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MAGNETIC CURVES OF THE REEB VECTOR FIELD OF A NORMAL ALMOST PARACONTACT THREE-MANIFOLD

Abstract. In this paper we first show that in any (oriented) Lorentzian three-manifold (M,g), magnetic fields F of positive constant length are in correspondence with the almost paracontact structures compatible with the Lorentzian metric g and with divergence-free Reeb vector field. Then, we report the results of [5], that is, we consider a normal almost paracontact metric structure, with divergence-free Reeb vector field, and describe the magnetic curves of the corresponding magnetic field.

Key words and phrases. Almost paracontact metric structures, normal structures, Killing vector fields, magnetic curves.

1. Introduction

Let (M,g) be a pseudo-Riemannian manifold and ∇ its Levi-Civita connection. A *magnetic field* on (M,g) is a closed two-form F on M. The *Lorentz force* ϕ corresponding to F, is a skew-symmetric $(1,1)$-tensor field uniquely determined by $g(\phi X,Y)=F(X,Y)$, for any vector fields X,Y on M. A smooth curve on M is called a *magnetic curve*, or *trajectory* for the magnetic field F if it is a solution of the *Lorentz equation* $\nabla_{\dot\gamma}\dot\gamma=\phi(\dot\gamma)$. As such, they are a natural generalization of geodesics of M, that satisfy the Lorentz equation in the absence of any magnetic field. However, it is relevant to note that the magnetic curves never reduce to geodesics [1, Prop. 2.1].

In the last decade, magnetic curves have been widely studied (see, for example, [1, 2], [6]–[12] and references therein). Such curves, under a geometric point of view, are often related to slant curves with respect to some natural Killing vector fields (see, for example, Proposition 1 and [15]). While, under a physical point of view, magnetic curves shape the magnetic flow in a background magnetic field, and describe the motion of charged particles in several physical scenarios (see, for example, [2]).

In particular, the three-dimensional case has been investigated, because it shows some special behaviors [4, 6, 8, 9, 11]. On an oriented three-dimensional pseudo-Riemannian manifold (M,g), the Lorentz force is provided via the cross product, and magnetic fields are in a one-to-one correspondence with divergence-free vector fields. As Killing vector fields are divergence-free, one may define a special class of magnetic fields called *Killing magnetic fields*, namely, the ones corresponding to Killing vector fields. This leads to investigate magnetic curves related to some Killing vector fields which appear naturally in the geometry of the three-dimensional pseudo-Riemannian manifold itself.

In the contact Riemannian case, Cabrerizo et al. in [6] have studied the magnetic fields determined by the Reeb vector field of any Sasakian three-manifold, determining the magnetic trajectories and proving that they are helices with axis the Reeb vector field itself. In dimension three, paracontact structures are the Lorentzian counterpart to contact Riemannian structures.

This paper is based on the Chapter 5 of [13] and on the joint paper [5]. More precisely, in Section 2 we report some basic information about magnetic curves, in particular in dimension three. In Section 3 we show that in any (oriented) Lorentzian three-manifold (M,g) of signature $(++-)$, magnetic fields F_V with $g(V,V) = constant > 0$ are in one-to-one correspondence with the almost paracontact structures compatible with the Lorentzian metric g and with divergence-free Reeb vector field. Moreover, the magnetic field F_V is a Killing magnetic field if and only if the corresponding almost paracontact structure is normal. These results motivate the study of the magnetic curves in almost paracontact metric three-manifolds. Then, in Section 4 we report the results of [5], that is, we describe the magnetic curves corresponding to the magnetic field $F = qF_\xi, q \in \mathbb{R} \setminus \{0\}$, associated with the Reeb vector field ξ of a normal almost paracontact metric three-manifold with ξ divergence-free. We observe that the class of normal almost paracontact metric manifolds with divergence-free Reeb vector field is very large. In particular, it includes paraSasakian and paracosymplectic manifolds. Moreover, with respect to the contact metric case, the study here is at the same time more complex and interesting, because a metric compatible with an almost paracontact three-structure is Lorentzian and so, the vectors $\dot\gamma$ and $\nabla_{\dot\gamma}\dot\gamma$ can have any causal character. Next, as an application, explicit descriptions are given for the magnetic curves of the standard left-invariant paraSasakian structure of the hyperbolic Heisenberg group and of a model of paracosymplectic three-manifold.

2. Magnetic curves

The magnetic curves on a pseudo-Riemannian manifold (M,g) are trajectories of charged particles, moving on M under the action of a magnetic field. A *magnetic field* on (M,g) is a closed two-form F on M, to which one may associate a skew-symmetric $(1,1)$-tensor field ϕ on M, called the *Lorentz force*, uniquely determined by $g(\phi X,Y) = F(X,Y)$, for any vector fields X,Y on M. Let us remark that ϕ is metrically equivalent to F, so that no information is lost when ϕ is considered instead of F, and ϕ and F are then said to be physically equivalent.

