International conference

Geometric Structures on Riemannian Manifolds

June 25-26, 2015, Bari, Italy

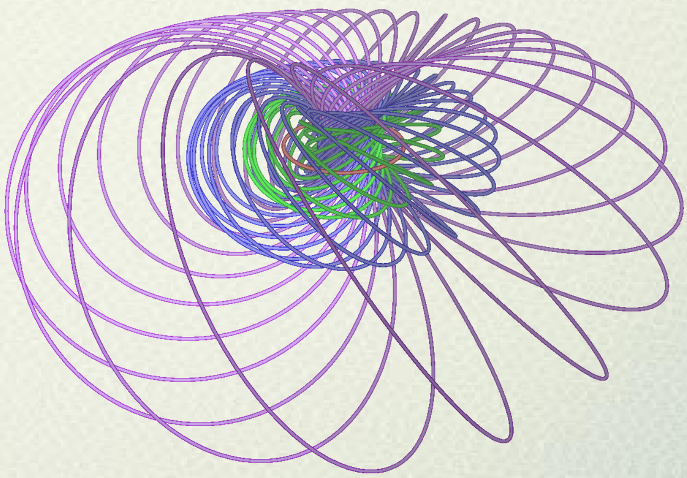
Dipartimento di Matematica, Università degli Studi di Bari Aldo Moro

Invited speakers

Giovanni Calvaruso	Università del Salento, Lecce
Beniamino Cappelletti Montano	Università di Cagliari
Alfonso Carriazo	Universidad de Sevilla, Spain
Giuseppe De Cecco	Lecce
Cătălin Gherghe	University of Bucharest, Romania
Costantino Medori	Università di Parma
Marian Ioan Munteanu	University of Iaşi, Romania
Liviu Ornea	University of Bucharest, Romania
Robert Wolak	Jagiellonian University, Cracow, Poland

Scientific Committee

Maria Falcitelli	Università di Bari
Anna Maria Fino	Università di Torino
Stefano Marchiafava	Roma



Organizing Committee

Luigia Di Terlizzi	Università di Bari
Giulia Dileo	Università di Bari
Antonio Lotta	Università di Bari
Francesca Verroca	Università di Bari

Sponsor

DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DEGLI STUDI DI BARI ALDO MORO

INdAM GNSAGA

Fondazione Cassa di Risparmio di Puglia

URL:http://www.dm.uniba.it/~lotta/Geometric_Structures.html

Aims

The *Seminario Matematico* is a society of members of mathematics-related departments of the University and the Politecnico in Turin. Its scope is to promote study and research in all fields of mathematics and its applications, and to contribute to the diffusion of mathematical culture.

The *Rendiconti* is the official journal of the Seminario Matematico. It publishes papers and invited lectures. Papers should provide original contributions to research, while invited lectures will have a tutorial or overview character. Other information and the contents of recent issues are available at `http://seminariomatematico.dm.unito.it/rendiconti/`

Instructions to Authors

Authors should submit an electronic copy (preferably in the journal's style, see below) via

e-mail: `rend_sem_mat@unito.it`

English is the preferred language, but papers in Italian or French may also be submitted. The paper should incorporate the name and affiliation of each author, postal and e-mail addresses, a brief abstract, and MSC classification numbers. Authors should inform the executive editor as soon as possible in the event of any change of address.

The final decision concerning acceptance of a paper is taken by the editorial board, based upon the evaluation of an external referee. Papers are typically processed within two weeks of arrival, and referees are asked to pass on their reports within two months.

Authors should prepare their manuscripts in LaTeX, with a separate BibTeX file whenever possible. To avoid delays in publication, it is essential to provide a LaTeX version once a paper has been accepted. The appropriate style file and instructions can be downloaded from the journal's website. The editorial board reserves the right to refuse publication if the file does not meet these requirements.

Reprints are not normally offered.

GEOMETRIC STRUCTURES ON RIEMANNIAN MANIFOLDS-BARI

Università e Politecnico di Torino

CONTENTS

RENDICONTI DEL SEMINARIO MATEMATICO-UNIVERSITÀ
E POLITECNICO DI TORINO

GEOMETRIC STRUCTURES ON RIEMANNIAN MANIFOLDS-BARI, JUNE 2015

Autorizzazione del Tribunale di Torino N. 2962 del 6.VI.1980

DIRETTORE RESPONSABILE

Alberto Collino

QUESTO FASCICOLO È STAMPATO CON IL CONTRIBUTO DI:
UNIVERSITÀ DEGLI STUDI DI TORINO
POLITECNICO DI TORINO

Geom. Struc. on Riem. Man.-Bari
Vol. 73/1, 3–4 (2015), 7 – 8

PREFACE

This volume gathers a large part of the plenary lectures and posters of the international Conference “Geometric Structures on Riemannian Manifolds” held at the Department of Mathematics of University “Aldo Moro” - Bari, 25-26 June 2015.

The Conference aimed at dealing with developments of recent results on differential geometry and topology of Riemannian manifolds endowed with Hermitian, contact and other related structures.

Leading researchers involved in differential geometry attended the Conference coming from several countries. The plenary lectures were held by: G. Calvaruso (University of Salento, Lecce), B. Cappelletti Montano (University of Cagliari), A. Carriazo (University of Sevilla, Spain) , G. De Cecco (Lecce) , C. Gherghe (University of Bucharest, Romania), C. Medori (University of Parma), M.I. Munteanu (University of Iasi, Romania), L. Ornea (University of Bucharest, Romania), B. Sahin (Inonu University, Malatya, Turkey), R. Wolak (Jagiellonian University, Cracow, Poland) and posters were presented by P. Alegre Rueda (Pablo de Olivade University, Sevilla, Spain), G. Bini-D. Iacono (University of Milano and University of Bari), R. Caddeo-P. Piu (University of Cagliari), A. De Nicola (University of Coimbra, Portugal), E. Loiudice (University of Bari), V. Martín-Molina (University of Sevilla), P. Mutlu-Z. Şentürk (University of Istanbul, Turkey), A. I. Nistor (University of Iasi, Romania), A. Perrone (University of Salento), H. Tadano (Osaka University, Japan), S. Yanan-B. Sahin (Adiyaman University and Inonu University, Turkey).

The issues were related to geometric structures and the respective subjects developed by the research group led by Anna Maria Pastore, following Aldo Cossu glorious tradition in Differential Geometry. Indeed, the organizers, with the very glad agreement of all participants, have proposed to dedicate the Conference to Anna Maria in the occasion of her 70th birthday.

The workshop was financially supported by INDAM-GNSAGA, Fondazione Cassa di Risparmio di Puglia, and the Department of Mathematics (University of Bari), under the sponsorship of Regione Puglia and Università degli Studi di Bari “Aldo Moro”. To all these Institutions, we express our special thankfulness.

We would like also to express our heartfelt thanks to all participants for their contribution to the success of the Conference. In particular to:

- all the plenary speakers and the poster session contributors;

- the Director of the Department of Mathematics Professor Francesco Altomare and the President of the Science and Technology School Professor Paolo Spinelli. Both of them outlined the various aspects of Anna Maria leading activities, constantly carried on at the University of Bari;

- the members of the Organizing Committee: Luigia Di Terlizzi, Giulia Dileo, Antonio Lotta, Francesca Verroca.

Finally, we would like to thank the Editor in chief Professor Marino Badiale, who made possible the publication of this volume on the prestigious journal Rendiconti del Seminario Matematico - Università e Politecnico di Torino.

Guest Editors

Maria Falcitelli
Anna Maria Fino
Stefano Marchiafava

Geom. Struc. on Riem. Man.-Bari
Vol. 73/1, 3–4 (2015), 9 – 20

G. Bini - D. Iacono

DIFFEOMORPHISM CLASSES OF CALABI-YAU VARIETIES

Abstract. In this article we investigate diffeomorphism classes of Calabi-Yau threefolds. In particular, we focus on those embedded in toric Fano manifolds. Along the way, we give various examples and conclude with a curious remark regarding mirror symmetry.

1. Introduction.

A longstanding problem in geometry is the classification of geometric objects up to isomorphism. For example, from a topological point of view, we are interested in classifying objects up to homeomorphism. In Differential Geometry, the classification is up to diffeomorphism and in complex geometry, we look for a classification up to (analytic) isomorphism.

This is the starting point for the construction of the moduli space. The main goal is the classification of families of these geometric objects (up to equivalence) so that the classifying space, the so called *moduli space*, is a reasonable geometric space. Roughly speaking, the moduli space is a parameter space that classifies these objects, in the sense that its points parametrise the geometric objects that we are considering. One of the easiest examples is the collection of all the lines (through the origin) in three dimensional space. The space that classifies this collection is well known and has a nice geometric structure: it is the projective plane (a smooth and compact manifold). As another example, we can consider the space that classifies, up to isomorphism, smooth rational curves of genus zero with 3 distinct marked points. It turns out that this space is just a point, since any triple of distinct points on a projective line can be sent in a distinct triple by an automorphism.

Unfortunately, the general situation turns out to be very complicated. In complex dimension one, we would like to classify all smooth curves, i.e., Riemann surfaces up to isomorphism. The classification can be carried out by using the genus g of the curve. For $g \geq 1$ the moduli space $\mathcal{M}_g$ is well understood and has a rich geometric structure. We also observe that in this case all the objects are projective, i.e., all smooth curves embedded in projective space.

In dimension two, the classification of compact complex surfaces is more involved than that in dimension one. It turns out that it is convenient to classify birational classes of surfaces. Then, for every birational class there is a unique minimal model, that has to be classified.

In dimension higher or equal than three, the classification is quite far from being complete. The idea is to generalize the technique used for dimension two and this has developed the so called Minimal Model Program. This classification is not concluded yet and already in dimension three there are many technical issues that have to be understood such as the uniqueness of the minimal model.

Motivated by a better understanding of this classification, we are interested in the role played by Calabi-Yau manifolds. First of all, the classification of 3-dimensional algebraic varieties has still some gaps due to the lack of understanding of Calabi-Yau threefolds. Moreover, the moduli space of Calabi-Yau varieties has received attention by theoretical physicists, since these geometric objects are important for mirror symmetry, cohomological quantum field theory and string theory, branches of physics dealing with general relativity and quantum mechanics.

In dimension one, Calabi-Yau curves are genus 1 curves and they are all homeomorphic each other. These are not isomorphic and they are classified by the so called j-invariant. In dimension two, Calabi-Yau surfaces are called K3 surfaces and they are all homeomorphic each other. Also in this case they are not all isomorphic; moreover, there exist K3 surfaces that are not projective. We also remark that K3 surfaces are extensively studied and they play a central role in algebraic and complex geometry.

In higher dimension, the classification of Calabi-Yau manifolds is quite hard and many questions are still open also in the topological setting. For example, the topology of Calabi-Yau manifolds is not uniquely determined for dimension greater or equal than three. It is also not known if there are only finitely many topological types of Calabi-Yau threefolds.

From the differential point of view, C.T.C. Wall described the invariants that determine the diffeomorphism type of closed, smooth, simply connected 6-manifolds with torsion free cohomology [18]. In particular, the Hodge data, the triple intersection in cohomology and the second Chern class completely determine the diffeomorphism type of a simply connected Calabi-Yau threefold. Recently, A. Kanazawa, P. M. H. Wilson [12], refined the theorem by Wall, providing some inequalities on the invariants, which hold in the case of Calabi-Yau threefolds.

The setting is very complicated. An interesting task is to find new examples of Calabi-Yau threefolds. The best known example of Calabi-Yau threefolds is the smooth quintic hypersurface in projective space $\mathbb{P}^4$, which is defined by a homogeneous polynomial of degree five. Actually, this example can be generalized to construct the majority of all known Calabi-Yau varieties. Indeed, projective space $\mathbb{P}^4$ is a particular example of smooth toric Fano varieties and these manifolds play a fundamental role in the construction of examples of Calabi-Yau. Once we have a toric Fano manifold, there always exists a submanifold of codimension one that is a Calabi-Yau manifold (see Section 2.2).

The toric set-up is an algebraic property that can be analyzed in terms of combinatorial algebra. Indeed, the classification of smooth toric Fano varieties of dimension n up to isomorphism turns out to be equivalent to the classification of combinatorial objects, namely some special polytopes in $\mathbb{R}^n$ up to linear unimodular transformation.

In [1], V. Batyrev describes a combinatorial criterion in terms of reflexive polyhedra for a hypersurface in a toric variety to be Calabi-Yau. He also investigates mirror symmetry in terms of an exchange of a dual pair of reflexive lattice polytopes. Moreover, he also provides the complete biregular classification of all 4-dimensional smooth toric Fano varieties: there are exactly 123 different types [2].

Using a computer program, M. Kreuzer and H. Sharke are able to describe all the reflexive polyhedra that exist in dimension four. They are about $500,000,000$ [13]. In particular, they find more than 30,000 topological distinct Calabi-Yau threefolds with distinct pairs of the Hodge numbers (a,b), where $h^{1,1}(X) = a$ and $h^{1,2}(X) = b$ (see Section 2.1). Furthermore, in [3] the authors find 210 reflexive 4-polytopes defining 68 topologically different Calabi-Yau varieties of dimension 3 with the Hodge number $a = 1$.

Therefore, it is interesting to investigate the set-up of toric Fano manifolds and try to answer some questions that naturally arise. For example, if the Hodge numbers (a,b) of two Calabi-Yau manifolds X_1 and X_2 are different, then they are not homeomorphic. It is interesting to understand the converse. If X_1 and X_2 have the same Hodge numbers, we wonder if they are homeomorphic or even diffeomorphic or isomorphic.

First of all, we deal with Calabi-Yau manifolds X_1 and X_2 contained in the same toric Fano manifold. In this specific context, we are able to prove that if X_1 and X_2 are deformation equivalent as abstract manifolds, then they are deformation equivalent as embedded manifolds.

Then, we review the Theorems by C.T.C. Wall (Theorem 1) and by A. Kanazawa, P. M. H. Wilson (Theorem 2), and we investigate some examples of simply connected Calabi-Yau manifolds with Hodge number $a = 1$.

From the point of view of moduli spaces, it is an interesting problem to understand the behaviour of the moduli space of Calabi-Yau manifolds. In dimension two, the moduli space of K3 surfaces is an irreducible 20 dimensional space and many properties are known. For Calabi-Yau threefolds, it is not known whether the moduli space is irreducible or not: M. Reid's conjecture predicts that this space should not behave too bad [15].

Then, instead of studying the moduli space of all Calabi-Yau threefolds, M.-C. Chang and H.I. Kim propose to investigate the space $M_{m,c}$ [6] that classifies Calabi-Yau threefolds with fixed invariants m and c, which are related to the invariant used by Wall (see Section 5). In this context, we describe an example of Calabi-Yau threefold and its mirror lying in the same $M_{5,50}$. In particular, we provide an example of two Calabi-Yau threefolds lying in $M_{5,50}$ that are neither diffeomorphic nor deformation equivalent.

With the aim of providing an introduction to the subject, Section 2 is devoted to recalling some preliminaries on Calabi-Yau manifolds and toric Fano manifolds. In Section 3, we compare the embedded deformations of a Calabi-Yau manifold in a toric Fano manifold with the abstract ones. Section 4 recalls Wall's Theorem on the invariants that determine the diffeomorphism type of closed, smooth, simply connected 6-manifolds with torsion free cohomology. We also describe some examples. In Section 5, we make some remarks on the relation between Calabi-Yau and mirror symmetry.

Notation. Throughout the paper, we will work over the field of complex numbers.

2. Preliminaries

In this section we recall the main defintions of Calabi-Yau and toric Fano manifolds.

2.1. Calabi-Yau Manifolds

Let X be a complex manifold and denote by T_X its holomorphic tangent bundle. X is a Calabi-Yau manifold of dimension n if it is a projective manifold with trivial canonical bundle and without holomophic p-forms, i.e., $K_X := \Omega_X^n \cong O_X$ and $H^0(X,\Omega_X^p) = 0$ for p in between 0 and n.

If X has dimension 3, we have $\Omega_X^3 \cong O_X$. Since Ω_X^1 is isomorphic to the dual of T_X, this implies that $\Omega_X^2 \cong T_X$, and, by duality, that $H^0(X,\Omega_X^2) = H^2(X,O_X) = H^1(X,O_X) = H^0(X,\Omega_X^1) = 0$.

Denoting by $h^{i,j}(X) = \dim_{\mathbb{C}} H^j(X,\Omega_X^i)$ and fixing $h^{1,1}(X) = a$ and $h^{1,2}(X) = b$, we can collect the information above in the so-called *Hodge diamond*:

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & a & & 0 & \\
1 & & b & & b & & 1 \\
 & 0 & & a & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1. & & &
\end{array}$$

This shows that the topological type of Calabi-Yau manifold is not uniquely determined for dimension 3. If X_1 and X_2 are two Calabi-Yau threefolds with different a and b then they cannot be homeomorphic.

Next, consider the case where the Hodge numbers (a,b) are the same. Let X_1 and X_2 be two Calabi-Yau threefolds, with the same Hodge numbers a and b, i.e., $h^{1,1}(X_i) = a$ and $h^{1,2}(X_i) = b$, for $i = 1,2$. Then, we wonder if X_1 and X_2 are diffeomorphic. Indeed, if the Calabi-Yau threefolds are diffeomorphic, then they have the same numbers Hodge numbers (a,b) but nothing is known about the other implication.

To understand the problem, we focus our attention on the class of Calabi-Yau manifolds embedded in toric Fano manifolds.

2.2. Toric Fano manifolds

Let F be a smooth toric Fano variety of dimension n. A Fano manifold is a projective manifold F, whose anticanonical line bundle $-K_F := \wedge^n T_F$ is ample.

If F is also a toric variety, then $-K_F$ is very ample (so base point free) [14, Lemma 2.20]. Therefore, by Bertini's Theorem [9, Corollary III.10.9], the generic section of $O_F(-K_F)$ gives a smooth connected hypersurface $X \subseteq F$, such that $X \in |-K_F|$. Thus, X is a smooth Calabi-Yau variety [7, Proposition 11.2.10]. This shows that once we have a toric Fano manifold, then there always exists a submanifold of codimension 1 that is a Calabi-Yau manifold.

In particular, if F has dimension 4, X is a smooth complex Calabi-Yau threefold. This is actually one of the most fruitful way to construct examples of Calabi-Yau threefolds [3].

EXAMPLE 1. The projective space $\mathbb{P}^4$ is a smooth toric Fano manifold of dimension 4. The general quintic hypersurface is a smooth Calabi-Yau threefold. This is the most extensively studied example of Calabi-Yau threefold. In this case, it can be proved that $a = 1$ and $b = 101$.

In Proposition 1, we investigate the infinitesimal deformations of smooth complex Calabi-Yau threefolds, which are obtained as anticanonical hypersurfaces in a Fano manifold.

3. Abstract vs Embedded Deformations

In this section, we review the notion of deformations of a submanifold X in a manifold F. In particular, we are interested in the infinitesimal deformations of X as an abstract manifold and in the embedded deformations of X in F. For more details, we refer the reader to [17, Sections 2.4 and 3.2].

We denote by Def_X the functor of infinitesimal deformations of X as an abstract variety, i.e.,

$$\mathrm{Def}_X : \mathbf{Art} \to \mathbf{Set},$$

where $\mathrm{Def}_X(A)$ is the set of isomorphism classes of commutative diagrams:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d] & X_A \arrow[d, "p_A"] \\ \mathrm{Spec}(\mathbb{K}) \arrow[r] & \mathrm{Spec}(A), \end{tikzcd}$$

where i is a closed embedding and p_A is a flat morphism.

REMARK 1. In our setting, X is smooth, then all the fibers of p_A are diffeomorphic by Ereshman's Theorem. Thus, an infinitesimal deformation of X is nothing else than a deformation of the complex structure over the same differentiable structure of X. In particular, if X_1 and X_2 are deformation equivalent then they are diffeomorphic, i.e., $X_1 \sim_{def} X_2 \Longrightarrow X_1 \cong_{dif} X_2$. The converse is not true: $X_1 \cong_{dif} X_2 \not\Longrightarrow X_1 \sim_{def} X_2$. There are examples of diffeomorphic Calabi-Yau threefolds that are not deformation equivalent [8, 16].

REMARK 2. If X is a Calabi-Yau manifold, then Bogomolov-Tian-Todorov Theorem implies that the functor Def_X is smooth. This property implies that the moduli space is smooth at the point corresponding to X.

We denote by H_X^F the functor of infinitesimal embedded deformations of X in F, i.e.,

$$H_X^F : \mathbf{Art} \to \mathbf{Set},$$

where $H_X^F(A)$ is the set of commutative diagram:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d] & X_A \subset F \times \mathrm{Spec}(A) \arrow[d, "p_A"] \\ \mathrm{Spec}(\mathbb{K}) \arrow[r] & \mathrm{Spec}(A), \end{tikzcd}$$

where i is a closed embedding, $X_A \subset F \times \mathrm{Spec}(A)$ and p_A is a flat morphism induced by the projection $F \times \mathrm{Spec}(A) \to \mathrm{Spec}(A)$.

In particular, the following forgetful morphism of functors is well defined:

$$\phi : H_X^F \to \mathrm{Def}_X;$$

moreover, the image of an infinitesimal deformation of X in F is the isomorphism class of the deformation of X, viewed as an abstract deformation [17, Section 3.2.3].

EXAMPLE 2. Let $n \geq 4$ and X be the general anticanonical hypersurface in $\mathbb{P}^n$. Note that $\mathbb{P}^n$ is a smooth toric Fano variety and X a smooth Calabi-Yau manifold.

For every Calabi-Yau manifold X in a projective space $\mathbb{P}^n$, the embedded deformations of X in $\mathbb{P}^n$ are unobstructed [10, Corollary A.2].

Therefore, the functor Def_X and the morphism ϕ are both smooth and this implies that $H_X^{\mathbb{P}^n}$ is also smooth [17, Corollary 2.3.7].

In particular, this implies that all the infinitesimal deformations of the general anticanonical hypersurface X as an abstract variety are obtained as embedded deformations of X inside $\mathbb{P}^n$. The following proposition shows that a similar property is true for any smooth toric Fano variety and not only for $\mathbb{P}^n$.

PROPOSITION 1. *Let F be a smooth toric Fano variety and denote by X a smooth connected hypersurface in F such that $X \in |-K_F|$. Then, the forgetful morphism*

$$\phi : H_X^F \to \mathrm{Def}_X$$

is smooth.

Proof. The varieties F and X are both smooth, so we have the exact sequence

$$0 \to T_X \to T_{F|X} \to N_{X/F} \to 0$$

that induces the following exact sequence in cohomology, namely:

$$\cdots \to H^0(X, N_{X/F}) \xrightarrow{\alpha} H^1(X, T_X) \to H^1(X, T_{F|X}) \to H^1(X, N_{X/F}) \xrightarrow{\beta} H^2(X, T_X) \to \cdots.$$

The morphism α is the map induced by ϕ on the tangent spaces and β is an obstruction map for ϕ [17, Proposition 3.2.9]. Applying the standard smoothness criterion [17, Proposition 2.3.6], it is enough to prove that α is surjective and β is injective; in particular, it suffices to prove that $H^1(X, T_{F|X}) = 0$.

For this purpose, consider the exact sequence

$$0 \to O_F(-X) \to O_F \to O_X \to 0$$

and tensor it with T_F, thus yielding

$$0 \to T_F \otimes O_F(-X) \to T_F \to T_{F|X} \to 0$$

and the induced exact sequence in cohomology, namely:

$$\cdots \to H^1(F, T_F) \to H^1(F, T_{F|X}) \to H^2(F, T_F \otimes O_F(-X)) \to H^2(F, T_F) \to \cdots.$$

If F is a smooth toric Fano variety, then $H^i(F, T_F) = 0$, for all $i > 0$ [5, Proposition 4.2]. Since $O_F(-X) \cong O_F(K_F)$, we are reduced to prove that $H^2(F, T_F \otimes O_F(K_F)) = 0$. This follows from Lemma 1. $\square$

LEMMA 1. *Let F be a smooth toric Fano variety with* $\dim F > 3$. *Then the following holds:*

$$H^2(F, T_F \otimes O_F(K_F)) = 0.$$

Proof. As for projective space, there exists a generalized Euler exact sequence for the tangent bundle of toric varieties [7, Theorem 8.1.6]:

$$0 \to Pic(F) \otimes_{\mathbb{Z}} O_F \to \oplus_i O_F(D_i) \to T_F \to 0,$$

where $K_F = -\sum_i D_i$ [7, Theorem 8.2.3]. We note also that $Pic(F) \otimes_{\mathbb{Z}} O_F \cong O_F^{rank}$, where *rank* denotes the *rank* of $Pic(F)$. By tensoring with $O_F(K_F)$, we obtain

$$0 \to Pic(F) \otimes_{\mathbb{Z}} O_F(K_F) \to \oplus_i O_F(D_i + K_F) \to T_F \otimes O_F(K_F) \to 0$$

and so

$$\cdots \to H^2(F, Pic(F) \otimes_{\mathbb{Z}} O_F(K_F)) \to$$

$$\cdots \to \oplus_i H^2(F, O_F(D_i + K_F)) \to H^2(F, T_F \otimes O_F(K_F)) \to H^3(F, Pic(F) \otimes_{\mathbb{Z}} O_F(K_F)) \to \cdots$$

Since $-K_F$ is ample, by Kodaira vanishing Theorem, $H^j(F, O_F) = 0, j > 0$. Moreover, by Serre duality $H^j(F, O_F(K_F)) = 0, j \neq \dim F$. Therefore, if $\dim F > 3$, then $H^2(F, Pic(F) \otimes_{\mathbb{Z}} O_F(K_F)) = H^3(F, Pic(F) \otimes_{\mathbb{Z}} O_F(K_F)) = 0$ and

$$\oplus_i H^2(F, O_F(D_i + K_F)) \cong H^2(F, T_F \otimes O_F(K_F)).$$

By Serre duality, $H^2(F, O_F(D_i + K_F)) \cong H^{\dim F - 2}(F, O_F(-D_i))^\vee$, for all i.

Using the following exact sequence

$$0 \to O_F(-D_i) \to O_F \to O_{D_i} \to 0$$

and the fact that $H^j(F, O_F) = 0$, for $j > 0$, we conclude that $H^{\dim F-2}(F, O_F(-D_i)) \cong H^{\dim F-3}(D_i, O_{D_i})$, for all i.

Therefore, we are left to prove that $\oplus_i H^{\dim F-3}(D_i, O_{D_i}) = 0$.

Consider the following exact sequence on a toric variety [7, Theorem 8.1.4]

$$0 \to \Omega^1_F \to M \otimes_Z O_F \to \oplus_i O_{D_i} \to 0.$$

This induces

$$\cdots \to H^{\dim F-3}(F, M \otimes_Z O_F) \to \oplus_i H^{\dim F-3}(D_i, O_{D_i}) \to H^{\dim F-2}(F, \Omega^1_F) \to \cdots.$$

Since $H^j(F, O_F) = 0$, for $j > 0$ and $\dim F > 3$, we have $H^{\dim F-3}(F, M \otimes_Z O_F) = H^{\dim F-2}(F, \Omega^1_F) = 0$ [7, Theorem 9.3.2]. This implies $\oplus_i H^{\dim F-3}(D_i, O_{D_i}) = 0$. $\square$

REMARK 3. Proposition 1 shows that all the infinitesimal deformations of X as an abstract variety are obtained as infinitesimal deformations of X inside the smooth toric Fano manifold F. Moreover, since every deformation of a Calabi-Yau manifold is smooth (Bogomolov-Tian-Todorov Theorem), we conclude that the deformations of X inside F are also smooth.

4. Diffeomorphic Three-dimensional Calabi-Yau varieties

In this section, we focus on the diffeomorphism class of three dimensional Calabi-Yau manifolds.

In 1966, C.T.C. Wall described the invariants that determine the diffeomorphism type of closed, smooth, simply connected 6-manifolds with torsion free cohomology.

THEOREM 1. *[18] Diffeomorphism classes of simply-connected, spin, oriented, closed 6-manifolds X with torsion-free cohomology correspond bijectively to isomorphism classes of systems of invariants consisting of*

1. *free Abelian groups $H^2(X,\mathbb{Z})$ and $H^3(X,\mathbb{Z})$,*
2. *a symmetric trilinear form $\mu : H^2(X,\mathbb{Z})^{\otimes 3} \to H^6(X,\mathbb{Z}) \cong \mathbb{Z}$ defined by $\mu(x,y,z) := x \cup y \cup z$,*
3. *a linear map $p_1 : H^2(X,\mathbb{Z}) \to H^6(X,\mathbb{Z}) \cong \mathbb{Z}$, defined by $p_1(x) := p_1(X) \cup x$, where $p_1(X) \in H^4(X,\mathbb{Z})$ is the first Pontrjagin class of X, satisfying,*

for any $x, y \in H^2(X,\mathbb{Z})$, the following conditions

$$\mu(x,x,y) + \mu(x,y,y) \equiv 0 \pmod 2 \qquad 4\mu(x,x,x) - p_1(x) \equiv 0 \pmod{24}.$$

The symbol $\cup$ denotes the cup product of differential forms and the isomorphism $H^6(X,\mathbb{Z}) \cong \mathbb{Z}$ above is given by pairing a cohomology class with the fundamental class of X with natural orientation.

Let X be a Calabi-Yau threefold. In [12], the authors investigate the interplay between the trilinear form μ and the Chern classes $c_2(X)$ and $c_3(X)$ of X, providing the following numerical relation.

THEOREM 2. *[12] Let (X,H) be a very ample polarized Calabi-Yau threefold, i.e., $x=H$ is a very ample divisor on X. Then the following inequalities holds:*

$$-36\mu(x,x,x)-80 \le \frac{c_3(X)}{2} = h^{1,1}(X)-h^{2,2} \le 6\mu(x,x,x)+40. \tag{1}$$

Note, that if X is a Calabi-Yau threefold, then $p_1(X)=-2c_2(X)\in H^4(X,\mathbb{Z})$ and

$$\int_X c_3(X)=\chi(X)=\sum_{i=0}^{6}\dim H^i(X,\mathbb{R})=2h^{1,1}(X)-2h^{1,2}.$$

REMARK 4. By Wall's Theorem, if X is simply-connected, spin, oriented, closed 6-manifolds with torsion-free cohomology, then the diffeomorphism class is determined by the free Abelian groups $H^2(X,\mathbb{Z})$ and $H^3(X,\mathbb{Z})$, and the form μ and p_1. For any data we have a diffeormphism class. If X is Calabi Yau, then μ and p_1 have to satisfy the numerical conditions of Equation (1). Note that, having μ and p_1 on X that satisfy all the numerical conditions, it does not imply that X is a Calabi Yau.

In particular, let X_1 and X_2 be two simply connected Calabi-Yau threefolds with torsion-free cohomology and the same Hodge numbers $h^{1,1}(X)=a$ and $h^{1,2}(X)=b$. To be diffeomorphic, they should have the same μ and p_1, that satisfy the numerical conditions.

COROLLARY 1. *Let X be a Calabi-Yau threefold, with torsion-free cohomology and $h^{1,1}(X)=1$ and $h^{1,2}(X)=h^{2,1}(X)=b$, for some $b\in\mathbb{N}$; hence, we have $H^2(X,\mathbb{Z})\cong\mathbb{Z}$ and $H^3(X,\mathbb{Z})\cong\mathbb{Z}^{2+b}$. Fix a generator $H\in H^2(X,\mathbb{Z})$ and set $\mu(H,H,H)=m\in\mathbb{Z}$. Then the following holds:*

$$m\ge\frac{b-81}{36}.$$

Proof. Set $p_1(X)=-2c_2(X)\in H^4(X,\mathbb{Z})$; so there exists $c\in\mathbb{Z}$ such that $c_2(X)=cH^*$. Therefore, the linear form p_1 reduces to

$$p_1: H^2(X,\mathbb{Z})\to H^6(X,\mathbb{Z})\cong\mathbb{Z} \qquad p_1(xH):=-2c_2(X)\cup xH=-2cxH^*\cup H=-2cx.$$

The numerical constraints of Theorem 1 reduce to

$$mx^2y+mxy^2\equiv 0 \pmod 2 \qquad\qquad 2mx^3+cx\equiv 0 \pmod{12}$$

for any $x,y\in\mathbb{Z}$. The former condition is always verified while the latter congruence is equivalent to $2m+c\equiv 0 \pmod{12}$.

As for the numerical restriction of Theorem 2, Equation (1) reduces to

$$-36\mu(x,x,x)-80\le 1-b\le 6\mu(x,x,x)+40.$$

and so

$$-36m-80 \leq 1-b \leq 6m+40.$$

In particular,

$$b \leq 81+36m \qquad -39-6m \leq b;$$
$$m \geq \frac{b-81}{36} \qquad m \geq \frac{-39-b}{6}.$$

Since b is positive, they reduce to

$$m \geq \frac{b-81}{36}.$$

□

EXAMPLE 3. If $m=5$, then $b \leq 261$. In [11], Appendix 1, there are three examples that satisfy this condition, namely $b=51,101,156$. For $b=101$ we obtain the general quintic threefold in $\mathbb{P}^4$. Projective models for the remaining two are still mysterious, as indicated by the question mark in the table in [11].

5. Some remarks on the moduli space of Calabi-Yau manifolds and mirror symmetry

Let X be a Calabi-Yau variety. Denote by H a primitive ample divisor. As in [6], let $M_{m,c}$ be the space of polarized varieties (X,H) such that $H^3=m$ and $c_2(X)H=c$ for integers m and c. Little is known on the geometric structure of $M_{m,c}$. Some information can be found in [6].

Here we make the following remarks. Let X be a general quintic in $\mathbb{P}^4$. A hyperplane section on X is a (very) ample divisor H such that $H^3=5$. On the Calabi-Yau manifold X the Grothendieck-Riemann-Roch Theorem reads as follows:

$$\chi(H)=\frac{H^3}{6}+\frac{1}{12}c_2(X)H.$$

Since H is a divisor on X, we have

$$\chi(O_X)+\chi(O_H(H))=\chi(H).$$

The first term on the left-hand side is zero because X is a Calabi-Yau; the second term can be computed via Noether's formula, namely:

$$\chi(O_H(H))=\frac{K_H^2+c_2(H)}{12}.$$

A linear section of a quintic is a quintic surface in three-dimensional projective space. As well known, the second Betti number is 53, so the Euler characteristic is 55. Therefore, we get

$$\chi(O_H(H))=5.$$

Hence, we get
$$5 = \frac{H^3}{6} + \frac{1}{12}c_2(X)H,$$
which yields $c_2(X)H = 50$.

This means that the pair (X,H) belongs to the space $M_{5,50}$, where X is a quintic in $\mathbb{P}^4$ and H is a hyperplane section.

The Hodge numbers of X are given by $(a,b) = (1,101)$. The Hodge numbers of a mirror manifold X' are given by $(101,1)$.

PROPOSITION 2. *There exists a primitive ample divisor D on X' such that (X',D) belongs to $M_{5,50}$.*

Proof. In fact, as recalled in [4], a mirror of X can be found as a crepant resolution of a singular quintic in $\mathbb{P}^4$. Denote by D the pull-back of the hyperplane divisor on projective space. Clearly, $D^3 = 5$. Now, we need to compute
$$\chi(D) = \frac{D^3}{6} + \frac{1}{12}c_2(X')D.$$

Like before, we have
$$\chi(O_D(D)) = \frac{K_D^2 + c_2(D)}{12}.$$

Notice that
$$K_D^2 = D^2D = 5.$$

As mentioned before, the divisor D is the pull-back of the hyperplane divisor. We can take a member of it that does not intersect the blown up locus. Thus, $c_2(D)$ is again the Euler characteristic of a quintic surface in three-dimensional projective space. Therefore, the claim follows. □

In particular, this implies that the mirror X' of X is a smooth Calabi-Yau threefold with $a = 101$, $b = 1$, $m = 5$ and $c = 50$. So it lies in the same space $M_{5,50}$ but the Hodge numbers are exchanged. This implies that X and the mirror X' are neither diffeomorphic nor deformation equivalent.

EXAMPLE 4. For the general quintic threefold X in $\mathbb{P}^4$, we have $a = 1$, $b = 101$, $m = 5$ and $c = 50$, that satisfy the previous conditions. Therefore, X lies in the space $M_{5,50} = \{(X,H) \mid H^3 = 5, c_2(X)\cdot H = 50\}$ introduced in [6]. The mirror $\tilde{X}$ of X is a smooth Calabi-Yau threefolds with $a = 101$, $b = 1$, $m = 5$ and $c = 50$. So it lies in the same space $M_{5,50}$ but the Hodge numbers are exchanged: this implies that X and $\tilde{X}$ are neither diffeomorphic nor deformation equivalent!

