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RICCI NILSOLITONS ASSOCIATED TO GRAPHS AND EDGE-COLOURING

Abstract. We provide a method to attach to every simple graph a 2-step nilpotent Ricci nilsoliton.

1. Introduction

The purpose of this note is to provide a new method to attach to every simple graph a 2-step nilpotent *Ricci nilsoliton*, namely a simply connected 2-step nilpotent Lie group $(\mathcal{N}, g)$ endowed with a left-invariant metric g whose Ricci operator $Q : \mathfrak{n} \to \mathfrak{n}$ satisfies

$$Q = cI + D \text{ for some } c \in \mathbb{R} \text{ and } D \in Der(\mathfrak{n}), \tag{1}$$

where $\mathfrak{n}$ is the Lie algebra of $\mathcal{N}$.
Lauret proved that there is at most one Ricci soliton metric on a nilpotent Lie group, up to isometry and scaling [8, Theorem 3.5]. Ricci nilsolitons are noteworthy in connection with the problem of classifying the *homogeneous Einstein* Riemannian manifolds; indeed, the metric nilpotent Lie algebras $(\mathfrak{n}, <,>)$ satisfying (1) are exactly the nilradicals $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ of the standard metric Einstein solvable Lie algebras $\mathfrak{s}$ (endowed with the induced metric), see [8, Theorem 3.7]. For this reason, they are also called *Einstein nilradicals*. For more information and a list of recent relevant results concerning the classification of Einstein nilradicals see [9].
In [10] it is showed that the natural 2-step nilpotent Lie algebra $\mathfrak{n}_G$ attached to a simple graph G admits a Ricci soliton metric if and only if the graph is *positive*, i.e., all the entries of the vector

$$(3I + \mathrm{Adj}(L(G)))^{-1}(1, \dots, 1)$$

are positive; here $\mathrm{Adj}(L(G))$ is the adjacency matrix of the line graph of G. For instance, regular graphs are positive. This criterion has been exploited in [11] in order to prove that for every pair (p,q) of integers with $q \geq 21$ and $q-1 \leq p \leq \frac{1}{2}q^2 - \frac{5}{2}q + 9$, there do exist indecomposable 2-step nilpotent Lie algebras $\mathfrak{n}$ of type (p,q) which are *not* Einstein nilradicals, where type (p,q) means that $\dim \mathfrak{n} = p+q$ and $\dim [\mathfrak{n}, \mathfrak{n}] = p$.
We recall that $\mathfrak{n}_G$ is defined in a very simple way as follows:

$$\mathfrak{n}_G = V_0 \oplus W_0,$$

where V_0 is the real vector space of dimension $n =$ order of G spanned by the vertices $v_1, \dots, v_n$ of G and W_0 is the vector space of dimension $l =$ size of G, spanned by the

[1] This paper is dedicated to Professor Anna Maria Pastore on the occasion of her 70th birthday.

edges of G; the Lie bracket $V_0 \times V_0 \to W_0$ is determined declaring

$$[v_i, v_j] := \{v_i, v_j\}$$

provided the vertices v_i and v_j are adjacent and $i < j$, and setting $[v_i, v_j] = 0$ whenever v_i and v_j are not adjacent (cf. [2]). It should be remarked that, in general, the standard basis of $\mathfrak{n}_G$ made up by the vertices and the edges of a positive graph G is orthogonal but not orthonormal with respect to the Ricci soliton metric on the corresponding group $\mathcal{N}_G$ (cf. [4]).

Our construction provides instead, for *every* graph G, a bigger 2-step nilpotent Einstein nilradical $\mathfrak{n}$ containing $\mathfrak{n}_G$ as a *totally geodesic* subalgebra (see Theorem 1). This Lie algebra is constructed as

$$\mathfrak{n} = V \oplus W_G, \quad V := V_0 \otimes \mathbb{R}^{2N}, \tag{2}$$

where $N \in \mathbb{N}^* \cup \{\frac{1}{2}\}$ is uniquely determined by the degree sequence of the graph and W_G is a suitable subspace of $\mathfrak{so}(V)$, whose dimension s depends both on the degree sequence and the size of the graph. The Lie bracket $V \times V \to W_G$ of $\mathfrak{n}$ is defined in the canonical way by

$$< [X,Y], J > := < J(X), Y >, \qquad X, Y \in V, \quad J \in W_G, \tag{3}$$

where $<,>$ denotes both the inner product on V with respect to which the basis $\{v_i \otimes e_k\}$ is orthonormal and the inner product $< F, G >= -\mathrm{tr}(F \circ G)$ on $\mathfrak{so}(V)$.