A smooth parametrized curve $\gamma(t)$ in M is called *magnetic curve* of the magnetic field F if it satisfies the *Lorentz equation*

$$\nabla_{\dot\gamma}\dot\gamma = \phi(\dot\gamma), \tag{A.1}$$

where ∇ is the Levi-Civita connection of (M,g). It is well known that for any point $p \in M$ and for any vector $X_p \in T_pM$ there exists a unique geodesic $\gamma(t)$ such that $\gamma(0) = p$ and $\dot\gamma = X_p$. When $F \neq 0$, the same existence and uniqueness property can be stated for magnetic curves.

We observe that a magnetic curve γ has constant speed:

$$\frac{1}{2}\frac{d}{ds}g(\dot\gamma,\dot\gamma) = g(\nabla_{\dot\gamma}\dot\gamma,\dot\gamma) = g(\phi\dot\gamma,\dot\gamma) = F(\dot\gamma,\dot\gamma) = 0.$$

On a three-dimensional pseudo-Riemannian manifold (M,g) oriented by the volume form Ω, one may define a Lorentz cross product via Ω on M, putting

$$g(X\wedge Y,Z)=\Omega(X,Y,Z),$$

for any vector fields X,Y,Z on M. Moreover, in dimension three, the two-forms F are in a one-to-one correspondence with the vector fields V by

$$F(X,Y)=\Omega(V,X,Y),$$

for any vector fields X,Y on M. Since $F_V(X,Y):=\Omega(V,X,Y)$ satisfies

$$dF_V=\mathcal{L}_V\Omega=(\mathrm{div}\ V)\Omega,$$

then F_V is closed if and only if V is divergence-free. Therefore, magnetic fields are in a one-to-one correspondence with divergence-free vector fields. Besides, for any vector field V, with $\mathrm{div}V=0$, from $g(\phi X,Y)=F_V(X,Y)=\Omega(V,X,Y)=g(V\wedge X,Y)$, we get that the Lorentz force may be expressed as

$$\phi X=V\wedge X,$$

and so, Equation (A.1) becomes

$$\nabla_{\dot\gamma}\dot\gamma=V\wedge\dot\gamma. \tag{A.2}$$

Therefore, magnetic curves associated to V correspond to solutions of the ordinary differential equation (A.2). In particular, a large class of Lorentz forces may be obtained from Killing vector fields on M (since V Killing implies V divergence-free). We now state the following.

PROPOSITION 1. *A divergence-free vector field V on a pseudo-Riemannian three-manifold (M,g) is Killing if and only if $g(V,\dot\gamma)$ is constant for any magnetic curve $\gamma(s)$ of F_V.*

Proof. It is well known that V is a Killing vector field if and only if it satisfies

$$g(\nabla_X V,Y)+g(X,\nabla_Y V)=0,$$

for any tangent vector fields X,Y. Therefore, if V is a Killing vector field and γ is a magnetic curve, using Equation (A.2) we get that $g(V,\dot\gamma)$ is constant. In fact,

$$\frac{d}{ds}g(V,\dot\gamma)=g(\nabla_{\dot\gamma}V,\dot\gamma)+g(V,\nabla_{\dot\gamma}\dot\gamma)=g(\nabla_{\dot\gamma}V,\dot\gamma)+g(V,V\wedge\dot\gamma)=g(\nabla_{\dot\gamma}V,\dot\gamma)=0.$$

Conversely, for any vector field X, consider the magnetic curve γ such that $\dot\gamma(0)=X_p$, where $\gamma(0)=p$. Since $g(V,\dot\gamma)$ is constant we have

$$0=\frac{d}{ds}g(V,\dot\gamma)=g(\nabla_{\dot\gamma}V,\dot\gamma)+g(V,\nabla_{\dot\gamma}\dot\gamma)=g(\nabla_{\dot\gamma}V,\dot\gamma).$$

Hence, $g(\nabla_X V,X)(p)=g(\nabla_{X_p}V,X_p)=0$ for any $p\in M$. Consequently, we obtain

$$0=g(\nabla_{X+Y}V,X+Y)=g(\nabla_X V,Y)+g(\nabla_Y V,X)$$

and so V is Killing. $\square$

In the Riemannian case Proposition 1 corresponds to Lemma 4.1 of [6]. Under a geometric point of view, this proposition implies that magnetic curves are related to slant curves with respect to some natural Killing vector fields (see, for example, [15]).

3. Almost paracontact structures and magnetic fields

A.1. Almost paracontact metric structures

An *almost paracontact structure* on a $(2n+1)$-dimensional (connected) smooth manifold M is a triple (φ,ξ,η), where φ is a tensor of type $(1,1)$, ξ a vector field (called the *Reeb vector field* or the *characteristic vector field*) and η a one-form, satisfying

(i) $$\eta(\xi)=1, \qquad \varphi^2 = I_d - \eta\otimes\xi$$

(ii) the tensor field φ induces an almost paracomplex structure on the distribution $\mathcal{D}=\ker\eta$, that is, the eigendistributions $\mathcal{D}^+$ and $\mathcal{D}^-$, corresponding to the eigenvalues 1, -1 of φ respectively, have equal dimension n.