References

[1] BATYREV V., *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, J. Alg. Geom. **3** (1994), 493–535.

[2] BATYREV V., *On the classification of Toric Fano 4-folds*, Journal of Mathemtical Sciences, **94** no. 1 (1999), 1021–1050.

[3] BATYREV V. AND KREUZER M., *Constructing new Calabi-Yau 3-folds and their mirrors via conifold transitions*, Adv. Theor. Math. Phys. **14** no. 3 (2010), 879–898.

[4] BINI G., VAN GEEMEN B., KELLY T., *Mirror quintics, discrete symmetries and Shioda maps*, J. Alg. Geom. **21** (2012), 401–412.

[5] BIEN F. AND BRION M., *Automorphisms and local rigidity of regular varieties*, Compositio Math. **104** (1996), 1–26.

[6] CHANG M.-C. AND KIM H.I., *The Euler number of certain primitive Calabi-Yau threefolds*, Math. Proc. Cambridge Philos. Soc., **128** no. 1 (2000), 79–86.

[7] COX D.A., LITTLE J.B., SCHENCK H.K., *Toric varieties*, Graduate Studies in Mathematics, **124**, American Mathematical Society, Providence, RI, 2011.

[8] GROSS M., *The deformation space of Calabi-Yau n-folds can be ob- structed*, Mirror symmetry II, AMS/IP Stud. Adv. Math., **1**, Amer. Math. Soc., Providence, RI, 1997, 401–411.

[9] HARTSHORNE R., *Algebraic Geometry,* Graduate texts in mathematics, **52**, Springer-Verlag, New York Berlin, 1977.

[10] HUYBRECHTS D., *The tangent bundle of a Calabi-Yau manifold-deformations and restriction to rational curves*. Comm. Math. Phys., **171** no. 1 (1995), 139–158.

[11] KAPUSTKA G., *Projections od del Pezzo surfaces and Calabi-Yau threefolds*, e-print arXiv:1010.3895v5

[12] KANAZAWA A. AND WILSON P. M. H., *Trilinear forms and Chern classes of Calabi-Yau threefolds*, Osaka J. Math., **51** no. 1 (2014), 203–213.

[13] KREUZER M. AND SHARKE H. , *Complete classification of reflexive polyhedra in fourdimensions*, Adv. Theor. Math. Phys., **4** (2002), 1209–1230.

[14] ODA T., *Convex bodies and algebraic geometry*, Ergebnisse der Math., **15**, Springer-Verlag, New York Berlin, 1988.

[15] REID M. , *The moduli space of 3-folds with K=0 may nevertheless be irreducible*, Math. Annal., **278** (1987), 329–334.

[16] RUAN Y., *Topological sigma model and Donaldson-type invariants in Gromov theory*, Duke Math. Jour. **83** (1996), 461–500.

[17] SERNESI E., *Deformations of Algebraic Schemes.* Grundlehren der mathematischen Wissenschaften, **334**, Springer-Verlag, New York Berlin, 2006.

[18] WALL C.T.C., *Classification problems in differential topology V. On certain 6-manifolds*, Inv. Math. **1** (1966), 355–374.

AMS Subject Classification: 13D10, 14J32, 14J45

Gilberto BINI,
Dipartimento di Matematica “F.Enriques”, Università degli Studi di Milano,
Via C. Saldini 50, 20133 Milano, Italy
e-mail: `gilberto.bini@unimi.it`

Donatella IACONO,
Dipartimento di Matematica, Università degli Studi di Bari,
Via E. Orabona 4, 70125 Bari, Italy.
e-mail: `donatella.iacono@uniba.it`

Lavoro pervenuto in redazione il 11.11.2015 e accettato il 05.02.2016.

Geom. Struc. on Riem. Man.-Bari
Vol. 73/1, 3–4 (2015), 21 – 35

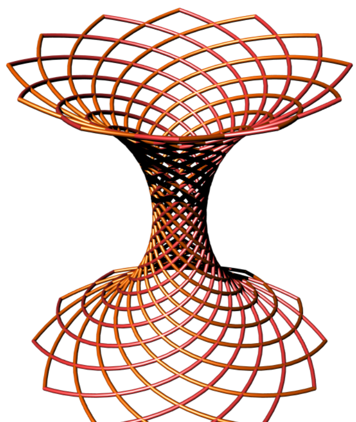
R. Caddeo - P. Piu

ELICHE SU SUPERFICI DI ROTAZIONE: VECCHIE E NUOVE [1]

Dedicato alla Professoressa Anna Maria Pastore

1. Introduzione

Come è ben noto, nello spazio ordinario $\mathbb{R}^3$, dopo i meridiani, i paralleli, le lossodromiche e le geodetiche, le curve notevoli probabilmente più interessanti di una superficie di rotazione sono le eliche. Curve di grande utilità e non prive di un certo fascino, come si può constatare osservando la struttura dell'*Albero della vita* scelto a simbolo dell'Expo 2015 di Milano. Il modello geometrico a lato è stato realizzato (in collaborazione con Gregorio Franzoni) tracciando due famiglie di curve asintotiche su un catenoide.



Storicamente, a nostra conoscenza, i primi studi sistematici, in senso moderno, delle eliche su superfici di rotazione risalgono alla fine del 1800 e si devono principalmente a Geminiano Pirondini (a Parma), a Gino Loria (a Genova, costretto al ritiro nel 1935 per le leggi razziali), e a Erich Salkowski (a Charlottenburg, attualmente un quartiere di Berlino). Più di recente, nel 1939, James K. Whittemore (a Harvard), partendo dalle *Lezioni* di Luigi Bianchi [3], ma apparentemente all'oscuro dei risultati precedenti, si cimenta nell'indagine sulle eliche delle superfici di rotazione, nell'intento di ricavare una loro parametrizzazione esplicita almeno su alcune quadriche.
In questi ultimi anni ci siamo imbattuti a più riprese in ambito riemanniano in eliche cilindriche sia nello studio delle geodetiche, sia nella determinazione delle curve biarmoniche nel senso di Eells e Sampson [10]. Esistono infatti, oltre a $\mathbb{R}^3$, altre varietà riemanniane tridimensionali in ogni punto delle quali c'è un asse attorno al quale si possono costruire superfici di rotazione, spazi che L. Bianchi in [4] annovera tra quelli

[1] Lavoro svolto nell'ambito del GNSAGA dell'INdAM e del P.R.I.N. 2010/11 - Varietà reali e complesse: geometria, topologia e analisi armonica - MIUR Italia.

sistatici. Tra questi c'è anche $\mathbb{H}_3$, il gruppo di Heisenberg, munito della metrica

$$g = dx^2 + dy^2 + \left(dz - \frac{1}{2}xdy + \frac{1}{2}ydx\right)^2.$$

In [2] M. Bekkar e T. Sari hanno dato esplicitamente il suo gruppo delle isometrie, nella forma

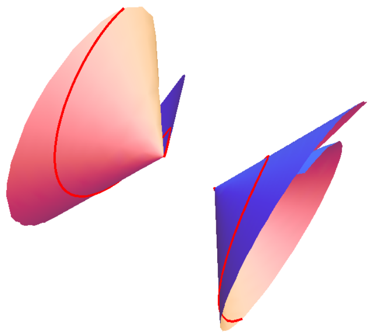
$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ \frac{-b\cos\vartheta + a\sin\vartheta}{2} & \frac{b\sin\vartheta + a\cos\vartheta}{2} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} a \\ b \\ c - \frac{ab}{2} \end{pmatrix} \qquad \vartheta, a, b, c \in \mathbb{R},$$

nella quale si vede subito che tale gruppo contiene il sottogruppo delle rotazioni attorno all'asse z.
Come vedremo nel § 4, anche in questi spazi appaiono eliche cilindriche sia nello studio delle geodetiche, sia nella determinazione delle curve che sono biarmoniche nel senso di Eells e Sampson ([10]).
Nei primi paragrafi richiameremo brevemente i momenti salienti - noti, ma forse non proprio ben noti - della evoluzione dello studio e della ricerca delle eliche delle superfici di rotazione in $\mathbb{R}^3$, per mettere in evidenza i principali contributi, dai primi di G. Pirondini ([16]), di G. Loria ([14]) e di E. Salkowski ([21]), sino a quello ([24]) di J.K. Whittemore. Grazie a quest'ultimo potremo rappresentare graficamente alcune delle eliche trovate.

2. Eliche: primi studi sulle superfici di rotazione e generalizzazioni

La famiglia delle eliche dello spazio ordinario può essere ampliata accettando tra i suoi membri, oltre alle eliche cilindriche a forma di molla aventi curvatura e torsione costanti, anche quelle che formano un angolo costante con una direzione fissata. Che questa generalizzazione sia ragionevole appare chiaro per via di un risultato classico, annunciato da M.A. Lancret nel 1802 e dimostrato da B. de Saint Venant nel 1845 (si veda [22], §§ 1-9, per i dettagli) che afferma: *condizione necessaria e sufficiente affinché una curva sia un'elica generalizzata è che il rapporto tra curvatura e torsione sia costante.*
Poiché tutte le eliche cilindriche verificano l'equazione $\kappa/\tau = costante$, si possono cercare quelle fra esse che soddisfano a una ulteriore condizione, per esempio a quella di appartenere anche ad una assegnata superficie di rotazione attorno a un asse parallelo alle generatrici del cilindro sul quale si trova l'elica. Fu A. Enneper nel 1866 a porre il problema di determinare curve che siano traiettorie oblique sia rispetto alle generatrici di un cono che a quelle di un cilindro. La questione venne da lui stesso risolta nel 1882 in [11]. In seguito G. Pirondini in [16] studiò diversi casi, in particolare le curve che sono eliche di due coni, e dedusse che:

- se si sviluppa su un piano il cilindro che proietta l'elica biconica parallelamente alla congiungente i vertici, la curva diviene una cicloide;
- se si fa ruotare l'elica attorno alla congiungente i vertici dei due coni si ottiene una superficie di quart'ordine; quindi l'elica, benché trascendente, appartiene ad una superficie algebrica.



In precedenza, nel 1878, E. Catalan aveva studiato *l'elica catenoidica* $(e^t, e^{-t}, t\sqrt{2})$, mostrando che la sua proiezione su un piano è un'iperbole equilatera.
Nel 1925 G. Loria presenta in [14] i risultati a lui noti, ottenuti applicando metodi classici della geometria analitica allo studio delle eliche su superfici di rotazione. In alcuni paragrafi vengono descritte le eliche delle quadriche di rotazione partendo dalla curva che l'elica proietta su un piano perpendicolare all'asse di rotazione. Tale descrizione consente di rappresentare le eliche su alcune quadriche:

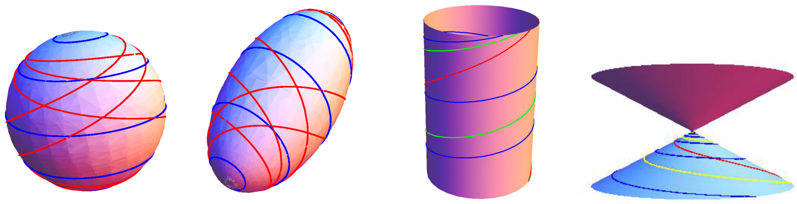


Figura 1: *Alcune eliche della sfera, dell'ellissoide, del cilindro e del cono.*

La nozione di elica generalizzata di $\mathbb{R}^3$ può essere estesa in diversi modi. In [20] viene considerata la stessa definizione in $\mathbb{R}^n$, con $n > 3$. In [12] Hayden si trasferisce in ambito riemanniano definendo elica generalizzata una curva per la quale un campo vettoriale lungo essa, parallelo secondo Levi-Civita, forma angoli costanti con tutti i vettori del riferimento di Frenet. Questa definizione è piuttosto restrittiva e consente di trovare "eliche solo nel caso in cui la dimensione n sia dispari. Inoltre, negli stessi articoli, viene generalizzato il teorema di Lancret dimostrando che la definizione è equivalente al fatto che i rapporti $\frac{k_2}{k_1}, \frac{k_4}{k_3}, \cdots$, tra le curvature k_i della curva, in numero pari, sono costanti. In [15] Monterde studia le curve di $\mathbb{R}^n$ per le quali tutti i rapporti $\frac{k_{i+1}}{k_i}$ tra le curvature sono costanti. In [1] l'autore propone un'altra definizione di elica generalizzata su uno spazio tridimensionale M a curvatura costante sostituendo la direzione fissa che appare nella usuale definizione di elica con un campo vettoriale di Killing ξ definito lungo la curva: *una curva $\gamma(s)$ in M è un'elica generalizzata se esiste, lungo γ, un campo di Killing $\xi(s)$ di lunghezza costante che forma con γ' un angolo costante e diverso da zero.* Un modello di elica in $\mathbb{S}^3 \subset \mathbb{R}^4$ è dato da

$$\gamma(t) = (\cos\varphi\cos at, \cos\varphi\sin at, \sin\varphi\cos bt, \sin\varphi\sin bt)$$

con $a^2\cos^2\varphi + b^2\sin^2\varphi = 1$. Si verifica facilmente che γ è contenuta nel toro piatto

$$x_1^2 + x_2^2 = \cos^2\varphi, \qquad x_3^2 + x_4^2 = \sin^2\varphi.$$

Figura 2: *proiezione stereografica dal polo Nord e dal polo Sud sul piano equatoriale di un'elica della sfera* $\mathbb{S}^3$

3. Il metodo di Whittemore

In [24] Whittemore, per trovare le eliche della sfera si serve della loro rappresentazione sferica. Questo è reso possibile dal fatto che sulla sfera (ma solo sulla sfera) ci sono delle relazioni differenziali tra curvatura e torsione che sono valide per tutte le curve[1]. Per estendere il suo studio ad altre superfici di rotazione egli adotta un metodo che permette di determinare facilmente le equazioni delle eliche delle superfici di rotazione aventi lo stesso asse di queste ultime.
Sia S una superficie di rotazione attorno all'asse z parametrizzata da

(1) $$S(u,v) = \{u\cos v, u\sin v, f(u)\},$$

e quindi con prima forma fondamentale $ds^2 = (1+f'^2)du^2 + u^2dv^2$. Sia poi $\gamma(s)$ un'elica di S parametrizzata con l'ascissa curvilinea s, cioè

(2) $$\gamma(s) = \{u(s)\cos v(s), u(s)\sin v(s), f(u(s))\}.$$

Essendo la curva γ unitaria, se essa forma un angolo acuto costante ϑ_0 con l'asse z, si ha che $u(s)$ e $v(s)$ devono soddisfare alle equazioni differenziali

$$\dot{u}^2(1+f'^2) + u^2\dot{v}^2 = 1 \qquad \text{e} \qquad f'\dot{u} = \cos\vartheta_0,$$

dove $\dot{u} = \frac{du}{ds}$ e $f' = \frac{df}{du}$. Da queste si ricava

$$u^2\dot{v}^2 = \sin^2\vartheta_0 - \frac{\cos^2\vartheta_0}{f'^2} = (f'^2\tan^2\vartheta_0 - 1)\dot{u}^2.$$

[1] Una notevole è l'equazione differenziale delle curve sferiche ([22] pag. 32, [5]):

$$\rho\tau + (\sigma\rho')' = 0, \qquad \rho = \frac{1}{k}, \quad \sigma = \frac{1}{\tau}, \quad \tau \neq 0, \quad k' \neq 0.$$

Quindi

$$\dot{v} = \sqrt{f'^2 \tan^2 \vartheta_0 - 1}\, \frac{\dot{u}}{u}$$

e, integrando,

$$v = \pm \int \sqrt{f'^2 \tan^2 \vartheta_0 - 1}\, \frac{du}{u} + K, \qquad K \in \mathbb{R}. \tag{3}$$

Nella (3) possiamo scegliere $K = 0$ e il segno positivo. Per $K \neq 0$ si ottiene la stessa elica ruotata attorno all'asse z, mentre al segno negativo corrisponde un'elica simmetrica rispetto al piano $x = 0$. In ultima analisi, la ricerca delle eliche di S si può ricondurre al calcolo dell'integrale (3), che in alcuni casi può essere effettuato. Vediamo dunque come, grazie alla (3), è possibile trovare eliche su alcune superfici di rotazione.

Il paraboloide. Consideriamo il paraboloide ottenuto ruotando attorno all'asse z la parabola di equazione $(u, 0, u^2/(2p))$. Sostituendo $f' = u/p$ nella (3) otteniamo

$$\begin{aligned} v(u) &= \frac{\tan \vartheta_0}{p} \int \sqrt{u^2 - p^2 \cot^2 \vartheta_0}\, \frac{du}{u} \\ &= \sqrt{\frac{\tan^2 \vartheta_0\, u^2}{p^2} - 1} - \arccos\left(\frac{p \cot \vartheta_0}{u}\right). \end{aligned} \tag{4}$$

Introducendo nella (4) le notazioni

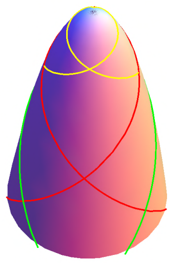
$$R = p \cot \vartheta_0, \qquad \cos \varphi(u) = \frac{R}{u}, \qquad t(u) = \tan \varphi,$$

troviamo

$$v(u) = \sqrt{\frac{u^2}{R^2} - 1} - \varphi = \sqrt{\frac{1}{\cos^2 \varphi} - 1} - \varphi = \tan \varphi - \varphi = t - \varphi\,.$$

Quindi le equazioni parametriche dell'elica del paraboloide considerato e la relativa rappresentazione grafica sono

$$\begin{cases} x(u) = u \cos \varphi \cos t + u \sin \varphi \sin t = R \cos t + Rt \sin t \\ y(u) = u \cos \varphi \sin t - u \sin \varphi \cos t = R \sin t - Rt \cos t \\ z(u) = \frac{R^2 + R^2 t^2}{2p} \end{cases}$$



L'iperboloide a una falda. Consideriamo poi l'iperboloide a una falda, ottenuto ruotando attorno all'asse z il ramo di iperbole di equazione $(u, 0, b\sqrt{u^2 - 1}), u > 0$. In questo caso quindi

$$f(u) = b\sqrt{u^2 - 1} \qquad \text{e} \qquad f'(u) = \frac{bu}{\sqrt{u^2 - 1}}.$$

Sostituendo f' nella (3), si può scrivere v in funzione di u:

$$v(u) = \int \sqrt{\frac{u^2(b^2\tan^2 C - 1) + 1}{u^2 - 1}}\ \frac{du}{u}.$$

Se si pone

$$\sqrt{b^2\tan^2 C - 1} = \frac{1}{a}$$

questa diviene

$$v = \frac{1}{a}\int \sqrt{\frac{u^2 + a^2}{u^2 - 1}}\ \frac{du}{u}.$$

Allora integrando si trova

$$v = \frac{1}{2a}\log\left(\frac{\sqrt{u^2+a^2}+\sqrt{u^2-1}}{\sqrt{u^2+a^2}-\sqrt{u^2-1}}\right) - \arctan\left[\frac{1}{a}\sqrt{\frac{u^2+a^2}{u^2-1}}\right].$$

Con le ulteriori sostituzioni

$$a = \tan\alpha \qquad \text{e} \qquad \varphi = \arctan\left[\frac{1}{a}\sqrt{\frac{u^2+a^2}{u^2-1}}\right]$$

si ottiene

$$\tan^2\varphi = \frac{1}{a^2}\,\frac{u^2+a^2}{u^2-1} = \frac{u^2\cos^2\alpha + \sin^2\alpha}{\sin^2\alpha}\ \frac{1}{u^2-1}$$

e quindi

$$v = \frac{\cot\alpha}{2}\log\frac{a\tan\varphi+1}{a\tan\varphi-1} - \varphi = \frac{\cot\alpha}{2}\log\left(\frac{\sin\alpha\sin\varphi+\cos\alpha\cos\varphi}{\sin\alpha\sin\varphi-\cos\alpha\cos\varphi}\right) - \varphi.$$

Risulta dunque che u e v sono

$$\begin{aligned} u(\varphi) &= \frac{\sin\alpha}{\sqrt{-\cos(\alpha+\varphi)\cos(\alpha-\varphi)}} \\ v(\varphi) &= \frac{\cot\alpha}{2}\ \log\left(-\frac{\cos(\alpha-\varphi)}{\cos(\alpha+\varphi)}\right) - \varphi, \end{aligned}$$

e possiamo disegnare le eliche sull'iperboloide a una falda (vedi Fig.3).

L'iperboloide a due falde. Consideriamo la componente dell'iperboloide a due falde ottenuta ruotando, attorno all'asse z, il meridiano di equazione $(u, 0, b\sqrt{u^2+1})$. Ancora con riferimento alla (3) abbiamo

$$f(u) = b\sqrt{u^2+1} \qquad \text{e} \qquad f'(u) = \frac{bu}{\sqrt{u^2+1}}.$$

Questo caso può essere trattato seguendo passo per passo quello precedente dell'iperboloide a una falda, e si trovano le soluzioni

$$\begin{aligned} u &= \frac{\sin\alpha}{\sqrt{\cos(\alpha+\varphi)\cos(\alpha-\varphi)}} \\ v &= \frac{\cot\alpha}{2}\log\left(\frac{\cos(\alpha-\varphi)}{\cos(\alpha+\varphi)}\right)-\varphi\,, \end{aligned}$$

che conducono alla relativa rappresentazione grafica

Figura 3: *Eliche su iperboloidi a una e a due falde.*

Il cono. Facendo ruotare attorno all'asse z la retta

$$\alpha(u) = (u, 0, mu), \qquad m \in \mathbb{R},$$

si trova il cono

$$X(u,v) = (u\cos v, u\sin v, f(u) = mu),$$

i cui coefficienti della prima forma fondamentale sono

$$E = 1+m^2, \qquad F = 0 \qquad G = u^2.$$

Mediante la (3) possiamo quindi determinare v, che risulta avere l'espressione

$$v = \int \frac{\sqrt{m^2\tan^2\vartheta_0 - 1}}{u}\,du = \sqrt{m^2\tan^2\vartheta_0 - 1}\,\log u + d.$$

La superficie a imbuto. Per la superficie a imbuto, ottenuta facendo ruotare attorno all'asse z la curva

$$\alpha(u) = (u, 0, \log u), \qquad u > 0,$$

cioè

$$X(u,v) = (u\cos v, u\sin v, \log u),$$

si ha

$$E = 1+\frac{1}{u^2}, \qquad F = 0, \qquad G = u^2.$$

Ancora mediante la (3) troviamo

$$v = \int \left(\frac{1}{u^2} \tan^2 \vartheta_0 - 1 \right)^{\frac{1}{2}} \frac{du}{u} = \tan \vartheta_0 \int \frac{(1 - u^2 \cot^2 \vartheta_0)^{\frac{1}{2}}}{u^2} \, du$$

Se poniamo $a = \tan \vartheta_0$ e $b^2 = \cot^2 \vartheta_0$ quest'ultimo integrale assume l'espressione

$$v = a \int \frac{(1 - b^2 u^2)^{\frac{1}{2}}}{u^2} \, du.$$

Integrando per parti si ha

$$\text{(5)} \qquad \begin{aligned} v &= a \left\{ (1 - b^2 u^2)^{\frac{1}{2}} \cdot \left(-\frac{1}{u} \right) - \int \left[\frac{b^2}{\sqrt{1 - b^2 u^2}} \right] du \right\} \\ &= -a \left[\frac{\sqrt{1 - b^2 u^2}}{u} + b \arcsin(bu) + d \right]. \end{aligned}$$

Sostituendo le espressioni delle costanti a e b^2 e ponendo $d = 0$ si trova

$$v = -\frac{\tan \vartheta_0 \sqrt{1 - (\cot^2 \vartheta_0) u^2}}{u} - (\tan \vartheta_0 \cot \vartheta_0) \arcsin[(\cot \vartheta_0) u].$$

Pertanto, semplificando,

$$v = \frac{\sqrt{\cos 2C}}{\cos \vartheta_0} + \arccos[(\cot \vartheta_0) u].$$

Vogliamo ora dare altri due esempi di superfici di rotazione notevoli che non sono quadriche in cui il metodo di Whittemore permette di trovare eliche con lo stesso asse di rotazione della superficie.

Il catenoide. Se γ è una curva contenuta nel catenoide

$$X(u, v) = (\cosh u \cos v, \cosh u \sin v, u)$$

si ha

$$\dot{\gamma} = \{ \sinh u \cos v \, \dot{u} - \cosh u \sin v \, \dot{v}, \sinh u \sin v \, \dot{u} + \cosh u \cos v \, \dot{v}, \; \dot{u} \}.$$

Se γ è unitaria e forma un angolo costante ϑ_0 con il versore $(0, 0, 1)$ dell'asse z, cioè se

$$\begin{cases} \|\dot{\gamma}\|^2 = (\sinh^2 u + 1) \dot{u}^2 + \cosh^2 u \, \dot{v}^2 = 1 \\ \dot{u} = \cos \vartheta_0, \end{cases}$$

si trova

$$\text{(6)} \qquad \begin{aligned} \dot{v}^2 \cosh^2 u &= 1 - (\sinh^2 u + 1) \cos^2 \vartheta_0 \\ &= \cos^2 \vartheta_0 (\tan^2 \vartheta_0 - \sinh^2 u) \end{aligned}$$

e quindi

$$\dot{v} = \cos\vartheta_0 \sqrt{\frac{\tan^2\vartheta_0}{\cosh^2 u} - \tanh^2 u},$$

ovvero

$$\frac{dv}{dt} = \sqrt{\frac{\tan^2\vartheta_0}{\cosh^2 u} - \tanh^2 u} \cdot \frac{du}{dt} \ .$$

Possiamo porre $\tan\vartheta_0 = a$, per cui

$$\frac{dv}{dt} = \sqrt{\frac{a^2 - \sinh^2 u}{\cosh^2 u}} \cdot \frac{du}{dt}.$$

Integrando si ottiene l'espressione di v, che è

$$\begin{aligned} v &= -\frac{1}{1+2a^2-\cosh(2u)} \cosh u \left\{ \sqrt{1+a^2}\, \operatorname{arctanh}\left[\frac{\sqrt{2+2a^2}\sinh u}{\sqrt{-1-2a^2+\cosh(2u)}}\right] \right. \\ &\cdot \left. \sqrt{-1-2a^2+\cosh(2u)} + \sqrt{a^2}\, \arcsin\left[\frac{\sinh u}{\sqrt{a^2}}\right] \sqrt{2+\frac{1}{a^2}-\frac{\cosh(2u)}{a^2}} \right\} \cdot \\ &\cdot \sqrt{2a^2 \operatorname{sech}^2 u - 2\tanh^2 u}, \end{aligned}$$

e quindi l'immagine centrale in Fig.4.

La pseudosfera. Consideriamo infine il caso della pseudosfera parametrizzata da

$$X[a](u,v) = a\left\{\cos u \sin v, \sin u \sin v, \cos v + \log\left[\tan\left(\frac{v}{2}\right)\right]\right\}.$$

Se $\gamma[a](t)$ è una curva della pseudosfera, cioè se

$$\gamma[a](t) = X[a][u(t), v(t)],$$

il suo vettore velocità è

$$\begin{aligned} \dot{\gamma}[a](t) &= a\{-\sin[u(t)]\sin[v(t)]\dot{u}(t) + \cos[u(t)]\cos[v(t)]\dot{v}(t), \\ &\quad \cos[u(t)]\sin[v(t)]\dot{u}(t) + \cos[v(t)]\sin[u(t)]\dot{v}(t), \\ &\quad \cos[v(t)]\cot[v(t)]\dot{v}(t)\}. \end{aligned}$$

Se γ è un'elica con velocità unitaria con asse parallelo all'asse z, devono essere verificate le condizioni

$$\begin{cases} \|\dot{\gamma}[a(t)]\|^2 = a^2(\sin^2[v(t)]\dot{u}(t)^2 + \cot^2[v(t)]\dot{v}(t) = 1 \\ a\cos[v(t)]\cot[v(t)]\dot{v}(t) = \cos\vartheta_0. \end{cases}$$

Ricavando $\dot{v}$ dalla seconda equazione del sistema e sostituendo nella prima otteniamo

$$\dot{u} = \sqrt{\frac{\cos^2 v - c^2}{a^2 \sin^2 v}\ \cot^2 v\ \dot{v}},$$

dove abbiamo posto $\cos\vartheta_0 = c$. L'integrazione dà

$$\begin{aligned} u \quad = \quad & \left\{\sqrt{1-2c^2+\cos(2v)} + \sqrt{2}\arctan\left[\frac{\sqrt{2}\sin v}{\sqrt{1-2c^2+\cos(2v)}}\right]\sin v\right\} \cdot \\ & \cdot \left(-\frac{1}{\sqrt{1-2c^2+\cos(2v)}}\sqrt{\frac{\cot^4 v}{c^2} - \cot^2 v \csc^2 v}\right)\tan v\,, \end{aligned}$$

e con questa espressione è possibile rappresentare le eliche della pseudosfera nell'immagine a destra della Figura 4.

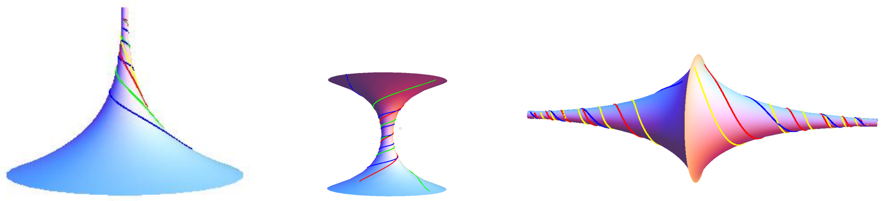


Figura 4: *Eliche sulla superficie a imbuto, sul catenoide e sulla pseudosfera.*

4. Curve biarmoniche nelle varietà di Bianchi-Cartan-Vranceanu

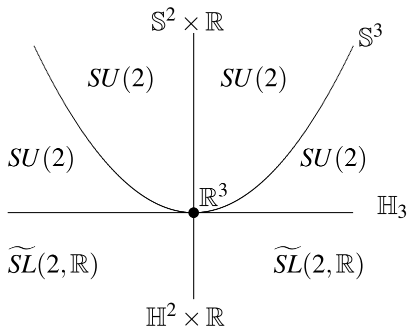
Come si può facilmente immaginare, le eliche appaiono in modo naturale anche in ambito riemanniano, con ruoli geometricamente rilevanti (si veda ad esempio [1] e [13]). Qui noi vogliamo segnalare la presenza di eliche in concomitanza con la ricerca delle curve biarmoniche nel senso di Eells e Sampson ([10]).
Su una varietà differenziabile tridimensionale M consideriamo la famiglia a due parametri di metriche riemanniane

$$(7) \qquad ds^2_{\ell,m} = \frac{dx^2+dy^2}{[1+m(x^2+y^2)]^2} + \left(dz + \frac{\ell}{2}\frac{ydx-xdy}{[1+m(x^2+y^2)]}\right)^2, \qquad \ell, m \in \mathbb{R},$$

dove $\{x,y,z\}$ è un sistema di coordinate locali su M. Queste metriche sono note da più di un secolo. Si trovano, in forme equivalenti, nella classificazione delle metriche tridimensionali data da L. Bianchi nel 1897 (vedi [4]); successivamente furono studiate nella forma (7) da É. Cartan, in [9], e da G. Vranceanu, in [23].
La loro rilevanza geometrica si basa sul fatto seguente: *la famiglia di metriche (7) include tutte le metriche di uno spazio tridimensionale omogeneo il cui gruppo delle isometrie ha dimensione* 4 *o* 6*, ad eccezione di quella a curvatura sezionale costante*

negativa.
In particolare, tra gli spazi che corrispondono ai diversi valori di ℓ e m ci sono 6 delle 8 strutture fondamentali di Thurston, che, a parte $\mathbb{R}^3$ ($\ell = m = 0$), possono essere rappresentate come punti del piano (ℓ, m) dei parametri ([18]).

• Se $\ell = 0$, allora M è il prodotto tra una superficie S con curvatura Gaussiana costante $4m$ e la retta reale $\mathbb{R}$.
• Se $4m - \ell^2 = 0$, allora M ha curvatura sezionale costante non negativa.
• Se $\ell \neq 0$ e $m > 0$, allora M è localmente $SU(2)$.
• Se $\ell \neq 0$ e $m < 0$, allora M è localmente $\widetilde{SL}(2,\mathbb{R})$, il rivestimento universale di $SL(2,\mathbb{R})$.
• Per $m = 0$ e $\ell \neq 0$ si ottiene una metrica invariante a sinistra sul gruppo di Heisenberg $\mathbb{H}_3$.



Le curve biarmoniche $\gamma : I \subset \mathbb{R} \to (N, h)$ di una varietà riemanniana sono soluzioni di un'equazione differenziale del quart'ordine

$$\nabla^3_{\gamma'}\gamma' - R(\gamma', \nabla_{\gamma'}\gamma')\gamma' = 0,$$

dove ∇ è la connessione di Levi-Civita associata alla metrica h di N, mentre R è il relativo operatore di curvatura di Riemann. Esse risolvono un problema variazionale e sono una naturale generalizzazione delle geodetiche. Le geodetiche sono curve biarmoniche, come si vede subito, ma ci sono curve biarmoniche che non sono geodetiche, e che sono chiamate curve *biarmoniche proprie*.
Sia $\{T = T_iE_i, N = N_iE_i, B = B_iE_i\}$ il riferimento di Frenet tangente a M lungo γ, rappresentato tramite la base ortonormale

$$E_1 = \left[1 + m(x^2 + y^2)\right] \frac{\partial}{\partial x} - \frac{\ell y}{2}\frac{\partial}{\partial z}; \ E_2 = \left[1 + m(x^2 + y^2)\right] \frac{\partial}{\partial y} + \frac{\ell x}{2}\frac{\partial}{\partial z}; \ E_3 = \frac{\partial}{\partial z}.$$

Allora si ha il seguente

TEOREMA 1. ([8]) *Sia $(M, ds^2_{\ell,m})$ una varietà di Bianchi-Cartan-Vranceanu. Una curva $\gamma : I \to (M, ds^2_{\ell,m})$ parametrizzata con l'ascissa curvilinea è una curva biarmonica propria se e solo se*

$$\text{(8)} \qquad \begin{cases} k = costante \neq 0 \\ \tau = costante \\ N_3 = 0 \\ k^2 + \tau^2 = \frac{\ell^2}{4} - (\ell^2 - 4m)B_3^2. \end{cases}$$

In particolare, le curve biarmoniche proprie sono eliche.

OSSERVAZIONE 1. ([8]) Le biarmoniche proprie:

- giacciono sul *cilindro circolare*

$$S = \{(x,y,z) \in M : (x-c)^2 + (y-d)^2 = b^2 \sin^2 \alpha_0\}$$

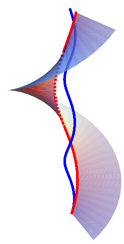
  e sono geodetiche di S. La superficie S è invariante per traslazioni lungo l'asse z, che sono isometrie per le metriche di Bianchi-Cartan-Vranceanu;

- sono geodetiche del *cilindro* $y = x \tan \beta_0 + a$ o $x = x_0$;

- nel caso $\ell^2 = 4m$, $\ell \neq 0$, cioè nel caso della sfera $\mathbb{S}^3$, è stato dimostrato in [6] che le curve biarmoniche proprie sono geodetiche del toro di Clifford, superficie $SO(2)$-invariante di $\mathbb{S}^3$.

Possiamo concludere che qualsiasi curva biarmonica propria delle varietà di Bianchi-Cartan-Vranceanu è una geodetica di una superficie che è invariante rispetto all'azione di un gruppo a 1-parametro di isometrie.

OSSERVAZIONE 2. ([7]) In ciascun punto $p \in \mathbb{H}_3$ i vettori tangenti alle curve biarmoniche per p formano un cono solido $\mathcal{C}_p$ in $T_p\mathbb{H}_3$. Per ogni vettore $X_p \in T_p\mathbb{H}_3 \setminus \mathcal{C}_p$, l'unica curva biarmonica γ uscente da p e tale che $\dot{\gamma}(p) = X_p$ è la geodetica determinata da p e da X_p. Invece ogni $X_p \in \mathcal{C}_p$ è simultaneamente tangente ad una geodetica e ad una curva biarmonica non geodetica. Nella figura a destra, se p è il punto più basso della curva $\gamma(t)$ bordo interno dell'elicoide, e se X_p è il vettore velocità di questa curva, risulta che γ è una curva biarmonica propria, mentre la curva che non giace sull'elicoide è anch'essa tangente a X_p in p ed è una geodetica.