In general, a 2-step nilpotent Lie algebra $\mathfrak{n} = V \oplus W$ constructed in this way starting from an Euclidean vector space $(V, <,>)$ of dimension q and a p-dimensional subspace $W \subset \mathfrak{so}(V)$, is usually called *standard of type* (p,q). Hence, using this terminology, our $\mathfrak{n}$ is of type $(s, 2Nn)$.

Requiring that $W = V^{\perp}$ and keeping on V and $\mathfrak{so}(V)$ the inner products $<,>$, a standard 2-step nilpotent Lie algebra $\mathfrak{n}$ is turned into a metric Lie algebra in a natural way, which we shall call a *standard metric* 2-step nilpotent Lie algebra. Certainly the left-invariant metric on the corresponding Lie group $\mathcal{N}$ is a Ricci soliton provided its Ricci tensor is *optimal*, in the sense that the restrictions of Q to V and to W are both scalar operators. This happens if and only if W admits a basis $U = \{J_i\}$ such that

$$\sum J_i^2 = -\rho Id, \quad < J_i, J_j >= \lambda \delta^i_j, \tag{4}$$

where λ and ρ are positive constants (cf. [3] or [6], where such a W is called a *uniform* subspace of $\mathfrak{so}(V)$). Indeed, it is known that $< QX, J >= 0$ for every $X \in V$ and $J \in W$, and moreover

$$Q|_V = \frac{1}{2} \sum F_i^2, \qquad Q|_W = \frac{1}{4} Id_W,$$

where $\{F_i\}$ is an arbitrary orthonormal basis of W (cf. e.g. [3, Prop. 3.1]).

In our case, we provide such a basis U_G of W_G parametrizing it by the edges of the graph G and a certain set of vertices; in building the corresponding operators J_i we make use of a fixed edge-colouring of the complete graph K_{2N} with $2N-1$ colours.

We remark that the construction of $\mathfrak{n}$ is performed in such a way that $\mathfrak{n} = \mathfrak{n}_G$ if and only if the graph G is regular.

In the last section we also propose a method for attaching to G a standard Einstein nilradical $\mathfrak{n}_{\mathbb{A}}$ having the same properties, constructed using a Cayley-Dickson algebra $\mathbb{A}$ of dimension 2^M, where M is the smallest non negative integer such that $2^{M-1} \geq N$. The type of $\mathfrak{n}_{\mathbb{A}}$ is $(s, 2^M n)$.

2. Preliminary remarks

Given a simple graph G, we shall denote by $V(G)$ its vertex set and by $E(G)$ its edge set; recall that $V(G)$ is a finite set and $E(G)$ is a set of subsets of $V(G)$ each having cardinality 2.

We shall denote by K_n the *complete graph* with n vertices, namely $V(K_n) = \{1, \ldots, n\}$ and $E(K_n)$ is the set of all subsets of $\{1, \ldots, n\}$ having two elements.

We recall that an *edge colouring* of a graph G is a way of assigning a colour to each edge, in such a way that adjacent edges have different colours. In other words, an edge-colouring is a map

$$C_\mu : E(G) \to \{1, \ldots, \mu\},$$

where μ is a positive integer, such that $C_\mu(\delta) \neq C_\mu(\delta')$ whenever $\delta \cap \delta' \neq \emptyset$. The *chromatic index* $\mathcal{X}'(G)$ of G is the minimum integer μ for which such a colouring C_μ exists.