As a consequence of (i), one gets $\varphi\xi=0$, $\eta\circ\varphi=0$. M is said to be an *almost paracontact manifold* if it is equipped with an almost paracontact structure. M is called an *almost paracontact metric manifold* if it is equipped with an almost paracontact metric structure, that is, if it admits a pseudo-Riemannian metric g (called *compatible metric*), such that

$$g(\varphi X,\varphi Y) = -g(X,Y)+\eta(X)\eta(Y). \tag{A.3}$$

Any compatible metric g with a given almost paracontact structure is of signature $(n+1,n)$. We note that condition (i) and Equation (A.3) imply that $\eta(X)=g(\xi,X)$ and condition (ii).

For an almost paracontact metric manifold, we can define the *fundamental two-form* Φ by $\Phi(X,Y)=g(X,\varphi Y)$, for all tangent vector fields X,Y. If $\Phi=d\eta$, then the manifold is called a *paracontact metric manifold* and g the *associated metric*.

An almost paracontact metric structure (φ,ξ,η,g) is said to be *normal* if it satisfies the condition $N^{(1)} := [\varphi,\varphi]-2d\eta\otimes\xi=0$, where $[\varphi,\varphi]$ is the Nijenhuis torsion tensor of φ. In dimension three, normal almost paracontact metric structures are characterized by condition (see [14],[15]):

$$(\nabla_X\varphi)Y = \alpha(g(\varphi X,Y)\xi-\eta(Y)\varphi X)+\beta(g(X,Y)\xi-\eta(Y)X), \tag{A.4}$$

or equivalently,

$$\nabla_X\xi = \alpha(X-\eta(X)\xi)+\beta\varphi X, \tag{A.5}$$

where $2\alpha=\mathrm{tr}\nabla\xi=\mathrm{div}\xi$ and $2\beta=\mathrm{tr}(\varphi\nabla\xi)$. A normal almost paracontact metric structure is called

- *quasi-paraSasakian* if $\alpha=0\neq\beta$ (in particular, β-paraSasakian when β is a constant, and when $\beta=-1$ the structure is paraSasakian);

- *paracosymplectic* if $\nabla\eta = \nabla\Phi = 0$. In the three-dimensional case, this is equivalent to the condition $\nabla\xi = 0$, that is, the structure is normal with $\alpha = \beta = 0$.

A.2. Paracontact magnetic fields

Let $(M, \varphi, \xi, \eta, g)$ be an almost paracontact metric three-manifold with $\operatorname{div}\xi = 0$. Then, the volume form is given by $\Omega = 3\eta \wedge \Phi$ and the magnetic field $F_\xi = \Phi$, as

$$F_\xi(X,Y) := \Omega(\xi, X, Y) = 3(\eta \wedge \Phi)(\xi, X, Y) = \Phi(X,Y).$$

So, the almost paracontact metric structure (φ, ξ, η, g), with $\operatorname{div}\xi = 0$, determines the set $\{F' = qF_\xi,\, q \in \mathbb{R} \setminus \{0\}\}$ of magnetic fields on the Lorentzian manifold (M, g). Vice versa, consider now on an oriented Lorentzian three-manifold (M, g), that we suppose of signature $(++-)$, a magnetic field F with corresponding divergence-free vector field V and so, with Lorentz force ϕ expressed by the Lorentz cross product $\phi X = V \wedge X$. This Lorentz cross product satisfies (see, for example, [8])

$$\begin{aligned} X \wedge (Y \wedge Z) &= g(X,Y)Z - g(X,Z)Y, \\ g(Z \wedge X, Z \wedge Y) &= g(Z,X)g(Z,Y) - g(Z,Z)g(X,Y). \end{aligned}$$

Then, by using these properties, the vector field V, the Lorentz force ϕ and the one-form $\eta_V = g(V, \cdot)$ satisfy:

$$\phi^2 X = V \wedge (V \wedge X) = g(V,V)X - g(V,X)V = g(V,V)X - \eta_V(X)V,$$

$$\begin{aligned} g(\phi X, \phi Y) &= g(V \wedge X, V \wedge Y) = g(V,X)g(V,Y) - g(V,V)g(X,Y) \\ &= \eta_V(X)\eta_V(Y) - g(V,V)g(X,Y). \end{aligned}$$