OSSERVAZIONE 3. ([19]) Per la metrica $ds^2_{\ell,0}$ del gruppo di Heisenberg $\mathbb{H}_3$, i sottogruppi a un parametro $\sigma(u) = \exp uX$, dove $X = aE_1 + bE_2 + cE_3$, sono eliche. Risulta infatti che se $\sigma(u)$ non è una geodetica, allora la curvatura e la torsione k e τ sono

$$k = \frac{\ell c\sqrt{a^2+b^2}}{a^2+b^2+c^2}, \qquad \tau = -\frac{\ell(a^2+b^2-c^2)}{2(a^2+b^2+c^2)}. \tag{9}$$

Inoltre i sottogruppi a un parametro $\sigma(u) = \exp uX$ sono biarmonici se e solo se sono geodetiche. Infatti, se $\sigma(u)$ non è una geodetica, allora k e τ sono sempre correlate dalla formula

$$k^2 + \tau^2 = \frac{\ell^2}{4} \tag{10}$$

e si ha $B_3 \neq 0$; di conseguenza la (8) non è soddisfatta.

5. Geodetiche delle varietà di Bianchi-Cartan-Vranceanu

Ricordiamo innanzi tutto un importante teorema di Levi-Civita.

TEOREMA 2. *Se X è un campo di Killing per la varietà riemanniana (M, g), allora le equazioni delle geodetiche $\gamma(t)$ ammettono l'integrale primo*

$$g(\dot{\gamma}, X) = costante.$$

Il Teorema 2, applicato al campo di Killing $E_3 = \frac{\partial}{\partial z}$, dà il seguente integrale primo

$$g(\dot{\gamma}, E_3) = costante$$

e pertanto γ è un'elica secondo la definizione di Barros ([1]). Vogliamo determinare le curve $\gamma(s)$ delle varietà di Bianchi-Cartan-Vranceanu che sono eliche secondo Barros. L'unico campo di Killing di lunghezza costante per la metrica (7) è dato da

$$\xi = \lambda E_3\,, \qquad \lambda \in \mathbb{R}.$$

Sia $\gamma(s) = (x(s), y(s), z(s))$ una curva parametrizzata con l'ascissa curvilinea e $\dot{\gamma}(s) = T(s)$ il campo vettoriale tangente. Se γ forma un angolo costante con ξ risulta che $T_3(s) \in (-1, 1)$ ed essendo $T(s)$ di norma unitaria esiste una costante $\alpha_0 \in (0, \pi)$ e un'unica (a meno di una costante additiva $2k\pi$) funzione differenziabile β tale che ([8])

$$T(s) = \sin\alpha_0 \cos\beta(s) E_1 + \sin\alpha_0 \sin\beta(s) E_2 + \cos\alpha_0 E_3.$$

Usando le formule di Frenet risulta che la curvatura di γ è

$$k = \big(\beta' + 2my \sin\alpha_0 \cos\beta - 2mx \sin\alpha_0 \sin\beta - \ell \cos\alpha_0\big) \sin\alpha_0,$$

mentre la torsione geodetica τ di γ è data da

$$\tau = -\big(\beta' + 2my \sin\alpha_0 \cos\beta - 2mx \sin\alpha_0 \sin\beta - \ell \cos\alpha_0\big) \cos\alpha_0 - \frac{\ell}{2}. \tag{11}$$

In conclusione si ha

$$\sin\alpha_0 \tau + \cos\alpha_0 k = -\frac{\ell}{2} \sin\alpha_0$$

Quindi le eliche generalizzate secondo Barros sono curve di Bertrand. Se $\ell = 0$, ossia se la varietà è una varietà prodotto ($\mathbb{S}^2 \times \mathbb{R}$ o $\mathbb{H}^2 \times \mathbb{R}$), esse sono eliche anche in senso classico. Vale inoltre la

PROPOSIZIONE 1. ([17]) *Nelle varietà di Bianchi-Cartan-Vranceanu, le geodetiche $\gamma(t)$ uscenti dall'origine e tali che $\dot{\gamma}(0) = (u, v, w)$ possono essere definite come l'intersezioni di due superfici (Fig. 5):*

- *un cilindro circolare con le generatrici parallele all'asse z, o un piano parallelo all'asse z;*
- *una superficie di rotazione attorno all'asse z.*

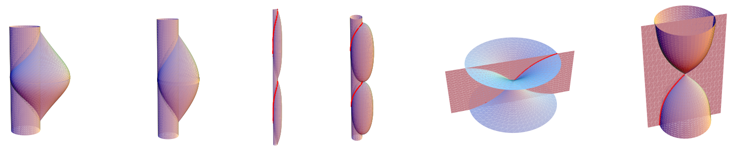


Figura 5: *Geodetiche nelle varietà di Bianchi-Cartan-Vranceanu.*

Riferimenti bibliografici

[1] BARROS M., *General helices and a theorem of Lancret*, Proceedings of the Am. Math. Soc. **125** (1997), 1503–1509.

[2] BEKKAR M., SARI T.; *Surfaces minimales réglées dans l'espace de Heisenberg* H_3. Rend. Sem. Mat. Univ. Politec. Torino, 50, **3** (1993), 243–254.

[3] BIANCHI L., *Lezioni di Geometria Differenziale*. Terza edizione interamente rifatta. Ed. Zanichelli, Bologna, 1927.

[4] BIANCHI L., *Gruppi continui e finiti*. Ed. Zanichelli, Bologna, 1928.

[5] CADDEO R. AND GRAY A., *Curve e Superfici*, Vol. 1, CUEC, Cagliari 2001.

[6] CADDEO R., MONTALDO S., ONICIUC C., *Biharmonic submanifolds of* $\mathbb{S}^3$, Internat. J. Math. **12** (2001), 867–876.

[7] CADDEO R., ONICIUC C., PIU P., *Explicit formulas for non-geodesic biharmonic curves of the Heisenberg group*, Rend. Sem. Mat. Univ. Politec. Torino **62** (2004), 265–277.

[8] CADDEO R., MONTALDO S., ONICIUC C., PIU P., *The Euler-Lagrange method for biharmonic curves*, Mediterr. J. Math. **3** no. 3–4 (2006), 449–465.

[9] CARTAN É., *Leçons sur la géométrie des espaces de Riemann*, Gauthier Villars, Paris 1946.

[10] EELLS J. AND SAMPSON J. H., *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160.

[11] ENNEPER A., *Zur theorie der Kurven doppelter Krümmung*, Math. Ann. **19** 1 (1881), 72–83.

[12] HAYDEN H. A., *On a generalized helix in a Riemannian n-space*, Proc. London Math. Soc. **32** (1931), 37–45.

[13] GLUCK H., *Geodesics in the unit tangent bundle of a round sphere*, Enseign. Math. (2) **34** 3–4, (1988), 233–246.

[14] LORIA G., *Curve sghembe speciali*, Vol. II, Zanichelli, Bologna 1925.

[15] MONTERDE J., *Curves with constant curvature ratios*, Bol. Soc. Mat. Mexicana (3) **13** 1 (2007), 177–186.

[16] PIRONDINI G., *Sulla teoria delle superficie di rivoluzione*, Ann. Mat. Pura App., Series 2 (1890-03-01), 165–212.

[17] PIU P. AND PROFIR M. M., *On the three-dimensional homogenous SO(2)-isotropic Riemannian manifolds*, An. Şt. Cuza, Iaşi. Mat. **57** 2 (2011), 361–376.

[18] PIU P., *Sur certains types de distributions non-intégrables totalement géodésiques*, Tesi di dottorato, Mulhouse 1988.

[19] PIU P. AND SANINI A., *One-parameter subgroups and minimal surfaces in the Heisenberg group*, Note Mat. **18** 1 (1999), 143–153.

[20] ROMERO-FUSTER M.C. AND SANABRIA-CODESAL E., *Generalized helices, twistings and flattenings of curves in n-space*, Mat. Contemp. **17** (1999), 267–280.

[21] SALKOWSKI E., *Das Aoust'sche Problem der Kurven theorie*, Sitzungsberg. der Berliner math. Ges., 1907.

[22] STRUIK D. J., *Lectures on Classical Differential Geometry*, Dover, New-York 1988.

[23] VRANCEANU G., *Leçons de géométrie différentielle*, Ed. Acad. Rep. Pop. Roum., vol I, Bucarest 1957.

[24] WHITTEMORE J. K., *Bertrand curves and helices*, Duke Math. J. **5** 1 (1940), 235–245.

AMS Subject Classification: 58E20, 31B30.

Renzo CADDEO, Paola PIU
Dipartimento di Matematica e Informatica, Università degli Studi di Cagliari
Via Ospedale 72, 09124 Cagliari, ITALY
e-mail: `caddeo@unica.it`, `piu@unica.it`

Lavoro pervenuto in redazione il 01.12.2015 e accettato il 04.03.2016.

Geom. Struc. on Riem. Man.-Bari
Vol. 73/1, 3–4 (2015), 37 – 50

G. Calvaruso[1]

HARMONICITY PROPERTIES OF PARACONTACT METRIC MANIFOLDS

Sommario. We shall describe some recent results concerning harmonicity properties of the Reeb vector field of a paracontact metric manifold, and their application to the study of paracontact Ricci solitons.

1. Introduction

Contact Riemannian structures are a natural odd-dimensional analogue to complex structures. Similarly, in pseudo-Riemannian settings, paracontact metric structures were introduced in [21] as an odd-dimensional counterpart to paraHermitian geometry.

Up to recent years, the study of paracontact metric manifolds essentially focused on the special case of paraSasakian manifolds. However, starting on 2009 with the work of Zamkovoy [29], a systematic study of paracontact metric structures began. Since then, paracontact metric manifolds have been studied under several different points of view, emphasizing similarities and differences with respect to the corresponding properties in the contact Riemannian case. Some recent results on paracontact and almost paracontact metric structures may be found in [4],[6]-[9],[15],[26] and references therein.

Harmonicity conditions of vector fields over pseudo-Riemannian manifolds have been intensively studied in recent years. We may refer to [3] and the monograph [18] and references therein for an overview on harmonicity properties of vector fields. Because of these results, it is a natural problem to investigate when the Reeb vector field of a paracontact metric manifold satisfies some harmonicity properties.

Given a (smooth, oriented, connected) semi-Riemannian manifold (M,g) and a unit vector field V on M, the *energy* of V is the energy of the corresponding smooth map $V : (M,g) \to (T_1M, g^s)$, where (T_1M, g^s) is the unit tangent bundle of (M,g), equipped with the Sasaki metric. V is said to be a *harmonic vector field* if $V : (M,g) \to (T_1M, g^s)$ is a critical point for the energy functional restricted to maps defined by unit vector fields.

The Reeb vector field ξ of a contact Riemannian manifold is harmonic if and only if ξ is a Ricci eigenvector [23]. This led to define *H-contact Riemannian manifolds* as contact metric manifolds, whose Reeb vector field is harmonic. Since their introduction, H-contact Riemannian manifolds have been intensively studied and their relations to other contact geometry properties are now well understood (see, for example, [2, Section 10.3.1], [16], [18, Chapter 4] and references therein). Correspondingly, *H-paracontact (metric) manifolds*, that is, paracontact metric manifolds whose Reeb

[1] Joint works with A. Perrone and D. Perrone.

vector field is harmonic, were introduced in [10]. It turns out that a paracontact metric manifold is H-paracontact if and only if the Reeb vector field is a Ricci eigenvector. Although formally similar to its contact Riemannian counterpart, this result needs a completely different approach, because of the deep differences arising between Riemannian and semi-Riemannian settings. In fact, in the Riemannian case, a self-adjoint operator admits an orthonormal basis of eigenvectors, but this property does not hold any more in pseudo-Riemannian settings.

Besides their intrinsic interest, the study of H-paracontact manifolds is also motivated by their relations with some other relevant geometric properties, like the Reeb vector field being an *infinitesimal harmonic transformation* or the existence of *paracontact Ricci solitons*. Under these points of view, a deep difference arises between the Riemannian case, where some strong rigidity results hold (see [24] and references therein), and the pseudo-Riemannian one, which allows several nontrivial interesting behaviours.

More precisely, the Reeb vector field of a contact metric manifold (M,φ,ξ,η,g) is an infinitesimal harmonic transformation (in particular, satisfies the Ricci soliton equation) if and only if the structure is both K-contact and Einstein. As such, it yields a trivial Ricci soliton, given by a Killing vector field together with an Einstein manifold. On the other hand, a positive answer was given in [8] to the open question about the existence of nontrivial paracontact Ricci solitons. A complete description of these objects can be achieved for the three-dimensional case, and their relationship with (κ,μ)-nullity condition arises.

The aim of the present paper is to illustrate these recent results, obtained in [7], [8] and [10], concerning harmonicity properties of the Reeb vector field of a paracontact metric manifold. The paper is organized in the following way. In Section 2 we report some basic information about paracontact metric manifolds and harmonicity properties of vector fields. The characterization of H-paracontact metric manifolds in terms of the Ricci operator is illustrated in Section 3. The relationship between H-paracontact metric manifolds and paracontact metric manifolds, whose Reeb vector field is 1-harmonic (equivalently, an infinitesimal harmonic transformation) is then discussed. In Section 4 we turn our attention to paracontact metric manifolds whose vector field determines a Ricci soliton, with particular regard to the study of nontrivial three-dimensional examples.

2. Preliminaries

2.1. Paracontact metric structures

An *almost paracontact structure* on a $(2n+1)$-dimensional (connected) smooth manifold M is a triple (φ,ξ,η), where φ is a $(1,1)$-tensor, ξ a global vector field and η a 1-form, such that

$$\varphi(\xi)=0, \qquad \eta\circ\varphi=0, \qquad \eta(\xi)=1, \qquad \varphi^2=Id-\eta\otimes\xi \tag{1}$$

and the restriction J of φ on the horizontal distribution $\ker\eta$ is an almost paracomplex structure (that is, the eigensubbundles D^+, D^- corresponding to the eigenvalues $1, -1$ of J have equal dimension n).

A pseudo-Riemannian metric g on M is *compatible* with the almost paracontact structure (φ, ξ, η) when

$$g(\varphi X, \varphi Y) = -g(X,Y) + \eta(X)\eta(Y). \tag{2}$$

In such a case, (φ, ξ, η, g) is said to be an *almost paracontact metric structure*. We can observe that by (1) and (A.3), $\eta(X) = g(\xi, X)$ for any compatible metric.

Any almost paracontact structure admits compatible metrics, which, by (A.3), have signature $(n+1, n)$. The *fundamental* 2*-form* Φ of an almost paracontact metric structure (φ, ξ, η, g) is defined by $\Phi(X,Y) = g(X, \varphi Y)$, for all tangent vector fields X, Y. If $\Phi = d\eta$, then the manifold (M, η, g) (or $(M, \varphi, \xi, \eta, g)$) is called a *paracontact metric manifold* and g the *associated metric*.

Throughout the paper, we shall denote with ∇ the Levi-Civita connection and by R the curvature tensor of g, taken with the sign convention

$$R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y].$$

An almost paracontact metric structure (φ, ξ, η, g) is said to be *normal* if

$$[\varphi, \varphi] - 2d\eta \otimes \xi = 0, \tag{3}$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion tensor of φ. A *paraSasakian manifold* is a normal paracontact metric manifold.

We recall that by definition ([15]), a *paracontact* (κ, μ)*-space* is a paracontact metric manifold $(M, \varphi, \xi, \eta, g)$, satisfying the curvature condition

$$R(X,Y)\xi = \kappa(\eta(X)Y - \eta(Y)X) + \mu(\eta(X)hY - \eta(Y)hX), \tag{4}$$

for all vector fields X, Y on M, where κ and μ are smooth functions, and $h := \frac{1}{2}\mathcal{L}_\xi \varphi$ is a $(1,1)$-tensor which plays an important role in the study of paracontact metric geometry. These manifolds generalize the paraSasakian ones, for which $\kappa = -1$ and μ is undetermined.

We also recall that any almost paracontact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ admits (at least, locally) a φ-*basis* [29], that is, a pseudo-orthonormal basis of vector fields of the form $\{\xi, E_1, .., E_n, \varphi E_1, .., \varphi E_n\}$, where $\xi, E_1, .., E_n$ are space-like vector fields and so, by (A.3), vector fields $\varphi E_1, .., \varphi E_n$ are time-like.

Observe that if $(M^3, \varphi, \xi, \eta, g)$ is a three-dimensional almost paracontact metric manifold, then any (local) pseudo-orthonormal basis of $\ker\eta$ determines a φ-basis, up to sign. In fact, if $\{e_2, e_3\}$ is a (local) pseudo-orthonormal basis of $\ker\eta$, with e_3, time-like, then (A.3) yields that $\varphi e_2 \in \ker\eta$ is time-like and orthogonal to e_2, so that $\varphi e_2 = \pm e_3$. Hence, $\{\xi, e_2, \pm e_3\}$ is a φ-basis.

We can provide a local description of all paracontact metric three-manifolds. Indeed, a local description was obtained in [7] for the much larger class of three-dimensional *natural* almost paracontact metric structures, for which one only requires the much weaker condition $\xi \in \ker d\eta$.

Let (φ,ξ,η,g) be a three-dimensional natural almost paracontact metric structure on M. Then,

$$2hX = \varphi(\nabla_X\xi) - \nabla_{\varphi X}\xi. \tag{5}$$

Let now $\{\xi, e, \varphi e\}$ denote a (local) φ-basis on M, with φe time-like. Then,

$$he = a_1 e + a_2\varphi e, \qquad h\varphi e = (-\varphi he) = -a_2 e - a_1\varphi e, \tag{6}$$

for some smooth functions a_1, a_2. Consequently,

$$\|h\|^2 = \mathrm{tr}h^2 = 2(a_1^2 - a_2^2). \tag{7}$$

In particular, by (6) and (7) we have that the following conditions are equivalent:

(i) $h^2 = 0$, that is, h is two-step nilpotent;

(ii) $\mathrm{tr}h^2 = 0$;

(iii) $a_2 = \varepsilon a_1 = \pm a_1$.

Since $\nabla_e\xi$ is orthogonal to ξ, there exist two smooth functions b_1, b_2, such that $\nabla_e\xi = b_1 e + b_2\varphi e$. So, (A.7) yields $\nabla_{\varphi e}\xi = (b_2 - 2a_1)e + (b_1 - 2a_2)\varphi e$. Moreover,

$$\nabla_\xi e = g(\nabla_\xi e,\xi)\xi + g(\nabla_\xi e, e)e - g(\nabla_\xi e,\varphi e)\varphi e = -g(\nabla_\xi e,\varphi e)\varphi e = a_3\varphi e,$$

where we put $a_3 := g(\nabla_\xi\varphi e, e)$. By similar computations and taking into account the compatibility of g, we obtain

$$\begin{cases} \nabla_e\xi = b_1 e + b_2\varphi e, & \nabla_{\varphi e}\xi = (b_2 - 2a_1)e + (b_1 - 2a_2)\varphi e, \\ \nabla_\xi e = a_3\varphi e, & \nabla_\xi\varphi e = a_3 e, \\ \nabla_e e = -b_1\xi + a_4\varphi e, & \nabla_{\varphi e}\varphi e = (b_1 - 2a_2)\xi + a_5 e, \\ \nabla_e\varphi e = b_2\xi + a_4 e, & \nabla_{\varphi e} e = (2a_1 - b_2)\xi + a_5\varphi e, \end{cases}$$

for some real smooth functions a_i, b_j. Equivalently, the Lie brackets of $\xi, e, \varphi e$ are described by

$$\begin{cases} [\xi, e] = -b_1 e + (a_3 - b_2)\varphi e, \\ [\xi,\varphi e] = (a_3 + 2a_1 - b_2)e + (2a_2 - b_1)\varphi e, \\ [e,\varphi e] = 2(b_2 - a_1)\xi + a_4 e - a_5\varphi e. \end{cases}$$

and must satisfy the Jacoby identity, which, by standard calculations, is proved to be equivalent to the following system of equations:

$$\begin{cases} \xi(b_2 - a_1) - 2(a_2 - b_1)(b_2 - a_1) = 0, \\ \xi(a_4) - e(a_3 + 2a_1 - b_2) - \varphi e(b_1) - a_4(2a_2 - b_1) - a_5(a_3 + 2a_1 - b_2) = 0, \\ \xi(a_5) + e(2a_2 - b_1) + \varphi e(b_2 - a_3) + a_4(b_2 - a_3) + a_5 b_1 = 0. \end{cases} \tag{8}$$

The above local description holds for any three-dimensional *natural* almost paracontact metric structure. In particular, the case of a paracontact metric is characterized by Equation $a_1 - b_2 = 1$, which, by the first Equation in (8), also yields $b_1 = a_2$. Therefore, we have the following result.

PROPOSITION 1. *Any three-dimensional paracontact metric manifold (M,φ,ξ,η,g) is locally described by*

$$\left\{\begin{array}{l}[\xi,e]=-a_2e+(a_3-a_1+1)\varphi e,\\ [\xi,\varphi e]=(a_3+a_1+1)e+a_2\varphi e,\\ [e,\varphi e]=-2\xi+a_4e-a_5\varphi e\end{array}\right. \tag{9}$$

with respect to a local φ-basis $\{\xi,e,\varphi e\}$, for some smooth functions $a_1,\dots,a_5$, satisfying

$$\left\{\begin{array}{l}\xi(a_4)-e(a_3+a_1)-\varphi e(a_2)-a_4a_2-a_5(a_3+a_1+1)=0,\\ \xi(a_5)+e(a_2)+\varphi e(a_1-a_3)+a_4(a_1-a_3-1)+a_5a_2=0.\end{array}\right. \tag{10}$$

2.2. Harmonicity properties of vector fields

We now report some basic information on harmonic vector fields over a pseudo-Riemannian manifold, referring to [3] and [18, Chapter 8] for more details.

Let (M,g) be an m-dimensional semi-Riemannian manifold, ∇ its Levi-Civita connection and V a smooth vector field on M. The *energy* of V is, by definition, the energy of the corresponding smooth map $V:(M,g)\to(TM,g^s)$, where g^s is the *Sasaki metric* (also referred to as the *Kaluza-Klein metric* in Mathematical Physics) on the tangent bundle TM of M. If M is compact, then

$$E(V)=\frac{1}{2}\int_M(\mathrm{tr}V^*g^s)dv=\frac{m}{2}\mathrm{vol}(M,g)+\frac{1}{2}\int_M g(V,V)dv,$$

(in the non-compact case, one works over relatively compact domains). Note that the energy of a vector field V, up to a constant, also corresponds to the *total bending* of V [28]. The Euler-Lagrange equation yields that a vector field V defines a harmonic map from (M,g) to (TM,g^s) if and only if its *tension field* $\tau(V)=\mathrm{tr}(\nabla dV)$ vanishes, that is, when

$$\mathrm{tr}[R(\nabla_{.}V,V)\cdot]=0 \qquad \text{and} \qquad \bar{\Delta}V=0.$$

Here, $\bar{\Delta}V:=-\mathrm{tr}\nabla^2V$ is the so called *rough Laplacian* of V. With respect to any local pseudo-orthonormal frame field $\{E_1,..,E_m\}$ on (M,g), with $\varepsilon_i=g(E_i,E_i)=\pm1$ for all indices $i=1,\dots,m$, it is given by

$$\bar{\Delta}V=\sum_i\varepsilon_i\left(\nabla_{\nabla_{E_i}E_i}V-\nabla_{E_i}\nabla_{E_i}V\right).$$

Next, for any real constant $r\neq0$, let $\mathfrak{X}^r(M)=\{V\in\mathfrak{X}(M):g(V,V)=r\}$ denote the set of tangent vector fields of constant length r. A vector field $V\in\mathfrak{X}^r(M)$ is called *harmonic* if it is a critical point for the energy functional $E|_{\mathfrak{X}^r(M)}$, restricted to vector fields of the same length. The Euler-Lagrange equation of this variational condition yields that V is harmonic if and only if

$$\bar{\Delta}V \quad \text{is collinear to} \quad V. \tag{11}$$

This characterization, obtained in the Riemannian case by G. Wiegmink [28] and C.M. Wood [27], was successively generalized in pseudo-Riemannian settings, to vector fields of constant length, if not light-like [3].

Let T_1M denote the *unit tangent sphere bundle* over M, and g^s the metric induced on T_1M by the Sasaki metric of TM. Then, the map $V:(M,g)\to(T_1M,g^s)$ is harmonic if V is a harmonic vector field and the additional condition

$$\mathrm{tr}[R(\nabla_\cdot V,V)\cdot]=0 \tag{12}$$

holds. In analogy with the contact metric case [23], we now introduce the following.

DEFINITION 1. A paracontact metric manifold (M,φ,ξ,η,g) is said to be *H-paracontact* if its Reeb vector field ξ is a harmonic vector field.

Let again (M^m,g) denote a pseudo-Riemannian manifold and $f:x\mapsto x'$ a point transformation in (M,g). If $\nabla(x)$ denotes the Levi-Civita connection at x and $\nabla'(x):=f^{-1}(\nabla(x'))$ [25], the *Lie difference* at x is defined as $\nabla'(x)-\nabla(x)$. The map f is said to be *harmonic* if $\mathrm{tr}(\nabla'(x)-\nabla(x))=0$.

Consider now a vector field V on M and the local one-parameter group of infinitesimal point transformations f_t generated by V. In this case, $(L_V\nabla)(x)=\nabla'(x)-\nabla(x)$ and so, V generates a group of harmonic transformations if and only if

$$\mathrm{tr}(L_V\nabla)=0.$$

In this case, V is said to be an *infinitesimal harmonic transformation* [22].

Infinitesimal harmonic transformations are also critical points for a suitable energy functional. In fact, if g^c denotes the *complete lift metric* of g to TM, which is of neutral signature (n,n), a vector field V on M defines a harmonic section $V:(M,g)\to(TM,g^c)$ if and only if V is an infinitesimal harmonic transformation [22].

Consequently, infinitesimal harmonic transformations are also called *1-harmonic vector fields*, because this harmonicity property is equivalent to the vanishing of the linear part of the tension field of the local one-parameter group of infinitesimal point transformations [17]. A vector field V is an infinitesimal harmonic transformation if and only if $\bar{\Delta}V=QV$, where Q denotes the Ricci operator (see for example [10],[13]).

3. *H*-paracontact metric manifolds

We start with the following result, obtained in [10].

THEOREM 1. [10] *Let (M,φ,ξ,η,g) be a $(2n+1)$-dimensional paracontact metric manifold. Then,*

$$\bar{\Delta}\xi=-4n\xi-Q\xi=\parallel\nabla\xi\parallel^2\xi-\mathrm{pr}_{|\ker\eta}Q\xi, \tag{13}$$

where $\parallel\nabla\xi\parallel^2=-(2n+\mathrm{tr}h^2)$ *and* $\mathrm{pr}_{|\ker\eta}$ *denotes the projection on* $\ker\eta$.

Differently from its analogue proved for the contact Riemannian case in [23], the above result could not be proved using the existence of an orthonormal basis of eigenvectors for the tensor h and so, it required a completely new and *ad hoc* argument.

Since ξ is harmonic if and only if $\bar{\nabla}\xi$ is collinear to ξ, as a direct consequence of the above result we get at once the following characterization.

THEOREM 2. *A paracontact metric manifold is H-paracontact if and only if the Reeb vector field ξ is an eigenvector of the Ricci operator.*

The characterization given in Theorem 2 implies that the class of H-paracontact manifolds is indeed very large. In fact, it is easy to check that paraSasakian and K-paracontact manifolds, paracontact (κ,μ)-spaces, three-dimensional homogeneous paracontact metric manifolds, η-Einstein paracontact manifolds all are examples of H-paracontact spaces. Moreover, it again follows from Theorem 2 that any paracontact metric structure, obtained applying a *$\mathcal{D}$-homothetic deformation* to an H-paracontact structure, is again paracontact. Thus, the property that ξ is harmonic is invariant under $\mathcal{D}$-homothetic deformations. We may refer to [10] for more details.

We now turn our attention to the case when ξ is an infinitesimal harmonic deformation. Observe that in general, a harmonic vector field needs not be 1-harmonic, nor conversely. This statement may be easily proved, for example, by comparing the classifications of harmonic and 1-harmonic left-invariant vector fields over three-dimensional Lorentzian Lie algebras, given respectively in [3] and [13].

However, in the case of the Reeb vector field of a paracontact metric manifold (M,φ,ξ,η,g), by Theorem 1 and equation $\rho(\xi,\xi) = -2n + \mathrm{tr}h^2$ for the Ricci curvature in the direction of ξ (see [29]), we get at once that

$$\bar{\Delta}\xi = Q\xi \quad \Longleftrightarrow \quad Q\xi = -2n\xi \quad \Longleftrightarrow \quad \mathrm{tr}h^2 = 0 \quad \text{and} \quad Q\xi \text{ is collinear to } \xi.$$

Therefore, taking into account Theorem 2, we have the following result.

THEOREM 3. *Let (M,φ,ξ,η,g) be a paracontact metric manifold. Then, the following properties are equivalent:*

1) $Q\xi = -2n\xi$*;*

2) ξ *is an infinitesimal harmonic transformation (equivalently,* 1*-harmonic);*

3) *M is H-paracontact and* $\mathrm{tr}h^2 = 0$.

By the above Theorem 3, if the Reeb vector field of a paracontact metric manifold is 1-harmonic, then it is harmonic. However, the converse does not hold, because of the additional condition $\mathrm{tr}h^2 = 0$. Several explicit examples of H-paracontact manifolds with $\mathrm{tr}h^2 \neq 0$ may be found in [10]. In particular, all paracontact (κ,μ)-spaces are H-paracontact; however, their Reeb vector field is 1-harmonic (that is, $\mathrm{tr}h^2 = 0$) only if $\kappa = -1$.

REMARK 1. Observe that the Reeb vector field of a contact *Riemannian* manifold is an infinitesimal harmonic transformation if and only if it is a Killing vector field [24], that is, when the contact Riemannian structure is K-contact.

Such a rigidity result does not hold for paracontact spaces. In fact, there exist paracontact metric manifolds for which ξ is 1-harmonic (for example, paracontact $(-1,\mu)$-spaces), which are not K-paracontact. Some explicit examples, related to the issue of the existence of nontrivial paracontact Ricci solitons, will be presented in the next Section.

4. Paracontact Ricci solitons

A *Ricci soliton* is a pseudo-Riemannian manifold (M,g), admitting a smooth vector field X, such that

$$\mathcal{L}_X g + \rho = \lambda g, \tag{14}$$

where $\mathcal{L}_X$, ρ and λ denote the Lie derivative in the direction of X, the Ricci tensor and a real number, respectively. A Ricci soliton is said to be *shrinking*, *steady* or *expanding*, according to whether $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively.

It is obvious that Einstein manifolds (together with $X = 0$ or, more generally, a Killing vector field X) satisfy the above equation. So, they are considered trivial Ricci solitons.

Ricci solitons were introduced in Riemannian Geometry [20] as the self-similar solutions of the *Ricci flow*, and play an important role in understanding its singularities. A wide survey on Riemannian Ricci solitons may be found in [14].

Recently, Ricci solitons have also been extensively studied in pseudo-Riemannian settings. For some recent results and further references on pseudo-Riemannian Ricci solitons, we may refer to [1],[5],[12] and references therein.

Given a class of pseudo-Riemannian manifolds (M,g), it is then a natural problem to solve Equation (14), especially when it holds for a smooth vector field playing a special role in the geometry of these manifolds. Under this point of view, the Reeb vector field ξ of a contact metric manifold would be a natural candidate.

However, in these settings, a strong rigidity result holds: the Reeb vector field of a contact Riemannian (or Lorentzian) manifold (M,φ,ξ,η,g) satisfies (14) if and only if (M,φ,ξ,η,g) is K-contact Einstein [11]. Thus, contact Riemannian or Lorentzian Ricci solitons are necessarily trivial.

As proved in [25], the study of Ricci solitons is closely related to the one of infinitesimal harmonic transformations. In fact, a vector field X determining a Ricci soliton (that is, satisfying (14)) is necessarily an infinitesimal harmonic transformation. The same argument, initially obtained in [25] for the Riemannian case, also applies to pseudo-Riemannian manifolds. We now introduce the following.

DEFINITION 2. A *paracontact Ricci soliton* is a paracontact metric manifold (M,φ,ξ,η,g), such that Equation (14) holds with $X=\xi$.

Consider now a paracontact Ricci soliton (M,φ,ξ,η,g). Then, in particular ξ is an infinitesimal harmonic transformation. Hence, Theorem 3 yields that M is H-paracontact and $Q\xi=-2n\xi$. Comparing with Equation (14), we then necessarily have that $\lambda=-2n$ and so, we have the following result.

THEOREM 4. *A paracontact Ricci soliton is H-paracontact, and is necessarily expanding.*

As we already recalled, in the contact Riemannian case, if ξ is an infinitesimal harmonic transformation, then ξ is Killing. Consequently, there do no exist nontrivial *contact* Ricci solitons.

On the other hand, the above Theorem 4 does not exclude the existence of nontrivial *paracontact* Ricci solitons. A careful analysis of the three-dimensional case shows that nontrivial paracontact Ricci solitons do exist.

As showed in Proposition 1, a three-dimensional paracontact metric manifold (M,φ,ξ,η,g) is completely described, with respect to a local φ-basis $\{\xi,e,\varphi e\}$, by Equations (9) and (10). Using these Equations, we can then obtain a complete description of the Levi-Civita connection and the curvature of such a manifold. First of all, the Levi-Civita connection is described by

$$(15)\quad \begin{array}{lll}\nabla_\xi\xi=0 & \nabla_e\xi=a_2e+(a_1-1)\varphi e, & \nabla_{\varphi e}\xi=-(a_1+1)e-a_2\varphi e,\\ \nabla_\xi e=a_3\varphi e, & \nabla_e e=-a_2\xi+a_4\varphi e, & \nabla_{\varphi e}e=(a_1+1)\xi+a_5\varphi e\\ \nabla_\xi\varphi e=a_3e, & \nabla_e\varphi e=(a_1-1)\xi+a_4e, & \nabla_{\varphi e}\varphi e=-a_2\xi+a_5e,\end{array}$$

Consequently, we find

$$\begin{aligned}\operatorname{tr}(\mathcal{L}_\xi\nabla)=&(\mathcal{L}_\xi\nabla)(\xi,\xi)+(\mathcal{L}_\xi\nabla)(e,e)-(\mathcal{L}_\xi\nabla)(\varphi e,\varphi e)\\ =&4\big(a_1^2-a_2^2\big)\xi\\ &+\big(e(a_2)+\varphi e(a_3+a_1)-\xi(a_5)+2a_1a_4+3a_2a_5+a_4(a_3+a_1+1)\big)e\\ &+\big(e(a_1-a_3)+\varphi e(a_2)+\xi(a_4)+2a_1a_5+3a_2a_4-a_5(a_3-a_1+1)\big)\varphi e.\end{aligned}$$

In particular, substituting $\xi(a_4)$ and $\xi(a_5)$ from (10), we conclude that $\operatorname{tr}(\mathcal{L}_\xi\nabla)=0$ if and only if $a_2=\varepsilon a_1=\pm a_1$ and $(e+\varepsilon\varphi e)(a_1)+2\varepsilon a_1a_4+2a_1a_5=0$. So, we proved the following result.

PROPOSITION 2. [8] *Let (M,φ,ξ,η,g) be a three-dimensional paracontact metric manifold. Then, the Reeb vector field ξ is an infinitesimal harmonic transformation if and only if the manifold is locally described by*

$$(16)\qquad \begin{cases}[\xi,e]=-\varepsilon a_1e+(a_3-a_1+1)\varphi e,\\ [\xi,\varphi e]=(a_3+a_1+1)e+\varepsilon a_1\varphi e,\\ [e,\varphi e]=-2\xi+a_4e-a_5\varphi e,\end{cases}$$

with respect to a local φ*-basis* $\{\xi, e, \varphi e\}$*, for some smooth functions* a_1, a_3, a_4, a_5*, satisfying*

$$\begin{cases} \xi(a_4) - e(a_3) + \varepsilon a_1 a_4 - a_5(a_3 - a_1 + 1) = 0, \\ \xi(a_5) - \varphi e(a_3) - \varepsilon a_1 a_5 - a_4(a_3 + a_1 + 1) = 0, \\ (e + \varepsilon\varphi e)(a_1) + 2\varepsilon a_1 a_4 + 2a_1 a_5 = 0. \end{cases} \tag{17}$$

Observe that if ξ is an infinitesimal harmonic transformation, then $a_2 = \varepsilon a_1$ and so, $\mathrm{tr} h^2 = 0$, compatibly with the result of Theorem 3.

