Geom. Struc. on Riem. Man.-Bari
Vol. 73/1, 3–4 (2015), 151 – 159

P. Mutlu and Z. Şentürk

ON WALKER TYPE IDENTITIES LOCALLY CONFORMAL KAEHLER SPACE FORMS

Abstract. The notion of locally conformal Kaehler manifold (l.c.K-manifold) in Hermitian Geometry has been introduced by I. Vaisman in 1976. In this work, we present results on l.c.K-space forms satisfying curvature identities named Walker type identities.

1. Introduction

Let (M,g,J) be a real $m=2n$-dimensional Hermitian manifold with the structure (J, g), where J is the almost complex structure and g is the Hermitian metric. Then

$$J^2=-Id, \qquad g(JX,JY)=g(X,Y),$$

for any vector fields X and Y tangent to M. The fundamental 2-form Ω is defined by

$$\Omega(X,Y)=g(JX,Y)=-\Omega(Y,X).$$

The manifold M is called a *locally conformal Kaehler manifold (an l.c.K-manifold)* if each point x in M has an open neighborhood U with a positive differentiable function $\rho : U \to \mathbb{R}$ such that

$$g^* = e^{-2\rho} g \mid_{\mathrm{U}}$$

is a Kaehlerian metric on U. Especially, if we can take $U=M$, then the manifold M is said to be *globally conformal Kaehler*.

A Hermitian manifold whose metric is locally conformal to a Kaehler metric is called an l.c.K-manifold. I. Vaisman gives its characterization as follows [10] :

A Hermitian manifold M is an l.c.K-manifold if and only if there exists on M a global closed 1-form α such that

$$d\Omega = 2\alpha \wedge \Omega\ ,$$

where α is called the *Lee form*.

A Hermitian manifold (M,g,J) is an l.c.K-manifold if and only if

$$\text{(A.1)} \qquad \nabla_k J_{ij} = -\beta_i g_{kj} + \beta_j g_{ki} - \alpha_i J_{kj} + \alpha_j J_{ki}\ ,$$

where

$$\beta_j = -\alpha_r J^r_j.$$

An l.c.K-manifold M is called *an l.c.K-space form* if the holomorphic sectional curvature of the section $\{X,JX\}$ at each point of M has a constant value. Let M(c) be an l.c.K-space form with constant holomorphic sectional curvature c, then the Riemannian curvature tensor R_{ijhk} with respect to g_{ij} has the form [8]

$$\begin{aligned} R_{hijk} &= \frac{c}{4}(g_{hk}g_{ij} - g_{hj}g_{ik} + J_{hk}J_{ij} - J_{hj}J_{ik} - 2J_{hi}J_{jk}) \\ &+ \frac{3}{4}(P_{hk}g_{ij} + P_{ij}g_{hk} - P_{hj}g_{ik} - P_{ik}g_{hj}) \\ &- \frac{1}{4}(\widetilde{P}_{hk}J_{ij} + \widetilde{P}_{ij}J_{hk} - \widetilde{P}_{hj}J_{ik} - \widetilde{P}_{ik}J_{hj} - 2\widetilde{P}_{hi}J_{jk} - 2\widetilde{P}_{jk}J_{hi}), \end{aligned} \tag{A.2}$$

where $\widetilde{P}_{ij} = -P_{ir}J_j^r$,

$$P_{ij} = -\nabla_i\alpha_j - \alpha_i\alpha_j + \frac{\|\alpha\|^2}{2}g_{ij} \tag{A.3}$$

is hybrid, *i.e.*,$P_{ir}J_j^r + P_{jr}J_i^r = 0$, $P_{ij} = P_{ji}$ and $\|\alpha\|$ denotes the length of Lee form.
Contracting (A.2) with g^{hk}, we have

$$S_{ij} = \frac{1}{4}\{(m+2)c + 3\,trP\}g_{ij} + \frac{3}{4}(m-4)P_{ij}\ , \tag{A.4}$$

where S denotes the Ricci tensor with respect to g.