We shall use the following basic result: $\mathcal{X}'(K_{2N}) = 2N - 1$ *for every positive integer* N. See e.g. [5]. Moreover, given a fixed colouring $C = C_{2N-1}$ of K_{2N}, for each colour $t \in \{1, \ldots, 2N-1\}$ we shall denote by C_t the set of ordered pairs $(k, k') \in \{1, \ldots, 2N\}^2$ such that

$$C(\{k, k'\}) = t \text{ and } k < k'.$$

Then, by definition

$$t \neq t' \Rightarrow C_t \cap C_{t'} = \emptyset. \tag{5}$$

Moreover, it easily established that each colour class $C^{-1}(t)$ has cardinality N, so that

$$|C_t| = N \qquad \text{for all } t \in \{1, \ldots, 2N-1\}. \tag{6}$$

3. Einstein nilradicals attached to graphs

Let $G = (V(G), E(G))$ be a simple graph of order $n := |V(G)|$ and size $l := |E(G)|$. We shall denote by d_v the degree of a vertex $v \in V(G)$ and by $\Delta(G)$ the maximum degree.

Let $V^o(G)$ the set of vertices having odd degree, and let $V'(G)$ be the set of vertices whose degree is different from $\Delta(G)$. We put

$$2m := |V^o(G)|, \qquad r := |V'(G)|.$$

Denote by $v_1,\dots,v_n$ the vertices of G. We shall label the vertices of $V^o(G)$ and those in $V'(G)$ as follows:

$$V^o(G)=\{v_{q_1},\dots,v_{q_{2m}}\},\qquad V'(G)=\{v_{p_1}\dots,v_{p_r}\}.$$

Now we define a pair (N,s) of numbers uniquely determined by the graph, determining the type of the standard metric 2-step nilpotent Einstein nilradical we are going to attach to G. First we set

$$h_i:=\left[\frac{\Delta(G)-d_{v_{p_i}}+\varepsilon}{2}\right]\qquad \text{for each } v_{p_i}\in V'(G),$$

where [ ] means the integer part, $\varepsilon\in\{0,1\}$ and $\varepsilon=1$ iff $\Delta(G)$ is odd.
Next let h be the greatest of the h_i and set $\bar h:=\sum_{i=1}^r h_i$.
Hence, we define

$$s:=\begin{cases} l+\bar h+m & \text{if } 0<2m<n\\ l+\bar h & \text{otherwise}\end{cases}$$

and

$$N:=\text{smallest number in } \mathbb{N}^*\cup\{\frac{1}{2}\} \text{ such that } \begin{cases} 2N>h+1 & \text{if } 0<2m<n\\ 2N>h & \text{otherwise.}\end{cases}$$

We remark that the graph is regular iff $h=0$ and this is the only case where $N=\frac{1}{2}$.
As an example, consider the case of a non regular subcubic graph, i.e., $\Delta(G)=3$; then we have $(N,s)=(1,l+r)$ in the case $V^o(G)=V(G)$ and $(N,s)=(2,l+m+r)$ otherwise.

Keeping this notation, we prove the following result.

THEOREM 1. *Let G be a simple graph of order n. Let $\mathfrak{n}_G$ be the natural nilpotent Lie algebra attached to G. Then there exists a standard metric 2-step nilpotent Einstein nilradical $\mathfrak{n}$ of type $(s,2Nn)$ containing $\mathfrak{n}_G$ as a totally geodesic subalgebra.*

Proof. We consider the vector space V_0 spanned by the vertices of G and $V:=V_0\otimes\mathbb{R}^{2N}$ with its standard basis $\{v_i\otimes e_k\}$. We shall order the elements of $\mathfrak{B}$ according to

$$v_i\otimes e_k<v_j\otimes e_{k'} \text{ iff } i<j \text{ or } i=j \text{ and } k<k'$$

and we shall denote by $<,>$ the inner product on V with respect to which $\mathfrak{B}$ is an orthonormal basis. Given a pair (u,w) of elements of $\mathfrak{B}$, with $u<w$, we shall denote by F^u_w the skew-symmetric endomorphism of V such that

$$u\mapsto w,\quad w\mapsto -u,$$

and whose kernel contains all other vectors in $\mathfrak{B}$ different from u and w. Of course with respect to the standard inner product $<F,G>:=-\mathrm{tr}(F\circ G)$, the F^u_w make up an orthogonal basis of $\mathfrak{so}(V)$; observe that $||F^u_w||^2=2$.

Now we construct an s-dimensional subspace W_G of $\mathfrak{so}(V)$ attached to the graph G. To this aim, we also fix an edge colouring C of the complete graph K_{2N} with $2N-1$ colours. In the case $N=\frac{1}{2}$ we understand the empty colouring. Hence we build a subset $U_G \subset \mathfrak{so}(V)$ according to the following recipes A), B) and eventually C) in the case $0<2m<n$.