If $g(V,V) = q^2 > 0$, for some smooth function $q \neq 0$, then the tensors

$$\Big(\varphi = -\frac{1}{q}\phi,\ \xi = \frac{1}{q}V,\ \eta = \frac{1}{q}\eta_V,\ g\Big)$$

satisfy condition (i) and Equation (A.3), and such conditions define an almost paracontact metric structure. Besides, if $q \neq 0$ is a real constant, then ξ is divergence-free, and the magnetic field F is given by

$$F(X,Y) = g(\phi X, Y) = -g(q\varphi X, Y) = q\Phi(X,Y) = qF_\xi.$$

We note that, given the magnetic field $F = F_V$, $g(V,V) = constant = q^2 > 0$, if we consider the magnetic field $F' = \lambda F = \lambda F_V$, $\lambda \in \mathbb{R} \setminus \{0\}$, then F' determines the same almost paracontact structure (φ, ξ, η) compatible with g and with $\operatorname{div}\xi = 0$. In fact,

$$\begin{aligned} &F' = \lambda F_V = \lambda\Omega(V,\cdot,\cdot) = F_{V'}, && V' = \lambda V, && g(V',V') = q'^2 = \lambda^2 q^2, \\ &\phi' X = V' \wedge X = \lambda V \wedge X = \lambda\phi X, && \varphi' = -\tfrac{1}{q'}\phi' = -\tfrac{1}{\lambda q}\lambda\phi = \varphi, && \\ &\xi' = \tfrac{1}{q'}V' = \xi, && \eta' = \tfrac{1}{q'}\eta_{V'} = \tfrac{1}{\lambda q}g(V',\cdot) = \tfrac{1}{q}g(V,\cdot) = \eta. && \end{aligned}$$

Moreover, if the almost paracontact structure (φ,ξ,η) compatible with g, and with ξ divergence-free, is normal then from Equation (A.5) we get $\nabla\xi=\beta\varphi$. This gives that ξ is Killing, and thus F_V is a Killing magnetic field. Vice versa, suppose that the magnetic field F_V, $g(V,V)=constant=q^2>0$, is Killing, then also the corresponding Reeb vector field ξ is Killing. Besides, it is well known that a unit Killing vector field is geodesic, and hence ξ satisfies $\nabla_\xi\xi=0$. Then, by using (4) of Proposition 3.1 of [3] we have that the tensor $h:=(1/2)\mathcal{L}_\xi\varphi$ vanishes. On the other hand, in dimension three, if $h=0$ then the almost paracontact structure is normal. In fact, first of all, we note that the distributions $\mathcal{D}^+$ and $\mathcal{D}^-$ are involutive, because they have dimension one. Then, for any $X,Y\in\Gamma(\mathcal{D}^+)$, or $X,Y\in\Gamma(\mathcal{D}^-)$, we have $d\eta(X,Y)=0$ and so

$$N^{(1)}(X,Y)=[\varphi,\varphi](X,Y)=\varphi^2[X,Y]+[\varphi X,\varphi Y]-\varphi[\varphi X,Y]-\varphi[X,\varphi Y]=0.$$

Moreover, for $X\in\Gamma(\mathcal{D}^+)$ and $Y\in\Gamma(\mathcal{D}^-)$, we get $2d\eta(X,Y)=-\eta[X,Y]$ and so

$$N^{(1)}(X,Y)=\varphi^2[X,Y]+[\varphi X,\varphi Y]-\varphi[\varphi X,Y]-\varphi[X,\varphi Y]+\eta[X,Y]\xi=0.$$

Similarly for $X\in\Gamma(\mathcal{D}^-)$ and $Y\in\Gamma(\mathcal{D}^+)$. Finally, for $X\in\Gamma(\mathcal{D}^\pm)$

$$\begin{aligned}N^{(1)}(X,\xi)&=\varphi^2[X,\xi]-\varphi[\pm X,\xi]-2d\eta(X,\xi)\xi=[X,\xi]\mp\varphi[X,\xi]\\&=\mp(\mathcal{L}_\xi\varphi)X=\mp 2h(X)=0.\end{aligned}$$

Analogously $N^{(1)}(\xi,X)=0$.

Summarizing, we proved the following

THEOREM 1. *In any (oriented) Lorentzian three-manifold (M,g) of signature $(++-)$, magnetic fields F_V with $g(V,V)=constant>0$, defined up to a constant of proportionality, are in one-to-one correspondence with the almost paracontact structures compatible with the Lorentzian metric g and with divergence-free Reeb vector field. Moreover, the magnetic field F_V is a Killing magnetic field if and only if the corresponding almost paracontact structure is normal.*

REMARK 1. By above proof we get that if an (oriented) Lorentzian three-manifold (M,g) of signature $(++-)$ admits a magnetic fields F_V, $g(V,V)=q^2>0$, for some smooth function $q\neq 0$, then it admits an almost paracontact structure (φ,ξ,η) compatible with the Lorentzian metric g.

Theorem 1 motivates the study of the magnetic curves in almost paracontact metric three-manifolds.

