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MOTION OF CHARGED PARTICLES IN A KILLING MAGNETIC FIELD IN $\mathbb{H}^2 \times \mathbb{R}$

Dedicated to Prof. Anna Maria Pastore on the occasion of her 70th birthday

Abstract. In this paper we study the equations of motion for charged particles in a magnetic field generated by the Killing vector fields of $\mathbb{H}^2 \times \mathbb{R}$. Furthermore, we obtain some explicit parametrizations of such magnetic curves.

1. Introduction

The importance of the study of the motion of charged particles in certain magnetic or electric fields is given by the good physical understanding of some dynamical processes. Usually, these fields are known as functions depending on position and time. In the following we will consider only the action of time-independent magnetic fields and vanishing electric fields, keeping the problems in the so-called magnetostatics theory.

From the point of view of Physics, according to [1], if we consider a particle of charge $\mathbf{e}$, under the action of the Lorentz force $\mathbf{F}$ generated by the magnetic field $\mathbf{B}$, the equation of motion (Lorentz equation) can be written as:

$$(*) \qquad \frac{d\mathbf{p}}{dt} = \mathbf{F} = \mathbf{e}(\mathbf{v} \times \mathbf{B}),$$

where $\mathbf{p}$ denotes the momentum of the particle and $\mathbf{v}$ is its velocity vector. We know that the momentum $\mathbf{p}$ is collinear to $\mathbf{v}$, more precisely we have the relation:

$$(**) \qquad \mathbf{p} = m\,\mathbf{v} \Big/ \sqrt{1 - \frac{v^2}{c^2}},$$

where $v = |\mathbf{v}|$ and c is the light speed. Because of equations $(*)$ and $(**)$ we observe that $\frac{d\mathbf{p}}{dt}$ is orthogonal to $\mathbf{p}$ and therefore $|\mathbf{p}|$ is constant. Consequently, the velocity v and the energy $\varepsilon = mc^2 / \sqrt{1 - \frac{v^2}{c^2}}$ are both constant. Inserting $(**)$ in $(*)$, the equation of motion becomes $\dfrac{d\mathbf{v}}{dt} = \phi(\mathbf{v}(t))$, where ϕ is a skew-symmetric operator such that $\phi(\mathbf{v})$ is orthogonal to $\mathbf{v}$.

This problem can be formulated in Riemannian geometry as follows. On a Riemannian manifold $(M, \tilde{g})$ a magnetic field is defined as a closed 2-form F. The corresponding Lorentz force ϕ is a skew-symmetric operator given by $\tilde{g}(\phi(X), Y) = F(X,Y)$, for all $X, Y \in \mathfrak{X}(M)$. Notice that F and ϕ are metrically equivalent, and moreover, they are physically equivalent, because ϕ is obtained from F raising its second index [1]. A magnetic trajectory is defined as a smooth curve γ on M which satisfies the Lorentz equation $\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = \phi(\dot{\gamma})$, where $\tilde{\nabla}$ is the Levi-Civita connection on M. If we deal with a null Lorentz force, the magnetic trajectories are the geodesics of the ambient space. Hence, the geodesics can be regarded as magnetic trajectories described by particles moving freely, only under the action of gravity.

The aim of this paper is to investigate the equations of motion for charged particles under the action of Killing magnetic fields in the homogeneous 3-space $\mathbb{H}^2 \times \mathbb{R}$, in order to complete the study started for such magnetic fields in the Euclidean 3-space [3] and in the product space $\mathbb{S}^2 \times \mathbb{R}$ [7].