PROPOSITION 1. *[9] If the tensor field P is hybrid and the trace of the tensor field P is constant in a 4-dimensional l.c.K-space form M(c), then M(c) is Einstein.*

THEOREM 1. *[9] A real m-dimensional $(m \neq 4)$ l.c.K-space form $M(c)$ in which the tensor field P is hybrid and the trace of the tensor field P is constant is Einstein if and only if the tensor field P is proportional to g.*

2. Preliminaries

Let (M,g) be an n-dimensional, $n \geq 3$, semi-Riemannian connected manifold of class C^∞ with Levi-Civita connection ∇. The Ricci operator $\mathcal{S}$ is defined by $g(\mathcal{S}X,Y) = S(X,Y)$, where X, Y $\in \Xi(M)$, $\Xi(M)$ being the Lie algebra of vector fields on M.
We define the endomorphisms $X \wedge_A Y$, $\mathcal{R}(X,Y)Z$ and $\mathcal{C}(X,Y)$ of $\Xi(M)$ by

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y\ , \tag{A.5}$$

$$\mathcal{R}(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\ , \tag{A.6}$$

$$\begin{aligned} \mathcal{C}(X,Y)Z &= \mathcal{R}(X,Y)Z \\ &- \frac{1}{n-2}(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1}X \wedge_g Y)Z\ , \end{aligned} \tag{A.7}$$

respectively, where X, Y, Z $\in \Xi(M)$, A is a symmetric (0,2)-tensor, κ the scalar curvature and $[X,Y]$ is the Lie bracket of vector fields X and Y. In particular we have $(X \wedge_g Y) = X \wedge Y$.

The Riemannian-Christoffel curvature tensor R, the Weyl conformal curvature tensor C and the (0,4)-tensor G of (M,g) are defined by

$$\begin{aligned} R(X_1,X_2,X_3,X_4) &= g(\mathcal{R}(X_1,X_2)X_3,X_4)\,, \\ C(X_1,X_2,X_3,X_4) &= g(\mathcal{C}(X_1,X_2)X_3,X_4)\,, \\ G(X_1,X_2,X_3,X_4) &= g((X_1 \wedge_g X_2)X_3,X_4)\,, \end{aligned} \tag{A.8}$$

respectively, where $X_1,X_2,X_3,X_4 \in \Xi(M)$.
A tensor $\mathcal{B}$ of type (1,3) on M is said to be a *generalized curvature tensor* if

$$\begin{aligned} \sum_{X_1,X_2,X_3} \mathcal{B}(X_1,X_2)X_3 &= 0\,, \\ \mathcal{B}(X_1,X_2)+\mathcal{B}(X_2,X_1) &= 0\,, \\ B(X_1,X_2,X_3,X_4) &= B(X_3,X_4,X_1,X_2)\,, \end{aligned} \tag{A.9}$$

where $B(X_1,X_2,X_3,X_4) = g(\mathcal{B}(X_1,X_2)X_3,X_4)$.
For symmetric (0,2)-tensor E and F we define their Kulkarni-Nomizu product $E \wedge F$ by

$$\begin{aligned} (E \wedge F)(X_1,X_2,X_3,X_4) &= E(X_1,X_4)F(X_2,X_3)+E(X_2,X_3)F(X_1,X_4) \\ &- E(X_1,X_3)F(X_2,X_4)-E(X_2,X_4)F(X_1,X_3). \end{aligned}$$

For a symmetric (0,2)-tensor E and a (0,k)-tensor T, $k \geq 2$, we define their Kulkarni-Nomizu product $E \wedge T$ by [3]

$$\begin{aligned} (E \wedge T)(X_1,X_2,X_3,X_4;Y_3,...,Y_k) &= E(X_1,X_4)T(X_2,X_3,Y_3,...,Y_k) \\ &+ E(X_2,X_3)T(X_1,X_4,Y_3,...,Y_k) \\ &- E(X_1,X_3)T(X_2,X_4,Y_3,...,Y_k) \\ &- E(X_2,X_4)T(X_1,X_3,Y_3,...,Y_k). \end{aligned}$$