Let (M,φ,ξ,η,g) be an almost paracontact metric three-manifold with $\mathrm{div}\xi=0$. For the magnetic field F with *strength* q, corresponding to the given almost paracontact metric structure, that is, for $F=qF_\xi$, $q\in\mathbb{R}\setminus\{0\}$, we have

$$g(\phi X,Y)=F(X,Y)=q\Phi(X,Y)=qg(X,\varphi Y)=-qg(\varphi X,Y),$$

for all tangent vector fields X,Y. Therefore, we get $\phi = -q\varphi$ and the Lorentz equation (A.1) becomes

$$\nabla_{\dot\gamma}\dot\gamma = -q\varphi\dot\gamma. \tag{A.6}$$

Furthermore, since $g(\xi \wedge X,Y) = F_\xi(X,Y) = \Phi(X,Y) = -g(\varphi X,Y)$, we have $\varphi X = -\xi \wedge X$ and so, the Lorentz force is expressed by

$$\phi X = q\xi \wedge X.$$

If we suppose that $\gamma(t)$ is non-geodesic, then from Equation (A.6) we have that $\gamma(t)$ can not be an integral curve of ξ (for the integral curves of ξ we have $\varphi\dot\gamma = \varphi\xi_{\gamma(t)} = 0$).

4. Magnetic curves in a normal almost paracontact metric 3D-manifold

The results of this section were obtained in the joint paper [5]. We shall only report the essential steps, for more details see the same paper [5].

Let (M,φ,ξ,η,g) be a normal almost paracontact metric three-manifold. Suppose that the two-form F_ξ is a magnetic field, that is, $\operatorname{div}\xi = 0$. Then, from Theorem 1 it follows that ξ is Killing. We now study the magnetic curves corresponding to the Killing magnetic field $F = qF_\xi$, $q \in \mathbb{R} \setminus \{0\}$.

Let γ denote such a (non-geodesic) magnetic curve. We know that γ has constant speed and, as ξ is a Killing vector field, by Proposition 1, we get that

$$a_0 := \eta(\dot\gamma) = g(\xi,\dot\gamma) \text{ is a constant.}$$

We now shall treat separately three cases, depending on the causality of γ.

Case I: γ is light-like.
First, we analyze the acceleration vector field. By Equation (A.6), we find

$$g(\nabla_{\dot\gamma}\dot\gamma,\nabla_{\dot\gamma}\dot\gamma) = q^2 g(-\varphi\dot\gamma,-\varphi\dot\gamma) = q^2(-g(\dot\gamma,\dot\gamma)+\eta(\dot\gamma)^2) = q^2\eta(\dot\gamma)^2 = q^2a_0^2. \tag{A.7}$$

If $\eta(\dot\gamma) = 0$, then γ is a *Legendre curve*. In this case, we find that $\nabla_{\dot\gamma}\dot\gamma$ is light-like and by Theorem 4.1 in [14] we conclude that γ is a geodesic. Therefore, from now on we shall assume that $a_0 = \eta(\dot\gamma) \neq 0$. From Equation (A.7) we now get $g(\nabla_{\dot\gamma}\dot\gamma,\nabla_{\dot\gamma}\dot\gamma) = q^2a_0^2 > 0$ and so, $\nabla_{\dot\gamma}\dot\gamma$ is space-like. Thus, reparametrizing γ, we can reduce to the case where the curvature satisfies $\kappa = \left\|\nabla_{\dot\gamma}\dot\gamma\right\| = 1$ (*pseudo-arc length parameter*), that is, $q^2a_0^2 = 1$. In fact, after an affine reparametrization, γ still remains a magnetic curve but with a different strength.

The normal vector field $N = \nabla_{\dot\gamma}\dot\gamma$ is space-like, $T = \dot\gamma$ is light-like and $g(N,T) = 0$. Hence, there exists a unique light-like vector field B, such that $g(B,N) = 0$ and $g(B,T) = 1$, so that $\{T,N,B\}$ is a null basis along the curve γ. We recall that the torsion τ of γ is given by $\tau := g(\nabla_{\dot\gamma}N, B)$. Writing $\nabla_{\dot\gamma}N = aT + bN + cB$, for some smooth real functions a, b, c, we get the second Frenet formula

$$\nabla_{\dot\gamma}N = \tau T - B. \tag{A.8}$$

We now determine τ. As $N = -q\varphi\dot\gamma$, we have

$$\nabla_{\dot\gamma}N = -q\nabla_{\dot\gamma}(\varphi\dot\gamma) = -q((\nabla_{\dot\gamma}\varphi)\dot\gamma + \varphi\nabla_{\dot\gamma}\dot\gamma) = -q((\nabla_{\dot\gamma}\varphi)\dot\gamma + \varphi N).$$

Using Equations (A.8) and (A.4) (with $\alpha = 0$), we find

$$\tau T - B = -\beta q\big(g(\dot\gamma,\dot\gamma)\xi - \eta(\dot\gamma)\dot\gamma\big) - q\varphi N = \beta q a_0 T - q\varphi N.$$

Thus, $q\varphi N = (-\tau + \beta q a_0)T + B$ and so, $q^2 g(\varphi N, \varphi N) = -2(\tau - \beta q a_0)g(T, B)$, that is, using Equation (A.3), $q^2 g(N, N) = 2(\tau - \beta q a_0)$, which yields

$$\tau = \frac{q^2}{2} + \beta q a_0 = \frac{q^2}{2} \pm \beta.$$

Then, τ is constant (and so, γ is a helix) if and only if β is constant, that is, when M is either β-paraSasakian or paracosymplectic.