2. Preliminaries

Let $\gamma : I \subset \mathbb{R} \to \mathbb{H}^2 \times \mathbb{R}$ be a smooth curve in the product space $\mathbb{H}^2 \times \mathbb{R}$, where $\mathbb{H}^2$ denotes the hyperbolic plane and $\mathbb{R}$ denotes the real line. In the following we work with the the upper half-plane model of the hyperbolic plane, that is we consider

$$\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2, y > 0\},$$

endowed with the metric $g_{\mathbb{H}} = \frac{dx^2+dy^2}{y^2}$, of constant Gaussian curvature -1. It is well known that the geodesics in $\mathbb{H}^2$ are half-circles perpendicular to the x-axis and straight vertical half-lines with the end on the x-axis. In these notations, the ambient space is given as the Riemannian product of the hyperbolic plane $(\mathbb{H}^2(-1), g_{\mathbb{H}})$ and the one dimensional Euclidean space endowed with the usual metric. The metric on the ambient space is therefore given by $\widetilde{g} = g_{\mathbb{H}} + dt^2$, where t is the global coordinate on $\mathbb{R}$. Denote by $\widetilde{\nabla}$ its corresponding Levi-Civita connection, whose non-vanishing Christoffel symbols are:

$$\widetilde{\Gamma}^1_{12} = \widetilde{\Gamma}^1_{21} = \widetilde{\Gamma}^2_{22} = -1/y, \quad \widetilde{\Gamma}^2_{11} = 1/y. \tag{A.1}$$

Setting s the arc-length parameter on γ, the curve is parametrized, in local coordinates (x,y,t), as

$$\gamma(s) = (x(s), y(s), t(s)), \tag{A.2}$$

such that

$$\frac{\dot{x}^2 + \dot{y}^2}{y^2} + \dot{t}^2 = 1. \tag{A.3}$$

Recall that the curve γ is a geodesic in $(\mathbb{H}^2 \times \mathbb{R}, \widetilde{g}, \widetilde{\nabla})$ if its velocity vector field is parallel along γ, namely

$$\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = 0. \tag{A.4}$$

Using the expressions of the Christoffel symbols from (A.1) we get that the components of γ satisfy the following system of ordinary differential equations:

$$\begin{cases} \ddot{x} - \frac{2\dot{x}\dot{y}}{y} = 0, \\ \ddot{y} + \frac{\dot{x}^2 - \dot{y}^2}{y} = 0, \\ \ddot{t} = 0, \end{cases} \tag{A.5}$$

with the initial conditions $x(0) = x_0,\ y(0) = y_0,\ t(0) = t_0,\ \dot{x}(0) = u_0,\ \dot{y}(0) = v_0,$ $\dot{t}(0) = \zeta_0$.

Solving this Cauchy problem, the explicit parametrizations of geodesics in $\mathbb{H}^2 \times \mathbb{R}$ are obtained as follows.

THEOREM 1. *The geodesics of the product space $\mathbb{H}^2 \times \mathbb{R}$ can be explicitly parametrized as:*

(a) $\gamma(s) = (x_0,\ y_0 e^{s\sqrt{1-\zeta_0^2}},\ \zeta_0 s + t_0)$,

(b) $\gamma(s) = (-d + r\cos\lambda(s),\ r\sin\lambda(s),\ \zeta_0 s + t_0)$,

where $r^2 := \frac{1-\zeta_0^2}{c^2}$, *with* $c \in \mathbb{R} \setminus \{0\}$, $\lambda(s) = 2\arctan(\delta e^{s\sqrt{1-\zeta_0^2}})$, *with* $\delta = \tan\frac{\lambda_0}{2}$. *The numbers* λ_0 *and* d *satisfy the initial conditions* $y_0 = r\sin\lambda_0$ *and* $d = y_0\cot\lambda_0 - x_0$.

Proof. From the last equation of (A.5) we immediately get

$$t(s) = \zeta_0 s + t_0,\ \ \zeta_0, t_0 \in \mathbb{R}. \tag{A.6}$$

In order to determine the other two coordinate functions, that is x and y, we distinguish two cases.

Case 1. $\dot{x} = 0$. Thus, $x(s) = x_0$, $x_0 \in \mathbb{R}$. Replacing x in the second equation we find that $\frac{\ddot{y}y - \dot{y}^2}{y}$ vanishes or, equivalently, $\frac{\dot{y}}{y}$ is constant. Now, taking into account that s is the arc-length parameter, and using (A.6), we obtain that $y(s)$ satisfies the ordinary differential equation $\frac{\dot{y}}{y} = \pm\sqrt{1-\zeta_0^2}$. It follows that $y(s) = y_0 e^{s\sqrt{1-\zeta_0^2}}$, concluding the proof of statement (a) in the theorem.