We now determine the Ricci tensor of any paracontact metric three-manifold whose Reeb vector field is an infinitesimal harmonic transformation. Using (15) with $a_2 = \varepsilon a_1$ and taking into account (17), standard calculations yield

$$\begin{cases} R(\xi, e)\xi = -\big(\varepsilon\xi(a_1) + 2a_1a_3 + 1\big)e - \big(\xi(a_1) + 2\varepsilon a_1 a_3\big)\varphi e, \\ R(\xi, \varphi e)\xi = \big(\xi(a_1) + 2\varepsilon a_1 a_3\big)e + \big(\varepsilon\xi(a_1) + 2a_1a_3 - 1\big)\varphi e, \\ R(e, \varphi e)\xi = 0, \\ R(e, \varphi e)e = \big(\varphi e(a_4) - e(a_5) + 1 - 2a_3 + a_4^2 - a_5^2\big)\varphi e, \end{cases} \tag{18}$$

which easily imply that, with respect to $\{\xi, e, \varphi e\}$, the Ricci tensor ρ is completely described by the matrix

$$\rho = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -B - \varepsilon A & A \\ 0 & A & B - \varepsilon A \end{pmatrix}, \qquad \begin{array}{l} A := \xi(a_1) + 2\varepsilon a_1 a_3, \\ B := e(a_5) - \varphi e(a_4) + 2a_3 - a_4^2 + a_5^2. \end{array} \tag{19}$$

Observe that by (19), we see that the Ricci operator Q satisfies $Q\xi = -2\xi$, compatibly with the characterization proved in Theorem 3.

Next, from (18) we have that $R(\xi, e, \xi, e) = -\varepsilon A - 1$. On the other hand, it is well known that in dimension three, the curvature tensor R satisfies

$$\begin{aligned} R(X,Y,Z,V) = {} & g(X,Z)\rho(Y,V) - g(Y,Z)\rho(X,V) + g(Y,V)\rho(X,Z) \\ & - g(X,V)\rho(Y,Z) - \frac{r}{2}\big(g(X,Z)g(Y,V) - g(Y,Z)g(X,V)\big), \end{aligned} \tag{20}$$

where r denotes the scalar curvature. In particular, for $X = Z = \xi$ and $Y = V = e$, we then get $R(\xi, e, \xi, e) = \rho(e,e) - 2 - \frac{r}{2}$ and so, $\rho(e,e) = -\varepsilon A + \frac{r}{2} + 1$. Comparing with (19), we then find $B = -\frac{r}{2} - 1$. Consequently, (19) becomes

$$\rho = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -\varepsilon A + \frac{r}{2} + 1 & A \\ 0 & A & -\varepsilon A - \frac{r}{2} - 1 \end{pmatrix}, \tag{21}$$

Next, for any paracontact metric three-manifold $(M, \varphi, \xi, \eta, g)$, if $h^2 = 0$, then applying (15) with $a_2 = \varepsilon a_1$, we easily find that, with respect to $\{\xi, e, \varphi e\}$,

$$\mathcal{L}_\xi g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\varepsilon a_1 & -2a_1 \\ 0 & -2a_1 & 2\varepsilon a_1 \end{pmatrix}. \tag{22}$$

Therefore, ξ satisfies equation (14) if and only if

$$\lambda = -2, \qquad A = \xi(a_1) + 2\varepsilon a_1 a_3 = 2a_1 \qquad \text{and} \qquad r = -6. \tag{23}$$

Thus, we proved the following result.

THEOREM 5. *A three-dimensional paracontact metric manifold* $(M, \varphi, \xi, \eta, g)$ *is a paracontact Ricci soliton if and only if the manifold is locally described by equations* (16),(17) *and* (23)*, with respect to a local* φ*-basis* $\{\xi, e, \varphi e\}$*, for some smooth functions* a_1, a_3, a_4, a_5*. (In particular, the Ricci soliton is necessarily expanding.)*

Before giving some explicit examples, we now clarify the relationship between three-dimensional paracontact Ricci solitons and (κ, μ)-spaces. Checking Equation (1) for an arbitrary three-dimensional paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ with ξ an infinitesimal harmonic transformation (that is, by Proposition 2, locally described by (16) and (17)), by standard calculations (see also [8]) we get the following.

THEOREM 6. *A three-dimensional paracontact metric manifold* $(M, \varphi, \xi, \eta, g)$ *is a nontrivial paracontact Ricci soliton if and only if it is a* (κ, μ)*-space, with* $\kappa = -1$ *and* $\mu = -2\varepsilon$*, of scalar curvature* $r = -6$*.*

4.1. Homogeneous 3D nontrivial paracontact Ricci solitons

As proved in [4], a (simply connected, complete) homogeneous paracontact metric three-manifold is isometric to some Lie group G equipped with a left-invariant paracontact metric structure (φ, ξ, η, g). Then, denoting by $\mathfrak{g}$ the Lie algebra of G, we have that $\xi \in \mathfrak{g}$, η is a 1-form over $\mathfrak{g}$ and $\ker(\eta) \subset \mathfrak{g}$. Moreover, starting from a φ-basis of tangent vectors at the base point of G, by left translations one builds a φ-basis $\{\xi, e, \varphi e\}$ of the Lie algebra $\mathfrak{g}$.

Suppose now that the left-invariant paracontact metric structure (φ, ξ, η, g) is a nontrivial Ricci soliton. Then, with respect to the φ-basis $\{\xi, e, \varphi e\}$ of the Lie algebra $\mathfrak{g}$, standard calculations yield that necessarily

$$[\xi, e] = -a_1 e + (2 - a_1)\varphi e, \quad [\xi, \varphi e] = (2 + a_1)e + a_1 \varphi e, \quad [e, \varphi e] = -2\xi, \tag{24}$$

and the Reeb vector field of this paracontact metric structure satisfies (14).

The Lie algebra described in (24) is not solvable, as $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Indeed, comparing (24) with the classification of left-invariant paracontact metric structures obtained in [4] (see also [7]), we conclude that (24) corresponds to the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ of the universal covering of $SL(2, \mathbb{R})$. Thus, we have the following result.

THEOREM 7. *A homogeneous paracontact metric three-manifold* $(M, \varphi, \xi, \eta, g)$ *is a nontrivial paracontact Ricci soliton if and only if* M *is locally isometric to* $SL(2, \mathbb{R})$*, equipped with the left-invariant paracontact metric structure described in* (24)*.*

4.2. Inhomogeneous 3D nontrivial paracontact Ricci solitons

Using Darboux coordinates, the following local description can be given for any three-dimensional paracontact metric structure.

PROPOSITION 3. [8] *Any three-dimensional paracontact metric structure* (φ,ξ,η,g)*, in terms of local Darboux cordinates* (x,y,z)*, is explicitly described by*

$$\xi = 2\partial_z, \quad \eta = \frac{1}{2}(dz - ydx),$$

$$g = \frac{1}{4}\begin{pmatrix} a & b & -y \\ b & c & 0 \\ -y & 0 & 1 \end{pmatrix}, \qquad \varphi = \begin{pmatrix} -b & -c & 0 \\ (a-y^2) & b & 0 \\ -by & -cy & 0 \end{pmatrix}$$

for some smooth functions a,b,c*, satisfying* $ac - b^2 - cy^2 = -1$*. In particular,*

i) the structure is paraSasakian if and only if the functions a,b,c *do not depend on z,*

ii) $h^2 = 0$ *(equivalently,* $\mathrm{tr}h^2 = 0$*) if and only if* $b_z^2 - a_z c_z = 0$*.*

The above result emphasizes the fact that differently from the contact metric case, for paracontact metric structures the condition $h^2 = 0$ does not imply $h = 0$.

We now consider $M = \mathbb{R}^3(x,y,z)$, equipped with the paracontact metric structure (φ,ξ,η,g) defined by

$$\xi = 2\partial_z, \quad \eta = \frac{1}{2}(dz - ydx), \tag{25}$$

$$g = \frac{1}{4}\begin{pmatrix} F & 1 & -y \\ 1 & 0 & 0 \\ -y & 0 & 1 \end{pmatrix}, \qquad \varphi = \begin{pmatrix} -1 & 0 & 0 \\ F-y^2 & 1 & 0 \\ -y & 0 & 0 \end{pmatrix}, \tag{26}$$

where

$$F = F(x,y,z) = f(x) + \alpha e^{2z} + \beta y + \gamma,$$

for a smooth function $f(x)$ and some real constant $\alpha \neq 0, \beta, \gamma$. We have the following result.

THEOREM 8. *Let* (φ,ξ,η,g) *be the paracontact metric structure described by* (25) *and* (26)*. Then,* $(\mathbb{R}^3,\varphi,\xi,\eta,g)$ *is a paracontact Ricci soliton. Moreover, for any* $\beta \neq 0$*, such a paracontact metric structure is not locally homogeneous.*

Dimostrazione. The paracontact metric structure defined by (25) and (26) is of the type described in Proposition 3, with $a = F$, $b = 1$ and $c = 0$. This structure is not paraSasakian, because $a_z = F_z = 2\alpha e^{2z} \neq 0$. On the other hand, since b,c are constant, one concludes at once that $b_z^2 - a_z c_z = 0$. Therefore, $h^2 = 0 \neq h$.

We now determine a global φ- basis $(\xi, E, \varphi E)$ on M, taking

$$E := \frac{1}{\sqrt{2}}\big(4\partial_x + (2y^2 - 2F + 1)\partial_y + 4y\partial_z\big)$$

and so,

$$\varphi E = \frac{1}{\sqrt{2}}\big(-4\partial_x + (1 - 2y^2 + 2F)\partial_y - 4y\partial_z\big).$$

By a standard calculation, we then get

$$\begin{cases} [\xi, E] = -4\alpha e^{2z}(E + \varphi E), \\ [\xi, \varphi E] = 4\alpha e^{2z}(E + \varphi E), \\ [E, \varphi E] = -2\xi + \sqrt{2}(\beta - 2y)(E + \varphi E). \end{cases} \tag{27}$$

We can now compare (27) with (9), obtaining $a_1 = a_2 = 4\alpha e^{2z}$, $a_3 = -1$ and $a_4 = -a_5 = \sqrt{2}(\beta - 2y)$. It is then easy to check that the conditions in (17) and (23) are satisfied. Hence, by Theorem 5, we conclude that $(M, \varphi, \xi, \eta, g)$ is a paracontact Ricci soliton.

Finally, $(\mathbb{R}^3, \varphi, \xi, \eta, g)$ is a locally homogeneous paracontact metric manifold if and only if $\beta = 0$, in which case we get again the situation already described in Theorem 7, namely, it is locally isometric to the Lie group $SL(2, \mathbb{R})$ of. On the other hand, whenever $\beta \neq 0$, we described a paracontact Ricci soliton which is not locally homogeneous.

□

Riferimenti bibliografici

[1] BROZOS-VAZQUEZ M., CALVARUSO G., GARCIA-RIO E., GAVINO-FERNANDEZ S., *Three-dimensional Lorentzian homogeneous Ricci solitons*, Israel J. Math. **188** (2012), 385–403.

[2] BLAIR D. E., *Riemannian geometry of contact and symplectic manifolds*, Progress in Math. 203, Birkhäuser Boston, 2nd ed., 2010.

[3] CALVARUSO G., *Harmonicity properties of invariant vector fields on three-dimensional Lorentzian Lie groups*, J. Geom. Phys. **61** (2011), 498–515.

[4] CALVARUSO G., *Homogeneous paracontact metric three-manifolds*, Illinois J. Math. **55** (2011), 697–718.

[5] CALVARUSO G. AND FINO A., *Ricci solitons and geometry of four-dimensional non-reductive homogeneous spaces*, Canad. J. Math. **64** (2012), 778–804.

[6] CALVARUSO G. AND MARTIN-MOLINA V., *Paracontact metric structures on the unit tangent sphere bundle*, Ann. Mat. Pura Appl. **194** (2015), 1359–1380.

[7] CALVARUSO G. AND PERRONE A., *Classification of 3D left-invariant almost paracontact metric structures*, Adv. Geom., to appear.

[8] CALVARUSO G. AND PERRONE A., *Ricci solitons in three-dimensional paracontact geometry*, J. Geom. Phys. **98** (2015), 1–12.

[9] CALVARUSO G., PERRONE A., MUNTEANU M. I., *Killing magnetic curves in three-dimensional almost paracontact manifolds*, J. Math. Anal. Appl. **426** (2015), 423–439.

[10] CALVARUSO G. AND PERRONE D., *Geometry of H-paracontact metric manifolds*, Publ. Math. Debrecen **86** (2015), 325–346.

[11] CALVARUSO G. AND PERRONE D., *H-contact semi-Riemannian manifolds*, J. Geom. Phys. **71** (2013), 11–21.

[12] CALVARUSO G. AND ZAEIM A., *A complete classification of Ricci and Yamabe solitons of non-reductive homogeneous* 4*-spaces*, J. Geom. Phys. **80** (2014), 15–25.

[13] CALVINO-LOUZAO E., SEOANE-BASCOY J., VAZQUEZ-ABAL M. E., VAZQUEZ-LORENZO R., *One-harmonic invariant vector fields on three-dimensional Lie groups*, J. Geom. Phys. **62** (2012), 1532–1547.

[14] CAO H.-D., *Recent progress on Ricci solitons*, arXiv:0908.2006v1, Adv. Lect. Math. (ALM) **11** (2009), 1-38.

[15] CAPPELLETTI MONTANO B., KUPELI ERKEN I., MURATHAN C., *Nullity conditions in paracontact geometry*, Diff. Geom. Appl. **30** (2012), 665–693.

[16] CHUN S. H., PARK J. H., SEKIGAWA K., *H-contact unit tangent sphere bundles of Einstein manifolds*, Quart. J. Math. **62** (2011), 59–69.

[17] DODSON C. T. J., TRINIDAD PEREZ M., VAZQUEZ-ABAL M. E., *Harmonic Killing vector fields*, Bull. Belg. Math. Soc. Simon Stevin **9** (2002), 481–490.

[18] DRAGOMIR S. AND PERRONE D., *Harmonic Vector Fields: Variational Principles and Differential Geometry*, Elsevier, Science Ltd, 2011.

[19] GIL-MEDRANO O. AND HURTADO A., *Spacelike energy of timelike unit vector fields on a Lorentzian manifold*, J. Geom. Phys. **51** (2004), 82–100.

[20] HAMILTON R. S., *The Ricci flow on surfaces*, Contemporary Mathematics **71** (1988), 237–261.

[21] KANEYUKI S. AND WILLIAMS F. L., *Almost paracontact and parahodge structures on manifolds*, Nagoya Math. J. **99** (1985), 173–187.

[22] NOUHAUD O., *Transformations infinitesimales harmoniques*, C. R. Acad. Sc. Paris, Sér. A **274** (1972), 573–576.

[23] PERRONE D., *Contact metric manifolds whose characteristic vector field is a harmonic vector field*, Diff. Geom. Appl. **20** (2004), 367–378.

[24] PERRONE D., *Geodesic Ricci solitons on unit tangent sphere bundles*, Ann. Glob. Anal. Geom. **44** (2013), 91–103.

[25] STEPANOV S. E. AND SHELEPOVA V. N., *A remark on Ricci solitons*, Mat. Zametki **86** (2009), 474–477 (in Russian). Translation in Math. Notes **86** (2009), 447–450.

[26] WELYCZKO J., *On Legendre Curves in* 3*-Dimensional Normal Almost Paracontact Metric Manifolds*, Results Math. **54** (2009), 377–387.

[27] WOOD C. M., *On the energy of a unit vector field*, Geom. Dedicata **64** (1997), 319–330.

[28] WIEGMINK G., *Total bending of vector fields on Riemannian manifolds*, Math. Ann. **303** (1995), 325–344.

[29] ZAMKOVOY S., *Canonical connections on paracontact manifolds*, Ann. Glob. Anal. Geom. **36** (2009), 37–60.

AMS Subject Classification: 53D10, 53C15, 53C50, 53C43, 53C25

Giovanni CALVARUSO,
Dipartimento di Matematica e Fisica "Ennio De Giorgi", Università del Salento
Via Prov. Lecce-Arnesano, 73100 Lecce (LE), ITALY
e-mail: `giovanni.calvaruso@unisalento.it`

Lavoro pervenuto in redazione il 06.11.2015 e accettato il 14.01.2016.

Geom. Struc. on Riem. Man.-Bari
Vol. 73/1, 3–4 (2015), 51 – 61

B. Cappelletti-Montano[1] - A. De Nicola[2] - I. Yudin[23]

EXAMPLES OF 3-QUASI-SASAKIAN MANIFOLDS

Dedicated to Prof. Anna Maria Pastore on the occasion of her 70th birthday

Abstract. We provide a general method to construct examples of quasi-Sasakian 3-structures on a $(4n+3)$-dimensional manifold. Moreover, among this class, we give the first explicit example of a compact 3-quasi-Sasakian manifold which is not the global product of a 3-Sasakian manifold and a hyper-Kähler manifold.

1. Introduction

The class of quasi-Sasakian manifolds was introduced by Blair in [1], and then studied by several authors (e.g. [14, 13, 10]) in order to unify the most important classes of almost contact metric manifolds, namely the Sasakian and coKähler ones, which are quasi-Sasakian manifolds of maximal and minimal rank, respectively. Moreover any quasi-Sasakian manifold is canonically endowed with a transversely Kähler foliation, so that they can be thought as an odd-dimensional analogue of Kähler manifolds.

When on a smooth manifold M there are defined three distinct quasi-Sasakian structures, with the same compatible metric, which are related to each other by certain relations similar to the quaternionic identities, one says that M is a 3-quasi-Sasakian manifold (see Section 2 for the precise definition). The class of 3-quasi-Sasakian manifolds was extensively studied a few years ago in [5] and [6], where several properties on 3-quasi-Sasakian manifolds, which do not hold for a general quasi-Sasakian structure, were proved. In particular, it was proved that the aforementioned quaternionic-like structure forces any 3-quasi-Sasakian manifold of non-maximal rank $4l+3$ to be the local Riemannian product of a 3-c-Sasakian manifold and a hyper-Kähler manifold. Therefore a natural question arises: are there examples of 3-quasi-Sasakian manifolds which are not the global product of a 3-c-Sasakian manifold and a hyper-Kähler manifold? In this article we give an affirmative answer to this problem. We present a general procedure to produce a large class of examples, and we prove that the 11-dimensional 3-quasi-Sasakian manifold in this class is not a global product of 3-Sasakian and hyper-Kähler manifolds.

All manifolds considered in this paper will be assumed to be smooth and connected. For wedge product, exterior derivative and interior product we use the conventions as in Goldberg's book [9].

[1] Partially supported by Prin 2010/11 – Varietà reali e complesse: geometria, topologia e analisi armonica – Italy.

[2] Partially supported by CMUC, funded by the European program COMPETE/FEDER, by FCT (Portugal) grants PEst-C/MAT/UI0324/2011.

[3] Partially supported by the exploratory research project FCT IF/00016/2013 in the frame of Programa Investigador.

2. Preliminaries

We start with a few preliminaries on almost contact metric manifolds, referring the reader to the monographs [2, 4] and to the survey [8] for further details.

An *almost contact metric structure* on a $(2n+1)$-dimensional manifold M is the data of a $(1,1)$-tensor ϕ, a vector field ξ, called *Reeb vector field*, a 1-form η and a Riemannian metric g satisfying

$$\phi^2 = -I + \eta \otimes \xi, \ \ \eta(\xi) = 1, \ \ g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{1}$$

for all $X, Y \in \Gamma(TM)$, where I denotes the identity mapping on TM. From (1) it follows that $g(X, \phi Y) = -g(\phi X, Y)$, so that we can define the 2-form Φ on M by $\Phi(X,Y) = g(X, \phi Y)$, which is called the *fundamental* 2*-form* of the almost contact metric manifold (M, ϕ, ξ, η, g).

The manifold is said to be *normal* if the tensor field $N_\phi := [\phi, \phi]_{FN} + 2d\eta \otimes \xi$ vanishes identically. Normal almost contact metric manifolds such that both η and Φ are closed are called *coKähler manifolds* and those such that $d\eta = c\Phi$ are called *c-Sasakian manifolds*, where c is a non-zero real number (for $c = 2$ one obtains the well-known *Sasakian manifolds*).

The notion of quasi-Sasakian structure was introduced by Blair in his Ph.D. thesis in order to unify those of Sasakian and coKähler structures. A *quasi-Sasakian manifold* is defined as a normal almost contact metric manifold whose fundamental 2-form is closed. A quasi-Sasakian manifold M is said to be of *rank* $2p$ (for some $p \leq n$) if $(d\eta)^p \neq 0$ and $\eta \wedge (d\eta)^p = 0$ on M, and to be of *rank* $2p+1$ if $\eta \wedge (d\eta)^p \neq 0$ and $(d\eta)^{p+1} = 0$ on M (cf. [1, 14]). Blair proved that there are no quasi-Sasakian manifolds of even rank. Just like Blair and Tanno implicitly did, we will only consider quasi-Sasakian manifolds of constant (odd) rank. If the rank of M is $2p+1$, then the module $\Gamma(TM)$ of vector fields over M splits into two submodules as follows: $\Gamma(TM) = \mathcal{E}^{2p+1} \oplus \mathcal{E}^{2q}$, $p + q = n$, where

$$\mathcal{E}^{2q} = \{X \in \Gamma(TM) \mid i_X d\eta = 0 \text{ and } i_X \eta = 0\}$$

and $\mathcal{E}^{2p+1} = \mathcal{E}^{2p} \oplus \langle \xi \rangle$, $\mathcal{E}^{2p}$ being the orthogonal complement of $\mathcal{E}^{2q} \oplus \langle \xi \rangle$ in $\Gamma(TM)$. These modules satisfy $\phi \mathcal{E}^{2p} = \mathcal{E}^{2p}$ and $\phi \mathcal{E}^{2q} = \mathcal{E}^{2q}$ ([14]).

We now come to the main topic of our paper, i.e. 3-quasi-Sasakian geometry, which is framed into the more general setting of almost 3-contact geometry. An *almost contact metric* 3*-structure* on a smooth manifold M is the data of three almost contact structures (ϕ_1, ξ_1, η_1), (ϕ_2, ξ_2, η_2), (ϕ_3, ξ_3, η_3) satisfying the following relations, for any even permutation (α, β, γ) of $(1,2,3)$,

$$\begin{gathered} \phi_\gamma = \phi_\alpha \phi_\beta - \eta_\beta \otimes \xi_\alpha = -\phi_\beta \phi_\alpha + \eta_\alpha \otimes \xi_\beta, \\ \xi_\gamma = \phi_\alpha \xi_\beta = -\phi_\beta \xi_\alpha, \ \ \eta_\gamma = \eta_\alpha \circ \phi_\beta = -\eta_\beta \circ \phi_\alpha, \end{gathered} \tag{2}$$

and a Riemannian metric g compatible with each of them. This definition was introduced, independently, by Kuo ([12]) and Udriste ([15]). In particular, they proved that necessarily $\dim(M) = 4n+3$. It is well known that in any almost 3-contact metric manifold the Reeb vector fields ξ_1, ξ_2, ξ_3 are orthonormal with respect to the compatible metric g and that the structural group of the tangent bundle is reducible to $Sp(n) \times I_3$.

Moreover, by putting $\mathcal{H} = \bigcap_{\alpha=1}^{3} \ker(\eta_\alpha)$ one obtains a $4n$-dimensional *horizontal distribution* on M and the tangent bundle splits as the orthogonal sum $TM = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V} = \langle \xi_1, \xi_2, \xi_3 \rangle$ is the *vertical distribution*.

DEFINITION 1. *A* quasi-Sasakian 3-structure *is an almost contact metric* 3-*structure* $\{(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)\}_{\alpha\in\{1,2,3\}}$ *on a smooth manifold M such that each almost contact metric structure is quasi-Sasakian. The manifold M will be called a* 3-quasi-Sasakian manifold.

In particular, a quasi-Sasakian 3-structure such that each structure is Sasakian is called a *Sasakian* 3-*structure* and the manifold is said to be a 3-*Sasakian manifold*. A quasi-Sasakian 3-structure such that each structure is coKähler is called a *cosymplectic* 3-*structure* and the manifold is said to be a 3-*cosymplectic manifold*.

Let us collect some known results on 3-quasi-Sasakian manifolds. The following theorem combines the results obtained in Theorems 3.4 and 4.2 of [5], and Theorem 3.7 of [6].

THEOREM 1. *Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a* 3-*quasi-Sasakian manifold. Then the* 3-*dimensional distribution $\mathcal{V}$ generated by ξ_1, ξ_2, ξ_3 is integrable. Moreover, $\mathcal{V}$ defines a Riemannian foliation with totally geodesic leaves on M, and for any even permutation (α, β, γ) of $(1,2,3)$ and for some $c \in \mathbb{R}$*

$$[\xi_\alpha, \xi_\beta] = c\xi_\gamma.$$

Moreover, $c = 0$ if and only if the structure is 3-*cosymplectic.*

Using Theorem 1 we may divide 3-quasi-Sasakian manifolds in two classes according to the behaviour of the leaves of the foliation $\mathcal{V}$: those 3-quasi-Sasakian manifolds for which each leaf of $\mathcal{V}$ is locally $SO(3)$ (or $SU(2)$) (which corresponds to take in Theorem 1 the constant $c \neq 0$), and those for which each leaf of $\mathcal{V}$ is locally an abelian group (this corresponds to the case $c = 0$).

3. Basic properties of 3-quasi-Sasakian manifolds

For a 3-quasi-Sasakian manifold one can consider the ranks, a priori distinct, of the three quasi-Sasakian structures $(\phi_1, \xi_1, \eta_1, g)$, $(\phi_2, \xi_2, \eta_2, g)$, $(\phi_3, \xi_3, \eta_3, g)$. The following theorem assures that these three ranks coincide.

THEOREM 2 ([5, 6]). *Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a* 3-*quasi-Sasakian manifold of dimension $4n+3$. Then the* 1-*forms η_1, η_2 and η_3 have the same rank $4l+3$, for some integer $l \leq n$, or* 1 *according to $[\xi_\alpha, \xi_\beta] = c\xi_\gamma$ with $c \neq 0$ or $c = 0$, respectively.*

According to Theorem 2, the common rank of η_1, η_2, η_3 is called the *rank* of the 3-quasi-Sasakian manifold $(M,\phi_\alpha,\xi_\alpha,\eta_\alpha,g)$

Furthermore, for any 3-quasi-Sasakian manifold of rank $4l+3$ one can consider the following distribution

$$\mathcal{E}^{4m} := \left\{X \in \Gamma(\mathcal{H}) \mid i_X d\eta_\alpha = 0,\ \alpha = 1,2,3\right\} \quad (l+m=n)$$

and its orthogonal complement $\mathcal{E}^{4l+3} := (\mathcal{E}^{4m})^\perp$. In [6] it was proved the following remarkable property of 3-quasi-Sasakian manifolds, which in general does not hold for a general quasi-Sasakian structure.

THEOREM 3. *Let $(M,\phi_\alpha,\xi_\alpha,\eta_\alpha,g)$ be a 3-quasi-Sasakian manifold of rank $4l+3$. Then the distributions $\mathcal{E}^{4l+3}$ and $\mathcal{E}^{4m}$ are integrable and define Riemannian foliations with totally geodesic leaves.*

In particular it follows that $\nabla\mathcal{E}^{4l+3}\subset\mathcal{E}^{4l+3}$ and $\nabla\mathcal{E}^{4m}\subset\mathcal{E}^{4m}$. The leaves of such foliations are 3-c-Sasakian manifolds (i.e., for each $\alpha \in \{1,2,3\}$, $d\eta_\alpha = c\Phi_\alpha$) and hyper-Kähler manifolds, respectively (cf. Theorem 5.4 and Theorem 5.6 of [6]). Thus we can state the following corollary.

COROLLARY 1. *Any 3-quasi-Sasakian manifold of rank $4l+3$, with $1 \leq l < n$, is the local product of a 3-c-Sasakian manifold and of a hyperKähler manifold.*

Another strong consequence of Theorem 3 is the following

COROLLARY 2. *Any 3-quasi-Sasakian manifold of maximal rank $4n+3$ is necessarily 3-c-Sasakian.*

Thus in the two extremal cases — maximal and minimal rank — the geometry of a 3-quasi-Sasakian manifold is well known. In the rank 1 case, the structure turns out to be 3-cosymplectic and we can refer the reader to [7] for the main properties of these geometric structures and non-trivial examples. In the rank $(4n+3)$ case, by applying a certain homothety one can obtain a 3-Sasakian structure.

Thus we shall deal with the non-trivial cases $\mathrm{rank}(M) \neq 1$, $\mathrm{rank}(M) \neq \dim(M)$.

4. A general construction

Let $(M',\phi'_\alpha,\xi'_\alpha,g')$ and (M'',J''_α,g'') be a 3-Sasakian and a hyper-Kähler manifold, respectively. Set $\dim(M') = 4l+3$ and $\dim(M'') = 4m$. We define a canonical 3-quasi-Sasakian structure on the product manifold $M := M' \times M''$ in the following way.

We define as Reeb vector fields $\xi_\alpha := \xi'_\alpha$, for each $\alpha \in \{1,2,3\}$. Next, let ϕ_α be the $(1,1)$-tensor field determined by

$$\phi_\alpha X := \begin{cases} \phi'_\alpha X, & \text{if } X \in \Gamma(TM') \\ J''_\alpha X, & \text{if } X \in \Gamma(TM''). \end{cases}$$

Finally, we consider the product metric $g := g' + g''$ and we define three 1-forms η_1, η_2, η_3 by $\eta_\alpha := g(\cdot, \xi_\alpha)$. From the definition it follows that the horizontal distribution $\mathcal{H} := \bigcap_{\alpha=1}^{3} \ker(\eta_\alpha)$ coincides with $\mathcal{H}' \oplus TM''$, where $\mathcal{H}'$ is the horizontal distribution of the 3-Sasakian manifold M'. Then on $\mathcal{H}$ the triple (ϕ_1, ϕ_2, ϕ_3) satisfies the quaternionic relations

$$\phi_\alpha \phi_\beta = -\phi_\beta \phi_\alpha = \phi_\gamma$$

for a cyclic permutation (α, β, γ) of $\{1,2,3\}$. On the other hand, $\phi_\alpha \xi_\beta = \phi'_\alpha \xi'_\beta = \xi'_\gamma = \xi_\gamma = -\phi_\beta \xi_\alpha$. Hence

$$\begin{gathered}
\phi_\gamma = \phi_\alpha \phi_\beta - \eta_\beta \otimes \xi_\alpha = -\phi_\beta \phi_\alpha + \eta_\alpha \otimes \xi_\beta, \\
\xi_\gamma = \phi_\alpha \xi_\beta = -\phi_\beta \xi_\alpha, \;\; \eta_\gamma = \eta_\alpha \circ \phi_\beta = -\eta_\beta \circ \phi_\alpha
\end{gathered}$$

and we conclude that $\{(\phi_\alpha, \xi_\alpha, \eta_\alpha)\}_{\alpha \in \{1,2,3\}}$ is an almost contact 3-structure on M. By the very definition of g and ϕ_α then we have that g is a compatible metric.

Let us show that $\{(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)\}_{\alpha \in \{1,2,3\}}$ is a 3-quasi-Sasakian structure on M. Notice that each fundamental 2-form $\Phi_\alpha := g(\cdot, \phi_\alpha \cdot)$ is given by

$$\Phi_\alpha(X,Y) := \begin{cases} \Phi'_\alpha(X,Y), & \text{if } X,Y \in \Gamma(TM') \\ 0, & \text{if } X \in \Gamma(TM'), \text{ if } Y \in \Gamma(TM'') \\ \Omega''_\alpha(X,Y), & \text{if } X,Y \in \Gamma(TM'') \end{cases}$$

where Φ'_α and Ω''_α denote the fundamental 2-forms of $(M', \phi'_\alpha, \xi'_\alpha, g')$ and (M'', J''_α, g''), respectively. By using the well-known formula

$$\begin{aligned}
d\Phi_\alpha(X,Y,Z) = {} & X(\Phi_\alpha(Y,Z)) + Y(\Phi_\alpha(Z,X)) + Z(\Phi_\alpha(X,Y)) \\
& - \Phi_\alpha([X,Y],Z) - \Phi_\alpha([Y,Z],X) - \Phi_\alpha([Z,X],Y)
\end{aligned}$$

we see that

$$d\Phi_\alpha(X,Y,Z) = \begin{cases} d\Phi'_\alpha(X,Y,Z), & \text{if } X,Y,Z \in \Gamma(TM') \\ 0, & \text{if } X,Y \in \Gamma(TM'), Z \in \Gamma(TM'') \\ 0, & \text{if } X \in \Gamma(TM'), Y,Z \in \Gamma(TM'') \\ d\Omega''_\alpha(X,Y,Z), & \text{if } X,Y,Z \in \Gamma(TM''). \end{cases}$$

Since Φ'_α and Ω''_α are closed, we conclude that also each Φ_α is closed. Moreover, in order to prove the normality of the 3-structure $\{(\phi_\alpha, \xi_\alpha, \eta_\alpha)\}_{\alpha \in \{1,2,3\}}$, it is enough to check the vanishing of N_{ϕ_α} on the couples of vector fields of this type:

$$N_{\phi_\alpha}(X',Y'), \quad N_{\phi_\alpha}(X',Y''), \quad N_{\phi_\alpha}(Y',Y''),$$

where X', Y' are vector fields on M' and X'', Y'' are vector fields on M''. Since $d\eta_\alpha = 0$ on TM'', using the definitions of ϕ_α and N_{ϕ_α},

$$\begin{gathered}
N_{\phi_\alpha}(X',Y') = N_{\phi'_\alpha}(X',Y') = 0, \\
N_{\phi_\alpha}(X'',Y'') = N_{J''_\alpha}(X'',Y'') = 0,
\end{gathered}$$

because M' is 3-Sasakian and M'' hyper-Kähler, and

$$N_{\phi_\alpha}(X',Y'') = \phi_\alpha^2[X',Y''] + [\phi_\alpha X', \phi_\alpha Y''] - \phi_\alpha[\phi_\alpha X', Y''] - \phi_\alpha[X', \phi_\alpha Y''] = 0$$

since each summand in the last equation is zero.

Therefore $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is a 3-quasi-Sasakian manifold with rank $4l+3 = \dim(M')$.

We say that $f\colon M \to M$ is a *3-quasi-Sasakian isometry* if it is an isometry of the Riemannian manifold (M,g) preserving each quasi-Sasakian structure, namely

$$f_* \circ \phi_\alpha = \phi_\alpha \circ f_*, \quad f_*\xi_\alpha = \xi_\alpha \tag{3}$$

for each $\alpha \in \{1,2,3\}$. Notice that from (3) it follows that

$$f^*\eta_\alpha = \eta_\alpha. \tag{4}$$

Indeed for any $X \in \Gamma(TM)$

$$f^*\eta_\alpha(X) = \eta_\alpha(f_*X) = g(f_*X, \xi_\alpha) = g(f_*X, f_*\xi_\alpha) = g(X, \xi_\alpha) = \eta_\alpha(X).$$

Given a free and properly discontinuous action of a discrete group (in particular, a free action of a finite group) G on a 3-quasi-Sasakian manifold M by 3-quasi-Sasakian isometries, the quotient M/G is a smooth manifold of the same dimension as M and inherits a 3-quasi-Sasakian structure from M.

Recall that $f\colon M'' \to M''$ is a hyper-Kähler isometry if f is an isometry of the Riemannian manifold (M'', g'') and

$$f_* \circ J''_\alpha = J''_\alpha \circ f_* \tag{5}$$

for each $\alpha \in \{1,2,3\}$. From (5) it follows that

$$f^*\Omega''_\alpha = \Omega''_\alpha.$$

Suppose G is a finite group that acts on M' by 3-Sasakian isometries and on M'' by hyper-Kähler isometries. Then G also acts on the product manifold $M' \times M''$ by $g \cdot (p', p'') = (g \cdot p', g \cdot p'')$, $g \in G$. It is easy to check that G preserves the 3-quasi-Sasakian structure on $M' \times M''$ defined above. If the action of G on $M' \times M''$ is free then the quotient $(M' \times M'')/G$ is a 3-quasi-Sasakian manifold.

As an application, we consider the 3-Sasakian manifold S^{4l+3}. We recall how the standard 3-Sasakian structure $(\phi'_\alpha, \xi'_\alpha, \eta'_\alpha, g')$ of the sphere is defined. Let us consider the sphere S^{4l+3} as an hypersurface in $\mathbb{H}^{l+1}$. Let (J_1, J_2, J_3) be the standard quaternionic structure of $\mathbb{H}^{l+1}$ that is upon identification of $T_x\mathbb{H}^{l+1}$ with $\mathbb{H}^{l+1}$ the operators J_1, J_2, J_3 act by multiplication with **i**, **j**, **k** on the left.