Finally, expressing ξ with respect the basis $\{T, N, B\}$, using Equation (A.8) we find $\xi = \frac{1}{2a_0}\,T + a_0 B$.

Case II: γ is time-like.

Let γ be parametrized by arc length, so that $g(\dot\gamma,\dot\gamma) = -1$. By Equation (A.6), we have

$$g(\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma) = q^2 g(\varphi\dot\gamma, \varphi\dot\gamma) = q^2(-g(\dot\gamma,\dot\gamma) + \eta(\dot\gamma)^2) = q^2(1 + \eta(\dot\gamma)^2) = q^2(1 + a_0^2).$$

It follows that $\nabla_{\dot\gamma}\dot\gamma$ is space-like. Moreover, there exists a real constant θ, such that $a_0 = \sinh\theta$, where θ is the hyperbolic angle between the space-like vector ξ and the time-like vector $\dot\gamma$.

The Frenet frame on γ is defined by the time-like vector $T = \dot\gamma$, the space-like vector $\nabla_{\dot\gamma}\dot\gamma = \kappa N$, and the space-like vector $B = T \wedge N$, where the positive function κ denotes the curvature of γ, and N is the unitary normal.

We get $\kappa = |q|\cosh\theta$. Hence, the curvature κ is a nonzero constant, and in particular $\kappa^2 = q^2(1 + a_0^2)$. Besides, we find $\tau = \beta - a_0 q$ and $\xi = -a_0 T + \frac{\kappa}{q}B$.

Case III: γ is space-like. Let γ be parametrized by arc length, that is, without loss of generality, we assume that $g(\dot\gamma,\dot\gamma) = 1$. Taking into account Equation (A.6), we get

$$g(\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma) = q^2 g(-\varphi\dot\gamma, -\varphi\dot\gamma) = q^2(-g(\dot\gamma,\dot\gamma) + \eta(\dot\gamma)^2) = q^2(-1 + \eta(\dot\gamma)^2) = q^2(a_0^2 - 1).$$

We shall distinguish three cases, depending on whether $a_0^2 > 1$ (that is, $\nabla_{\dot\gamma}\dot\gamma$ is space-like), $0 \le a_0^2 < 1$ ($\nabla_{\dot\gamma}\dot\gamma$ is time-like) or $a_0^2 = 1$ ($\nabla_{\dot\gamma}\dot\gamma$ is light-like).

Case III.a: $\nabla_{\dot\gamma}\dot\gamma$ is space-like.

Since $a_0^2 > 1$, there exists some constant θ, such that $a_0 = \varepsilon_0 \cosh\theta$. Besides, both the tangent $T = \dot{\gamma}$ and the normal $N = \frac{1}{\kappa}\nabla_{\dot{\gamma}}\dot{\gamma}$ are space-like vector fields and we get $\kappa = |q \sinh\theta|$. The binormal B is a time-like vector field defined by $B = T \wedge N$ and hence $\{T,N,B\}$ is the pseudo-orthonormal Frenet frame along the curve γ. Finally, we get that the torsion of γ is $\tau := g(\nabla_{\dot{\gamma}}N,B) = -(\beta + a_0 q)$ and $\xi = a_0 T - \frac{\kappa}{q}B$.

Case III.b: $\nabla_{\dot{\gamma}}\dot{\gamma}$ is time-like.

In such a case, $0 \leq a_0^2 < 1$, so there exists $\theta \in]0,\pi[$ such that $a_0 = \cos\theta$. As before, we determine the Frenet frame $\{T,N,B\}$, where $T = \dot{\gamma}$ is space-like, $N = \frac{1}{\kappa}\nabla_{\dot{\gamma}}\dot{\gamma}$ is time-like, $B = T \wedge N$ is space-like and $\kappa = |q \sin\theta|$, the torsion $\tau = \beta + a_0 q$ and $\xi = a_0 T - \frac{\kappa}{q}B$.

Case III.c: $\nabla_{\dot{\gamma}}\dot{\gamma}$ is light-like.