Case 2. $\dot{x} \neq 0$. From the first equation of (A.5) we get

$$\dot{x} = cy^2,\ c \in \mathbb{R}^*. \tag{A.7}$$

Combining this relation with the second equation of (A.5) we get

$$\frac{d}{ds}\left(\frac{\dot{y}}{y}\right) = -c\dot{x}. \tag{A.8}$$

Integrating once with respect to s, we have $\frac{\dot{y}}{y} = -c(x+d)$, $d \in \mathbb{R}$. Computing the square of this equality and taking into account the arc-length parametrization condition, as well as equations (A.6) and (A.7), we obtain a relation between the x and y coordinates, as follows

$$y^2 + (x+d)^2 = r^2,\ \ r^2 := \frac{1-\zeta_0^2}{c^2}.$$

Hence, there exists a function $\lambda(s)$ such that

$$x(s) = -d + r\cos\lambda(s), \quad y(s) = r\sin\lambda(s). \tag{A.9}$$

Again, as s is the arc-length, we get, up to the orientation of γ, that

$$\lambda(s) = 2\arctan(\delta e^{s\sqrt{1-\zeta_0^2}}), \tag{A.10}$$

where $\delta = \tan\frac{\lambda_0}{2}$ and λ_0 satisfies the initial conditions $x_0 = -d + r\cos\lambda_0,\ y_0 = r\sin\lambda_0$. Moreover, from these relations we get $d = y_0\cot\lambda_0 - x_0$. Hence, item (b) of the theorem is proved. $\square$

The geodesics of $\mathbb{H}^2 \times \mathbb{R}$, thought as a particular case of Bianchi-Cartan-Vrănceanu spaces, were described in [8]. It is important to observe that the hyperbolic plane $\mathbb{H}^2$ was modelled by the Poincaré disk.

3. Magnetic curves in $\mathbb{H}^2 \times \mathbb{R}$

Recall that a *magnetic field* may be regarded as a closed 2-form F and the Lorentz force corresponding to it in the ambient space $(\mathbb{H}^2 \times \mathbb{R}, \widetilde{g}, \widetilde{\nabla})$ is a $(1,1)-$type tensor field ϕ given by:

$$\widetilde{g}(\phi(X), Y) = F(X,Y), \tag{A.11}$$

where X, Y are vector fields in $\mathbb{H}^2 \times \mathbb{R}$. Moreover, the *magnetic trajectories* of the magnetic flow generated by the magnetic field F are given by those curves γ which are the solutions of the *Lorentz equation*:

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \phi(\dot\gamma). \tag{A.12}$$

As $\mathbb{H}^2 \times \mathbb{R}$ is a 3-dimensional manifold, 2-forms may be identified with the corresponding vector fields using the Hodge operator and the volume form. Hence, magnetic fields can be thought as *divergence free* vector fields. An important class is obtained when a Killing vector field generates the magnetic field. In this situation, the magnetic trajectories are known as *Killing magnetic curves*.

Let us fix a Killing vector field V. If F_V is the magnetic field corresponding to V, then the Lorentz force of the magnetic background $(\mathbb{H}^2 \times \mathbb{R}, \widetilde{g}, F_V)$ may be expressed in terms of the cross product $\wedge$ on $\mathbb{H}^2 \times \mathbb{R}$,

$$\phi(X) = V \wedge X \text{ for any } X \in \mathfrak{X}(\mathbb{H}^2 \times \mathbb{R}), \tag{A.13}$$

and the Lorentz equation (A.12) becomes

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma = V \wedge \dot\gamma. \tag{A.14}$$