For symmetric (0, 2)-tensor E and F we have [6]

$$Q(E,E \wedge F) = -Q(F,\bar{E}), \tag{A.10}$$

where $\bar{E} = \frac{1}{2}E \wedge E$. We also have [7]

$$E \wedge Q(E,F) = -Q(F,\bar{E}). \tag{A.11}$$

For a (0,k)-tensor field T, $k \geq 1$, a (0,2)-tensor field A and a generalized curvature tensor $\mathcal{B}$ on (M,g), we define the tensors $B \cdot T$ and $Q(A,T)$ by

$$\begin{aligned} (B \cdot T)(X_1,...,X_k;X,Y) &= -T(\mathcal{B}(X,Y)X_1,X_2,...,X_k) \\ &- ...-T(X_1,X_2,...,X_{k-1},\mathcal{B}(X,Y)X_k), \end{aligned} \tag{A.12}$$

$$\begin{aligned} Q(A,T)(X_1,...,X_k;X,Y) &= -T((X \wedge_A Y)X_1,X_2,...,X_k) \\ &- ...-T(X_1,X_2,...,X_{k-1},(X \wedge_A Y)X_k), \end{aligned} \tag{A.13}$$

respectively, where $X, Y, X_1, X_2, ..., X_k \in \Xi(M)$.

Putting in the above formulas $\mathcal{B} = \mathcal{R}$ or $\mathcal{B} = \mathcal{C}$, $T = R$ or $T = C$ or $T = S$, $A = g$ or $A = S$, we obtain the tensors $R \cdot R$, $R \cdot C$, $R \cdot S$, $C \cdot S$, Q (g, R), Q (S, R), Q (g, C), Q (g, S) and Q (S, C) respectively.

Let (M, g) be covered by a system of charts $\{W; x^k\}$. We define by g_{ij}, R_{hijk}, S_{ij}, $G_{hijk} = g_{hk}g_{ij} - g_{hj}g_{ik}$ and

$$\begin{aligned} C_{hijk} &= R_{hijk} - \frac{1}{n-2}(g_{hk}S_{ij} - g_{hj}S_{ik} + g_{ij}S_{hk} - g_{ik}S_{hj}) \\ &+ \frac{\kappa}{(n-1)(n-2)}G_{hijk}\ , \end{aligned} \tag{A.14}$$

the local components of the metric tensor g, the Riemannian-Christoffel curvature tensor R, the Ricci tensor S, the tensor G and the Weyl tensor C, respectively. Further, we denote by $S_{ij} = S_{ir}g^r_j$ and $S^j_i = g^{jr}S_{ir}$.

The local components of the (0,6)-tensors $R \cdot T$ and Q (g, T) on M are the following:

$$(R \cdot T)_{hijklm} = g^{rs}(T_{rijk}R_{shlm} + T_{hrjk}R_{silm} + T_{hirk}R_{sjlm} + T_{hijr}R_{sklm}), \tag{A.15}$$

$$\begin{aligned} Q(g,T)_{hijklm} &= -g_{mh}T_{lijk} - g_{mi}T_{hljk} - g_{mj}T_{hilk} - g_{mk}T_{hijl} \\ &+ g_{lh}T_{mijk} + g_{li}T_{hmjk} + g_{lj}T_{himk} + g_{lk}T_{hijm}\ , \end{aligned} \tag{A.16}$$

where T_{hijk} are the local components of the tensor T.

In this part we present some considerations leading to the definition of Deszcz Symmetric (Pseudosymmetric in the sense of Deszcz) and Ricci-pseudosymmetric manifolds.

A semi-Riemannian manifold (M, g) satisfying the condition $\nabla R = 0$ is said to be *locally symmetric*. Locally symmetric manifolds form a subclass of the class of manifolds characterized by the condition

$$R \cdot R = 0. \tag{A.17}$$

Semi-Riemannian manifolds fulfilling (A.17) are called *semisymmetric*. Here $R \cdot R$ is a (0,6)-tensor with components

$$\begin{aligned} (R \cdot R)_{hijklm} &= \nabla_m \nabla_l R_{hijk} - \nabla_l \nabla_m R_{hijk} \\ &= g^{rs}(R_{rijk}R_{shlm} + R_{hrjk}R_{silm} + R_{hirk}R_{sjlm} + R_{hijr}R_{sklm}). \end{aligned} \tag{A.18}$$

A more general class of manifolds than the class of semisymmetric manifolds is the class of Deszcz Symmetric manifolds.