As $a_0^2 = 1$, we can write $a_0 = \pm 1$. In this case, $T = \dot{\gamma}$ is space-like, $N = \nabla_{\dot{\gamma}}\dot{\gamma}$ is light-like, the curvature κ is not defined and there exists a unique light-like vector field B, such that $g(B,T) = 0$ and $g(B,N) = 1$. So, $\{T,N,B\}$ is a null basis along the curve γ. Finally, we obtain that the torsion τ is given by $\tau = \mp(\beta + a_0 q)$ and $\xi = a_0 T \pm \frac{1}{q}B$.

Summarizing, we obtain the following

THEOREM 2. *Let (M,φ,ξ,η,g) be a normal almost paracontact metric three-manifold with ξ divergence-free and $\gamma(s) : I \longrightarrow M$ be a non-geodesic magnetic curve associated to ξ with strength q. We put $2\beta = \mathrm{tr}(\varphi\nabla\xi)$, then we have the following:*

- *if $\gamma(s)$ is a light-like curve, parametrized by pseudo-arc length, then the acceleration is space-like, $\xi = \frac{1}{2a_0}T + a_0 B$, $\kappa = 1$ and $\tau = \frac{q^2}{2} \pm \beta$, where $a_0^2 q^2 = 1$;*
- *if $\gamma(s)$ is a unit speed time-like curve, then the acceleration is space-like, $\xi = -a_0 T + \frac{\kappa}{q}B$, $\kappa^2 = (1 + a_0^2)q^2$ and $\tau = \beta - a_0 q$;*
- *if $\gamma(s)$ is a unit speed space-like curve with a space-like acceleration, then $\xi = a_0 T - \frac{\kappa}{q}B$, $\kappa^2 = (a_0^2 - 1)q^2$ and $\tau = -(\beta + a_0 q)$;*
- *if $\gamma(s)$ is a unit speed space-like curve with a time-like acceleration, then $\xi = a_0 T - \frac{\kappa}{q}B$, $\kappa^2 = (1 - a_0^2)q^2$ and $\tau = \beta + a_0 q$;*
- *if $\gamma(s)$ is a unit speed space-like curve with a light-like acceleration, then $\xi = a_0 T \pm \frac{1}{q}N$, κ is not defined, $\tau = \mp(\beta + a_0 q)$ and $a_0 = \pm 1$;*

where κ and τ are, respectively, the curvature and the torsion of the curve γ, $a_0 := \eta(\dot{\gamma})$ is constant and (T,N,B) is the corresponding Frenet frame along γ.

Moreover, in all cases the curve $\gamma(s)$ is a helix if and only if β is constant, that is, M is either β-paraSasakian ($\beta \neq 0$) or paracosymplectic ($\beta = 0$).

Magnetic curves of the hyperbolic Heisenberg group H_h^3

Consider the Lie group $\mathbb{R}^3$ with the following group law

$$(x,y,z) \cdot (x',y',z') := (x + x', y + y', z + z' - xy' + x'y). \tag{A.9}$$

A basis of left-invariant vector fields is given by $\xi = 2\frac{\partial}{\partial z}$, $U = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z}$, $V = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial z}$. Next, consider the standard left-invariant paraSasakian structure (φ, ξ, η, g) defined by

$$\eta = \frac{1}{2}dz + (xdy - ydx), \qquad g = dx \otimes dx - dy \otimes dy + \eta \otimes \eta,$$

$$\varphi(\xi) = 0, \quad \varphi(U) = V, \quad \varphi(V) = U.$$

The Lie group $\mathbb{R}^3$, with the law group (A.9), equipped with the pseudo-Riemannian metric g, is called *hyperbolic Heisenberg group*, and we shall denote it by H_h^3.

Let $\gamma(s) = (x(s), y(s), z(s))$ be a magnetic curve of H_h^3, that is, a solution of the Lorentz equation $\nabla_{\dot\gamma}\dot\gamma = -q\varphi\dot\gamma$. We put $\dot\gamma = f_1 U + f_2 V + f_3 \xi$ where f_1, f_2, f_3 are smooth functions on H_h^3. In particular, $f_3 = g(\dot\gamma(s), \xi) = \eta(\dot\gamma) =: a_0$ is a constant, because ξ is Killing. Next, with respect to the φ-basis (ξ, U, V), we get

$$(\dot f_1 - 2f_2 a_0)U + (\dot f_2 - 2f_1 a_0)V = \nabla_{\dot\gamma}\dot\gamma = -q\varphi\dot\gamma = -q(f_1 V + f_2 U).$$

Therefore, the curve $\gamma(s)$ is a magnetic curve if and only if f_1, f_2 satisfy the following system of differential equations: $\dot f_1 = \lambda f_2$, $\dot f_2 = \lambda f_1$, where $\lambda := 2a_0 - q =$ is a constant.