Let N be the outer vector field normal to the sphere. Then one can prove that the vector fields

$$\xi'_\alpha := -J_\alpha N \tag{6}$$

are tangent to the sphere. Moreover, for any $X \in \Gamma(TS^{4l+3})$, we decompose $J_\alpha X$ in their components tangent and normal to the sphere,

$$J_\alpha X = \phi'_\alpha X + \eta'_\alpha(X)N, \tag{7}$$

so obtaining, for each $\alpha \in \{1,2,3\}$, a tensor field ϕ'_α and a 1-form η'_α on S^{4l+3}. Then one can check that the geometric structure $\{(\phi'_\alpha, \xi'_\alpha, \eta'_\alpha, g')\}_{\alpha\in\{1,2,3\}}$ is a 3-Sasakian structure on S^{4l+3}, being g' the Riemannian metric induced by the Riemannian metric g of $\mathbb{H}^{l+1} \cong \mathbb{R}^{4l+4}$.

Now we consider the isometry f of $\mathbb{H}^{l+1}$ given by the multiplication with $\mathbf{i}$ on the right. Notice that $f(S^{4l+3}) = S^{4l+3}$, because for any $x \in S^{4l+3}$ one has $\|f(x)\| = \|x\mathbf{i}\| = \|x\| = 1$. Hence f induces an isometry on (S^{4l+3}, g'), again denoted by f. Notice that the associativity of the product in $\mathbb{H}$ implies

$$f_* \circ J_\alpha = J_\alpha \circ f_*.$$

Thus f is a hyper-Kähler isometry. Moreover, for every $X \in \Gamma(TS^{4l+3})$, one has $g(f_*N, f_*X) = g(N,X) = 0$, so that $f_*N \in (TS^{4l+3})^\perp = \langle N\rangle$. Since $\|N\| = 1$ and f is an isometry, it follows that

$$f_*N = N.$$

Then by (6) and (7) we get

$$f_*\xi'_\alpha = -f_*J_\alpha N = -J_\alpha f_*N = -J_\alpha N = \xi'_\alpha,$$

and, for all $X \in \Gamma(TS^{4l+3})$,

$$\begin{aligned} f_*(\phi'_\alpha X) + \eta'_\alpha(X)N &= f_*(\phi'_\alpha X) + \eta'_\alpha(X) f_*N \\ &= f_*J_\alpha X \\ &= J_\alpha f_* X \\ &= \phi'_\alpha(f_*X) + \eta'_\alpha(f_*X)N, \end{aligned}$$

from which, taking the tangential and the normal components to the sphere, it follows that $f_* \circ \phi'_\alpha = \phi'_\alpha \circ f_*$ and $f^*\eta'_\alpha = \eta'_\alpha$. Thus f is a 3-Sasakian isometry of S^{4l+3}. Moreover, f^4 is the identity operator. Thus we get an action of $\mathbb{Z}_4$ on S^{4l+3} by 3-Sasakian isometries.

Let m be a positive integer. We denote the hyper-Kähler isometry of $\mathbb{H}^m$, $(q_1,\dots,q_m) \mapsto (q_1\mathbf{i},\dots,q_m\mathbf{i})$, by h. The map h induces a hyper-Kähler isometry on the torus $\mathbb{T}^{4m} = \mathbb{H}^m/\mathbb{Z}^{4m}$. Thus h generates an action of $\mathbb{Z}_4$ on $\mathbb{T}^{4m}$ by hyper-Kähler isometries. Note, that $\mathbb{Z}_4$ acts freely on S^{4l+3}, but has a fixed point $[0]$ in $\mathbb{T}^{4m}$. Nevertheless, the resulting action of $\mathbb{Z}_4$ on $S^{4l+3} \times \mathbb{T}^{4m}$ is free. We will denote the 3-quasi-Sasakian manifold $(S^{4l+3} \times \mathbb{T}^{4m})/\mathbb{Z}_4$ by $M_{l,m}$.

Concerning this example, in view of Corollary 1, an interesting question is the following: is $M_{l,m}$ the global product of a 3-Sasakian manifold of dimension $4l+3$ and a hyperKähler manifold of dimension $4m$?

In the next Section we shall show that the answer is negative, at least in the case $l = 1$ and $m = 1$. Namely we will prove that the 3-quasi-Sasakian manifold

$$M_{1,1} := (S^7 \times \mathbb{T}^4)/\mathbb{Z}_4$$

is not topologically equivalent to the product of a 7-dimensional compact 3-Sasakian manifold and a 4-dimensional compact hyper-Kähler manifold.

5. The manifold $M_{1,1} = (S^7 \times \mathbb{T}^4)/\mathbb{Z}_4$

Let M be a compact Riemannian manifold and G a finite group freely acting on M. Denote by ρ_M the corresponding group homomorphism from G to $\mathrm{Aut}(M)$. Then from the Hodge theory we obtain the isomorphism

$$H^*(M/G) \cong H^*(M)^G := \{x \in H^*(M) \,|\, \rho(a)^* x = x, \text{ for all } a \in G\}. \tag{8}$$

Indeed, every harmonic form on M/G lifts to a G-periodic harmonic form on M and every G-periodic form on M defines a periodic form on M/G. Here it is important that the projection $M \to M/G$ is a local diffeomorphism and the Laplacian $\triangle$ is defined locally.

Now, let M and N be two compact manifolds with G-action given by $\rho_M \colon G \to \mathrm{Aut}(M)$ and $\rho_N \colon G \to \mathrm{Aut}(N)$. We will write $\rho\colon G \to \mathrm{Aut}(M \times N)$ for the corresponding action on the product $M \times N$. If ω is a q-form on M and σ is a p-form on N, then $\mathrm{pr}_M^* \omega \wedge \mathrm{pr}_N^* \sigma$ is a $(p+q)$-form on $M \times N$. Moreover,

$$\rho(a)^* (\mathrm{pr}_M^* \omega \wedge \mathrm{pr}_N^* \sigma) = \mathrm{pr}_M^* \rho_M(a)^* \omega \wedge \mathrm{pr}_M^* \rho_N(a)^* \sigma$$

for $a \in G$. By Künneth theorem we have

$$H^k(M \times N) = \bigoplus_{p+q=k} H^q(M) \otimes H^p(N).$$

From the above we see that $H^q(M) \otimes H^p(N)$ is a G-invariant subspace of $H^k(M \times N)$. Therefore

$$H^k(M \times N)^G = \bigoplus_{q+p=k} (H^q(M) \otimes H^p(N))^G. \tag{9}$$

Let us now specialize to the case of $M = S^7$ and $N = \mathbb{T}^4$ with the action of $\mathbb{Z}_4$ on S^7 and $\mathbb{T}^4$ defined in the previous section. Note that since the isometry $f\colon S^7 \to S^7$ was orientation preserving, the induced action of $\mathbb{Z}_4$ on $H^7(S^7) \cong \mathbb{R}$ is trivial. It is also clear that $\mathbb{Z}_4$ acts trivially on $H^0(S^7) \cong \mathbb{R}$. Thus for any $0 \leq k \leq 4$

$$(H^0(S^7) \otimes H^k(\mathbb{T}^4))^{\mathbb{Z}_4} \cong H^k(\mathbb{T}^4)^{\mathbb{Z}_4}, \qquad (H^7(S^7) \otimes H^k(\mathbb{T}^4))^{\mathbb{Z}_4} \cong H^k(\mathbb{T}^4)^{\mathbb{Z}_4}. \tag{10}$$

Let us denote the Betti numbers of $M_{1,1}$ by b_k and we write $\tilde{b}_k$ for $\dim H^k(\mathbb{T}_4)^{\mathbb{Z}_4}$. Then, from (8), (9), and (10) it follows that

$$\begin{aligned} &b_0 = \tilde{b}_0 = 1, \quad b_1 = \tilde{b}_1, \quad b_2 = \tilde{b}_2, \quad b_3 = \tilde{b}_3, \quad b_4 = \tilde{b}_4, \qquad b_5 = b_6 = 0, \\ &b_7 = \tilde{b}_0 = 1, \quad b_8 = \tilde{b}_1, \quad b_9 = \tilde{b}_2, \quad b_{10} = \tilde{b}_3, \quad b_{11} = \tilde{b}_4 = 1. \end{aligned} \tag{11}$$

Now we compute $\tilde{b}_1$, $\tilde{b}_2$, and $\tilde{b}_3$. Note that from the above equations and Poincaré duality for $M_{1,1}$, we get $\tilde{b}_1 = b_1 = b_{10} = \tilde{b}_3$. Thus it is enough to compute $\tilde{b}_1$ and $\tilde{b}_2$. The cup product on $H^*(\mathbb{T}^4)$ induces the $\mathbb{Z}_4$-invariant isomorphism

$$\begin{aligned} \Lambda^* H^1(\mathbb{T}^4) &\longrightarrow H^*(\mathbb{T}^4) \\ [\alpha_1] \wedge \cdots \wedge [\alpha_k] &\longmapsto [\alpha_1 \wedge \cdots \wedge \alpha_k], \end{aligned}$$

where $\Lambda^* V$ stands for the exterior algebra of a vector space V. Thus

$$\tilde{b}_k = \dim(\Lambda^k H^1(\mathbb{T}^4))^{\mathbb{Z}_4} = \dim(\Lambda^k H^1(\mathbb{T}^4))^{h^*},$$

where $h\colon \mathbb{T}^4 \to \mathbb{T}^4$ was defined in the previous section. Let $x_{\mathbf{1}}, x_{\mathbf{i}}, x_{\mathbf{j}}, x_{\mathbf{k}}$ be the coordinate functions on $\mathbb{T}^4$ induced from $\mathbb{H}$. Let $\theta_{\mathbf{1}}$, $\theta_{\mathbf{i}}$, $\theta_{\mathbf{j}}$, and $\theta_{\mathbf{k}}$ be the dual 1-forms. Then the classes $[\theta_{\mathbf{1}}]$, $[\theta_{\mathbf{i}}]$, $[\theta_{\mathbf{j}}]$, and $[\theta_{\mathbf{k}}]$ give a basis of $H^1(\mathbb{T}^4)$. The matrix of h^* in this basis is

$$A := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The eigenvalues of A over $\mathbb{C}$ are i and $-i$. Since 1 is not among the eigenvalues there is no element in $H^1(\mathbb{T}^4)$ which is h^*-invariant. Thus $\tilde{b}_1 = 0$. The matrix of h^* in the basis

$$[\theta_{\mathbf{1}}] \wedge [\theta_{\mathbf{i}}], \quad [\theta_{\mathbf{1}}] \wedge [\theta_{\mathbf{j}}] + [\theta_{\mathbf{i}}] \wedge [\theta_{\mathbf{k}}], \quad [\theta_{\mathbf{1}}] \wedge [\theta_{\mathbf{j}}] - [\theta_{\mathbf{i}}] \wedge [\theta_{\mathbf{k}}]$$

$$[\theta_{\mathbf{1}}] \wedge [\theta_{\mathbf{k}}] + [\theta_{\mathbf{i}}] \wedge [\theta_{\mathbf{j}}], \quad [\theta_{\mathbf{1}}] \wedge [\theta_{\mathbf{k}}] - [\theta_{\mathbf{i}}] \wedge [\theta_{\mathbf{j}}], \quad [\theta_{\mathbf{j}}] \wedge [\theta_{\mathbf{k}}]$$

of $\Lambda^2 H^1(\mathbb{T}^4)$ is $\operatorname{diag}(1,-1,1,1,-1,1)$. Thus $\tilde{b}_2 = \dim(\Lambda^2 H^1(\mathbb{T}^4))^{h^*} = 4$. Using that $\tilde{b}_1 = 0 = \tilde{b}_3$, $\tilde{b}_2 = 4$, we get from (11)

$$b_0 = b_4 = b_7 = b_{11} = 1, \quad b_1 = b_3 = b_5 = b_6 = b_8 = b_{10} = 0, \quad b_2 = b_9 = 4.$$

Thus the Poincaré polynomial of $M_{1,1}$ is

$$P(t) := 1 + 4t^2 + t^4 + t^7 + 4t^9 + t^{11} = (1+t^7)(1+4t^2+t^4). \tag{12}$$

Suppose $M_{1,1} \cong M' \times M''$, where M' is a 7-dimensional 3-Sasakian manifold and M'' a 4-dimensional hyper-Kähler manifold. Denote by P' and P'' the Poincaré polynomial of M', respectively of M''. Then by Künneth theorem

$$P(t) = P'(t)P''(t). \tag{13}$$

We will write $p_1 \leq p_2$ for two polynomials with non-negative coefficients if all the coefficients of $p_2 - p_1$ are non-negative. We also write $p_1 < p_2$ if $p_1 \leq p_2$ and $p_1 \neq p_2$. It is obvious that if $p_1 \leq p_2$ then $p_1 p \leq p_2 p$, and if $p_1 < p_2$ then $p_1 p < p_2 p$ for any non-zero polynomial p with non-negative coefficients.

With this notation we have $P''(t) \geq 1 + t^7$, since M'' is a compact orientable 7-dimensional manifold.

Let us recall the following well-known result that follows from Enriques-Kodaira classification [11] of compact closed surfaces (see e.g. [3]).

THEOREM 4. *If M^4 is a compact four-dimensional hyper-Kähler manifold, then M^4 is either a K3 surface or a four dimensional torus.*

If $M'' \cong \mathbb{T}^4$, then $P''(t) = 1+4t+6t^2+4t^3+t^4$. If M'' is a K3 surface then $P''(t) = 1+22t^2+t^4$. Thus in both cases $P''(t) > 1+4t^2+t^4$. Therefore

$$P(t) = P'(t)P''(t) > (1+t^7)(1+4t^2+t^7) = P(t),$$

which gives a contradiction to our assumption $M_{1,1} \cong M' \times M''$. So we finally proved

THEOREM 5. *There exists an 11-dimensional compact 3-quasi-Sasakian manifold of rank 7 which is not a global product of a 7-dimensional 3-Sasakian manifold and a 4-dimensional hyper-Kähler manifold.*

References

[1] BLAIR D. E., *The theory of quasi-Sasakian structures*, J. Differential Geom. **1** (1967), 331–345.

[2] BLAIR D. E., *Riemannian geometry of contact and symplectic manifolds*, 2nd edition, Birkhäuser, Boston, 2010.

[3] BOYER C. P., *A note on hyper-Hermitian four-manifolds*, Proc. Amer. Math. Soc. **102** (1988), 157–164.

[4] BOYER C. P. AND GALICKI, K., *Sasakian geometry*, Oxford University Press, 2008.

[5] CAPPELLETTI-MONTANO B., DE NICOLA A., DILEO G., *3-quasi-Sasakian manifolds*, Ann. Glob. Anal. Geom. **33** (2008), 397–409. Erratum: Ann. Glob. Anal. Geom. **35** (2009), 445–448.

[6] CAPPELLETTI-MONTANO B., DE NICOLA A., DILEO G., *The geometry of 3-quasi-Sasakian manifolds*, Internat. J. Math. **20** (2009), 1081–1105.

[7] CAPPELLETTI-MONTANO B., DE NICOLA A., YUDIN I., *Topology of 3-cosymplectic manifolds*, Q. J. Math. **64** (2013), 59–82.

[8] CAPPELLETTI-MONTANO B., DE NICOLA A., YUDIN I., *A survey on cosymplectic geometry*, Rev. Math. Phys. **25** (2013), 1343002 (55 pages).

[9] GOLDBERG S., *Curvature and homology*, Pure and Applied Mathematics, Vol. XI, Academic Press, New York, 1962.

[10] KANEMAKI S., *Quasi-Sasakian manifolds*, Tôhoku Math. J. **29** (1977), 227–233.

[11] KODAIRA K., *On the structure of compact complex analytic surfaces. I*, Amer. J. Math. **86** (1964), 751–798.

[12] KUO Y. Y., *On almost 3-structure*, Tôhoku Math. J. **22** (1970), 325-332.

[13] OLSZAK Z., *Curvature properties of quasi-Sasakian manifolds*, Tensor **38** (1982), 19–28.

[14] TANNO S., *Quasi-Sasakian structures of rank $2p+1$*, J. Differential Geom. **5** (1971), 317–324.

[15] UDRISTE C., *Structures presque coquaternioniennes*, Bull. Math. Soc. Sci. Math. R. S. Roumanie **13** (1969), 487–507.

AMS Subject Classification: 53C25, 53D35

Beniamino CAPPELLETTI-MONTANO,
Dipartimento di Matematica e Informatica, Universtà degli Studi di Cagliari
Via Ospedale 72, 09124 Cagliari, ITALY

e-mail: b.cappellettimontano@gmail.com

Antonio DE NICOLA,
CMUC, Department of Mathematics, University of Coimbra
3001-501 Coimbra, PORTUGAL
e-mail: antondenicola@gmail.com

Ivan YUDIN,
CMUC, Department of Mathematics, University of Coimbra
3001-501 Coimbra, PORTUGAL
e-mail: yudin@mat.uc.pt

Lavoro pervenuto in redazione il 12.11.2015 e accettato il 16.12.2015.

Geom. Struc. on Riem. Man.-Bari
Vol. 73/1, 3–4 (2015), 63 – 76

A. Carriazo[1] - P. Alegre[1] - C. Özgür - S. Sular

NEW EXAMPLES OF GENERALIZED SASAKIAN-SPACE-FORMS

Abstract. In this paper we study when a non-anti-invariant slant submanifold of a generalized Sasakian-space-form inherits such a structure, on the assumption that it is totally geodesic, totally umbilical, totally contact geodesic or totally contact umbilical. We obtain some general results (including some obstructions) and we also offer some explicit examples.

1. Introduction.

The study of generalized Sasakian-space-forms has been quickly developed since the first two named authors (jointly with David E. Blair) defined such a manifold in [1] as an almost contact metric manifold (M,ϕ,ξ,η,g) whose curvature tensor is given by

$$R = f_1R_1 + f_2R_2 + f_3R_3, \tag{1}$$

where f_1, f_2, f_3 are differential functions on M and

$$\begin{aligned} R_1(X,Y)Z &= g(Y,Z)X - g(X,Z)Y, \\ R_2(X,Y)Z &= g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z, \\ R_3(X,Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi, \end{aligned}$$

for any vector fields X,Y,Z. We denote it by $M(f_1, f_2, f_3)$. Actually, we can refer to the recent papers [2], [3], [4], [5], [6], [7], [18], [20] and [25].

But, as in any new subject, one of the most important things is the search for new examples. In this sense, a natural question arises now: if M is a submanifold isometrically immersed in a generalized Sasakian-space-form $\widetilde{M}(F_1,F_2,F_3)$, when does it inherit a generalized Sasakian-space-form structure, with functions f_1, f_2, f_3?

This is a non-trivial question, because two things have to be inherited from the ambient space. Firstly, the almost contact metric structure. Hence, we must work with some special classes of submanifolds, tangent to the structure vector field ξ. A natural election seems to be that of non-anti-invariant slant submanifolds (for some general background on the theory of slant submanifolds in almost contact metric manifolds, we recommend the survey paper [12]). But, secondly, in order to be a generalized Sasakian-space-form, the curvature tensor R of the submanifold M has to be written in a very special way. By using Gauss equation we have:

$$\begin{aligned} &F_1\widetilde{R}_1(X,Y,Z,W) + F_2\widetilde{R}_2(X,Y,Z,W) + F_3\widetilde{R}_3(X,Y,Z,W) \\ &= R(X,Y,Z,W) + g(\sigma(X,Z),\sigma(Y,W)) - g(\sigma(X,W),\sigma(Y,Z)), \end{aligned} \tag{2}$$

[1]The first two named authors are partially supported by the PAI group FQM-327 (Junta de Andalucía, Spain) and the MINECO-FEDER projects MTM2011-22621 and MTM2014-52197-P.

for any vector fields X,Y,Z,W tangent to M. Therefore, we must somehow control the second fundamental form σ of the immersion in order to obtain the correct writing for R.

Thus, after a preliminaries section containing some definitions and formulae for later use, in Section 3 we study the raised question for totally geodesic, totally umbilical, totally contact geodesic and totally contact umbilical non-anti-invariant slant submanifolds of a generalized Sasakian-space-form. We obtain some general results (Theorems 1, 2 and 3) as well as some obstructions (Theorems 4 and 5), and we also construct some explicit examples.

2. Preliminaries.

In this section, we recall some general definitions and basic formulas which we will use later. For more background on almost contact metric manifolds, we recommend the reference [8]. Anyway, we will recall some more specific notions and results in the following sections, when needed.

An odd-dimensional Riemannian manifold $(\widetilde{M},g)$ is said to be an *almost contact metric manifold* if there exist on $\widetilde{M}$ a $(1,1)$ tensor field ϕ, a vector field ξ (called the *structure vector field*) and a 1-form η such that $\eta(\xi)=1$, $\phi^2(X)=-X+\eta(X)\xi$ and $g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y)$, for any vector fields X,Y on $\widetilde{M}$. In particular, in an almost contact metric manifold we also have $\phi\xi=0$ and $\eta\circ\phi=0$.

Such a manifold is said to be a *contact metric* manifold if $d\eta=\Phi$, where $\Phi(X,Y)=g(X,\phi Y)$ is called the *fundamental 2-form* of $\widetilde{M}$. If, in addition, ξ is a Killing vector field, then $\widetilde{M}$ is said to be a *K-contact manifold*. It is well-known that a contact metric manifold is a K-contact manifold if and only if

$$\widetilde{\nabla}_X\xi=-\phi X, \tag{3}$$

for any vector field X on $\widetilde{M}$. On the other hand, the almost contact metric structure of $\widetilde{M}$ is said to be *normal* if $[\phi,\phi](X,Y)=-2d\eta(X,Y)\xi$, for any X,Y, where $[\phi,\phi]$ denotes the Nijenhuis torsion of ϕ, given by $[\phi,\phi](X,Y)=\phi^2[X,Y]+[\phi X,\phi Y]-\phi[\phi X,Y]-\phi[X,\phi Y]$. A normal contact metric manifold is called a *Sasakian manifold*. It can be proved that an almost contact metric manifold is Sasakian if and only if

$$(\widetilde{\nabla}_X\phi)Y=g(X,Y)\xi-\eta(Y)X,$$

for any X,Y.

In [23], J. A. Oubiña introduced the notion of a trans-Sasakian manifold. An almost contact metric manifold $\widetilde{M}$ is a *trans-Sasakian manifold* if there exist two functions α and β on M such that

$$(\widetilde{\nabla}_X\phi)Y \;=\; \alpha(g(X,Y)\xi-\eta(Y)X)+\beta(g(\phi X,Y)\xi-\eta(Y)\phi X), \tag{4}$$

for any X,Y on $\widetilde{M}$. If $\beta=0$, $\widetilde{M}$ is said to be an α-*Sasakian manifold*. Sasakian manifolds appear as examples of α-Sasakian manifolds, with $\alpha=1$. If $\alpha=0$, $\widetilde{M}$ is said to be a β-*Kenmotsu manifold*. Kenmotsu manifolds are particular examples with $\beta = 1$. If both α and β vanish, then M is a *cosymplectic manifold*.

In particular, from (4) it is easy to see that the following equations hold for a trans-Sasakian manifold:

$$\widetilde{\nabla}_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{5}$$

$$d\eta = \alpha\Phi.$$

Let now M be a submanifold of an almost contact metric manifold $(\widetilde{M}, \phi, \xi, \eta, g)$, tangent to the structure vector field ξ. We will denote also by g the induced metric on M and, if F is a differentiable function on $\widetilde{M}$, we will denote also by F the composition $F \circ x$, where $x : M \to \widetilde{M}$ is the corresponding immersion. We will write the Gauss and Weingarten formulas for this immersion as

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \tag{6}$$

$$\widetilde{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \tag{7}$$

for any X, Y (resp. U) tangent (resp. normal) to M. It is well-known that

$$g(A_U X, Y) = g(\sigma(X,Y), U). \tag{8}$$

If we define

$$(\overline{\nabla}_X \sigma)(Y,Z) = \nabla^{\perp}_X \sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \tag{9}$$

for any X, Y, Z tangent to M, then Codazzi's equation is given by

$$(\widetilde{R}(X,Y)Z)^{\perp} = (\overline{\nabla}_X \sigma)(Y,Z) - (\overline{\nabla}_Y \sigma)(X,Z), \tag{10}$$

where $(\widetilde{R}(X,Y)Z)^{\perp}$ denotes the normal component of $\widetilde{R}(X,Y)Z$.

For any vector field X tangent to M, we write

$$\phi X = TX + NX,$$

where TX (resp. NX) is the tangential (resp. normal) component of ϕX. Similarly, for any vector field U normal to M, we denote by tU the tangential component of ϕU, and it is well-known that

$$g(NX, U) = -g(X, tU). \tag{11}$$

If $\widetilde{M}$ is K-contact, from (3) and (6) we have

$$\nabla_X \xi = -TX, \tag{12}$$

$$\sigma(X,\xi) = -NX, \tag{13}$$

for any X on M. Similarly, if $\widetilde{M}$ is a trans-Sasakian manifold, if follows from (5) and (6) that:

$$\nabla_X \xi = -\alpha TX + \beta(X - \eta(X)\xi), \tag{14}$$
$$\sigma(X,\xi) = -\alpha NX. \tag{15}$$

The submanifold M is said to be *invariant* (resp. *anti-invariant*) if ϕX is tangent (resp. normal) to M, for any tangent vector field X, i.e., $N \equiv 0$ (resp. $T \equiv 0$). On the other hand, M is said to be *slant* if for any $x \in M$ and any $X \in T_x M$, linearly independent on ξ, the angle between ϕX and $T_x M$ is a constant $\theta \in [0,\pi/2]$, called the *slant angle* of M in $\widetilde{M}$ [21]. Invariant and anti-invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \pi/2$ respectively. A slant immersion which is neither invariant nor anti-invariant is called a *proper* slant immersion. In [9], it was proved that a submanifold M, tangent to the structure vector field ξ of an almost contact metric manifold, is θ-slant if and only if $T^2 = -\cos^2\theta(I - \eta \otimes \xi)$. Moreover, for any vector fields X,Y tangent to such a submanifold, we have:

$$g(TX,TY) = \cos^2\theta(g(X,Y) - \eta(X)\eta(Y)), \tag{16}$$

$$g(NX,NY) = \sin^2\theta(g(X,Y) - \eta(X)\eta(Y)). \tag{17}$$

If M is a non-anti-invariant θ-slant submanifold (i.e., $\theta \in [0,\pi/2)$), then it was proved in [10] that $(\overline{\phi},\xi,\eta,g)$ is an almost contact metric structure on M, where $\overline{\phi} = (\sec\theta)T$. If, in addition, the equation

$$(\nabla_X T)Y = \cos^2\theta(g(X,Y)\xi - \eta(Y)X) \tag{18}$$

is satisfied for any X,Y tangent to M, it was pointed out also in [10] that

$$(\nabla_X \overline{\phi})Y = \cos\theta(g(X,Y)\xi - \eta(Y)X),$$

which means that M is a $\overline{\alpha}$-Sasakian manifold, with $\overline{\alpha} = \cos\theta$. Actually, it was shown in [9] that slant submanifolds satisfying equation (18) play a very important role in that theory. Slant submanifolds in trans-Sasakian manifolds have been investigated in [15, 16, 17].

With respect to the behavior of its second fundamental form, a submanifold is said to be *totally geodesic* if σ vanishes identically, and it is called *totally umbilical* if

$$\sigma(X,Y) = g(X,Y)H \tag{19}$$

for any tangent vector fields X,Y, where H denotes the mean curvature vector. Any totally geodesic submanifold is totally umbilical, and the converse is true if and only if $H = 0$, i.e., if and only if the submanifold is minimal. But there are some other kinds of submanifolds more interesting in almost contact Riemannian geometry. Hence, a submanifold M of an almost contact metric manifold $(\widetilde{M},\phi,\xi,\eta,g)$ is said to be *totally contact geodesic* if

$$\sigma(X,Y) = \eta(X)\sigma(Y,\xi) + \eta(Y)\sigma(X,\xi), \tag{20}$$

for any X, Y tangent to M, and it is called *totally contact umbilical* if there exists a normal vector field V such that

$$\sigma(X,Y) = (g(X,Y) - \eta(X)\eta(Y))V + \eta(X)\sigma(Y,\xi) + \eta(Y)\sigma(X,\xi), \tag{21}$$

for any X, Y on M. Once again, any totally contact geodesic submanifold is totally contact umbilical (with $V = 0$). If $\widetilde{M}$ is K-contact or trans-Sasakian, it is easy to see that $V = ((m+1)/m)H$, where $m+1$ is the dimension of M. Therefore, in such two cases, if M is totally contact umbilical, then it is totally contact geodesic if and only if it is minimal.

3. Slant submanifolds of a generalized Sasakian-space-form.

In this section we obtain some new examples of generalized Sasakian-space-forms, by working with a non-anti-invariant θ-slant submanifold M of a generalized Sasakian-space-form $\widetilde{M}(F_1, F_2, F_3)$, under certain conditions. We always consider on the submanifold the induced almost contact metric structure $(\overline{\phi}, \xi, \eta, g)$ described in the previous section. Of course, the key tool to relate curvature tensors on both the submanifold and the ambient manifold is Gauss' equation (2). Actually, it is clear that

$$\widetilde{R}_i(X,Y)Z = R_i(X,Y)Z, \quad i = 1,3,$$

for any tangent vector fields X, Y, Z. On the other hand, with respect to $\widetilde{R}_2$, we have

$$\widetilde{R}_2(X,Y,Z,W) = g(X,TZ)g(TY,W) - g(Y,TZ)g(TX,W) + 2g(X,TY)g(TZ,W),$$

for any X, Y, Z, W tangent to M. But, since $\overline{\phi} = (\sec\theta)T$, the above equation can be written as

$$\widetilde{R}_2(X,Y,Z,W) = \cos^2\theta R_2(X,Y,Z,W),$$

and so Gauss' equation turns into:

$$\begin{aligned} &F_1R_1(X,Y,Z,W) + \cos^2\theta F_2R_2(X,Y,Z,W) + F_3R_3(X,Y,Z,W) \\ &= R(X,Y,Z,W) + g(\sigma(X,Z),\sigma(Y,W)) - g(\sigma(X,W),\sigma(Y,Z)). \end{aligned} \tag{22}$$

Therefore, we can obtain the following result:

THEOREM 1. *Let M be a θ-slant submanifold of a generalized Sasakian-space-form $\widetilde{M}(F_1, F_2, F_3)$, with $\theta \in [0, \pi/2)$.*

i) If M is totally geodesic, then it is a generalized Sasakian-space-form, with functions $f_1 = F_1$, $f_2 = \cos^2\theta F_2$, $f_3 = F_3$.

ii) If M is totally umbilical, then it is a generalized Sasakian-space-form, with functions $f_1 = F_1 + ||H||^2$, $f_2 = \cos^2\theta F_2$, $f_3 = F_3$.

Proof. Statement $i)$ follows directly from (22), because in this case $\sigma \equiv 0$. To prove statement $ii)$, we just have to take into account that

$$\begin{aligned} g(\sigma(X,Z),\sigma(Y,W)) &= g(X,Z)g(Y,W)||H||^2, \\ g(\sigma(X,W),\sigma(Y,Z)) &= g(X,W)g(Y,Z)||H||^2, \end{aligned}$$

for any X,Y,Z,W tangent to M, and so

$$g(\sigma(X,Z),\sigma(Y,W)) - g(\sigma(X,W),\sigma(Y,Z)) = -||H||^2 R_1(X,Y,Z,W). \tag{23}$$

Therefore, the aimed writing (1) for R is obtained from (22) and (23). □

Of course, a particular case in Theorem 1 is that of invariant submanifolds ($\theta = 0$). In such a case, $\overline{\phi} = \phi$ so statement $i)$ is more than expected. Moreover, if $\widetilde{M}$ is K-contact or (α,β) trans-Sasakian with $\alpha \neq 0$ at any point of M, then it follows from either (13) or (15) and the totally geodesic condition that $N \equiv 0$, which means that the invariant case is the only one under such two conditions. Nevertheless, if $\alpha = 0$, we can find nice examples of proper slant submanifolds satisfying statement $i)$ of Theorem 1. To show them, let us consider a function $f > 0$ on $\mathbb{R}$ and a θ-slant submanifold M_2 of a complex-space-form $\widetilde{M}_2(c)$, where $\theta \in [0,\pi/2)$. It is clear that $\mathbb{R} \times_f M_2$ is a submanifold isometrically immersed in $\mathbb{R} \times_f \widetilde{M}_2$, and it was proved in [1] that this manifold can be endowed with a natural structure of β-Kenmotsu generalized Sasakian-space-form, with $\beta = f'/f$ and functions

$$F_1 = \frac{c - 4f'^2}{4f^2}, \quad F_2 = \frac{c}{4f^2}, \quad F_3 = \frac{c - 4f'^2}{4f^2} + \frac{f''}{f}. \tag{24}$$

Moreover, it is easy to see that $\mathbb{R} \times_f M_2$ is also a θ-slant submanifold, and it follows from [14, Theorem 1] that it is totally geodesic in $\mathbb{R} \times_f \widetilde{M}_2$ if M_2 is a totally geodesic submanifold of $\widetilde{M}_2$. Therefore, by using [13, Example 2.1] we have:

EXAMPLE 1. For any differentiable function $f > 0$ on $\mathbb{R}$ and any $\theta \in [0,\pi/2)$,

$$x(t,u,v) = (t, u\cos\theta, u\sin\theta, v, 0)$$

defines a 3-dimensional, totally geodesic, θ-slant submanifold M in $\mathbb{R} \times_f \mathbb{C}^2$. Thus, by virtue of Theorem 1 and (24), we obtain that M is a generalized Sasakian-space-form, with functions:

$$f_1 = -\frac{f'^2}{f^2}, \quad f_2 = 0, \quad f_3 = -\frac{f'^2}{f^2} + \frac{f''}{f}.$$

The above example can be easily extended to some others with higher dimensions.

Concerning statement $ii)$ of Theorem 1, let us first point out that, if $\widetilde{M}$ is K-contact or trans-Sasakian, then it follows from either (13) or (15) and (19) that M

should be minimal, and so totally geodesic. On the other hand, it was proved in [3] that, if M is a connected and totally umbilical submanifold, and $F_2 \neq 0$ at any point of M, then M is also an invariant manifold. In such a case, statement *ii*) of Theorem 1 was already obtained in [3, Theorem 6.2].

With respect to totally contact geodesic submanifolds, we have to impose the additional condition of being trans-Sasakian to the ambient manifold in order to obtain a suitable writing for R:

THEOREM 2. *Let M be a θ-slant submanifold of an (α,β) trans-Sasakian generalized Sasakian-space-form $\widetilde{M}(F_1,F_2,F_3)$, with $\theta \in [0,\pi/2)$. If M is totally contact geodesic, then it is a generalized Sasakian-space-form, with functions*

$$f_1 = F_1, \quad f_2 = \cos^2\theta F_2, \quad f_3 = F_3 + \alpha^2 \sin^2\theta.$$

Proof. As before, we start at (22). Now, if M is totally contact geodesic, then it follows from (15) and (20) that

$$\sigma(X,Y) \;=\; -\alpha\eta(X)NY - \alpha\eta(Y)NX, \tag{25}$$

for any X,Y tangent to M. Therefore, a direct calculation from (17) and (25) gives:

$$g(\sigma(X,Z),\sigma(Y,W)) - g(\sigma(X,W),\sigma(Y,Z)) = -\alpha^2\sin^2\theta R_3(X,Y,Z,W), \tag{26}$$

for any X,Y,Z,W tangent to M. The result is then obtained by putting (26) in (22). □

In particular, if the ambient manifold $\widetilde{M}$ is a Sasakian-space-form with constant ϕ–sectional curvature c, the functions in Theorem 2 would be given by

$$f_1 = \frac{c+3}{4}, \quad f_2 = \cos^2\theta\frac{c-1}{4}, \quad f_3 = \frac{c-1}{4} + \sin^2\theta, \tag{27}$$

taking into account that such a manifold is Sasakian (i.e., $\alpha = 1$ and $\beta = 0$).