In order to define a cross product on $\mathbb{H}^2 \times \mathbb{R}$, from the parametrization (A.2) we compute first

$$\dot\gamma = \dot x\frac{\partial}{\partial x} + \dot y\frac{\partial}{\partial y} + \dot t\frac{\partial}{\partial t},$$

and since $\frac{\partial}{\partial x}\perp\frac{\partial}{\partial y}$, we can choose an orthonormal basis $\left\{e_1=y\dfrac{\partial}{\partial x},\ e_2=y\dfrac{\partial}{\partial y},\ \dfrac{\partial}{\partial t}\right\}$ such that the cross product $\wedge$ is defined as follows:

$$e_1\wedge e_2:=\frac{\partial}{\partial t},\ \ \frac{\partial}{\partial t}\wedge e_1:=e_2,\ \ \frac{\partial}{\partial t}\wedge e_2:=-e_1. \tag{A.15}$$

Hence, using the definition (A.15), we may easily find:

$$\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}=\frac{1}{y^2}\frac{\partial}{\partial t},\ \ \frac{\partial}{\partial t}\wedge\frac{\partial}{\partial x}=\frac{\partial}{\partial y},\ \ \frac{\partial}{\partial t}\wedge\frac{\partial}{\partial y}=-\frac{\partial}{\partial x}. \tag{A.16}$$

In this section we explicitly determine the normal magnetic curves associated to Killing vector fields in $\mathbb{H}^2\times\mathbb{R}$, where we use the upper half-plane model of $\mathbb{H}^2$. Recall that a basis of Killing vector fields in $\mathbb{H}^2\times\mathbb{R}$ [6] is given by:

$$\left\{\frac{\partial}{\partial t},\ \frac{\partial}{\partial x},\ \frac{x^2-y^2+1}{2}\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y},\ x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right\}.$$

In the next two subsections we will consider the translational Killing vector fields, $V_1=\frac{\partial}{\partial t}$ and $V_2=\frac{\partial}{\partial x}$ respectively, in order to classify the magnetic curves corresponding to the magnetic fields defined by V_1 and V_2.

A.1. Magnetic curves associated with the Killing vector field $V_1=\frac{\partial}{\partial t}$

The Killing magnetic field F_1 corresponding to the Killing vector field $V_1=\frac{\partial}{\partial t}$ can be expressed as

$$F_1\left(\frac{\partial}{\partial t},-\right)=0\quad\text{and}\quad F_1\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)=\frac{1}{y^2}.$$

This is a consequence of (A.11), (A.13) and (A.16).

In the following we give the explicit parametrization of Killing magnetic curves generated by the magnetic field F_1.

THEOREM 2. *Let $\gamma: I\subset\mathbb{R}\to\mathbb{H}^2\times\mathbb{R}$ be a smooth curve parametrized by arc-length. The curve γ is the Killing magnetic curve corresponding to the magnetic field defined by the Killing vector field $V_1=\frac{\partial}{\partial t}$ if and only if it is parametrized as:*

(a) a geodesic line $\gamma(s)=(x_0,y_0,t_0\pm s)$ in a point $(x_0,y_0)\in\mathbb{H}^2$,

(b) an Euclidean line $\gamma(s)=(x_0\pm sy_0,y_0,t_0)$ parallel to $Ox-$axis,

(c) a helix $\gamma(s)=\left(x_0+\dfrac{y_0(1+\sin\theta)\tan\theta\sin(s\cos\theta)}{1+\sin\theta\cos(s\cos\theta)},\ \dfrac{y_0(1+\sin\theta)}{1+\sin\theta\cos(s\cos\theta)},\ s\cos\theta+t_0\right)$,

(d) a hyperbolic circle $\gamma(s)=\left(x_0-\dfrac{sy_0}{s^2+1},\dfrac{y_0}{s^2+1},t_0\right)$, which is also an Euclidean circle centered at $(x_0,\frac{y_0}{2})$ and of radius $\frac{y_0}{2}$,

where $x_0, t_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}_+^*$ and θ represents the constant angle made by the curve γ with the real axis $\mathbb{R}$.