• If $\lambda = 0$, we find that $\gamma(s) = \big(x_0 + c_1 s,\ y_0 + c_2 s,\ z_0 + 2(a_0 + c_1 y_0 - c_2 x_0)s\big)$, for some real constants c_1, c_2, x_0, y_0, z_0, that is, it is a straight line that can be either space-like, time-like or light-like, as $g(\dot\gamma, \dot\gamma) = c_1^2 - c_2^2 + a_0^2$.

• If $\lambda \neq 0$, we find that $\gamma(s) = (x(s), y(s), z(s))$, where

$$\begin{cases} x(s) = x_0 + \frac{c_1}{\lambda}\sinh(\lambda s) + \frac{c_2}{\lambda}\cosh(\lambda s), \\ y(s) = y_0 + \frac{c_1}{\lambda}\cosh(\lambda s) + \frac{c_2}{\lambda}\sinh(\lambda s), \\ z(s) = z_0 + 2\big(a_0 + \frac{c_1^2 - c_2^2}{\lambda}\big) + \frac{2}{\lambda}(c_1 y_0 - c_2 x_0)\sinh(\lambda s) + \frac{2}{\lambda}(c_2 y_0 - c_1 x_0)\cosh(\lambda s), \end{cases}$$

for some real constants c_1, c_2, x_0, y_0, z_0, and it can be either space-like, time-like or light-like as $g(\dot\gamma, \dot\gamma) = c_1^2 - c_2^2 + a_0^2$.

Examples of magnetic curves in the paracosymplectic case

Let $M = \mathbb{R}_- \times \mathbb{R}^2$ with the standard Cartesian coordinates (x, y, z), we put $\partial_x = \frac{\partial}{\partial_x}$, $\partial_y = \frac{\partial}{\partial_y}$ and $\partial_z = \frac{\partial}{\partial_z}$. Consider on M the paracosymplectic structure (φ, ξ, η, g) defined by

$$\varphi\partial_x = \partial_y, \qquad \varphi\partial_y = \partial_x, \qquad \varphi\partial_z = 0, \qquad \partial_z = \xi,$$

$$\eta = dz, \qquad g = -x(dx \otimes dx - dy \otimes dy) + \eta \otimes \eta.$$

Now, let $\gamma(s) = \big(x(s), y(s), z(s)\big)$ be a magnetic curve of M, that is, a solution of $\nabla_{\dot\gamma}\dot\gamma = -q\varphi\dot\gamma$ with $x(s) < 0$. We put $\dot\gamma = \big(\dot x(s),\ \dot y(s),\ \dot z(s)\big) = \big(f_1(s),\ f_2(s),\ f_3(s)\big)$, where f_1, f_2, f_3 are smooth functions on M. In particular, as ξ is Killing, $\dot z(s) = g(\dot\gamma(s), \xi) = \eta(\dot\gamma) =: a_0$ is a constant. So, we get $z(s) = a_0 s + z_0$, for some constant z_0. Next, we explicitly compute the two members of the Lorentz equation with respect to the basis $(\partial_x, \partial_y, \partial_z)$. Since

$$\dot{f}_1\partial_x + \dot{f}_2\partial_y + f_1\nabla_{\dot\gamma}\partial_x + f_2\nabla_{\dot\gamma}\partial_y + a_0\nabla_{\dot\gamma}\xi = \nabla_{\dot\gamma}\dot\gamma = -q\varphi\dot\gamma = -q(f_2\partial_x + f_1\partial_y),$$

we have that $\gamma(s)$ is a magnetic curve if and only if f_1, f_2 satisfy the following system of differential equations:

$$\dot{f}_1 + \frac{f_1^2}{2x} + \frac{f_2^2}{2x} = -qf_2, \quad \dot{f}_2 + \frac{f_1 f_2}{x} = -qf_1.$$

We note that if $f_2 = 0$, then the second equation gives $f_1 = 0$ and so, γ is a geodesic. In order to find explicit solutions, we can consider special cases, and so, we find that examples of magnetic curves of M are:

- $\gamma(s) = \big(a_1,\ -2qa_1 s + a_2,\ a_0 s + z_0\big)$, for any real constants $a_1 < 0, a_0, a_2, z_0$, and it can be either light-like, time-like or space-like, as $g(\dot\gamma, \dot\gamma) = a_0^2 + 4q^2 a_1^3$.
- $\gamma(s) = \big(-\sqrt{k_2 - k_1 e^{-\varepsilon q s}},\ -\varepsilon\sqrt{k_2 - k_1 e^{-\varepsilon q s}},\ a_0 s + z_0\big)$, for any real constants $a_0, k_1 \neq 0$ and k_2, z_0, and it is space-like, as $g(\dot\gamma, \dot\gamma) = a_0^2$.

We close this paper with the following.

Open Problem: to describe the magnetic curves corresponding to the magnetic field $F = qF_\xi, q \in \mathbb{R} \setminus \{0\}$, associated with the Reeb vector field ξ, of an arbitrary almost paracontact metric manifold (with $\mathrm{div}\xi = 0$).

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