A) For each edge $\delta=\{v_p,v_q\},\ p<q$ we consider the following operator J_δ:

$$J_\delta := \sum_{k=1}^{2N} F^{v_p\otimes e_k}_{v_q\otimes e_k}.$$

B) For each vertex $v=v_{p_i}\in V'(G)$ we define the following h_i operators $J_{v,t}$:

$$J_{v,t} := \sqrt{2}\sum_{(k,k')\in C_t} F^{v\otimes e_k}_{v\otimes e_{k'}} \qquad t=1,\dots,h_i.$$

C) In the case where $0<2m<n$, we also define, for each vertex $w=v_{q_j}$ in $V^o(G)$ with $j=1,\dots,m$, an operator J_w by

$$J_w := \sum_{(k,k')\in C_{h+1}} F^{w\otimes e_k}_{w\otimes e_{k'}} + \sum_{(k,k')\in C_{h+1}} F^{w'\otimes e_k}_{w'\otimes e_{k'}}, \qquad \text{where } w' := v_{q_{j+m}}.$$

Let U_G be the set consisting of the s operators of type J_δ, $J_{v,t}$ and J_w. We claim that U_G satisfies (4). Observe first that

$$||J||^2 = 4N \qquad \text{for all } J\in U_G.$$

This is clear for the operators J_δ; as regards the operators $J_{t,v}$ and the J_w, this follows from (6). Moreover, (5) also guarantees that the elements of U_G are pairwise orthogonal. Finally, observe that

$$\sum_{\delta\in E(G)} J_\delta^2 \equiv -\mathrm{diag}(u,u,\dots,u), \quad u=(d_{v_1},\dots,d_{v_n})$$

where $\equiv$ means that the right-hand side square matrix of order $2Nn$ represents the operator on the left with respect to $\mathfrak{B}$. On the other hand, denoting by $\{\bar{e}_i\}$ the canonical basis of $\mathbb{R}^n$, for each $v=v_{p_i}\in V'(G)$ we have:

$$J_{v,t}^2 \equiv -\mathrm{diag}(2\bar{e}_{p_i},2\bar{e}_{p_i},\dots,2\bar{e}_{p_i})$$

and finally for each of the vertices $w=v_{q_j}$, $j=1,\dots,m$ of $V^o(G)$ we have

$$J_w^2 \equiv -\mathrm{diag}(\bar{e}_{q_j}+\bar{e}_{q_{j+m}},\bar{e}_{q_j}+\bar{e}_{q_{j+m}},\dots,\bar{e}_{q_j}+\bar{e}_{q_{j+m}}).$$

According to the definition of the h_i, we conclude that

$$\sum_{J\in U_G} J^2 = -(\Delta(G)+\nu)Id_V,$$

where $\nu \in \{0,1\}$ and $\nu = 1$ iff $0 < 2m < n$ and $\Delta(G)$ is odd.

We have thus proved that the standard metric 2-step nilpotent Lie algebra $\mathfrak{n} = V \oplus W_G$, where $W_G = \mathrm{span}(U_G)$ is an Einstein nilradical.

Concerning the last claim, denoting by V_1 the subspace of V spanned by $\{v_1 \otimes e_1, \ldots, v_n \otimes e_1\}$ and by W_E the subspace of W_G spanned by the operators J_δ, $\delta \in E(G)$, then the linear isomorphism

$$\mathfrak{n}_G \to V_1 \oplus W_E$$

determined by

$$v_i \mapsto v_i \otimes e_1, \qquad \delta \mapsto J_\delta$$

is a Lie algebra isomorphism between $\mathfrak{n}_G$ and the Lie subalgebra $\mathfrak{n}_1 := V_1 \oplus W_E$ of $\mathfrak{n}$. Finally, it is readily verified that for every $X, Y \in \mathfrak{n}_1$ and $Z \in \mathfrak{n}_1^\perp$ we have

$$< [X,Z], Y >= 0$$

yielding that $\mathfrak{n}_1$ is a totally geodesic subalgebra of $\mathfrak{n}$. □

Observe that the construction performed in the proof actually works also for every integer $N' \geq N$, providing a standard Einstein nilradical of type $(s, 2N'n)$; our choice aims at providing a totally geodesic embedding $\mathcal{N}_G \hookrightarrow \mathcal{N}$ with minimum codimension.

