound [32] for Einstein manifolds with positive Ricci curvature.

When the soliton has positive Ricci curvature, the diameter bound (A.22) above may be written in terms of the range of the potential function due to the relation (A.21).

COROLLARY 2 ([40]). *Let (M,g) be an n-dimensional compact shrinking Ricci soliton satisfying* (A.2)*. Suppose that the soliton has positive Ricci curvature. Then*

$$\operatorname{diam}(M,g) \leqslant 2\sqrt{\frac{2(f_{\max}-f_{\min})}{\lambda}}+\sqrt{\frac{8(f_{\max}-f_{\min})+(n-1)\pi^2}{\lambda}}.$$

On the other hand, by making some normalization on the potential function, we may give another diameter bound for compact shrinking Ricci solitons. Recall from Proposition 1 that any potential function f on a gradient shrinking Ricci soliton satisfies (A.19). By adding some constant on f, we may normalize f such that

$$R+|\nabla f|^2 = 2\lambda f. \tag{A.23}$$

By combining Theorem 6 and the normalizing condition (A.23), we may give the following upper diameter bound for compact shrinking Ricci solitons in terms of the maximum value of the scalar curvature:

COROLLARY 3 ([39]). *Let (M,g) be an n-dimensional compact shrinking Ricci soliton satisfying* (A.2)*. Suppose that the soliton is normalized in sense of* (A.23)*. Then*

$$\operatorname{diam}(M,g) \leqslant \frac{\pi}{\sqrt{\lambda}}\sqrt{n-1+\frac{4R_{\max}}{\pi\lambda}}.$$

4. An application to the Hitchin–Thorpe inequality

In this section, we shall introduce some validity of the Hitchin–Thorpe inequality for four-dimensional compact shrinking Ricci solitons. J. Thorpe [43] and N. Hitchin [22] proved independently that, if a four-dimensional compact Riemannian manifold M admits an Einstein metric, then the Euler number $\chi(M)$ and the signature $\tau(M)$ must satisfy the inequality

$$2\chi(M) \geqslant 3|\tau(M)|.$$

This inequality is known as the *Hitchin–Thorpe inequality* and has various geometric implications. For instance, if a four-dimensional compact Riemannian manifold M does not satisfy the inequality, then M never admit any Einstein metric. On the other hand, C. LeBrun [24] proved that there are infinitely many four-dimensional compact simply connected Riemannian manifolds which do not admit any Einstein metric, but nevertheless satisfy the *strict Hitchin–Thorpe inequality*

$$2\chi(M) > 3|\tau(M)|.$$

Just as in Einstein manifolds, we may expect some topological obstructions to the existence of four-dimensional compact Ricci solitons. The validity of the Hitchin–Thorpe inequality for four-dimensional compact shrinking Ricci solitons was first investigated by L. Ma [30] assuming some upper bounds on the L^2-norm of the scalar curvature.

THEOREM 21 (L. Ma [30]). *Let (M,g) be a four-dimensional compact shrinking Ricci soliton satisfying* (A.2). *If the scalar curvature satisfies*

$$\int_M R^2 \leqslant 24\lambda^2 \mathrm{vol}(M,g),$$

then the soliton must satisfy the Hitchin–Thorpe inequality.

M. Fernández-López and E. García-Río [13] investigated the validity of the Hitchin–Thorpe inequality assuming some upper diameter bounds in terms of the Ricci curvature.

THEOREM 22 (M. Fernández-López and E. García-Río [13]). *Let (M,g) be a four-dimensional compact non-trivial shrinking Ricci soliton satisfying* (A.2). *If*

$$\mathrm{diam}(M,g) \leqslant \max\left\{ \sqrt{\frac{2}{C-\lambda}}, \sqrt{\frac{2}{\lambda-c}}, 2\sqrt{\frac{2}{C-c}} \right\},$$

then the soliton must satisfy the Hitchin–Thorpe inequality.

The validity of the Hitchin–Thorpe inequality for four-dimensional compact shrinking Ricci solitons may also be obtained by assuming some lower diameter bounds in terms of the range of the scalar curvature.

COROLLARY 4 ([40]). *Let (M,g) be a four-dimensional compact shrinking Ricci soliton satisfying* (A.2). *If*

$$\left(2+\sqrt{4+\frac{3}{2}\pi^2}\right)\frac{\sqrt{R_{\max}-R_{\min}}}{\lambda}\leqslant \operatorname{diam}(M,g), \tag{A.24}$$

then the soliton must satisfy the Hitchin–Thorpe inequality.

REMARK 11. In Corollary 4 above, if the soliton has constant scalar curvature, then the soliton appears as an Einstein manifold and the assumption (A.24) above is trivially satisfied. Hence, Corollary 4 may be regarded as a natural generalization of the Hitchin–Thorpe inequality [22, 43] for Einstein manifolds with positive Ricci curvature.

On the other hand, if the soliton is normalized in sense of (A.23), we may give the following sufficient condition for four-dimensional compact shrinking Ricci solitons to satisfy the Hitchin–Thorpe inequality:

COROLLARY 5 ([39]). *Let (M,g) be a four-dimensional compact shrinking Ricci soliton satisfying* (A.2). *Suppose that the soliton is normalized in sense of* (A.23). *If*

$$\sqrt{\frac{R_{\max}}{\lambda^2}\left(4\pi+\frac{\pi^2}{2}\right)}\leqslant \operatorname{diam}(M,g),$$

then the soliton must satisfy the Hitchin–Thorpe inequality.

5. Addendum

After this paper was submitted, the author obtained the following lower diameter bound for compact shrinking Ricci solitons in terms of the scalar curvature:

THEOREM 23 ([42]). *Let (M,g) be an n-dimensional compact non-trivial shrinking Ricci soliton satisfying* (A.2). *Then*

$$\operatorname{diam}(M,g)\geqslant \frac{R_{\max}-n\lambda}{2\lambda\sqrt{R_{\max}-R_{\min}}}. \tag{A.25}$$

Note that, if the maximum value $R_{\max}$ of the scalar curvature is sufficiently large, then the diameter bound (A.25) above may be sharper than the diameter bound (A.18) obtained by Y. Chu and Z. Hu [9] and A. Futaki, H. Li and X.-D. Li [16].

Moreover, by combining Theorem 19 and Theorem 21, we may provide the following new sufficient condition for four-dimensional compact non-trivial shrinking Ricci solitons to satisfy the Hitchin–Thorpe inequality:

THEOREM 24 ([42]). *Let (M,g) be a four-dimensional compact non-trivial shrinking Ricci soliton satisfying* (A.2). *If*

$$\operatorname{diam}(M,g) \leqslant \frac{2\sqrt{2}(\sqrt{2}-1)\pi}{\sqrt{R_{\max}-R_{\min}}},$$

then the soliton must satisfy the Hitchin–Thorpe inequality.

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