A semi-Riemannian manifold (M, g) is said to be *Deszcz Symmetric* [2] if at every point of M the condition

$$R \cdot R = L_R Q(g, R) \tag{A.19}$$

holds on the set $\mathcal{U}_R = \{x \in M \mid R - \frac{\kappa}{n(n-1)}G \neq 0\ at\ x\}$, where L_R is some function on $\mathcal{U}_R$. There exist various examples of Deszcz Symmetric manifolds which are not semisymmetric.

A semi-Riemannian manifold is said to be *Ricci-semisymmetric* if we have $R \cdot S = 0$ on M.

A semi-Riemannian manifold (M, g) is said to be *Ricci-pseudosymmetric* ([2], [5]) if at every point of M the condition

$$R \cdot S = L_S Q(g, S) \tag{A.20}$$

holds on the set $\mathcal{U}_S = \{x \in M \mid S - \frac{\kappa}{n} g \neq 0 \; at \; x\}$, where L_S is some function on $\mathcal{U}_S$. The class of Ricci-pseudosymmetric manifolds is an extension of the class of Ricci-semisymmetric manifolds as well as of the class of pseudosymmetric manifolds. Every pseudosymmetric manifold is Ricci-pseudosymmetric. The converse statement is not true. Evidently, every Ricci-semisymmetric $(R \cdot S = 0)$ is Ricci-pseudosymmetric. There exist various examples of Ricci-pseudosymmetric manifolds which are not pseudosymmetric.

3. On Walker Type Identities Locally Conformal Kaehler Space Forms

In this section, we present results on l.c.K-space forms satisfying curvature identities named Walker type identities.

LEMMA 1. *[1] For a symmetric (0,2)-tensor A and a generalized curvature tensor $\mathcal{B}$ on a semi-Riemannian manifold (M,g)*, $n \geq 3$, *we have*

$$Q(A, \mathcal{B})_{hijklm} + Q(A, \mathcal{B})_{jklmhi} + Q(A, \mathcal{B})_{lmhijk} = 0. \tag{A.21}$$

It is well-known that the following identity

$$(R \cdot R)_{hijklm} + (R \cdot R)_{jklmhi} + (R \cdot R)_{lmhijk} = 0 \tag{A.22}$$

holds on any semi-Riemannian manifold. The equation (A.22) is called *the Walker type identity*.

On any semi-Riemannian manifold (M, g), $n \geq 4$, the following three identities are equivalent to each other [4]:

$$(R \cdot C)_{hijklm} + (R \cdot C)_{jklmhi} + (R \cdot C)_{lmhijk} = 0, \tag{A.23}$$

$$(C \cdot R)_{hijklm} + (C \cdot R)_{jklmhi} + (C \cdot R)_{lmhijk} = 0 \tag{A.24}$$

and

$$(R \cdot C - C \cdot R)_{hijklm} + (R \cdot C - C \cdot R)_{jklmhi} + (R \cdot C - C \cdot R)_{lmhijk} = 0. \tag{A.25}$$

The equations (A.23) - (A.25) are named *the Walker type identities*. We also can consider the following Walker type identity

$$(C \cdot C)_{hijklm} + (C \cdot C)_{jklmhi} + (C \cdot C)_{lmhijk} = 0. \tag{A.26}$$

THEOREM 2. *Let M(c) be a 4-dimensional l.c.K-space form, such that the tensor field P is hybrid and the trace of the tensor field P is constant. Then the Walker type identities (A.23) - (A.25) and (A.26) hold on M(c).*

Proof. In view of (A.15) , we have

$$(R\cdot C)_{hijklm}=g^{rs}(C_{rijk}R_{shlm}+C_{hrjk}R_{silm}+C_{hirk}R_{sjlm}+C_{hijr}R_{sklm}), \tag{A.27}$$

$$(C\cdot R)_{hijklm}=g^{rs}(R_{rijk}C_{shlm}+R_{hrjk}C_{silm}+R_{hirk}C_{sjlm}+R_{hijr}C_{sklm}). \tag{A.28}$$