4. A construction involving Cayley-Dickson algebras

In this section we provide an alternative description of an Einstein nilradical $\mathfrak{n}$ attached to a graph, satisfying the conditions of Theorem 1. We recall that the Cayley-Dickson algebras constitute an infinite sequence of real algebras

$$\mathbb{A}_0 \subset \mathbb{A}_1 \subset \mathbb{A}_2 \subset \ldots \subset \mathbb{A}_M \subset \ldots$$

where $\mathbb{A}_0 = \mathbb{R}$, $\mathbb{A}_1 = \mathbb{C}$, $\mathbb{A}_2 = \mathbb{H}$ (quaternions), $\mathbb{A}_3 = \mathbb{O}$ (octonions), etc. Their construction can be described recursively as follows; set $\mathbb{A}_0 = \mathbb{R}$ and define $\mathbb{A}_{n+1} = \mathbb{A}_n \times \mathbb{A}_n$. Given $(a,b),(c,d)$ in $\mathbb{A}_{n+1}$, their product is defined by

$$(a,b)(c,d) = (ac - db^*,\, a^*d + cb),$$

where the *conjugate* of (a,b) is also defined recursively by $(a,b)^* := (a^*, -b)$. We have $\dim_{\mathbb{R}} \mathbb{A}_M = 2^M$; moreover, $\mathbb{A}_M$ admits a standard basis

$$i_0 := 1,\, i_1, \ldots, i_{2^M-1}$$

such that $i_k^* = -i_k$ for $k > 0$, and the corresponding multiplication table is

$$i_p i_q = \gamma(p,q)\, i_{p \oplus q}, \tag{7}$$

where, for each $p, q \in \mathcal{G}_M := \{0, \ldots, 2^M - 1\}$, $\gamma(p,q) = \pm 1$, and $\oplus$ is the group operation in $\mathcal{G}_M$ defined by $p \oplus q$:=bit-wise *exclusive* OR of the binary representations of

p and q; the group $(\mathcal{G}_M, \oplus)$ is canonically isomorphic to $\mathbb{Z}_2^M$. In particular, $i_k^2 = -1$ for each $k \in \mathcal{G}_M$, $k > 0$. For more information and a detailed study of the function $\gamma\colon \mathcal{G}_M \times \mathcal{G}_M \to \{-1, 1\}$ we refer the reader to [1].

For each $x \in \mathbb{A}$, its *real part* is defined as usual by $Re(x) = x_0$, provided $x = \sum_{p \in \mathcal{G}_M} x_p i_p$. With respect to the inner product $< x, y >= Re(x^* y)$, the algebra $\mathbb{A}_M$ is known to satisfy the *adjoint properties*:

$$< xy, z >=< y, x^* z >, \qquad < x, yz >=< xz^*, y > .$$

Moreover, we remark that using the multiplication of $\mathbb{A}$ one can define an edge-colouring $C_{\mathbb{A}}$ of the complete graph K_{2^M} with vertex set $\{0, \dots, 2^M - 1\}$ as follows:

$$C_{\mathbb{A}}(\{k, k'\}) = t \quad \text{iff} \quad i_t i_{k'} = \pm i_k$$

for every $t \in \{1, \dots, 2^M - 1\}$ and $k, k' \in \{0, \dots, 2^M - 1\}$ with $k \neq k'$.

Consider now a graph G and the pair (N, s) defined above, and take the smallest non negative integer M such that $2^{M-1} \geq N$. In this case we consider a standard metric 2-step nilpotent Lie algebra

$$\mathfrak{n}_{\mathbb{A}} := \mathbb{A}^n \oplus W_G,$$

where $\mathbb{A} := \mathbb{A}_M$ and $\mathbb{A}^n$ is considered as a $n2^M$-dimensional Euclidean vector space in a natural way, declaring the natural basis $\{i_k \bar{e}_i\}$ to be orthonormal (here $\{\bar{e}_i\}$ is the canonical basis of $\mathbb{R}^n$ and $i_k \bar{e}_i$ is the vector of $\mathbb{A}^n$ whose only non null entry, indexed i, is equal to i_k).