Proof. Replacing $V = \partial_t$ in the expression of the Lorentz equation (A.14), we get

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \partial_t \wedge \dot\gamma. \tag{A.17}$$

We have already computed the left side of (A.17), namely

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \left(\ddot x - \frac{2\dot x\dot y}{y}\right)\partial_x + \left(\ddot y + \frac{\dot x^2 - \dot y^2}{y}\right)\partial_y + \ddot t\partial_t \ .$$

Now, using the formulas (A.16), we find the right side of (A.17), that is

$$\partial_t \wedge \dot\gamma = -\dot y\partial_x + \dot x\partial_y \ .$$

It follows that the magnetic curve $\gamma = (x, y, t)$ satisfies the following system of ordinary differential equations:

$$\begin{cases} \ddot x - \frac{2\dot x\dot y}{y} = -\dot y, \\ \ddot y + \frac{\dot x^2 - \dot y^2}{y} = \dot x, \\ \ddot t = 0. \end{cases} \tag{A.18}$$

Our purpose is to solve this system.

The condition (A.3) yields that there exists a function $\theta(s) \in [0, \pi]$ such that $\dot t = \cos\theta(s)$ and $\frac{\dot x^2 + \dot y^2}{y^2} = \sin^2\theta(s)$. From the last equation in (A.18) we see that $\dot t$ is constant; thus, the function θ is constant. Hence, the expression of the third coordinate of γ is given by

$$t(s) = s\cos\theta + t_0, \quad t_0 \in \mathbb{R}. \tag{A.19}$$

Notice that θ represents the constant angle made by the unit tangent to the curve $\dot\gamma$ and the real line ∂_t.

The first two coordinates of γ are related by the condition

$$\dot x^2 + \dot y^2 = y^2\sin^2\theta. \tag{A.20}$$

In this equation we easily observe that there exists a smooth function $\alpha(s) \in [0, 2\pi)$ such that

$$\dot x(s) = y(s)\sin\theta\cos\alpha(s), \tag{A.21a}$$

$$\dot y(s) = y(s)\sin\theta\sin\alpha(s). \tag{A.21b}$$

Taking the second derivative in (A.21a) and (A.21b) and replacing all these expressions in the first two equations of (A.18), we get:

$$\sin\theta\sin\alpha(s)\dot\alpha(s) + \sin^2\theta\sin\alpha(s)\cos\alpha(s) = \sin\theta\sin\alpha(s), \tag{A.22a}$$

$$\sin\theta\cos\alpha(s)\dot\alpha(s) + \sin^2\theta\cos^2\alpha(s) = \sin\theta\cos\alpha(s). \tag{A.22b}$$

At this point we distinguish several cases for the angle θ and the function $\alpha(s)$.

(a) $\sin\theta = 0$. From (A.21a) and (A.21b) we get that $\dot{x} = \dot{y} = 0$, hence $x(s) = x_0$, $y(s) = y_0$, where $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}^*_+$. From (A.19) it follows that the last component of γ is given by
$t(s) = \pm s + t_0$, proving item (a) of the theorem.

(b) $\cos\theta \neq 0$, $\sin\theta \neq 0$ and $\sin\alpha(s) = 0$, which implies that $\cos\alpha(s)\dot{\alpha}(s) = 0$, and from (A.22b) we have that $\sin\theta = \cos\alpha(s) = 1$, thus $\cos\theta = 0$. Hence, from (A.19) we have $t(s) = t_0$, $t_0 \in \mathbb{R}$, from (A.21b) $y(s) = y_0$, $y_0 \in \mathbb{R}^+$, and from (A.21a) $x(s) = sy_0 + x_0$, $x_0 \in \mathbb{R}$, proving item (b).

Notice that the situation $\sin\theta \neq 0,\ \cos\alpha(s) = 0$ yields a contradiction.