Now again, there are nice examples of proper slant submanifolds satisfying Theorem 2. To show some of them, let $(\mathbb{R}^{2n+1},\phi,\xi,\eta,g)$ denote the manifold $\mathbb{R}^{2n+1}$ with its usual Sasakian structure given by

$$\eta \;=\; 1/2(dz - \sum_{i=1}^{n} y^i dx^i), \quad \xi \;=\; 2\frac{\partial}{\partial z},$$

$$g \;=\; \eta\otimes\eta + 1/4\left(\sum_{i=1}^{n}(dx^i\otimes dx^i + dy^i\otimes dy^i)\right),$$

$$\phi(\sum_{i=1}^{n}(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i}) + Z\frac{\partial}{\partial z}) \;=\; \sum_{i=1}^{n}(Y_i\frac{\partial}{\partial x^i} - X_i\frac{\partial}{\partial y^i}) + \sum_{i=1}^{n} Y_i y^i\frac{\partial}{\partial z},$$

where (x^i, y^i, z), $i = 1 \dots n$ are the cartesian coordinates. It is well-known that, with this structure, $\mathbb{R}^{2n+1}$ is a Sasakian-space-form with constant ϕ–sectional curvature -3 (see for example [8]). Then, by virtue of Theorem 2 and (27), any totally contact geodesic θ-slant ($\theta \in [0, \pi/2)$) submanifold of $\mathbb{R}^{2n+1}(-3)$ is a generalized Sasakian-space-form with functions:

$$f_1 = 0, \quad f_2 = f_3 = -\cos^2\theta. \tag{28}$$

Actually, we do have examples of such a submanifold:

EXAMPLE 2. For any $\theta \in [0, \pi/2)$,

$$x(u, v, t) \ = \ 2(u\cos\theta, u\sin\theta, v, 0, t)$$

defines a 3-dimensional minimal θ-slant submanifold in $\mathbb{R}^5(-3)$ [9]. Moreover, it was proved in [11] that all these submanifolds are totally contact geodesic.

EXAMPLE 3. For any $\theta \in [0, \pi/2)$,

$$x(u, v, w, s, t) \ = \ 2(u, 0, w, 0, v\cos\theta, v\sin\theta, s\cos\theta, s\sin\theta, t)$$

defines a 5-dimensional minimal θ-slant submanifold in $\mathbb{R}^9(-3)$ [9]. As in Example 2, it can be checked that all these submanifolds are also totally contact geodesic.

We can now ask what is the structure of new generalized Sasakian-space-forms given by above examples. In fact, it was proved in [9] that all of them satisfy equation (18) and so, as we pointed out in the preliminaries section, they are $\overline{\alpha}$-Sasakian manifolds, with $\overline{\alpha} = \cos\theta$. In this way, we can obtain examples of $\overline{\alpha}$-Sasakian generalized Sasakian-space-forms for any constant value $0 < \overline{\alpha} \leq 1$. Let us mention how functions f_1, f_2, f_3 given by (28) satisfy Theorem 4.2 of [2], saying that if $M(f_1, f_2, f_3)$ is a connected $\overline{\alpha}$-Sasakian generalized Sasakian-space-form, with dimension greater than or equal to 5 (which is the case of submanifolds given in Example 3) then, f_1, f_2, f_3 are constant functions such that $f_1 - \overline{\alpha}^2 = f_2 = f_3$.

On the other hand, submanifolds given in Example 1 also satisfy Theorem 2, because it follows from (15) and (20) that a submanifold of a β-Kenmotsu manifold is totally contact geodesic if and only if it is totally geodesic.

Moreover, with the techniques we have been using in this paper, we can also obtain a new obstruction result for totally contact geodesic slant submanifolds:

COROLLARY 1. *Let M be a θ-slant submanifold of a Sasakian-space-form $\widetilde{M}(c)$, with $\theta \in [0, \pi/2)$. If M is of dimension greater than or equal to 5, it satisfies (18) and it is totally contact geodesic, then M is invariant or $c = -3$.*

Proof. Under these conditions, it follows from Theorem 2 and the above remarks that M is a $\overline{\alpha}$-Sasakian generalized Sasakian-space-form with $\overline{\alpha} = \cos\theta$ and functions f_1, f_2, f_3 satisfying (27). But, by applying Theorem 4.2 of [2], we have $f_2 = f_3$, which implies that $\sin^2\theta(c+3)/4 = 0$, and so $\theta = 0$ or $c = -3$. □

Let us point out that the condition of M satisfying (18) is not strange at all. Actually, it was proved in [9] that any 3-dimensional proper slant submanifold of a K-contact manifold does. On the other hand, the result of $\widetilde{M}(c)$ being a Sasakian-space-form with $c=-3$ means that, if it is complete and simply connected, it must be $\mathbb{R}^{2n+1}$ with its usual Sasakian structure (by virtue of the well-known classic result of S. Tanno given in [24]).

Now, in order to study what happens with totally contact umbilical θ-slant submanifolds, we state the following lemma:

LEMMA 1. *Let M be an $(m+1)$-dimensional θ-slant submanifold of an (α,β) trans-Sasakian generalized Sasakian-space-form $\widetilde{M}(F_1,F_2,F_3)$, with $\theta\in[0,\pi/2)$. If M is totally contact umbilical, then*

$$\begin{aligned} R(X,Y)Z \;&= (F_1+||V||^2)R_1(X,Y)Z \\ &+\cos^2\theta F_2R_2(X,Y)Z \\ &+(F_3+\alpha^2\sin^2\theta+||V||^2)R_3(X,Y)Z \\ &+\alpha\{\eta(X)g(Z,tV)Y-\eta(Y)g(Z,tV)X \\ &+g(X,Z)\eta(Y)tV-g(Y,Z)\eta(X)tV \\ &+g(X,tV)\eta(Z)Y-g(Y,tV)\eta(Z)X \\ &+g(X,Z)g(Y,tV)\xi-g(Y,Z)g(X,tV)\xi\}, \end{aligned} \tag{29}$$

for any vector fields X,Y,Z tangent to M, where $V=((m+1)/m)H$.

Proof. It follows from (15) and (21) that

$$\sigma(X,Y) \;=\; (g(X,Y)-\eta(X)\eta(Y))V-\alpha\eta(X)NY-\alpha\eta(Y)NX, \tag{30}$$

for any X,Y tangent to M. Thus, equation (29) follows from (11), (17), (22) and (30), through a quite long computation. ☐

Lemma 1 shows that, in general, a totally contact umbilical slant submanifold does not inherit the aimed structure from the ambient manifold. Nevertheless, with some additional conditions, it does:

THEOREM 3. *Let M be an $(m+1)$-dimensional θ-slant submanifold of an (α,β) trans-Sasakian generalized Sasakian-space-form $\widetilde{M}(F_1,F_2,F_3)$, with $\theta\in[0,\pi/2)$. Let us suppose that M is totally contact umbilical and at least one of the following conditions holds:*

i) $\alpha=0$, *ii) M is minimal,* *iii) M is invariant.*

Then, M is a generalized Sasakian-space-form, with functions

$$f_1=F_1+((m+1)^2/m^2)\,||H||^2,\; f_2=\cos^2\theta F_2,\; f_3=F_3+\alpha^2\sin^2\theta+((m+1)^2/m^2)\,||H||^2.$$

Proof. It is clear that, in both cases $i)$ and $ii)$, the last terms of (29) vanish. It also happens in case $iii)$, because M being invariant implies $t \equiv 0$. □

Let us now see what can we say about conditions $i) - iii)$ of the above theorem. Firstly, if condition $i)$ holds, then we have the following direct corollary:

COROLLARY 2. *Let M be an $(m+1)$-dimensional totally contact umbilical θ-slant submanifold of a β-Kenmotsu generalized Sasakian-space-form $\widetilde{M}(F_1,F_2,F_3)$, with $\theta \in [0,\pi/2)$. Then, M is a generalized Sasakian-space-form, with functions*

$$f_1 = F_1 + ((m+1)^2/m^2)\,||H||^2, \quad f_2 = \cos^2\theta F_2, \quad f_3 = F_3 + ((m+1)^2/m^2)\,||H||^2.$$

Secondly, if condition $ii)$ of Theorem 3 holds, then M is a totally contact geodesic submanifold, and so the corresponding result was already obtained in Theorem 2. Thirdly, with some additional conditions, we can see that condition $iii)$ holds. We obtain this result as a particular case of the following theorem:

THEOREM 4. *Let M be a connected, totally contact umbilical submanifold, tangent to the structure vector field of an (α,β) trans-Sasakian generalized Sasakian-space-form $\widetilde{M}(F_1,F_2,F_3)$. If $\dim M > 3$ and*

$$F_2 \neq -\alpha^2 \tag{31}$$

at any point of M, then M is either invariant or anti-invariant.

Proof. A direct computation from (9) and (21) gives

$$\begin{aligned}(\overline{\nabla}_X\sigma)(Y,Z) &= (g(Y,Z)-\eta(Y)\eta(Z))\nabla^{\perp}_X V\\ &+g(Y,\nabla_X\xi)\sigma(Z,\xi)+g(Z,\nabla_X\xi)\sigma(Y,\xi)\\ &-g(Y,\nabla_X\xi)\eta(Z)V-g(Z,\nabla_X\xi)\eta(Y)V\\ &+\eta(Y)\sigma(Z,\nabla_X\xi)+\eta(Z)\sigma(Y,\nabla_X\xi)\\ &+\eta(Y)(\overline{\nabla}_X\sigma)(Z,\xi)+\eta(Z)(\overline{\nabla}_X\sigma)(Y,\xi),\end{aligned} \tag{32}$$

for any vector fields X,Y,Z tangent to M. Thus, by using (14), (15) and (32), Codazzi's equation can be written as

$$\begin{aligned}(\widetilde{R}(X,Y)Z)^{\perp} &= g(Y,Z)\nabla^{\perp}_X V - g(X,Z)\nabla^{\perp}_Y V\\ &+\alpha^2 g(Z,TX)NY-\alpha^2 g(Z,TY)NX\\ &-\alpha\beta g(X,Z)NY+\alpha\beta g(Y,Z)NX\\ &-2\alpha^2 g(X,TY)NZ,\end{aligned} \tag{33}$$

for any tangent X,Y,Z, orthogonal to ξ. Since $\dim M > 3$, given a tangent vector field X, orthogonal to ξ, we can choose an unit tangent vector field Y such that it is orthogonal to X, ϕX and ξ. Then, for such a choice, equation (33) reduces to

$$(\widetilde{R}(X,Y)Y)^{\perp} \;=\; \nabla^{\perp}_X V + \alpha\beta NX. \tag{34}$$

But, as $\widetilde{M}$ is a generalized Sasakian-space-form,

$$(\widetilde{R}(X,Y)Y)^{\perp} = F_2(\widetilde{R}_2(X,Y)Y)^{\perp} = 0. \tag{35}$$

Therefore, from (34) and (35) we deduce that

$$\nabla^{\perp}_X V = -\alpha\beta NX, \tag{36}$$

for any X orthogonal to ξ, and from (33) and (36) it follows that

$$\begin{aligned}(\widetilde{R}(X,Y)Z)^{\perp} &= +\alpha^2 g(Z,TX)NY - \alpha^2 g(Z,TY)NX \\ &\quad -2\alpha^2 g(X,TY)NZ \\ &= -\alpha^2(\widetilde{R}_2(X,Y)Z)^{\perp},\end{aligned} \tag{37}$$

for any tangent X,Y,Z, orthogonal to ξ. But, once again, as $\widetilde{M}$ is a generalized Sasakian-space-form,

$$(\widetilde{R}(X,Y)Z)^{\perp} = F_2(\widetilde{R}_2(X,Y)Z)^{\perp}. \tag{38}$$

Thus, from (37) and (38) we obtain that

$$(F_2+\alpha^2)(\widetilde{R}_2(X,Y)Z)^{\perp} = 0,$$

which gives that $(\widetilde{R}_2(X,Y)Z)^{\perp} = 0$, for any tangent X,Y,Z orthogonal to ξ, since we are working under the assumption of $F_2 \neq -\alpha^2$ at any point of M. In particular, for any X,Y orthogonal to ξ, we have:

$$(\widetilde{R}_2(X,Y)X)^{\perp} = 3g(X,TY)NX = 0. \tag{39}$$

Since M is connected, (39) implies that either $T \equiv 0$ (i.e., M is anti-invariant) or $N \equiv 0$ (i.e., M is invariant) and the proof concludes. □

If $\widetilde{M}$ has dimension greater than or equal to 5 (which it what happens if M is a slant submanifold with dimension greater than 3), then it was proved by J. C. Marrero in [22] that it is either an α-Sasakian or a β-Kenmotsu manifold. In the first case, it was proved in [2] that α, F_1, F_2, F_3 are constant functions such that $F_1 - \alpha^2 = F_2 = F_3$. Therefore, condition (31) is equivalent to $F_1 \neq 0$. Thus, Theorem 4 is a generalization to trans-Sasakian manifolds of a classical result of I. Ishihara and M. Kon, given in [19] for a Sasakian-space-form with constant ϕ-sectional curvature $c \neq -3$, because in such a space $F_1 = (c+3)/4$. In the second case, i.e., if $\widetilde{M}$ is a β-Kenmotsu manifold, condition (31) just means that F_2 does not vanish on M. Hence, Theorem 4 also implies that, if we want to look for non-invariant slant submanifolds satisfying Corollary 2, then we should ask F_2 to vanish. We can obtain such a β-Kenmotsu generalized Sasakian-space-form by considering a warped product $\mathbb{R} \times_f \mathbb{C}^n$.

For 3-dimensional slant submanifolds in a 5-dimensional ambient manifold, we have that at least one of conditions *ii*) and *iii*) of Theorem 3 holds, with no assumptions about functions F_1, F_2, F_3:

THEOREM 5. *Let M be a connected 3-dimensional totally contact umbilical θ-slant submanifold of a 5-dimensional (α,β) trans-Sasakian manifold $\widetilde{M}$, with $\theta \in [0,\pi/2)$. Then M is minimal or invariant.*

Proof. Let M be a 3-dimensional θ-slant submanifold of an (α,β) trans-Sasakian manifold. Then, it was proved in [17] that

$$A_{NX}Y = A_{NY}X + \alpha \sin^2\theta(\eta(X)Y - \eta(Y)X), \tag{40}$$

for any vector fields X,Y tangent to M, which means that

$$A_{NX}Y = A_{NY}X \tag{41}$$

for any tangent X,Y, orthogonal to ξ. If, in addition, M is totally contact umbilical, a direct computation by using (8), (15) and (21) gives

$$A_{NX}Y = g(NX,V)Y - \alpha g(NX,NY)\xi. \tag{42}$$

Thus, from (41) and (42), we obtain that

$$g(NX,V)Y = g(NY,V)X,$$

which means that

$$g(NX,V) = 0, \tag{43}$$

for any tangent vector field X, orthogonal to ξ. But, as $\dim\widetilde{M} = 5$, we know that, if M is not invariant, then $T_p^{\perp}(M)$ is spanned at any point $p \in M$ by

$$\{(NX)_p |\ X \text{ is orthogonal to } \xi\}.$$

Therefore, equation (43) implies that, if M is not invariant, then $V \equiv 0$ and so it is minimal. □

Theorem 5 can be extended to an $(m+1)$-dimensional totally contact umbilical θ-slant submanifold M of an $(2m+1)$-dimensional (α,β) trans-Sasakian generalized Sasakian-space-form, such that

$$(\nabla_X T)Y = \alpha\cos^2\theta(g(X,Y)\xi - \eta(Y)X) + \beta(g(TX,Y)\xi - \eta(Y)TX), \tag{44}$$

for any vector fields X,Y tangent to M, because it was proved in [17] that equation (44) is equivalent to (40), and so the above proof works. Actually, slant submanifolds satisfying (44) play a similar role in trans-Sasakian manifolds to those satisfying (18) in Sasakian ones.

Finally, we can give more information for a totally contact umbilical anti-invariant submanifold M, tangent to the structure vector field of an α-Sasakian generalized Sasakian-space-form $\widetilde{M}(F_1,F_2,F_3)$. From (14) and (32) we have

$$\begin{aligned}(\overline{\nabla}_X\sigma)(Y,Z) &= (g(Y,Z) - \eta(Y)\eta(Z))\nabla_X^{\perp}V \\ &\quad + \eta(Y)(\overline{\nabla}_X\sigma)(Z,\xi) + \eta(Z)(\overline{\nabla}_X\sigma)(Y,\xi),\end{aligned} \tag{45}$$

for any vector fields X,Y,Z tangent to M. Assume that M has parallel second fundamental form. Then from (45) we have

$$(g(Y,Z)-\eta(Y)\eta(Z))\nabla^{\perp}_{X}V=0,$$

which gives us $\nabla^{\perp}_{X}V=0$. Hence M has parallel mean curvature vector and we can state the following result:

THEOREM 6. *Let M be a totally contact umbilical anti-invariant submanifold, tangent to the structure vector field of an α-Sasakian generalized Sasakian-space-form $\widetilde{M}(F_1,F_2,F_3)$. If M has parallel second fundamental form, then the mean curvature vector of M is parallel.*

Acknowledgements: The authors want to express their gratitude to the referees of this paper for their valuable comments, which helped to improve it.

References

[1] ALEGRE P., BLAIR D. E. AND CARRIAZO A., *Generalized Sasakian-space-forms*, Israel J. Math. **141** (2004), 157–183.

[2] ALEGRE P. AND CARRIAZO A., *Structures on generalized Sasakian-space-forms*, Differential Geom. Appl. **26** (2008), 656–666.

[3] ALEGRE P. AND CARRIAZO A., *Submanifolds of generalized Sasakian space forms*, Taiwanese J. Math. **13** (2009), 923–941.

[4] ALEGRE P. AND CARRIAZO A., *Generalized Sasakian-space-forms and conformal changes of the metric*, Results Math. **59** (2011), 485–493.

[5] ALEGRE P., CARRIAZO A., KIM Y. H. AND YOON D. W., *B.-Y. Chen's inequality for submanifolds of generalized space forms*, Indian J. Pure Appl. Math. **38** (2007), 185–201.

[6] AL-GHEFARI R., AL-SOLAMY F. R. AND SHAHID M. H., *CR-submanifolds of generalized Sasakian space forms*, JP J. Geom. Topol. **6** (2006), 151–166.

[7] AL-GHEFARI R., AL-SOLAMY F. R. AND SHAHID M. H., *Contact CR-warped product submanifolds in generalized Sasakian space forms*, Balkan J. Geom. Appl. **11** (2006), 1–10.

[8] BLAIR D. E., *Riemannian Geometry of Contact and Symplectic Manifolds*, Birkhäuser, Boston 2002.

[9] CABRERIZO J. L., CARRIAZO A., FERNÁNDEZ L. M. AND FERNÁNDEZ M., *Slant submanifolds in Sasakian manifolds*, Glasgow Math. J. **42** (2000), 125–138.

[10] CABRERIZO J. L., CARRIAZO A., FERNÁNDEZ L. M. AND FERNÁNDEZ M., *Structure on a slant submanifold of a contact manifold*, Indian J. Pure Appl. Math. **31** (2000), 857–864.

[11] CARRIAZO A., *Subvariedades slant en variedades de contacto*, Ph.D. Thesis, University of Seville 1998.

[12] CARRIAZO A., *New developments in slant submanifolds theory*, In *Applicable Mathematics in the Golden Age*, pp. 339–356, Narosa Publising House, New Delhi 2002.

[13] CHEN B.-Y., *Geometry of Slant Submanifolds*, Katholieke Universiteit Leuven, Leuven 1990.

[14] CHEN B.-Y., *On warped product immersions*, J. Geom. **82** (2005), 36–49.

[15] GUPTA R. S., HAIDER S. M. K. AND SHAHID M. H., *Slant submanifolds of a Kenmotsu manifold*, Rad. Mat. **12** (2004), 205–214.

[16] GUPTA R. S., HAIDER S. M. K. AND SHAHID M. H., *Slant submanifolds of cosymplectic manifolds*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **50** (2004), 33–50.

[17] GUPTA R. S., HAIDER S. M. K. AND SHARFUDDIN A., *Slant submanifolds of a trans-Sasakian manifold*, Bull. Math. Soc. Sci. Math. Romanie (N.S.) **47(95)** (2004), 45–57.

[18] HONG S. AND TRIPATHI M. M., *On Ricci curvature of submanifolds of generalized Sasakian space forms*, Int. J. Pure Appl. Math. Sci. **2** (2005), 173–201.

[19] ISHIHARA I. AND KON M., *Contact totally umbilical submanifolds of a Sasakian space form*, Ann. Mat. Pura Appl. **114** (1977), 351–364.

[20] KIM U. K., *Conformally flat generalized Sasakian-space-forms and locally symmetric generalized Sasakian-space-forms*, Note Mat. **26** (2006), 55–67.

[21] LOTTA A., *Slant submanifolds in contact geometry*, Bull. Math. Soc. Roumanie **39** (1996), 183–198.

[22] MARRERO J. C., *The local structure of trans-Sasakian manifolds*, Ann. Mat. Pura Appl. **162** (1992), 77–86.

[23] OUBIÑA J. A., *New classes of almost contact metric structures*, Publ. Math. Debrecen **32** (1985), 187–193.

[24] TANNO S., *Sasakian manifolds with constant ϕ–holomorphic sectional curvature*, Tôhoku Math. J. **21** (1969), 501–507.

[25] YOON D. W. AND CHO K. S., *Inequality for warped products in generalized Sasakian space forms*, Int. Math. J. **5** (2004), 225–235.

AMS Subject Classification: 53C25, 53C40

Alfonso CARRIAZO,
Department of Geometry and Topology, Faculty of Mathematics, University of Seville
c/ Tarfia s/n, 41012 - Sevilla, SPAIN
e-mail: carriazo@us.es

Pablo ALEGRE,
Department of Economy, Quantitative Methods and Economy History, Statistics and Operational Research Area, Pablo de Olavide University
Ctra. de Utrera, km. 1, 41013 - Sevilla, SPAIN
e-mail: psalerue@upo.es

Cihan ÖZGÜR, Sibel SULAR,
Department of Mathematics, Balikesir University
Campus of Cagis, 10145 - Balikesir, TURKEY
e-mail: cozgur@balikesir.edu.tr, csibel@balikesir.edu.tr

Lavoro pervenuto in redazione il 13.01.2016 e accettato il 14.03.2016.

Geom. Struc. on Riem. Man.-Bari
Vol. 73/1, 3–4 (2015), 77 – 87

G. De Cecco[1]

RENÉ THOM: IL CONCETTO DI BORDO E IL BORDO DI UN CONCETTO

Abstract. In this article R. Thom's thought about the concept of boundary is briefly examined: we know because we manage to distinguish the contour of both things and concepts. The author then presents the difference, from a mathematical standpoint, between "boundary" and "frontier".

1. Introduzione

Innanzitutto auguri ad Anna Maria, da parte degli amici e colleghi dell'Università di Lecce. Io conosco Anna Maria dagli anni '70, quando veniva a Lecce, guidando un gruppo di collaboratrici, che chiamavamo "le ragazze di Bari" e questo nome è rimasto a lungo. In quel tempo sono stati ospiti a Lecce, invitati particolarmente da Ida Cattaneo, i protagonisti della Geometria Differenziale di allora: T. Willmore, J. Koszul, A. Lichnerowicz, Y. Choquet-Bruhat, W. Klingenberg, A. Dold, ..., che si fermavano per 8-10 giorni e tenevano brevi cicli di seminari in cui spesso, partendo da nozioni di carattere introduttivo, arrivavano a trattare argomenti di ricerca.[2]

Mi fa piacere essere presente a questa manifestazione, accogliendo, pur con grande preoccupazione, l'invito a tenere una conferenza di tipo divulgativo.

Figura 1: Convegno INdAM 1981

[1] Già professore dell'Università del Salento.

[2] D. Pallara *La Matematica nell'Università di Lecce* in *Per una storia della scienza e della tecnologia nel Salento dall'Unità d'Italia a oggi,* (a cura di A. Rossi, A.L. Denitto, G. Sava, G. Belmonte, L. Ruggiero, A. Castellano), Congedo Editore, Galatina (Lecce), 2011.

2. La questione della divulgazione

Preciso innanzitutto che il mio intervento qui non è divulgazione, poiché voi non siete "volgo". Si tratta invece di condivisione, poiché, usando una terminologia tecnica, la sorgente e il ricevente del messaggio sono omogenei: infatti ho con tutti voi un linguaggio comune e i concetti sono più o meno noti[1].

Altra cosa è parlare a chi è digiuno di matematica, il cosiddetto "vulgus" cioè un anonimo pubblico, che ignora il linguaggio, la validità e i limiti del sapere scientifico. Si pone quindi il problema:

È possibile divulgare la matematica, sottinteso, in modo corretto?

Penso che non tutta la matematica si possa divulgare (nel senso prima precisato) senza correre il rischio di banalizzare il tutto e di dare una immagine errata. Mi viene in mente per es. tutto ciò che ha bisogno in modo essenziale di un formalismo spinto.

È vero che Hilbert, nel famoso Congresso di Matematica del 1900, ebbe a dire che un problema, capace di sviluppare nuovi campi di conoscenza, deve potersi spiegare a chiunque, ma è pur vero che egli si riferiva all'enunciazione del problema, non alle sue eventuali soluzioni.

Dobbiamo però non rinunciare subito, ma impegnarci a rendere il problema e le soluzioni comprensibili: la divulgazione come trasmissioni di conoscenze è necessaria e benemerita, poiché le persone dovrebbero continuare ad apprendere.

Ma che vuol dire "corretta divulgazione"?

E. De Giorgi (1928-1996) - che la maggior parte di voi ha conosciuto, almeno di nome, uno dei più illustri matematici del secolo scorso, che ha risolto il XIX problema di Hilbert[2]- nell'insegnamento invitava ad

> *esporre principalmente le idee non le procedure, a mettere in evidenza ciò che si è dimostrato, ciò che si è solo ipotizzato, evitando che di una teoria siano date interpretazioni più ampie, senza scivolare nel sensazionalismo o nella grossolanità, dicendo chiaramente che i problemi aperti sono in numero di gran lunga maggiore di quelli risolti. Una congettura non provata, un errore non banale in una dimostrazione possono essere elementi più stimolanti alla ricerca che perfette dimostrazioni.*

Si tratta insomma di sfruttare anche l'insuccesso!

Dopo questa lunga premessa apparirà più comprensibile perché ho scelto di parlare di René Thom (1923-2002), certamente uno dei grandi matematici del secolo scorso, Medaglia Fields nel 1958. Egli ha messo in evidenza l'importanza del pensiero matematico (in particolare la visione geometrica) come strumento di lettura del mondo reale. In particolare voleva studiare modelli che descrivono la "rottura della

[1]Cfr. G. De Cecco,*Comunicazione dei saperi: la questione della divulgazione scientifica* in *L'utopia: alla ricerca del senso della storia. Scritti in onore di Cosimo Quarta* (a cura di G. Schiavone con la collaborazione di D. Martina), Mimesis Edizioni, Milano-Udine 2015, pp. 509-525.

[2]Cfr.*E. De Giorgi: hanno detto di lui...* (a cura di G. De Cecco, M.L. Rosato), Quad. 5/2004, Univ. Lecce, SIBA, Edizioni del Grifo. Il lavoro che risolve il XIX problema è *Sulla differenziabilità e l'analicità delle estremali degli integrali multipli regolari*, Mem. Acc. Sc. Torino, Cl. Sci. Fis. Nat. (3),3,25-43.

continuità" e quindi i concetti di bordo e di frontiera, partendo dall'osservazione che noi conosciamo perché riusciamo a distinguere i confini, i contorni delle cose e dei concetti.

Le sue idee collegano la matematica ad altri rami del sapere e sono diffusamente trattate nei volumetti *Paraboles et catastrophes* e *Prédire n'est pas expliquer*, che riportano due interviste all'autore[3].

Del secondo ho curato il commento insieme a Giuseppe Del Re, chimico ed epistemologo dell'Università di Napoli, e ad Arcangelo Rossi, storico e filosofo della scienza dell'Università del Salento[4].

3. Thom e la metafisica

Thom approfondisce il concetto di "bordo" partendo dalla metafisica di Aristotele come è chiaramente espresso in questo passo[5]:

> *La notion de bord me parait aujourd'hui d'autant plus importante que j'ai plongé dans la métaphysique aristotelicienne. Pour Aristote, un être, en géneral, c'est ce qui est là, séparé. Il possède un bord, il est séparé de l'espace ambiant. En somme, le bord de la chose, c'est sa forme. Le concept, lui aussi, a un bord: c'est la définition de ce concept.*

In un altro passo Thom dice che l'atto è il bordo della potenza, come la forma è il bordo della materia.

Non deve meravigliare che l'ispirazione per una significativa teoria matematica nasca dalla metafisica. Infatti alla base di un discorso scientifico ci sono sempre, impliciti o espliciti, presupposti di ordine filosofico, che spesso sfuggono allo stesso scienziato, come osserva il matematico e storico della scienza G. Israel[6]:

> *Il successo della scienza è legato al fatto di aver costruito una fisica a partire da una metafisica, e non viceversa. Mentre Aristotele diceva di osservare i fenomeni meccanici per come sono, senza astrazioni, e da quell'osservazione materiale giungeva ad una metafisica, la scienza moderna*

[3]R. Thom, *Paraboles et catastrophes* (intervista di G. Giorello e S. Martini), Éd. Champs Flammarion, no 186, 1983; R. Thom, *Prédire n'est pas expliquer* (intervista di E. Noël), éd. Champs Flammarion, no. 288, 1993.

[4]*René Thom: prevedere non è spiegare* (a cura di G. De Cecco, G. Del Re, A. Rossi), Traduzione e commento di G. Del Re con la collaborazione di G. Bonomi, Quad. 3/2008, Dip. Mat. Univ. Salento, SIBA.

[5]Da *Prédire n'est pas expliquer*, pag.22: *Da quando mi sono immerso nello studio della metafisica aristotelica, il concetto di bordo mi sembra ancora più importante. Per Aristotele, un essere, in generale, è ciò che c'è lì, separato: esso possiede un bordo ed è separato dall'ambiente. In sintesi, il bordo della cosa è la sua forma. Anche un concetto ha un bordo, che è la sua definizione.*

Cfr. anche R. Thom, *Les intuitions primordiales de l'aristotélisme*, Reviu Thomiste, Juillet-September 1988, Toulouse, pp. 393-409.

[6]*Divulgazione scientifica e cultura*, F. Nardini intervista G. Israel, in *Conversazioni su Scienza e Fede*, a cura del Centro di Documentazione Interdisciplinare di Scienza e Fede della Pontificia Università della Santa Croce, Lindau, Torino 2012, pp. 204-223.

ha seguito la via inversa: da una metafisica di partenza ha dedotto una fisica. L'operazione di astrazione di un problema dalla realtà (ad esempio immaginare un sistema privo di attrito o assimilare un corpo a un punto materiale) è correlato ad una concezione platonica del mondo, il che esclude la possibilità di relegare la filosofia alla categoria di non scienza.

André Weil in uno scritto del 1960 intitolato *De la métaphysique aux mathématiques* così scrive[7]:

> *Rien n'est plus fécond, tous les mathématiciens le savent, que ces obscures analogies, ces troubles reflets d'une théorie à une autre, ces furtives caresses, ces brouilleries inexplicables ; rien aussi ne donne plus de plaisir au chercheur. Un jour vient où l'illusion se dissipe; le pressentiment se change en certitude ; les théories jumelles révèlent leur source commune avant de disparaître.* [...] *La métaphysique est devenue mathématique, prête à former la matière d'un traité dont la beauté froide ne saurait plus nous émouvoir.*

Che la Matematica abbia legami con la Filosofia non è una novità: la metafisica ha influenzato le ricerche in matematica da Pitagora a Grothendieck. Ricordo per esempio che alla base del calcolo delle variazioni c'è un principio generale d'economia (di sapore metafisico) espresso chiaramente da L. Euler (1707-1783) così[8] (nell'Appendice all'opera *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes*[9] del 1743):

> *Cum enim mundi universi fabrica sit perfectissima, atque a Creatore sapientissimo absoluta, nihil omnino in mundo contingit, in quo non maximi minimive ratio quaepiam eluceat.*

Il libro di S. Hildebrandt e A. Tromba *Principi di minimo*[10] illustra molto bene l'evoluzione di questi concetti[11], avvalorando la convinzione che in generale i matematici sono con Platone: le idee, le strutture matematiche vengono prima delle cose.

---

[7] *Come* sanno tutti i matematici, nulla è più fecondo di queste oscure analogie, questi indistinti riflessi tra una teoria e l'altra, queste carezze furtive, queste indecifrabili foschie; e nulla dà maggiore piacere allo studioso. Poi, un giorno, l'illusione svanisce, il presentimento diventa certezza, le teorie gemelle rivelano la loro origine comune primadi svanire *[...]* La metafisica è diventata matematica, pronta a formare la materia di un trattato la cui fredda bellezza non saprà più emozionarci.
Come si vede qui si tratta però non della "vera" metafisica, ma di un insieme di vaghe analogie.

[8] *Essendo la costruzione del mondo la più perfetta possibile, come quella di un Creatore infinitamente saggio, in natura nulla avviene che non presenti proprietà di massimo o di minimo.*

[9] Questo è il primo libro di testo sul Calcolo delle variazioni, testo giustamente celebre nella storia della matematica.

[10] S. Hildebrandt e A. Tromba *Principi di minimo. Forme ottimali in natura*, Scuola Normale Superiore Pisa, 2006; titolo orignale *The Parsimonious Universe. Shape and Form in the Natural World*, Springer-Verlag New York, Inc. 1996.

[11] Si consideri l'importanza che ha avuto in questa evoluzione il *principio di minima azione* di P-L Moreau de Maupertuis (1698-1759), formulato nel 1744 e riportato estensivamente nell'opera *Les lois du moviments et des répos, déduites d'un principe de métaphysique* del 1746.

Come E. De Giorgi, Thom ha sempre sostenuto che l'essenza della matematica non sta nel calcolo e che, spesso nelle fasi iniziali di introduzione di modelli, l'aspetto qualitativo è preminente rispetto a quello quantitativo.

Queste idee mi sembra opportuno ribadire oggi, dopo che negli ultimi decenni, in particolare in pedagogia, si è privilegiata una visione "funzionalista", che ha svuotato la matematica della sua funzione educativa e culturale.

Ce que limite le vrai, ce n'est pas le faux, c'est l'insignifiant[12]- dice Thom; a lui è attribuito anche il detto *Se devo scegliere tra rigore e significato, non esito un istante a scegliere il secondo*.

Thom di fronte ai virtuosismi tecnici e all'eccessivo astrattismo volge la sua attenzione alle basi intuitive della sua disciplina; infatti nella scienza moderna accade di frequente che un formalismo, quasi sempre elegante, a stento lascia scoprire la realtà nascosta dietro le formule. Anche E. Cartan[13] (padre di Henri, maestro di Thom) aveva espresso questa convinzione[14]:

> *Les services éminents qu'a rendus et que rendra encore le calcul différentiel absolu de Ricci et Levi-Civita ne doivent pas nous empcher d'éviter les calculs trop exclusivement formels, ou les débauches d'indices masquent une réalité géométrique souvent très simple. C'est cette réalité que j'ai cherché à mettre partout en évidence.*

Thom riconosce che l'idea aristotelica di "forma" lo ha guidato nella costruzione della sua originale "teoria delle catastrofi" (che dal punto di vista matematico è uno studio delle singolarità di applicazioni differenziabili), il cui nome, privato del suo significato etimologico e tecnico, è stato frainteso. Infatti il significato originario della parola "catastrofe" è "capovolgimento" (da *katà=giù* e *strèpho=vòlgo*).

A proposito della nascita della teoria delle catastrofi egli dice[15]

> *J'ai pour cela repris les travaux d'un mathèmaticien américain, mort récemment, Hassler Whitney. Partant de là, j'ai pu développer la classification des modes par lesquels on peut envoyer un espace dans un autre. Ce que j'ai découvert dans cette direction était assez intéressant. J'ai pu parvenir à quelques classifications. J'ai même plongé dans la physique, lorsque j'était professeur à Strasbourg: je voulais vérifier des idées*

[12]*Ciò che limita il vero, non è il falso, è l'insignificante*

[13]E. Cartan, *Leçons sur la géométrie des espaces de Riemann*, Gauthiers-Villars, 1928.

[14]*I servizi eminenti che ha reso e renderà ancora il calcolo differenziale assoluto di Ricci e Levi-Civita non devono impedirci di evitare i calcoli troppo esclusivamente formali, dove l'uso eccessivo degli indici maschera una realtà geometrica spesso molto semplice. È questa realtà che io ho cercato di mettere dappertutto in evidenza.*

[15]Da *Prédire n'est pas expliquer*, pag. 22: *Richiamandomi alle ricerche svolte da un matematico americano, morto recentemente, Hassler Whitney (1907-1989), ho potuto sviluppare la classificazione dei modi con cui si può applicare uno spazio in un altro. Ciò che scoprii in questa direzione fu abbastanza interessante. Pervenni ad alcune classificazioni. Mi immersi anche nella fisica quando ero professore a Strasburgo: volevo verificare alcune idee matematiche con l'ottica geometrica. ciò che trovai non mancava d'interesse. La teoria delle catastrofi nacque da tutto questo lavoro.*

> *mathématiques par l'optique géométrique: ce que j'y ai trouvé ne manquait pas d'intérêt. La théorie des catastrophes est née de ce travail.*

Quando Thom si propone di studiare una proprietà geometrica del mondo ideale della Matematica, non perde di vista la realtà concreta della sua esperienza; la sta solo leggendo in un particolare modo, che fa parte in ultima analisi del suo approccio conoscitivo.