In this case, the s operators making up the basis U_G of W_G will be all of type

$$< J_a(X), Y >:= Re({}^t X^* (aY)) \tag{8}$$

where a is a $n \times n$ matrix with entries in $\mathbb{A}$.

The s matrices are defined as follows:

A) For each edge $\delta = \{v_p, v_q\}$, $p < q$, we consider the matrix $a_\delta := E_{pq} - E_{qp}$ whose entries indexed by (p, q) and (q, p) are respectively 1 and -1 and all the other entries are equal to zero.

B) For each vertex $v = v_{p_i}$ in $V'(G)$, $i = 1, \dots, r$ we define the h_i matrices:

$$a_{v,t} := \mathrm{diag}(0, \dots, 0, \sqrt{2}\, i_t, 0, \dots, 0), \qquad t = 1, \dots, h_i$$

each having exactly one non null entry indexed p_i.

C) In the case where $0 < 2m < n$, for each vertex $w = v_{q_j}$, $j = 1, \dots, m$ we introduce the matrix

$$a_w := \mathrm{diag}(0, \dots, 0, i_{h+1}, 0, \dots, i_{h+1}, 0, \dots, 0),$$

where the two non null entries are indexed respectively q_j and q_{j+m}.

Observe that for each of these matrices, the fact that the real bilinear form on $\mathbb{A}^n$ on the right hand side of (8) is skew-symmetric is guaranteed by the adjoint properties of $\mathbb{A}$. Taking into account the isomorphism $\mathbb{A}^n \cong V_0 \otimes \mathbb{R}^{2^M}$ such that

$$i_k \bar{e}_i \mapsto v_i \otimes e_{k+1},$$

the corresponding operators differ slightly from those defined in the proof of Theorem 1, since the twist function γ comes into play; namely one gets:

$$\begin{aligned}
J_\delta &= \sum_{k=1}^{2^M} F^{v_p \otimes e_k}_{v_q \otimes e_k} \\
J_{v,t} &= \sqrt{2} \sum_{(k,k') \in C_t} \gamma(t,k') F^{v \otimes e_k}_{v \otimes e_{k'}} \\
J_w &= \sum_{(k,k') \in C_{h+1}} \gamma(h+1,k') F^{w \otimes e_k}_{w \otimes e_{k'}} + \sum_{(k,k') \in C_{h+1}} \gamma(h+1,k') F^{w' \otimes e_k}_{w' \otimes e_{k'}}
\end{aligned}$$

where $C = C_{\mathbb{A}}$ is the colouring of K_{2^M} considered above. Again W_G is a uniform subspace of $\mathfrak{so}(\mathbb{A}^n)$, yielding that $\mathfrak{n}_{\mathbb{A}}$ is an Einstein nilradical.

EXAMPLE 1. Let G be the graph obtained from the complete graph K_6 by removing all the edges $\{i,j\}$ with i,j both in $\{1,2,3,4\}$. It can be checked directly that this graph is not positive (cf. also the classification in [7, Table 1]), so that $\mathfrak{n}_G$ is not an Einstein nilradical. In this case $(N,s) = (2,18)$ so that $\mathbb{A} = \mathbb{H}$ and $\mathfrak{n}_{\mathbb{H}} = \mathbb{H}^6 \oplus W_G$, where W_G is determined via (8) by the following 18 square matrices of order 6 with entries in $\mathbb{H}$:

$$\begin{aligned}
&E_{56} - E_{65}, E_{i5} - E_{5i}, E_{i6} - E_{6i}, \quad i = 1, \dots, 4, \\
&\mathrm{diag}(\sqrt{2}i, 0,0,0,0,0),\ \mathrm{diag}(\sqrt{2}j, 0,0,0,0,0), \\
&\mathrm{diag}(0,\sqrt{2}i, 0,0,0,0),\ \mathrm{diag}(0,\sqrt{2}j, 0,0,0,0), \\
&\mathrm{diag}(0,0,\sqrt{2}i, 0,0,0),\ \mathrm{diag}(0,0,\sqrt{2}j, 0,0,0), \\
&\mathrm{diag}(0,0,0,\sqrt{2}i, 0,0),\ \mathrm{diag}(0,0,0,\sqrt{2}j, 0,0), \\
&\qquad\qquad\qquad\qquad\qquad \mathrm{diag}(0,0,0,0,k,k).
\end{aligned}$$

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