(c) In the general situation $\cos\theta \neq 0$, $\sin\theta \neq 0$, $\sin\alpha(s) \neq 0$ and $\cos\alpha(s) \neq 0$ we get that α is a solution for the ordinary differential equation:

$$\dot{\alpha}(s) = 1 - \sin\theta\cos\alpha(s). \tag{A.23}$$

Integrating (A.23) and for an appropriate choice of the integration constant, we obtain the expression of the function α:

$$\alpha(s) = 2\arctan\left(\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\tan\frac{s\cos\theta}{2}\right). \tag{A.24}$$

Replacing the expression (A.24) of α in (A.21b) and solving the ordinary differential equation, we get the coordinate function $y(s)$ of γ as:

$$y(s) = \frac{y_0(1+\sin\theta)}{1+\sin\theta\cos(s\cos\theta)}. \tag{A.25}$$

Next, computing the expression of $\cos\alpha$ using (A.24) and replacing it together with the expression (A.25) of y in (A.21a) and solving the obtained ordinary differential equation in x, we get (when $\cos\theta \neq 0$)

$$x(s) = x_0 + \frac{y_0(1+\sin\theta)\tan\theta\sin(s\cos\theta)}{1+\sin\theta\cos(s\cos\theta)}. \tag{A.26}$$

Thus, combining (A.19), (A.25) and (A.26), case (c) of the theorem is proved.

(d) Finally, we study the remained case $\cos\theta = 0$. Consequently, we get $t = t_0$ with $t_0 \in \mathbb{R}$ and $\dot{x}^2 + \dot{y}^2 = y^2$. As before, there exists a function α such that

$$\dot{x}(s) = y(s)\cos\alpha(s), \tag{A.27a}$$

$$\dot{y}(s) = y(s)\sin\alpha(s). \tag{A.27b}$$

In a similar manner as in the previous situation, we get that α satisfies the ordinary differential equation

$$\dot{\alpha}(s) = 2\sin^2\left(\frac{\alpha(s)}{2}\right). \tag{A.28}$$

If $\sin\alpha = 0$, then item (b) of the theorem is obtained again.

When $\sin\alpha \neq 0$, we find the solution of (A.28), $\alpha(s) = -2\arctan\left(\frac{1}{s-s_0}\right)$, where s_0 is such that $\tan\frac{\alpha(0)}{2} = \frac{1}{s_0}$. Solving the ordinary differential equations (A.27a) and (A.27b), one gets the parametrization of γ from item (d), concluding the direct implication.

The converse part is a simple verification of the fact that the parametrizations from items (a)-(d) define magnetic curves, namely they are solutions of the Lorentz equation (A.17).

□

These magnetic curves may be characterized in the following manner:

THEOREM 3. *The non-geodesic Killing magnetic curves in the magnetic background* $(\mathbb{H}^2\times\mathbb{R},\widetilde{g},F_1)$ *are the geodesics of the cylindrical surface* $C\times\mathbb{R}$*, where the curve* C *represents a hyperbolic circle of curvature* $\bar{\kappa} = \pm\frac{1}{\sin\theta}$ *in* $\mathbb{H}^2$.

For similar results see e.g. [2, 4, 5].

A.2. Magnetic curves associated with the Killing vector field $V_2 = \frac{\partial}{\partial x}$

The Killing magnetic field F_2 corresponding to the Killing vector field $V_2 = \frac{\partial}{\partial x}$ is given by

$$F_2\left(\frac{\partial}{\partial x},-\right) = 0 \quad \text{and} \quad F_2\left(\frac{\partial}{\partial y},\frac{\partial}{\partial t}\right) = \frac{1}{y^2}\ .$$

The equations of motion of charged particles evolving in the magnetic background $(\mathbb{H}^2\times\mathbb{R},\widetilde{g},F_2)$ corresponding to Killing magnetic vector field V_2 are given in the following.

THEOREM 4. *Let* $\gamma : I\subset\mathbb{R}\to\mathbb{H}^2\times\mathbb{R}$ *be a smooth curve parametrized by arc-length. The curve* γ *is Killing magnetic curve corresponding to the magnetic field defined by the Killing vector field* $V_2 = \frac{\partial}{\partial x}$ *if it satisfies the following equations:*

$$\text{(A.29)}\qquad \begin{cases} \dot{x}(s) = \beta y^2(s), \\ \dot{t}(s) = -\frac{1}{y(s)} + \alpha, \\ \dot{y}^2(s) + \beta^2 y^4(s) + (\alpha^2-1)y^2(s) - 2\alpha y(s) + 1 = 0, \quad \alpha,\beta\in\mathbb{R}. \end{cases}$$