Ecco come "vede" una "singolarità"[16]:

> *Les espaces que l'on considère généralement sont des espaces homogènes, localement homogènes. Ces espaces sont ce que nous appelons des variétés. L'espace euclidien est une variété. Mais les singularités apparaissent lorsque l'on soumet en quelque sorte l'espace à une contrainte. La manche de ma veste, si je la comprime, je fais apparaître des plis. C'est une situation génerale. Cela ne relève pas de la mécanique des matériaux. J'énonce en réalité un théorème abstrait: lorqu'un espace est soumis à una contrainte, c'est-à-dire lorsqu'on le projecte sur quelque chose de plus petit que sa propre dimension, il accepte la contrainte, sauf en un certain nombre de points où il concentre, si l'on peut dire, toute son individualité première. Et c'est dans la présence de ces singularités que se fait la résistance. Le concept de singularité, c'est le moyen de subsumer en un point toute une structure globale.*

Riprendendo l'esempio della manica della giacca: come per togliere le pieghe bisogna stendere il braccio, così Thom introduce il concetto di "dispiegamento universale", che è un modo di dispiegare tutta l'informazione racchiusa nella singolarità. Egli collega questo alla coppia aristotelica potenza/atto.

Insomma Thom ha osservato con nuovi occhi alcuni aspetti che sembrano schontati, ha messo in discussione fatti comunemente accettati come ovvii e "naturali". Questo ha anche colpito H. Hopf, che nella presentazione dell'opera di Thom per la Medaglia Fields (nel Congresso internazionale di Edimburgo nel 1958)[17] dice espressamente che[18]:

[16]Da *Prédire n'est pas expliquer*, pag.23: *Gli spazi che generalmente si considerano sono spazi omogenei, localmente omogenei. Questi spazi sono quelli che noi chiamiamo varietà. Lo spazio euclideo è una varietà. Le singolarità appaiono quando in qualche modo si sottopone lo spazio ad un vincolo. La manica della mia giacca, se la comprimo, fa comparire delle pieghe. è una situazione generale. Questo non dipende dalla meccanica dei materiali. Enuncio in realtà un teorema astratto: quando uno spazio viene sottoposto ad un vincolo, vale a dire quando lo si proietta su qualcosa di più piccolo della sua dimensione, esso accetta il vincolo salvo in un certo numero di punti in cui concentra, per così dire, tutta la sua individualità primaria. Ed è nella presenza di queste singolarità che si ha la resistenza. Il concetto di singolarità è il modo di assumere in un punto tutta una struttura globale.*

[17]H. Hopf, *The work of R. Thom*, Proc. Internat. Congress Math. 1958, Cambridge Univ. Press, New York 1960.

[18]*...le sue fondamentali idee, della cui grande semplicità ho io parlato prima, sono di natura squisitamente geometrica ed intuitiva. Queste idee hanno significativamente arricchito la Matematica e tutto indica che l'impatto delle idee di Thom–che trovano ora la loro espressione in lavori noti o in preparazione–è lungi dall'essere esaurito.*

> *...seine grundlegenden Ideen, von deren grossartiger Einfachheit ich vorhin gesprochen habe, sind von durchaus geometrisch-anschaulicher Natur. Diese Ideen haben die Mathematik wesentlich bereichert, und alles deutet darauf hin, dass die Wirkung Thomscher Ideen — mögen sie nun in den schon bekannten oder in noch ungeschriebenen Arbeiten zum Ausdruck kommen — noch lange nicht erschöpft ist.*

4. Il concetto di frontiera e di bordo

Prima di andare avanti vorrei fermarmi un momento sul concetto di frontiera e di bordo, concetti che hanno analogie, ma che sono sostanzialmente distinti, anche se spesso vengono identificati, usando lo stesso simbolismo. Anche nella lingua italiana essi non sono equivalenti: nessuno direbbe la "frontiera di un quadro", ma il "bordo di un quadro", così diciamo le "frontiere della scienza", non "i bordi della scienza".

"Frontiera" indica qualcosa che sta di fronte ad un'altra e sottintende un'idea di passaggio (per es. la frontiera di uno Stato); "bordo" è legato al concetto di orlo, una parte esterna che sta intorno ad una parte centrale e quindi estremità di una cosa, rottura di continuità.

Ritornando alla Matematica, il concetto di frontiera appartiene alla Topologia generale, mentre quello di bordo appartiene alla Topologia algebrica e alla Geometria differenziale.

La frontiera di un insieme X dipende dallo spazio ambiente E, come si vede con esempi. La indicheremo perciò $\mathcal{F}_E(X)$.

Sia

$$X = \{x \in \mathbb{R}^2 | \operatorname{dist}(O,x) < 1\}$$

allora

$$\begin{array}{ll} X \subset \mathbb{R}^2 = E & \Rightarrow \mathcal{F}_E(X) = \mathbb{S}^1 \\ X \subset \mathbb{R}^3 = E & \Rightarrow \mathcal{F}_E(X) = X \\ X = E & \Rightarrow \mathcal{F}_E(X) = \emptyset. \end{array}$$

Se X è un insieme aperto o chiuso dal punto di vista topologico, allora

$$\mathcal{F}_E(\mathcal{F}_E(X)) = \mathcal{F}_E(X).$$

Poiché $\mathcal{F}_E(X)$ è un chiuso, in ogni caso si ha una sorta d'idempotenza:

$$\mathcal{F}_E^h(X) = \mathcal{F}_E^2(X) \qquad h \geq 2.$$

Sia ora X una *varietà n-dimensionale con bordo*. Il bordo ∂X non dipende dall'immersione di X in uno spazio ambiente; i punti del bordo sono *intrinsecamente* differenti dai punti di X, privata del bordo ∂X.

Lo spazio topologico ∂X è una (n-1)-varietà priva di bordo, cioè

$$\partial(\partial X) = \emptyset.$$

Questo risultato chiarisce bene la differenza tra i due concetti, frontiera e bordo.

Ricordiamo che se X è uno spazio compatto e $\partial X = \emptyset$, allora X è detto *combinatorialmente chiuso*. Questo traduce il concetto intuitivo di curva chiusa, di recipiente chiuso, cioè con il coperchio. Una semisfera (cioè una scodella) è un insieme chiuso dal punto di vista topologico, ma una 2-varietà non chiusa, poiché il suo bordo (una circonferenza) è non vuoto.

5. Il cobordismo

Il matematico N. Steenrod (1910-1971) aveva posto il seguente problema (riportato in un articolo di Eilenberg del 1949):

Quali sono le condizioni affinché una varietà sia bordo di un'altra?

Thom ha affrontato la questione generalizzandola:

Quando due varietà sono bordo comune di una stessa varietà?

Se una delle due è l'insieme vuoto, si ritorna al problema originario.

Lascio la parola allo stesso Thom, che spiega come è nata la teoria del cobordismo[19]:

> *Il s'agissait de savoir quand deux variétés constituent justement le bord commun d'une même variété; c'est un problème qui, à première vue, peut sembler assez gratuit. Mais si on y réfléchit un peu, on s'aperoit que c'est le cas particulier d'un problème qui présente également un aspect philoshique. Nous avons deux espaces, deux variétés différentes et l'on cherche, en quelque sorte, à les réunir avec une espèce de déformation continue. La meilleure faon est, en définitive, la construction d'un "cobordisme" entre les deux variétés. A l'aide d'idées de ce type j'ai pu développer toute une technique sur les applications différentiables, grâce à laquelle j'ai réussi à résoudre, au moins en théorie, le problème de reconnaître si deux variétés sont cobordantes, en le réduisant en termes purement algébriques.*

La via seguita da Thom è quella classica: tradurre il fatto qualitativo in uno quantitativo, più precisamente tradurre il problema topologico in uno algebrico, con la segreta speranza che questo sia di più facile soluzione; in ogni caso aver stabilito un'analogia è di grande interesse. Pensiamo per esempio alla Geometria analitica.

[19] Da *Paraboles et catastrophes*, pag.22: *Si trattava di sapere quando due varietà costituiscono appunto il bordo comune di una stessa varietà: è un problema che a prima vista può sembrare abbastanza gratuito. Ma se ci riflettiamo un po', ci accorgiamo che è caso particolare di un problema che ha anche un suo risvolto filosofico. Abbiamo due spazi, due varietà differenti, e si cerca in qualche modo di congiungerle con una specie di deformazione continua. Il modo migliore risulta essere la costruzione di un "cobordismo" fra le due varietà. Con l'aiuto di questo tipo di idee, ho potuto sviluppare tutta una tecnica sulle applicazione differenziabili, attraverso la quale sono riuscito a risolvere, almeno teoricamente, il problema di riconoscere se le due varietà sono cobordanti, riducendolo in termini puramente algebrici.*

Non sempre però le asserzioni sono invertibili, per cui ritornare indietro spesso non è significativo.

Thom nel lavoro *Quelques propriètés globales des variétés differentiables*, (Comment. Math. Helv. 28 (1954), 17–86) dà una risposta abbastanza completa al problema, già studiato dalla scuola russa di Topologia differenziale, in particolare da L. Pontrjagin e B.A. Rokhlin, che avevano iniziato la teoria del "cobordismo", chiamata nella letteratura russa "omologia intrinseca". Thom facendo intervenire tutto ciò che allora si conosceva in Topologia algebrica, usando procedimenti analoghi a quelli che si incontrano nella Teoria di Morse, riesce ad invertire alcuni teoremi di Pontrjagin. Il teorema finale, che dà risposta esaustiva alla questione, fa intervenire risultati anche di Milnor, Averbuh, Wall:

Teorema. *Una varietà n-dimensionale differenziabile compatta e orientata è bordo di una varietà (n+1)-dimensionale se e solo se i suoi numeri di Stiefel-Whitney e i suoi numeri di Pontrjagin sono nulli.*

La definizione dei numeri di Stiefel-Whitney e dei numeri di Pontrjagin non è semplice e poi non sono in grado di andare nei dettagli del teorema. Si tratta di classi dell'anello caratteristico di Stiefel-Whitney e di Pontrjagin aventi dimensione adeguata.

Ora esporrò brevemente soltanto l'idea geometrica di cobordismo e come è partito Thom nell'affrontare il problema.

Ci occuperemo soltanto del caso elementare, cioè di varietà non singolari. Il caso di varietà singolari ci fu presentato da A. Dold in un ciclo di seminari negli anni 70[20].

Consideriamo varietà n-dimensionali senza bordo, compatte ed orientate, non necessariamente connesse; per esempio una 1-varietà può essere unione finita di circonferenze ciascuna disgiunta da tutte le altre.

Se M è una varietà con una data orientazione, indichiamo con $-M$ la stessa varietà con l'orientazione opposta;

se M e N sono due n-varietà, indichiamo con $M+N$ l'unione disgiunta di copie di M e N, così $M-N$ sarà $M+(-N)$

$M \cdot N$ è la varietà di dimensione $2n$, prodotto cartesiano di M e N con l'orientazione indotta[21].

Se per una n-varietà M esiste una $(n+1)$-varietà Λ compatta orientata con bordo $\partial\Lambda$ che è una copia di M, si dice che M è *bordante* (cioè M è bordo di qualcosa).

Se $M-N$ è bordante, diciamo che M e N sono *cobordanti*, in simboli $M \sim N$.

Se M è bordante, diciamo anche che M è cobordante alla varietà nulla O, in simbololi $M \sim O$.

[20] A. Dold, *Metodi moderni di topologia algebrica,* lezioni raccolte da M. Bordoni, F. Cacciafesta, A. Del Fra, S. Marchiafava, G. Romani, Quaderni dei Gruppi di Ricerca Matematica del CNR, Ist. Mat. "G. Castelnuovo", Un. Roma, 1973.

[21] I simboli $\setminus, \cup, \times$ sono simboli insiemistici, mentre $-, +, \cdot$ sono simboli algebrici che tengono conto dell'orientazione della varietà.

Affinché gli esempi possano essere visualizzati con l'intuizione ordinaria, considereremo varietà di dimensione 1 e 2.

1. Se $M = S$ è una circonferenza ed $N = \emptyset$, allora il cobordismo tra M ed N, $M \sim N$, può realizzarsi tramite una semisfera Σ tale che $\partial\Sigma = S$.

2. Siano S_1 e S_2 due circonferenze complanari i cui corrispondenti dischi chiusi siano disgiunti. Allora è possibile considerare una superficie omeomorfa ad un cilindro Γ avente come bordo $S_1 - S_2$. Risulta allora che $S_1 \sim S_2$ tramite Γ.

3. Se $M = S_1 + S_2$ (dove S_1 e S_2 sono due circonferenze disgiunte per es. del piano $z = 0$), e $N = S$ (circonferenza per esempio del piano $z = 1$), allora $M \sim N$ tramite una superficie Λ a forma di "pantaloni" dove S è la circonferenza della vita mentre S_1 e S_2 sono le circonferenze dei bordi delle gambe.

4. Se sulla circonferenza S costruiamo la superficie Σ con bordo S e consideriamo la superficie Λ dell'esempio precedente, possiamo concludere che $\Lambda \cup \Sigma$ è un cilindro Γ. Inversamente un cilindro Γ si può decomporre in due superficie Σ e Λ tramite il "cobordismo" di due opportune curve.

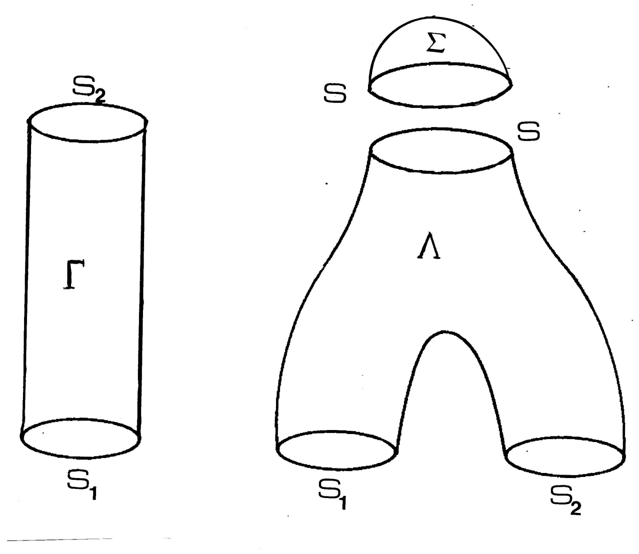


Figura 2: Cobordismo

6. Algebra di Thom

La relazione di equivalenza ”$\sim$” definisce *classi di cobordismo* e la suddivisione in classi è compatibile con l’addizione, che induce una struttura di gruppo nell’insieme Ω^n delle classi (delle varietà cobordanti n-dimensionali); la classe delle varietà bordanti costituisce l’elemento neutro.

La divisione in classi è compatibile anche con la moltiplicazione, che dota

$$\Omega = \Omega^0 + \Omega^1 + \Omega^2 + \cdots$$

di una struttura algebrica di anello, la cosiddetta *algebra di Thom*, utilizzata per lo studio in particolare delle varietà differenziabili.

Un fatto notevole è che certi invarianti della struttura differenziale sono anche invarianti per la relazione di cobordismo e la conoscenza di Ω^n permette di ottenere fra questi invarianti relazioni inaspettate.

AMS Subject Classification: 00A30

Giuseppe DE CECCO
Via Montefeltro 2, 00139 Roma, ITALY
e-mail: `giuseppe.dececco@unisalento.it`

Lavoro pervenuto in redazione il 01.09.2015 e accettato il 21.09.2015.

Geom. Struc. on Riem. Man.-Bari
Vol. 73/1, 3–4 (2015), 89 – 100

L. Di Terlizzi - G. Dileo

SOME PARACONTACT METRIC STRUCTURES ON CONTACT METRIC MANIFOLDS[1]

Abstract. We consider contact metric manifolds such that the Jacobi operator anticommutes with the structure tensor field φ. These manifolds admit two paracontact metric structures compatible with the contact form η. We describe some geometric properties of these structures.

1. Introduction

Various results in the recent literature revealed remarkable interplays between contact metric and paracontact metric structures [3, 4, 5]. Such a relation has been particularly investigated on contact metric (κ,μ)-spaces, that is contact metric manifolds (M,φ,ξ,η,g) such that the Riemannian curvature satisfies

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \tag{1}$$

for every vector fields X,Y on M, and for some real constants κ,μ. Here $2h$ is the Lie derivative of the structure tensor φ in the direction of ξ. As shown in [3], a non-Sasakian contact metric (κ,μ)-space, for which $\kappa < 1$, admits two paracontact metric structures (φ_i,ξ,η,g_i), $i = 1,2$, where

$$\varphi_1 = \frac{1}{\sqrt{1-\kappa}}\varphi h, \quad \varphi_2 = \frac{1}{\sqrt{1-\kappa}}h, \quad g_i := d\eta(\cdot,\varphi_i\cdot) + \eta\otimes\eta.$$

The triplet $(\varphi_1,\varphi_2,\varphi)$ provides an almost bi-paracontact structure. In particular, in [3] the author proves that the curvature tensors of the semi-Riemannian metrics g_1 and g_2 satisfy nullity conditions formally similar to (1). The structure (φ_2,ξ,η,g_2) is deeply investigated in [4], where it is shown that this structure, called canonical, induces on the underlying contact manifold (M,η) a sequence of compatible contact and paracontact metric structures satisfying nullity conditions.

In this note we investigate the existence of paracontact metric structures on the class of contact metric manifolds such that the Jacobi operator $l := R(\cdot,\xi)\xi$ anticommutes with φ. This class includes contact metric manifolds with vanishing Jacobi operator, called M_l-manifolds, and more generally contact metric Jacobi $(0,\mu)$-spaces, for which $l = \mu h$, for some real constant μ. Notice that contact metric $(0,\mu)$-spaces are special Jacobi $(0,\mu)$-spaces (see Section 3 for more details).

We prove that any contact metric manifold (M,φ,ξ,η,g) such that $\varphi l + l\varphi = 0$ admits two paracontact metric structures given by (h,ξ,η,g_1) and $(\varphi h,\xi,\eta,g_2)$, where

$$g_1 := d\eta(\cdot,h\cdot) + \eta\otimes\eta, \qquad g_2 := d\eta(\cdot,\varphi h\cdot) + \eta\otimes\eta.$$

[1] This paper is dedicated to Professor Anna Maria Pastore with deep gratitude for her teachings.

We study the basic properties of these geometric structures, determining the Jacobi operators $l_1 := R_1(\cdot,\xi)\xi$ and $l_2 := R_2(\cdot,\xi)\xi$ defined by the curvature tensors R_1 and R_2 of the semi-Riemannian metrics g_1 and g_2 respectively. It is worth remarking that in both cases $l = 0$ and $l = 4h$, the structure (h,ξ,η,g_1) provides an example of paracontact metric structure such that the symmetric operator $h_1 := \frac{1}{2}\mathcal{L}_\xi h$ is not vanishing and satisfies $h_1^2 = 0$. These structures are of special interest in the context of paracontact metric geometry, since they have no counterpart in contact metric geometry [2, 10, 11].

2. Preliminaries

An *almost contact manifold* is a $(2n+1)$-dimensional smooth manifold M endowed with a structure (φ,ξ,η), where φ is a $(1,1)$-tensor field, ξ a vector field and η a 1-form, such that

$$\varphi^2 = -I + \eta\otimes\xi, \quad \eta(\xi) = 1,$$

implying that $\varphi\xi = 0$, $\eta\circ\varphi = 0$ and φ has rank $2n$. An almost contact manifold admits a compatible metric, that is a Riemannian metric g such that, for every $X,Y \in \Gamma(TM)$,

$$g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y).$$

Then $\eta = g(\cdot,\xi)$, and the tangent bundle of M splits into the orthogonal sum $TM = \mathcal{D}\oplus[\xi]$, where $\mathcal{D}$ is the $2n$-dimensional distribution defined as $\mathrm{Ker}(\eta)$ or, equivalently, as $\mathrm{Im}(\varphi)$. The manifold (M,φ,ξ,η,g) is called an *almost contact metric manifold*. It is said to be a *contact metric manifold* if $d\eta(X,Y) = g(X,\varphi Y)$ for every vector fields X,Y; in this case the 1-form η turns out to be a *contact form*, in the sense that $\eta\wedge(d\eta)^n \neq 0$ everywhere on M. A *Sasakian manifold* is defined as a contact metric manifold for which the tensor field $N_\varphi := [\varphi,\varphi] + 2d\eta\otimes\xi$ vanishes identically.

On a contact metric manifold one can define the $(1,1)$-tensor field $h := \frac{1}{2}\mathcal{L}_\xi\varphi$. This operator satisfies $h\xi = 0$ and is symmetric with respect to g, so that $\eta\circ h = 0$. Furthermore, it anticommutes with φ and satisfies

$$\nabla_X\xi = -\varphi X - \varphi hX \tag{2}$$

where ∇ is the Levi-Civita connection of g. Denoting by R the Riemannian curvature tensor, we shall consider the *Jacobi operator* $l := R(\cdot,\xi)\xi$. It is known that l satisfies the following equations [1]:

$$\nabla_\xi h = \varphi - h^2\varphi - \varphi l, \tag{3}$$

$$\frac{1}{2}\left(-l + \varphi l\varphi\right) = h^2 + \varphi^2. \tag{4}$$

We recall now the basic notions on paracontact geometry [16]. An *almost paracontact structure* on a $(2n+1)$-dimensional manifold M is given by a $(1,1)$-tensor field $\tilde\varphi$, a vector field $\tilde\xi$ and a 1-form $\tilde\eta$, such that

(i) $\tilde\varphi^2 = I - \tilde\eta\otimes\tilde\xi$, $\tilde\eta(\tilde\xi) = 1$,

(ii) $\tilde{\varphi}$ induces an almost paracomplex structure on each fibre of the distribution $\mathcal{D} := \mathrm{Ker}(\tilde{\eta})$, i.e. the eigendistributions $\mathcal{D}^{+}_{\tilde{\varphi}}$ and $\mathcal{D}^{-}_{\tilde{\varphi}}$ corresponding to the eigenvalues $+1$ and -1 of $\tilde{\varphi}|_{\mathcal{D}}$, respectively, have dimension equal n.

From the definition it follows that $\tilde{\varphi}\tilde{\xi} = 0$, $\tilde{\eta} \circ \tilde{\varphi} = 0$ and $\tilde{\varphi}$ has constant rank $2n$. Any almost paracontact manifold M admits a semi-Riemannian metric $\tilde{g}$ satisfying

$$\tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) = -\tilde{g}(X,Y) + \tilde{\eta}(X)\tilde{\eta}(Y)$$

for all $X,Y \in \Gamma(TM)$, which necessarily has signature $(n+1,n)$. Then $(M,\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g})$ is called an *almost paracontact metric manifold*. If furthermore $d\tilde{\eta}(X,Y) = \tilde{g}(X,\tilde{\varphi}Y)$ for all $X,Y \in \Gamma(TM)$, then $(M,\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g})$ is called a *paracontact metric manifold*, in which case $\tilde{\eta}$ is a contact form. A *para-Sasakian manifold* is defined as a paracontact metric manifold such that the tensor field $N_{\tilde{\varphi}} := [\tilde{\varphi},\tilde{\varphi}] - 2d\tilde{\eta} \otimes \tilde{\xi}$ vanishes identically.

On a paracontact metric manifold one can consider the $(1,1)$-tensor field $\tilde{h}$ defined by $\tilde{h} := \frac{1}{2}\mathcal{L}_{\tilde{\xi}}\tilde{\varphi}$, which satisfies $\tilde{h}\tilde{\xi} = 0$. Furthermore, $\tilde{h}$ is symmetric and anticommutes with $\tilde{\varphi}$. Denoting by $\tilde{\nabla}$ the Levi-Civita connection of $\tilde{g}$, one has

$$\tilde{\nabla}_X \tilde{\xi} = -\tilde{\varphi}X + \tilde{\varphi}\tilde{h}X. \tag{5}$$

If $\tilde{R}$ is the curvature tensor of $\tilde{g}$, the operator $\tilde{l} := \tilde{R}(\cdot,\tilde{\xi})\tilde{\xi}$ satisfies

$$\tilde{\nabla}_{\tilde{\xi}}\tilde{h} = -\tilde{\varphi} + \tilde{h}^2\tilde{\varphi} - \tilde{\varphi}\tilde{l}, \tag{6}$$

$$\frac{1}{2}\left(\tilde{l} + \tilde{\varphi}\tilde{l}\tilde{\varphi}\right) = \tilde{h}^2 - \tilde{\varphi}^2. \tag{7}$$

3. Contact metric structures with $\varphi l + l\varphi = 0$

THEOREM 1. *Let (M,φ,ξ,η,g) be a contact metric manifold. Then, the following conditions are equivalent:*

i) $\varphi l + l\varphi = 0$,

ii) (h,ξ,η) *is an almost paracontact structure,*

iii) $(\varphi h,\xi,\eta)$ *is an almost paracontact structure.*

If any of the above conditions holds, then M is endowed with two paracontact metric structures (h,ξ,η,g_1) and $(\varphi h,\xi,\eta,g_2)$, where for every $X,Y \in \Gamma(TM)$,

$$g_1(X,Y) = d\eta(X,hY) + \eta(X)\eta(Y), \tag{8}$$

$$g_2(X,Y) = d\eta(X,\varphi hY) + \eta(X)\eta(Y). \tag{9}$$

Proof. From (4) it follows that $\varphi l + l\varphi = 0$ if and only if $h^2 + \varphi^2 = 0$, or equivalently $h^2 = I - \eta \otimes \xi$. This is also equivalent to $(\varphi h)^2 = I - \eta \otimes \xi$. Indeed, since h anticommutes with φ, then $(\varphi h)^2 = -\varphi^2 h^2 = h^2$.

We suppose now that i) holds. Then, being $h^2 = I - \eta \otimes \xi$, the symmetric operator h admits non-zero eigenvalues $+1$ and -1 of multiplicity n. Indeed, since h anticommutes with φ, if X is an eigenvector with eigenvalue $+1$, then φX is an eigenvector with eigenvalue -1. Therefore, h induces an almost paracomplex structure on $\mathcal{D}$ and (h, ξ, η) is an almost paracontact structure. Now define g_1 as in (8) so that

$$g_1(X,Y) = g(X, \varphi hY) + \eta(X)\eta(Y). \tag{10}$$

Being φh symmetric with respect to g, g_1 is symmetric. Furthermore, it is compatible with h since, being $i_\xi d\eta = 0$, we have

$$g_1(hX, hY) = d\eta(hX, h^2Y) = -d\eta(Y, hX) = -g_1(X,Y) + \eta(X)\eta(Y).$$

Finally, $g_1(X, hY) = d\eta(X, h^2Y) = d\eta(X,Y)$, so that (h, ξ, η, g_1) is a paracontact metric structure. Analogously, if i) holds the tensor field φh induces an almost paracomplex structure on $\mathcal{D}$, since it satisfies $(\varphi h)^2 = I - \eta \otimes \xi$, it is symmetric and anticommutes with φ. Hence, $(\varphi h, \xi, \eta)$ is an almost paracontact structure. In this case, one can define a metric g_2 as in (9), so that

$$g_2(X,Y) = -g(X, hY) + \eta(X)\eta(Y). \tag{11}$$

One can easily see that g_2 is symmetric and compatible with φh. On the other hand, $g_2(X, \varphi hY) = d\eta(X,Y)$, so that $(\varphi h, \xi, \eta, g_2)$ is a paracontact metric structure. $\square$

REMARK 1. Notice that if $(M, \varphi, \xi, \eta, g)$ is a contact metric manifold satisfying $\varphi l + l\varphi = 0$, then the triplet $(\varphi h, h, \varphi)$ is an almost bi-paracontact structure on M, in the sense of the definition given in [3]. In particular, one has

$$\mathcal{D}^{\pm}_{\varphi h} = \{X + \varphi X \,|\, X \in \mathcal{D}^{\pm}_{h}\}, \qquad \mathcal{D}^{\pm}_{h} = \{X + \varphi X \,|\, X \in \mathcal{D}^{\mp}_{\varphi h}\},$$

where $\mathcal{D}^{+}_{\varphi h}$ and $\mathcal{D}^{-}_{\varphi h}$ denote the eigendistributions of φh corresponding to the eigenvalues $+1$ and -1 respectively, while $\mathcal{D}^{+}_{h}$ and $\mathcal{D}^{-}_{h}$ are the eigendistributions of h corresponding to the eigenvalues $+1$ and -1. Furthermore, one can take a local orthonormal frame $\{\xi, e_i, \varphi e_i\}$, $i = 1, \ldots, n$, such that $he_i = e_i$ and consequently $h\varphi e_i = -\varphi e_i$. Then the orthogonal vector fields $e_i + \varphi e_i$, $i = 1, \ldots, n$, span the eigendistribution $\mathcal{D}^{+}_{\varphi h}$, while the orthogonal vector fields $e_i - \varphi e_i$, $i = 1, \ldots, n$, span the eigendistribution $\mathcal{D}^{-}_{\varphi h}$.

A first class of almost contact metric manifolds satisfying $\varphi l + l\varphi = 0$ is obviously given by contact metric manifolds with vanishing Jacobi operator, also known as M_l-manifolds in literature. This class is particularly large. For instance, the normal bundle of a Legendre submanifold of a Sasakian manifold admits a contact metric structure with $l = 0$, see [1, Theorem 9.16]. Contact metric 3-manifolds with $l = 0$ are studied in [13, 8, 9].

In [7] Ghosh and Sharma introduced a new class of contact metric manifolds. They define a Jacobi (κ, μ)-contact space as a contact metric manifold such that the Jacobi operator l satisfies

$$l = -\kappa\varphi^2 + \mu h, \tag{12}$$

for some constants κ and μ. In particular, this class includes contact metric (κ,μ)-spaces, for which the Riemannian curvature tensor satisfies (1). Now, if l is given by (12), then $\varphi l + l\varphi = 2\kappa\varphi$ and $\mathtt{trace}(l) = 2n\kappa$, so that

$$\varphi l + l\varphi = 0 \Leftrightarrow \kappa = 0 \Leftrightarrow \mathtt{trace}(l) = 0.$$

In [6] Cho and Inoguchi provide new examples of 3-dimensional Jacobi (κ,μ)-contact spaces which are neither contact (κ,μ)-spaces nor M_l-manifolds. They consider non-unimodular 3-dimensional Lie groups $G(\alpha,\gamma)$, α,γ real constants with $\alpha \neq 0$, equipped with a left invariant contact metric structure (φ,ξ,η,g), whose Lie algebra $\mathfrak{g}(\alpha,\gamma)$ is spanned by a basis $\{\xi,e_1,e_2\}$, with $e_2 = \varphi e_1$, satisfying the commutation relations

$$[\xi,e_1] = -\gamma e_2, \quad [\xi,e_2] = 0, \quad [e_1,e_2] = \alpha e_2 + 2\xi,$$

(see also [14]). Every Lie group $G(\alpha,\gamma)$ is a Jacobi (κ,μ)-contact space, with $\kappa = -\frac{1}{4}(\gamma^2 - 4)$ and $\mu = \gamma + 2$. Furthermore, except for the Sasakian case $G(\alpha,0)$, $G(\alpha,\gamma)$ is not a contact (κ,μ)-space. In particular the Lie groups $G(\alpha,2)$ are Jacobi $(0,4)$-contact spaces, for which l is not vanishing and anticommutes with φ.

In [6] the authors also provide an example of a non-homogeneous Jacobi (κ,μ)-contact space, which is not a (κ,μ)-space. This manifold M, described by Perrone in [15], is the open submanifold $\{(x,y,z) \in \mathbb{R}^3 | x \neq 0\}$ of $\mathbb{R}^3$, endowed with the contact metric structure (φ,ξ,η,g), where $\xi = \partial/\partial z$, $\eta = xydx + dz$, and φ is defined on the global frame fields

$$e_1 = -\frac{2}{x}\frac{\partial}{\partial y}, \quad e_2 = \frac{\partial}{\partial x} - \frac{4z}{x}\frac{\partial}{\partial y} - xy\frac{\partial}{\partial z}, \quad e_3 = \xi$$

by $\varphi\xi = 0$, $\varphi e_1 = e_2$, $\varphi e_2 = -e_1$. Finally g is the Riemannian metric with respect to which $\{e_1,e_2,e_3\}$ is orthonormal. In this case the Jacobi operator is given by $l = 4h$, so that M is a Jacobi $(0,4)$-contact space.

Now, given a contact metric manifold (M,φ,ξ,η,g) such that $\varphi l + l\varphi = 0$, with associated paracontact metric structures (h,ξ,η,g_1) and $(\varphi h,\xi,\eta,g_2)$, we denote by h_1 and h_2 the tensor fields defined by

$$h_1 := \frac{1}{2}\mathcal{L}_\xi h, \quad h_2 := \frac{1}{2}\mathcal{L}_\xi(\varphi h).$$

The operator h_1 is symmetric with respect to g_1 and anticommutes with h, while h_2 is symmetric with respect to g_2 and anticommutes with φh. We shall denote by ∇^1 and ∇^2 the Levi-Civita connections of g_1 and g_2, respectively, with curvature tensors R_1 and R_2. The associated Jacobi operators are defined by

$$l_1 := R_1(\cdot,\xi)\xi, \quad l_2 := R_2(\cdot,\xi)\xi.$$

LEMMA 1. *Let (M,φ,ξ,η,g) be a contact metric manifold such that $\varphi l + l\varphi = 0$. Then the following equations hold:*

$$\nabla_\xi h = -\varphi l, \tag{13}$$

$$lh - hl = 0. \tag{14}$$

Proof. Equation (13) immediately follows from (3), being $h^2 = I - \eta \otimes \xi$. Next, applying (13) and $\nabla_\xi(h^2) = 0$, we have

$$lh - hl = \varphi(\nabla_\xi h)h - h\varphi(\nabla_\xi h) = \varphi\big((\nabla_\xi h)h + h(\nabla_\xi h)\big) = \varphi\nabla_\xi(h^2) = 0.$$

□

PROPOSITION 1. *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric manifold with $\varphi l + l\varphi = 0$, and associated paracontact metric structures (h, ξ, η, g_1) and $(\varphi h, \xi, \eta, g_2)$. Then*

$$h_1 = \varphi + \varphi h - \frac{1}{2}\varphi l, \qquad h_2 = -h + \frac{1}{2}l. \tag{15}$$

Proof. Applying (13) and (2), we have

$$\begin{aligned} 2h_1 X &= [\xi, hX] - h[\xi, X] \\ &= (\nabla_\xi h)X - \nabla_{hX}\xi + h(\nabla_X \xi) \\ &= -\varphi l X + \varphi h X + \varphi h^2 X - h\varphi X - h\varphi h X \\ &= 2\varphi X + 2\varphi h X - \varphi l X. \end{aligned}$$

As regards h_2, we have

$$h_2 = \frac{1}{2}\big((\mathcal{L}_\xi \varphi)h + \varphi(\mathcal{L}_\xi h)\big) = h^2 + \varphi h_1 = h^2 + \varphi^2 + \varphi^2 h - \frac{1}{2}\varphi^2 l = -h + \frac{1}{2}l.$$

□

4. The paracontact metric structure (h, ξ, η, g_1).

LEMMA 2. *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric manifold such that $\varphi l + l\varphi = 0$. Then the operators h_1 and l_1 associated to the structure (h, ξ, η, g_1) satisfy the following identities:*

(a) $lh_1 + h_1 l = 0$,

(b) $h_1\varphi + \varphi h_1 = 2\varphi^2$,

(c) $h_1\varphi - \varphi h_1 = 2h - l$,

(d) $h_1^2 = \frac{1}{4}l^2 - hl$,

(e) $l_1 h + hl_1 = -2h - 2l + \frac{1}{2}l^2 h$.

Proof. Identity (a) follows from (15), taking into account the fact that l anticommutes with φ and commutes with h. Identities (b), (c) and (d) also follow from (15). Finally, from (7) we get $l_1 + hl_1 h = -2h^2 + 2h_1^2$, and using (d) we get (e). □

PROPOSITION 2. *Let* (M,φ,ξ,η,g) *be a contact metric manifold such that* $l = 0$. *Then* (h,ξ,η,g_1) *is a paracontact metric structure such that* $h_1 \neq 0$ *and* $h_1^2 = 0$.