Using (A.14) in (A.27) we obtain

$$\begin{aligned}(R\cdot C)_{hijklm} &= (R\cdot R)_{hijklm}-\frac{1}{(m-2)}\Big[R_{hklm}S_{ij}-R_{jhlm}S_{ik}+R_{jilm}S_{hk}\\ &- R_{kilm}S_{hj}-R_{hjlm}S_{ik}+R_{ijlm}S_{hk}+R_{khlm}S_{ij}-R_{iklm}S_{hj}\\ &+ g_{ij}S^s_kR_{shlm}+g_{hk}S^s_jR_{silm}+g_{hk}S^s_iR_{sjlm}+g_{ij}S^s_hR_{sklm}\\ &- g_{ik}S^s_jR_{shlm}-g_{hj}S^s_kR_{silm}-g_{ik}S^s_hR_{sjlm}-g_{hj}S^s_iR_{sklm}\Big]\\ &+ \frac{\kappa}{(m-1)(m-2)}\Big[R_{hklm}g_{ij}-R_{jhlm}g_{ik}+R_{jilm}g_{hk}\\ &- R_{kilm}g_{hj}+R_{ijlm}g_{hk}-R_{hjlm}g_{ik}+R_{hklm}g_{ij}-R_{iklm}g_{hj}\Big]\\ &= (R\cdot R)_{hijklm}-\frac{1}{(m-2)}\Big[g_{ij}(A_{khlm}+A_{hklm})+g_{hk}(A_{jilm}+A_{ijlm})\\ &- g_{ik}(A_{jhlm}+A_{hjlm})-g_{hj}(A_{kilm}+A_{iklm})\Big],\end{aligned} \tag{A.29}$$

where

$$A_{mijk}=S^s_mR_{sijk}. \tag{A.30}$$

Applying, in the same way, (A.14) in (A.28) we get

$$\begin{aligned}(C\cdot R)_{hijklm} &= (R\cdot R)_{hijklm}-\frac{1}{(m-2)}Q(S,R)_{hijklm}\\ &+ \frac{\kappa}{(m-1)(m-2)}Q(g,R)_{hijklm}\\ &- \frac{1}{(m-2)}(g_{lh}A_{mijk}-g_{mh}A_{lijk}-g_{li}A_{mhjk}+g_{mi}A_{lhjk}\\ &+ g_{lj}A_{mkhi}-g_{mj}A_{lkhi}-g_{kl}A_{mjhi}+g_{km}A_{ljhi}).\end{aligned} \tag{A.31}$$

Substituting (A.4) into (A.29) and (A.31) we get

$$(R\cdot C) = (R\cdot R)-\frac{\beta}{(m-2)}T \tag{A.32}$$

and

$$
\begin{aligned}
(C\cdot R) &= (R\cdot R) - \frac{(m-2)\alpha - \beta\, trP}{(m-2)(m-1)} Q(g,R) - \frac{\beta}{(m-2)} Q(P,R) \\
&- \frac{\beta}{(m-2)} \acute{T}, \qquad \text{(A.33)}
\end{aligned}
$$

where T and $\acute{T}$ are the (0,6)-tensor fields whose local components are given by

$$
\begin{aligned}
T_{hijklm} &= g_{ij}(E_{khlm}+E_{hklm}) + g_{hk}(E_{jilm}+E_{ijlm}) \\
&- g_{ik}(E_{jhlm}+E_{hjlm}) - g_{hj}(E_{kilm}+E_{iklm}), \qquad \text{(A.34)}
\end{aligned}
$$

$$
\begin{aligned}
\acute{T}_{hijklm} &= \Big(g_{lh}E_{mijk} - g_{mh}E_{lijk} - g_{li}E_{mhjk} + g_{mi}E_{lhjk} \\
&+ g_{lj}E_{mkhi} - g_{mj}E_{lkhi} - g_{kl}E_{mjhi} + g_{km}E_{ljhi} \Big), \qquad \text{(A.35)}
\end{aligned}
$$