Proof. We proceed in the same manner as in the previous section. Computing the cross product $\frac{\partial}{\partial x}\wedge\dot{\gamma}$ we obtain

$$V_2\wedge\dot{\gamma} = \frac{\dot{y}}{y^2}\frac{\partial}{\partial t} - \dot{t}\frac{\partial}{\partial y}\ .$$

Then, the Lorentz equation

$$\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = V_2\wedge\dot{\gamma},$$

becomes

$$\text{(A.30)} \qquad \begin{cases} \ddot{x} - \frac{2\dot{x}\dot{y}}{y} = 0, \\ \ddot{y} + \frac{\dot{x}^2 - \dot{y}^2}{y} + \dot{t} = 0, \\ \ddot{t} - \frac{\dot{y}}{y^2} = 0. \end{cases}$$

Combining (A.30) with the arc-length parametrization condition (A.3) we get the formulas (A.29).

□

Final remarks. In order to determine explicitly the expression of the y-coordinate, we have to make use of elliptic functions, in the same manner as in [7].

To close this section, let us discuss some particular cases involving the integration constants α and β.

Case A. $\beta = 0$, or equivalently $\dot{x} = 0$. Hence $x(s) = x_0$, $x_0 \in \mathbb{R}$. The coordinate y satisfies the following ordinary differential equation:

$$\text{(A.31)} \qquad \dot{y}^2(s) + (\alpha^2 - 1)y^2(s) - 2\alpha y(s) + 1 = 0.$$

It is clear that for $\alpha \leq -1$ there is no solution (since $y > 0$). For $\alpha > -1$ we distinguish three situations, namely

- if $|\alpha| < 1$, then $y(s) = \frac{1}{1-\alpha^2}\Big(\cosh(s\sqrt{1-\alpha^2}) - \alpha\Big)$; hence $y(s) \geq \frac{1}{1+\alpha}$; we find $t(s) = \alpha\, s - 2\arctan\left(\sqrt{\frac{1+\alpha}{1-\alpha}} \tanh\frac{s\sqrt{1-\alpha^2}}{2}\right)$;
- if $\alpha = 1$, then $y(s) = \frac{1}{2}\left(1 + (s-s_0)^2\right)$ and $t(s) = s - \sqrt{2}\arctan\Big(\sqrt{2}(s-s_0)\Big)$, $s_0 \in \mathbb{R}$;
- if $\alpha > 1$, then $y(s) = \frac{1}{\alpha^2-1}\Big(\alpha - \cos(s\sqrt{\alpha^2-1})\Big)$; hence y is bounded, namely $y(s) \in \left(\frac{1}{\alpha+1}\,, \frac{1}{\alpha-1}\right)$ for all s; then $t(s) = \alpha\, s - 2\arctan\left(\sqrt{\frac{\alpha+1}{\alpha-1}} \tan\frac{s\sqrt{\alpha^2-1}}{2}\right)$.

Case B. $\beta \neq 0$, $\alpha = 0$. The function y satisfies the equation

$$\text{(A.32)} \qquad \dot{y}^2(s) + \beta^2 y^4(s) - y^2(s) + 1 = 0.$$

Obviously, if $\beta^2 > \frac{1}{4}$ there is no solution. If $\beta^2 = \frac{1}{4}$ then $y(s) = \sqrt{2}$. The other two components are $t(s) = t_0 - \frac{s}{\sqrt{2}}$ and $x(s) = x_0 \pm s$. When $0 < \beta^2 < \frac{1}{4}$ the solution involves elliptic functions.

Case C. Let us look for horizontal magnetic curves, that is for which $t = t_0$, constant. We immediately deduce that $y(s) = \frac{1}{\alpha}$ and hence we should have $\alpha > 0$. Using the first equation in (A.29) we find the expression of x, that is $x(s) = x_0 + \frac{\beta}{\alpha^2}\, s$. The last equations lead to a relation between α and β, namely $\alpha^2 = \beta^2$.

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