$E_{hijk} = P_h^s R_{sijk}$, $\alpha = \frac{1}{4}\{(m+2)c + 3\, trP\}$ and $\beta = \frac{3}{4}(m-4)$.
Then by direct computation, one obtains:

$$\sum_{(X_1,X_2),(X_3,X_4),(X,Y)} (T+\acute{T})(X_1,X_2,X_3,X_4,X,Y) = 0.$$

Hence (A.23) (equivalently (A.24), (A.25)) holds if and only if

$$\frac{\beta}{(m-2)} \sum_{(X_1,X_2),(X_3,X_4),(X,Y)} T(X_1,X_2,X_3,X_4,X,Y) = 0.$$

In particular, if $m = 4$, then $\beta = 0$, so (A.23) holds.
Further, we note that (A.14) turns into $C = R - \frac{2c+trP}{4} G$. This gives

$$
\begin{aligned}
C\cdot C &= C\cdot (R - \frac{2c+trP}{4} G) = C\cdot R \\
&= (R - \frac{2c+trP}{4} G)\cdot R = R\cdot R - \frac{2c+trP}{4} Q(g,R). \qquad \text{(A.36)}
\end{aligned}
$$

Now using (A.21) and (A.22) we complete the proof. □

THEOREM 3. *Let M(c) be an m-dimensional* $(m > 4)$ *l.c.K-space form. If the tensor field P is proportional to g and the trace of the tensor field P is constant, then the Walker type identities (A.23) - (A.25) and (A.26) hold on M(c).*

Proof. In view of Theorem 1., the m-dimensional l.c.K-space form $M(c)$ is Einstein, so (A.23) - (A.25) hold on $M(c)$. Substituting $S = \frac{1}{4}\Big\{(m+2)c + \frac{6(m-2)}{m} trP\Big\} g$ into (A.14), we have

$$C = R - \frac{1}{4(m-1)} \left\{(m+2)c + \frac{6(m-2)}{m} trP\right\} G.$$

This gives

$$\begin{aligned} C\cdot C &= C\cdot\Big(R-\frac{1}{4(m-1)}\Big\{(m+2)c+\frac{6(m-2)}{m}trP\Big\}\,G\Big)\\ &= C\cdot R\\ &= \Big(R-\frac{1}{4(m-1)}\Big\{(m+2)c+\frac{6(m-2)}{m}trP\Big\}\,G\Big)\cdot R\\ &= R\cdot R-\frac{1}{4(m-1)}\Big\{(m+2)c+\frac{6(m-2)}{m}trP\Big\}\,Q(g,R). \end{aligned}$$

Using (A.21) and (A.22), we get the result. □

LEMMA 2. *Let M(c) be an m-dimensional* $(m>4)$ *l.c.K-space form such that the tensor field P is hybrid. Then, we have*

$$(m-2)\Big((R\cdot C)_{hijklm}+(R\cdot C)_{jklmhi}+(R\cdot C)_{lmhijk}\Big)$$

(A.37) $$=-\beta\Big((g\wedge(R\cdot P))_{hijklm}+(g\wedge(R\cdot P))_{jklmhi}+(g\wedge(R\cdot P))_{lmhijk}\Big).$$

Proof. Substituting (A.4) into (A.14), we obtain

(A.38) $$C=R-\frac{\beta}{m-2}(g\wedge P)-\frac{\alpha(m-2)-\beta\,trP}{(m-1)(m-2)}G,$$

where $\alpha=\frac{1}{4}\{(m+2)c+3\,trP\}$ and $\beta=\frac{3}{4}(m-4)$.
From (A.38), we get

(A.39) $$R\cdot C=R\cdot R-\frac{\beta}{m-2}g\wedge(R\cdot P).$$

Using (A.22) the proof is completed. □

LEMMA 3. *If one of the Walker type identities (A.23) - (A.25) holds on an m-dimensional* $(m>4)$ *l.c.K-space form M(c) and the tensor field P is hybrid, then on M(c) we have*

(A.40) $$(g\wedge(R\cdot P))_{hijklm}+(g\wedge(R\cdot P))_{jklmhi}+(g\wedge(R\cdot P))_{lmhijk}=0.$$

Proof. Lemma 2. completes the proof. □

THEOREM 4. *On any Ricci-pseudosymmetric l.c.K-space form M(c),* $(m>4)$ *such that the tensor field P is hybrid, the Walker type identities (A.23) - (A.25) hold on* $\mathcal{U}_s\subset M$.

Proof. In view of (A.20) and (A.4), m-dimensional Ricci-pseudosymmetric $(m>4)$ l.c.K-space forms satisfy

(A.41) $$R\cdot P=L_S Q(g,P).$$

Using (A.41) and (A.37), we obtain the following identity on $\mathcal{U}_S$

$$(m-2)\Big((R\cdot C)_{hijklm}+(R\cdot C)_{jklmhi}+(R\cdot C)_{lmhijk}\Big)$$
$$=-\beta L_S\Big((g\wedge Q(g,P))_{hijklm}+(g\wedge Q(g,P))_{jklmhi}+(g\wedge Q(g,P))_{lmhijk}\Big).$$

Making use of (A.11) and (A.21), we obtain on $\mathcal{U}_S$

$$(m-2)\Big((R\cdot C)_{hijklm}+(R\cdot C)_{jklmhi}+(R\cdot C)_{lmhijk}\Big)$$
$$=-\beta L_S\Big(Q(P,G)_{hijklm}+Q(P,G)_{jklmhi}+Q(P,G)_{lmhijk}\Big)=0.$$

Hence (A.23) (equivalently (A.24), (A.25)) holds on M(c). □

Acknowledgements. The authors would like to thank the referee for his/her valuable comments.

References

[1] DEPREZ J., DESZCZ R., VERSTRAELEN L., *Examples of pseudo-symmetric conformally flat warped products*, Chinese J. Math. **17** (1989), 51–65.

[2] DESZCZ R., *On pseudosymmetric spaces*, Bull. Soc. Math. Belg. Sér. **A 44** (1992), 1–34.

[3] DESZCZ R. AND GŁOGOWSKA M., *Some nonsemisymmetric Ricci-semisymmetric warped product hypersurfaces*, Publ. Inst. Math.(Beograd) (N.S.) **72** 86 (2002), 81–93.

[4] DESZCZ R., GŁOGOWSKA M., VERSTRAELEN L., *On some generalized Einstein metric conditions on hypersurfaces in semi-Riemannian space forms*, Colloq. Math. **96** (2003), 149–166.

[5] DESZCZ R. AND HOTLOS M., *Remarks on Riemannian manifolds satisfying a certain curvature condition imposed on the Ricci tensor*, Prace. Nauk. Pol. Szczec. **11** (1989), 23–34.

[6] DESZCZ R., VERSTRAELEN L., VRANCKEN L., *On the symmetry of warped product spacetimes*, Gen. Relativity Gravitation **23** (1991), 671–681.

[7] DESZCZ R. AND VERSTRAELEN L., *Hypersurfaces of semi-Riemannian conformally flat manifolds*, Geometry and Topology of Submanifolds, III, World Sci., River Edge, NJ 1991, 131–147.

[8] KASHIWADA T., *Some properties of locally conformal Kaehler manifolds*, Hokkaido Math. J. **8** (1978), 191–198.

[9] MATSUMOTO K., *Locally conformal Kaehler manifolds and their submanifolds*, Mem. Secţ. Ştiinţ. Acad. Română Ser. IV **14** 2 (1991), 7–49.

[10] VAISMAN I., *On locally conformal almost Kaehler manifolds*, Israel J. of Math. **24** (1976), 338–351.

AMS Subject Classification: 53B20, 53B25, 53B30, 53B50, 53C25

Pegah MUTLU, Zerrin ŞENTÜRK ,
Mathematics Engineering Department, Istanbul Technical University
Maslak, TR-34469, Istanbul, TURKEY
e-mail: `sariaslani@itu.edu.tr, senturk@itu.edu.tr`

Lavoro pervenuto in redazione il 13.11.2015 e accettato il 16.02.2016.