Proof. From (15) we have $h_1 = \varphi + \varphi h$, which is not vanishing since φ is skew-symmetric with respect to g, while φh is symmetric. On the other hand, from (d) of Lemma 2, $h_1^2 = 0$. □

PROPOSITION 3. *Let* (M,φ,ξ,η,g) *be a Jacobi* $(0,4)$*-contact space. Then* (h,ξ,η,g_1) *is a paracontact metric structure such that* $h_1 \neq 0$ *and* $h_1^2 = 0$.

Proof. The Jacobi operator of the contact metric structure (φ,ξ,η,g) is $l = 4h$. From (15) we have $h_1 = \varphi - \varphi h \neq 0$, while (d) of Lemma 2 implies $h_1^2 = 0$. □

REMARK 2. In both cases $l = 0$ and $l = 4h$, from (e) of Lemma 2, one can easily verify that $l_1 h + h l_1 = -2h \neq 0$, and in particular $l_1 \neq 0$.

In the following, we shall obtain an explicit expression for the Jacobi operator l_1. First, we determine the relation between the Levi-Civita connections of the Riemannian metrics g and g_1.

THEOREM 2. *Let* (M,φ,ξ,η,g) *be a contact metric manifold such that* $\varphi l + l\varphi = 0$. *Then for every* $X,Y \in \Gamma(TM)$*, the Levi-Civita connections* ∇^1 *and* ∇ *satisfy:*

$$\begin{aligned}(16)\qquad \nabla^1_X Y = {} & \nabla_X Y - \eta(Y)hX - \eta(X)hY - \eta(X)\eta(Y)\xi - \frac{1}{2}g(lX,Y)\xi \\ & + \frac{1}{2}\varphi h\big((\nabla_X\varphi)Y + (\nabla_X\varphi h)Y - R(\xi,X)Y\big) + g(X + hX - \varphi hX, Y)\xi.\end{aligned}$$

Proof. First, we prove that for every $X,Y \in \Gamma(TM)$ and $U \in \Gamma(\mathcal{D})$, we have

$$\begin{aligned}(17)\qquad g(\nabla^1_X Y,U) &= g(\nabla_X Y,U) - \eta(Y)g(hX,U) - \eta(X)g(hY,U) \\ &\quad + \frac{1}{2}g\big(\varphi h((\nabla_X\varphi)Y + (\nabla_X\varphi h)Y - R(\xi,X)Y),U\big).\end{aligned}$$

Let us consider $X,Y \in \Gamma(TM)$ and $Z \in \Gamma(\mathcal{D})$. Applying the Koszul formula, (10) and the fact that φh is symmetric, we have

$$\begin{aligned}2g_1(\nabla^1_X Y,Z) &= X(g_1(Y,Z)) + Y(g_1(Z,X)) - Z(g_1(X,Y)) \\ &\quad + g_1([X,Y],Z) + g_1([Z,X],Y) - g_1([Y,Z],X) \\ &= X(g(Y,\varphi hZ)) + Y(g(X,\varphi hZ)) - Z(g(X,\varphi hY)) \\ &\quad - Z(\eta(X)\eta(Y)) + g([X,Y],\varphi hZ) + g([Z,X],\varphi hY) \\ &\quad + \eta([Z,X])\eta(Y) - g([Y,Z],\varphi hX) - \eta([Y,Z])\eta(X) \\ &= 2g(\nabla_X Y,\varphi hZ) + g(Y,\nabla_X(\varphi hZ)) + g(X,\nabla_Y(\varphi hZ)) \\ &\quad - g(X,\nabla_Z(\varphi hY)) - g(\nabla_X Z,\varphi hY) - g(\nabla_Y Z,\varphi hX) \\ &\quad + g(\nabla_Z Y,\varphi hX) - 2d\eta(Z,X)\eta(Y) - 2d\eta(Z,Y)\eta(X) \\ &= 2g(\nabla_X Y,\varphi hZ) - 2g(Z,\varphi X)\eta(Y) - 2g(Z,\varphi Y)\eta(X) \\ &\quad + g((\nabla_Y\varphi h)Z,X) + g((\nabla_X\varphi h)Z,Y) - g((\nabla_Z\varphi h)Y,X).\end{aligned}$$

Now, recall that in a contact metric manifold we have ([12])

$$g(R(\xi,X)Y,Z) = g((\nabla_X\varphi)Y,Z) - g(X,(\nabla_Y\varphi h)Z) + g(X,(\nabla_Z\varphi h)Y).$$

Therefore, applying again (10), the symmetry of φh and $(\varphi h)^2(Z) = Z$, we have

$$\begin{aligned}
2g(\nabla^1_X Y,\varphi hZ) &= 2g(\nabla_X Y,\varphi hZ) - 2g(Z,\varphi X)\eta(Y) - 2g(Z,\varphi Y)\eta(X)\\
&\quad - g(R(\xi,X)Y,Z) + g((\nabla_X\varphi)Y,Z) + g((\nabla_X\varphi h)Y,Z)\\
&= 2g(\nabla_X Y,\varphi hZ) - 2g(hX,\varphi hZ)\eta(Y) - 2g(hY,\varphi hZ)\eta(X)\\
&\quad + g\big(\varphi h(-R(\xi,X)Y + (\nabla_X\varphi)Y + (\nabla_X\varphi h)Y),\varphi hZ\big)
\end{aligned}$$

which implies (17). Now, we prove that for every $X,Y \in \Gamma(TM)$

$$g(\nabla^1_X Y,\xi) = g(\nabla_X Y,\xi) - \eta(X)\eta(Y) + g\big(X + hX - \varphi hX - \frac{1}{2}lX, Y\big). \tag{18}$$

Indeed, using (5) and the first identity of (15), we get

$$\nabla^1_X\xi = -hX + hh_1X = -hX - \varphi X + h\varphi X - \frac{1}{2}h\varphi lX.$$

Hence, applying also (2), we get

$$\begin{aligned}
g(\nabla^1_X Y,\xi) &= g_1(\nabla^1_X Y,\xi) = X(g_1(Y,\xi)) - g_1(Y,\nabla^1_X\xi)\\
&= X(g(Y,\xi)) - g(\varphi hY,\nabla^1_X\xi) - \eta(Y)\eta(\nabla^1_X\xi)\\
&= g(\nabla_X Y,\xi) + g(Y,-\varphi X - \varphi hX) + g\big(hX + \varphi X + \varphi hX - \frac{1}{2}\varphi hlX,\varphi hY\big)\\
&= g(\nabla_X Y,\xi) - \eta(X)\eta(Y) + g\big(X + hX - \varphi hX - \frac{1}{2}lX, Y\big).
\end{aligned}$$

From (17) and (18) we get the result. □

PROPOSITION 4. *Let (M,φ,ξ,η,g) be a contact metric manifold such that $\varphi l + l\varphi = 0$. The Jacobi operator l_1 is given by:*

$$l_1 = \varphi^2 + 2\varphi + 2\varphi h - lh + l\varphi - \frac{1}{4}l^2 + \frac{1}{2}h\varphi\nabla_\xi l. \tag{19}$$

If M is a Jacobi $(0,\mu)$-contact space, then

$$l_1 = \Big(1 + \mu - \frac{1}{4}\mu^2\Big)\varphi^2 + 2\varphi + (2-\mu)\varphi h. \tag{20}$$

In particular, if $l = 0$, then $l_1 = \varphi^2 + 2\varphi + 2\varphi h$.

Proof. Recall that for a contact metric structure $\nabla_\xi\varphi = 0$. Hence, from (16) and using also (13), we have

$$\nabla^1_\xi = \nabla_\xi - h + \frac{1}{2}\varphi hl. \tag{21}$$

Now, from (21) and the first identity in (15), applying also $\nabla_\xi\varphi = 0$, (13), (14), $hh_1 + h_1h = 0$, and identities (a), (c) in Lemma 2, we get

$$\begin{aligned}
\nabla^1_\xi h_1 &= \nabla_\xi h_1 - hh_1 + h_1h + \frac{1}{2}(\varphi hlh_1 - h_1\varphi hl) \\
&= \varphi\nabla_\xi h - \frac{1}{2}\varphi\nabla_\xi l - 2hh_1 + \frac{1}{2}(\varphi h_1 - h_1\varphi)hl \\
&= l - \frac{1}{2}\varphi\nabla_\xi l - 2h\Big(\varphi + \varphi h - \frac{1}{2}\varphi l\Big) - l + \frac{1}{2}hl^2 \\
&= -\frac{1}{2}\varphi\nabla_\xi l + 2\varphi h + 2\varphi - \varphi hl + \frac{1}{2}hl^2.
\end{aligned}$$

On the other hand, from (6) we know that $\nabla^1_\xi h_1 = -h + h_1^2 h - hl_1$, and using (d) of Lemma (2), we have

$$2\varphi h + 2\varphi - \varphi hl + \frac{1}{2}hl^2 - \frac{1}{2}\varphi\nabla_\xi l = -h + \Big(\frac{1}{4}l^2 - hl\Big)h - hl_1.$$

Applying h on both sides of the last identity we get (19). Finally, if $l = \mu h$, applying (13) in (19), we obtain (20). □

If (M,φ,ξ,η,g) is a contact metric manifold such that $\varphi l + l\varphi = 0$, consider local orthonormal frames of type $\{\xi, e_i, \varphi e_i\}$, $i = 1,\dots,n$, such that $he_i = e_i$. Then, setting $u_i := e_i + \varphi e_i$ and $v_i := e_i - \varphi e_i$, one easily verifies that $\{\xi, u_i, v_i\}$, $i = 1,\dots,n$, is a local orthogonal frame with respect to g_1, such that each u_i is space-like and each v_i is time-like. Indeed, being $\varphi hu_i = u_i$, we have $g_1(u_i,u_i) = g(u_i,\varphi hu_i) = g(u_i,u_i) = 2$. Analogously, $\varphi hv_i = -v_i$ implies that $g_1(v_i,v_i) = -2$.

If M is a Jacobi $(0,\mu)$-contact space, being $l = \mu h$ one has the following ξ-sectional curvatures with respect to the Riemannian metric g:

$$K(\xi,e_i) = \mu, \qquad K(\xi,\varphi e_i) = -\mu, \qquad K(\xi,u_i) = K(\xi,v_i) = 0.$$

We compute now the sectional curvatures with respect to the semi-Riemannian metric g_1 of the nondegenerate 2-planes spanned by ξ and the vector fields u_i or v_i.

PROPOSITION 5. *Let (M,φ,ξ,η,g) be a Jacobi $(0,\mu)$-contact space. Consider a local orthonormal frame $\{\xi,e_i,\varphi e_i\}$, $i = 1,\dots,n$, such that $he_i = e_i$. Then the semi-Riemannian metric g_1 has ξ-sectional curvatures*

$$K_1(\xi,u_i) = 1 - 2\mu + \frac{1}{4}\mu^2, \qquad K_1(\xi,v_i) = -3 + \frac{1}{4}\mu^2.$$

In particular, if $l = 0$ then $K_1(\xi,u_i) = 1$ and $K_1(\xi,v_i) = -3$.

Proof. The sectional curvature of a nondegenerate 2-plane spanned by ξ and a vector field X is $K_1(\xi,X) = \frac{g_1(l_1X,X)}{g_1(X,X)}$. Applying (20), we have

$$l_1u_i = \Big(1 - 2\mu + \frac{1}{4}\mu^2\Big)u_i + 2\varphi u_i, \qquad l_1v_i = \Big(-3 + \frac{1}{4}\mu^2\Big)v_i + 2\varphi v_i,$$

which get the result. □

5. The paracontact metric structure $(\varphi h, \xi, \eta, g_2)$

THEOREM 3. *Let* $(M, \varphi, \xi, \eta, g)$ *be a contact metric manifold such that* $\varphi l + l\varphi = 0$*. Then for every* $X, Y, Z \in \Gamma(TM)$ *the Levi-Civita connections* ∇^2 *and* ∇ *satisfy:*

$$\begin{aligned}(22)\quad g(\nabla^2_X Y, Z) = g(\nabla_X Y, Z) - \eta(X)g(\varphi hY, Z) - \eta(Y)g(\varphi hX, Z) - \frac{1}{2}g((\nabla_{hZ}h)Y, X)\\ + \frac{1}{2}g\big((\nabla_X h)Y + (\nabla_Y h)X, hZ\big) - \frac{1}{2}\eta(Z)g(\varphi lX, Y).\end{aligned}$$

Proof. First, from (5) and the second identity in (15), we get

$$\nabla^2_X \xi = -\varphi hX + \varphi hh_2X = -\varphi hX - \varphi X + \frac{1}{2}\varphi hlX. \tag{23}$$

Hence, applying (11), (23) and (2), we have

$$\begin{aligned}g(\nabla^2_X Y, \xi) &= g_2(\nabla^2_X Y, \xi) = X(g_2(Y, \xi)) - g_2(Y, \nabla^2_X \xi)\\ &= X(g(Y, \xi)) + g(\nabla^2_X \xi, hY)\\ &= g(\nabla_X Y, \xi) - g(Y, \varphi X + \varphi hX) - g\big(\varphi hX + \varphi X - \frac{1}{2}\varphi hlX, hY\big)\\ &= g(\nabla_X Y, \xi) - \frac{1}{2}g(\varphi lX, Y).\end{aligned}$$

From (11), $g_2(\nabla^2_X Y, hZ) = -g(\nabla^2_X Y, h^2 Z)$ for every $X, Y, Z \in \Gamma(TM)$, and thus

$$g_2(\nabla^2_X Y, hZ) = -g(\nabla^2_X Y, Z) + \eta(Z)\Big(g(\nabla_X Y, \xi) - \frac{1}{2}g(\varphi lX, Y)\Big). \tag{24}$$

On the other hand, using the Jacobi identity and (11)

$$\begin{aligned}2g_2(\nabla^2_X Y, hZ) \;=\;& X(g_2(Y, hZ)) + Y(g_2(hZ, X)) - (hZ)(g_2(X, Y))\\ &+ g_2([X, Y], hZ) - g_2([Y, hZ], X) + g_2([hZ, X], Y)\\ =\;& -X(g(Y, h^2Z)) - Y(g(hZ, hX)) + (hZ)(g(X, hY))\\ &- (hZ)(\eta(X)\eta(Y)) - g([X, Y], h^2Z) + g([Y, hZ], hX)\\ &- \eta([Y, hZ])\eta(X) - g([hZ, X], hY) + \eta([hZ, X])\eta(Y)\\ =\;& -2g(\nabla_X Y, h^2Z) - g(Y, \nabla_X(h^2Z)) - g(X, \nabla_Y(h^2Z))\\ &+ g(X, \nabla_{hZ}(hY)) + g(\nabla_X(hZ), hY) + g(\nabla_Y(hZ), hX)\\ &- g(\nabla_{hZ}Y, hX) + 2d\eta(Y, hZ)\eta(X) - 2d\eta(hZ, X)\eta(Y)\\ =\;& -2g(\nabla_X Y, Z) + 2\eta(Z)g(\nabla_X Y, \xi)\\ &+ 2g(Y, \varphi hZ)\eta(X) - 2g(hZ, \varphi X)\eta(Y)\\ &- g(Y, (\nabla_X h)hZ) - g(X, (\nabla_Y h)hZ) + g(X, (\nabla_{hZ}h)Y).\end{aligned}$$

Comparing the above equation with (24), we obtain (22). □

PROPOSITION 6. *Let (M,φ,ξ,η,g) be a contact metric manifold such that $\varphi l + l\varphi = 0$. The Jacobi operator l_2 is given by:*

$$l_2 = -lh - \frac{1}{4}l^2 + \frac{1}{2}h\varphi\nabla_\xi l. \tag{25}$$

If M is a Jacobi $(0,\mu)$-contact space, then

$$l_2 = \mu\Big(1 - \frac{1}{4}\mu\Big)\varphi^2. \tag{26}$$

In particular, if $l = 0$, then $l_2 = 0$.

Proof. First we prove that

$$\nabla^2_\xi = \nabla_\xi + \frac{1}{2}\varphi hl. \tag{27}$$

Indeed, applying (22), (13) and (2), for every $Y,Z \in \Gamma(TM)$ we have

$$\begin{aligned} g(\nabla^2_\xi Y,Z) &= g(\nabla_\xi Y - \varphi hY, Z) - \frac{1}{2}g(\nabla_{hZ}(hY),\xi) + \frac{1}{2}g(-\varphi lY + h(\varphi Y + \varphi hY), hZ) \\ &= g(\nabla_\xi Y - \varphi hY, Z) + \frac{1}{2}g(hY, -\varphi hZ - \varphi Z) + \frac{1}{2}g(\varphi hlY + \varphi Y + \varphi hY, Z) \\ &= g(\nabla_\xi Y, Z) + \frac{1}{2}g(\varphi hlY, Z). \end{aligned}$$

From (27) we have $\nabla^2_\xi h_2 = \nabla_\xi h_2 + \frac{1}{2}(\varphi hlh_2 - h_2\varphi hl)$, where the operator $h_2 = -h + \frac{1}{2}l$ commutes with l and h, and anticommutes with φ. Therefore,

$$\nabla^2_\xi h_2 = -\nabla_\xi h + \frac{1}{2}\nabla_\xi l - h_2\varphi hl = \varphi l + \frac{1}{2}\nabla_\xi l - \Big(\frac{1}{2}l - h\Big)\varphi hl = \frac{1}{2}\left(\nabla_\xi l + \varphi hl^2\right).$$

On the other hand, from (6) we have $\nabla^2_\xi h_2 = -\varphi h + h_2^2\varphi h - \varphi hl_2$, and thus

$$\nabla^2_\xi h_2 = -\varphi h + \Big(h^2 - lh + \frac{1}{4}l^2\Big)\varphi h - \varphi hl_2 = \varphi h\Big(-lh + \frac{1}{4}l^2 - l_2\Big).$$

Comparing the two expressions of $\nabla^2_\xi h_2$, we have

$$\varphi h\Big(-lh + \frac{1}{4}l^2 - l_2\Big) = \frac{1}{2}\left(\nabla_\xi l + \varphi hl^2\right),$$

and applying φh to both sides in the above equation, we get (25). Finally, if $l = \mu h$, we easily obtain (26). □

Finally, let us consider a Jacobi $(0,\mu)$-contact space (M,φ,ξ,η,g), and fix a local orthonormal frame $\{\xi, e_i, \varphi e_i\}$, $i = 1,\ldots,n$, such that $he_i = e_i$. Notice that $g_2(e_i,e_i) = -1$ and $g_2(\varphi e_i,\varphi e_i) = 1$, and thus each e_i is time-like, while each φe_i is space-like with respect to g_2. Using (26), one can compute the sectional curvatures for g_2 of the non-degenerate 2-planes spanned by ξ and the vector fields e_i or φe_i, showing that

$$K_2(\xi,e_i) = K_2(\xi,\varphi e_i) = -\mu\Big(1 - \frac{1}{4}\mu\Big).$$

References

[1] BLAIR D. E., *Riemannian geometry of contact and symplectic manifolds. Second Edition*. Progress in Mathematics **203**, Birkhäuser, Boston 2010.

[2] CALVARUSO G. AND MARTÍN-MOLINA V., *Paracontact metric structures on the unit tangent sphere bundle*, Ann. Mat. Pura Appl. **194** (2015), 1359–1380.

[3] CAPPELLETTI-MONTANO B., *Bi-paracontact structures and Legendre foliations*, Kodai Math. J. **33** 3 (2010), 473–512.

[4] CAPPELLETTI-MONTANO B. AND DI TERLIZZI L., *Geometric structures associated to a contact metric (κ,μ)-space*, Pacific J. Math. **246** 2 (2010), 257–292.

[5] CAPPELLETTI-MONTANO B., KÜPELI ERKEN I., MURATHAN C., *Nullity conditions in paracontact geometry*, Differential Geom. Appl. **30** 6 (2012), 665–693.

[6] CHO J. T. AND INOGUCHI J., *Characteristic Jacobi operator on contact Riemannnian 3-manifolds*, Differ. Geom. Dyn. Syst. **17** (2015), 49–71.

[7] GHOSH A. AND SHARMA R., *A generalization of K-contact and (κ,μ)-contact manifolds*, J. Geom. **103** 3 (2012), 431–443.

[8] GOULI-ANDREOU F., *On contact metric 3-manifolds with $R(X,\xi)\xi = 0$*, Algebras Groups Geom. **17** 4 (2000), 393–400.

[9] KOUFOGIORGOS T. AND TSICHLIAS C., *Three dimensional contact metric manifolds with vanishing Jacobi operator*, Beiträge Algebra Geom. **50** 2 (2009), 563–573.

[10] MARTÍN-MOLINA V., *Local classification and examples of an important class of paracontact metric manifolds*, Filomat **29** 3 (2015), 507–515.

[11] MARTÍN-MOLINA V., *Paracontact metric manifolds without a contact metric counterpart*, Taiwanese J. Math. **19** 1 (2015), 175–191.

[12] OLSZAK Z., *On contact metric manifolds*, Tôhoku Math. J. **31** (1979), 247-253.

[13] PERRONE D., *Contact Riemannian manifolds satisfying $R(X,\xi)\cdot R = 0$*, Yokohama Math. J. **39** 2 (1992), 141–149.

[14] PERRONE D., *Homogeneous contact Riemannian 3-manifolds*, Illinois J. Math. **13** (1997), 243–256.

[15] PERRONE D., *Weakly ϕ-symmetric contact metric spaces*, Balkan J. Geom. Appl. **7** (2002), 67–77.

[16] ZAMKOVOY S., *Canonical connections on paracontact manifolds*, Ann. Global Anal. Geom. **36** 1 (2009), 37–60.

AMS Subject Classification: 53C15, 53C50, 53D10, 53C25.

Luigia DI TERLIZZI, Giulia DILEO
Dipartimento di Matematica, Università degli Studi di Bari Aldo Moro
Via E. Orabona 4, 70125 Bari, ITALY
e-mail: `luigia.diterlizzi@uniba.it, giulia.dileo@uniba.it`

Lavoro pervenuto in redazione il 05.11.2015 e accettato il 10.02.2016.

Geom. Struc. on Riem. Man.-Bari
Vol. 73/1, 3–4 (2015), 101 – 116

J. I. Inoguchi - M. I. Munteanu

NEW EXAMPLES OF MAGNETIC MAPS INVOLVING TANGENT BUNDLES

Dedicated to Prof. Anna-Maria Pastore
with the occasion of her anniversary

Abstract. We produce new examples of magnetic maps, having as either source or target manifold the tangent bundle of a Riemannian manifold equipped with several Riemannian metrics. In particular we study when the canonical projection, a vector field and the tangent map are, respectively, magnetic maps.

1. Preliminaries

In a tentative to generalize the notion of magnetic trajectory on a Riemannian manifold, the authors define in [13] the notion of *magnetic maps*. As we see, both magnetic curves and harmonic maps can be obtained as particular situations of magnetic maps.

1.1. Magnetic maps

Let $f : N \longrightarrow M$ be a smooth map between two Riemannian manifolds (N,h) of dimension n and (M,g) of dimension m. Let ξ be a global divergence free vector field on N and ω be a 1-form on M. For the moment suppose that N is compact. The energy of f (or the Dirichlet integral of f) is known as

$$E(f) = \frac{1}{2}\int_N |df|^2 dv_h,$$

where dv_h denotes the volume element on N and $|df|$ is the Hilbert Schmidt norm of the differential df given in a point $p \in N$ by

$$|df_p|^2 = \sum_{i=1}^{n} g_{f(p)}\left(f_{*,p}e_i, f_{*,p}e_i\right).$$

Here $\{e_i;\ i = 1,\dots,n\}$ is an arbitrary orthonormal basis for T_pN.
A smooth map $f : (N,h) \longrightarrow (M,g)$ which is a critical point of $E(f)$ is called a *harmonic map* (see e.g. [10, 25]).
Let us now define the following functional for f associated to ξ and ω:

$$LH(f) = E(f) + \int_N \omega(df(\xi))dv_h. \tag{1}$$

Take a smooth variation $\{\mathcal{F}_\varepsilon\}_{\varepsilon\in I}$ of f, that is a smooth map $\mathcal{F} : N \times I \longrightarrow M$, such that $\mathcal{F}(p,0) = f(p)$. Here I is an open interval containing 0. For the sake of simplicity we use to write $f_\varepsilon(p) = \mathcal{F}(p,\varepsilon)$.

DEFINITION 1. The map f is called *magnetic* with respect to ξ and ω if it is a critical point of the Landau Hall integral $LH(f)$, i.e., the first variation $\frac{d}{d\varepsilon}LH(f_\varepsilon)\big|_{\varepsilon=0}$ is zero for any f_ε.

REMARK 1. In analogy to the definition of harmonic maps, one may replace "N-compact" by the condition "compact support variation".

Let (N,h), (M,g), ξ and ω as before. In [13] the authors prove the following.

THEOREM 1. *Let $f:(N,h)\longrightarrow(M,g)$ be a smooth map. Then f is a magnetic map with respect to ξ and ω if and only if it satisfies the Lorentz equation, that is*

$$\tau(f)=\phi(f_*\xi), \tag{2}$$

where $\tau(f):=\mathrm{trace}_h\nabla df$ is the tension field of f. The endomorphism ϕ, called the Lorentz force associated to the potential 1-form ω, is defined by $g(\phi(X),Y)=d\omega(X,Y)$, for all X,Y tangent to M.

Sometimes, equation (2) will be called the *magnetic equation*. Recall that on a Riemannian manifold (M,g) a *magnetic field* is defined by a closed 2-form F and the *Lorentz force* associated to F is a $(1,1)$ tensor field ϕ on M given by $g(\phi X,Y)=F(X,Y)$. The *magnetic trajectories* of F are curves γ satisfying the *Lorentz equation* $\nabla_{\gamma'}\gamma'=\phi\gamma'$. This equation is a particular case of equation (2) when N is an interval of $\mathbb{R}$ and $\xi=\frac{d}{dt}$, where t is the global coordinate on N. Magnetic curves were intensively studied in the last years by several geometers (including the authors of this article) in different ambient spaces.

REMARK 2. The Lorentz equation (2) was obtained from a variational principle assuming that the domain is compact and the 2-form F is exact. Since it has a tensorial character, one can define a magnetic map $f:(N,h)\longrightarrow(M,g)$ without the assumptions N compact and F exact (but only closed). Moreover, the condition "ξ is divergence free" will be also removed.

More precisely, let ξ be a global vector field on N, F be a magnetic field on M and ϕ the Lorentz force associated to F. Similarly to magnetic curves, we may also introduce a *strength* (i.e., a real number) in the equation. Hence, we give the following.

DEFINITION 2. We say that f is a *magnetic map* with strength $q\in\mathbb{R}$ associated to ξ and F if the Lorentz equation

$$\tau(f)=q\,\phi(f_*\xi) \tag{3}$$

is satisfied.

1.2. Tangent bundle of a Riemannian manifold

Let (M,g) be a Riemannian manifold of dimension n and $\pi:T(M)\longrightarrow M$ its tangent bundle. Denote by ∇ the Levi-Civita connection of g. For each $u\in T(M)$ we have the following decomposition of the tangent space $T_{\mathfrak{u}}T(M)$ (in $\mathfrak{u}$ at $T(M)$)

$$T_{\mathfrak{u}}T(M) = V_{\mathfrak{u}}T(M) \oplus H_{\mathfrak{u}}T(M),$$

where $V_{\mathfrak{u}}T(M) = \ker \pi_{*,\mathfrak{u}}$ is the vertical space and $H_{\mathfrak{u}}T(M)$ is the horizontal space at $\mathfrak{u}$ obtained by using ∇. A curve $\tilde{\gamma} : I \longrightarrow T(M)$, $t \longmapsto (\gamma(t), V(t))$ is *horizontal* if the vector field $V(t)$ is parallel along $\gamma = \pi \circ \tilde{\gamma}$. A vector on $T(M)$ is *horizontal* if it is tangent to a horizontal curve and *vertical* if it is tangent to a fiber. Locally, take a chart (U, x^i), $i = 1, \ldots, n = \dim(M)$ in $p \in M$, and consider the induced chart $(\pi^{-1}(U), x^i, y^i)$ on $T(M)$. If $\Gamma^k_{ij}(x)$ are the Christoffel symbols, then $\delta_i := \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^k_{ij}(x) y^j \frac{\partial}{\partial y^k}$ in $\mathfrak{u}$, for $i = 1, \ldots, n$, span $H_{\mathfrak{u}}T(M)$, while $\dot{\partial}_i := \frac{\partial}{\partial y^i}$, for $i = 1, \ldots, n$, span the vertical space $V_{\mathfrak{u}}T(M)$. We have defined the horizontal (resp. vertical) distributions HTM (resp. VTM) and the direct sum decomposition

$$TTM = HTM \oplus VTM$$

of the tangent bundle of $T(M)$. If $X \in \mathfrak{X}(M)$, denote by X^H (resp. X^V) the horizontal (resp. the vertical) lift of X to $T(M)$. See for more details [8].

Two classical examples of Riemannian metrics on $T(M)$ are well known, namely the Sasaki metric and the Cheeger-Gromoll metric, respectively. See for example [7, 12, 17, 23]. These metrics are only two possible choices inside a wide family of Riemannian metrics on $T(M)$, known as Riemannian *g-natural metrics*. A large number of papers related to this topic have been published so far, but we emphasize only few of them: [1, 3, 16, 20].

The **Sasaki metric** is defined uniquely by the following relations

$$g_S(X^H, Y^H) = g_S(X^V, Y^V) = g(X, Y) \circ \pi, \; g_S(X^H, Y^V) = 0, \tag{4}$$

for all X, Y tangent to M. We give here, for later use, the expression of the Levi-Civita connection ${}^S\nabla$ of the Sasaki metric in terms of an adapted local basis defined above:

$$\begin{cases} {}^S\nabla_{\dot{\partial}_i} \dot{\partial}_j = 0, & {}^S\nabla_{\delta_i} \delta_j = \Gamma^h_{ij} \delta_h - \frac{1}{2} R^h_{0ij} \dot{\partial}_h, \\ {}^S\nabla_{\delta_i} \dot{\partial}_j = \Gamma^h_{ij} \dot{\partial}_h + \frac{1}{2} R^h_{i0j} \delta_h, & {}^S\nabla_{\dot{\partial}_i} \delta_j = \frac{1}{2} R^h_{j0i} \delta_h, \end{cases} \tag{5}$$

where $R^h_{kij} \frac{\partial}{\partial x^h} = R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^k}$ and "0" stands for the contraction with $\mathfrak{u}$, namely $R^h_{0ij} = R^h_{kij} y^k$. See e.g. [4, Chapter 9] and [15].

The **Cheeger-Gromoll metric** is given by

$$\begin{cases} g_{CG(p,\mathfrak{u})}(X^H, Y^H) = g_p(X, Y), \; g_{CG(p,\mathfrak{u})}(X^H, Y^V) = 0, \\ g_{CG(p,\mathfrak{u})}(X^V, Y^V) = \frac{1}{1+2t} \left[g_p(X, Y) + g_p(X, \mathfrak{u}) g_p(Y, \mathfrak{u}) \right], \end{cases} \tag{6}$$

for all X, Y tangent to M at $p = \pi(\mathfrak{u})$. We have denoted by t the energy density in $(p, \mathfrak{u})$ on $T(M)$, that is $\frac{1}{2}\, g_p(\mathfrak{u}, \mathfrak{u})$. Again, we give, for later use, the expression of the

Levi-Civita connection ${}^{CG}\nabla$ of the Cheeger-Gromoll metric:

$$(7)\qquad \begin{cases} {}^{CG}\nabla_{\dot{\partial}_i}\dot{\partial}_j = \frac{1}{1+2t}\left(g_{ij}C - g_{i0}\dot{\partial}_j - g_{j0}\dot{\partial}_i\right) + \frac{1}{(1+2t)^2}\left(g_{ij}+g_{i0}g_{j0}\right)C, \\ {}^{CG}\nabla_{\delta_i}\delta_j = \Gamma^h_{ij}\delta_h - \frac{1}{2}R^h_{0ij}\dot{\partial}_h, \quad {}^{CG}\nabla_{\dot{\partial}_i}\delta_j = \frac{1}{2(1+2t)}R^h_{j0i}\delta_h, \\ {}^{CG}\nabla_{\delta_i}\dot{\partial}_j = \Gamma^h_{ij}\dot{\partial}_h + \frac{1}{2(1+2t)}R^h_{i0j}\delta_h. \end{cases}$$

See e.g. [12, 24]. The vector field C is the Liouville vector field that will be defined in what follows. In fact, two global vector fields may be defined on $T(M)$:

(a) The *geodesic spray* ξ of the connection ∇ is the unique tangent vector at $(p,\mathfrak{u})$ which is horizontal and satisfies $\pi_{*,(p,\mathfrak{u})}\xi_{(p,\mathfrak{u})} = \mathfrak{u}$. As consequence, any integral curve $(x(t),y(t))$ of ξ through the point $(p,\mathfrak{u})$ obeys $\dot{x}(0) = \mathfrak{u}$ and $\nabla_{\dot{x}(0)}y = 0$. Therefore, if $\mathfrak{u}$ is a tangent vector at p to M and $\gamma : t \mapsto \gamma(t)$ is the geodesic through $p = \gamma(0)$ with $\dot{\gamma}(0) = \mathfrak{u}$, it is the projection under π of the integral curve $\tilde{\gamma} : t \mapsto \tilde{\gamma}(t)$ of ξ through $\mathfrak{u}$.

(b) The Liouville vector field C on $T(M)$ is the infinitesimal generator of the flow given by homotheties on each fiber, that is $(t,\mathfrak{u}) \in \mathbb{R}\times T_pM \mapsto e^t\mathfrak{u} \in T_pM$. The vector field C is the unique vertical vector field on $T(M)$ satisfying $K_{(p,\mathfrak{u})}C_{(p,\mathfrak{u})} = \mathfrak{u}$, where K is the connection map. See e.g. [8]. It is also called, sometimes, *the radial vector field* on $T(M)$.

2. Canonical projection $\pi : T(M) \longrightarrow M$ as magnetic map

Let (M,g) be a Riemannian manifold of dimension n and let $T(M)$ be its tangent bundle. In [2] the authors find a necessary and sufficient condition for the harmonicity of the canonical projection from $T(M)$ equipped with an arbitrary Riemannian g-natural metric to (M,g). In particular they prove that if the Riemannian g-natural metric is such that the horizontal and the vertical distributions are orthogonal, then $\pi : T(M) \longrightarrow M$ is harmonic. This is the case of the two metrics we have already mentioned, that is the Sasaki metric g_S and the Cheeger-Gromoll metric g_{CG}.

Let ξ be a global vector field on $T(M)$, F a magnetic field on M whose Lorentz force is ϕ and $q \neq 0$ an arbitrary real number. Then π is a magnetic map with strength q with respect to ξ and F if and only if $\pi_*\xi \in \ker\phi$. In particular, if ξ is vertical (as the Liouville vector field C) then π is magnetic with respect to that ξ and any magnetic field F.

In the following we will consider a nonlinear connection on $T(M)$. More precisely, let (M,g) be a Riemannian manifold of dimension n and $\pi : T(M) \longrightarrow M$ its tangent bundle, as before. The vertical subspaces (respectively the vertical distribution) on $T(M)$ depend only on the differential structures of M and $T(M)$. Thus, their definition is the same as in Section 1.2, that is at $\mathfrak{u} \in T_pM$, we have

$$(8)\qquad V_{\mathfrak{u}}T(M) := \ker\, \pi_{*,\mathfrak{u}}.$$

A *horizontal distribution* HTM on $T(M)$ is a supplementary distribution to VTM, that is

$$T_{\mathfrak{u}}TM = V_{\mathfrak{u}}TM \oplus H_{\mathfrak{u}}TM.$$

A *nonlinear connection* on $T(M)$ is a vector bundle morphism $v : TTM \longrightarrow VTM$ such that $v \circ \iota = Id_{VTM}$, where $\iota : VTM \longrightarrow TTM$ is the canonical inclusion. Hence, the kernel of the morphism v is the horizontal subbundle HTM. See, for details, e.g. [6, Part I] and the references therein. In the following, the names "horizontal" and "vertical" have the obvious meaning.

It is clear that, due to (8), the restriction $\pi_{*,\mathfrak{u}} : H_{\mathfrak{u}}TM \longrightarrow T_pM$, where $p = \pi(\mathfrak{u})$, is an isomorphism of vector spaces. Its inverse map is called the *horizontal lift* induced by the nonlinear connection.

A local chart (U, φ, x) on M induces on $T(M)$ a local chart $(\pi^{-1}(U), \Phi, (x,y))$ with respect to which we have a local adapted frame in HTM defined by the following vector fields

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x,y)\frac{\partial}{\partial y^j}, \text{ for } i = 1, \ldots, n. \tag{9}$$

They represent the horizontal lifts of the canonical basis $\frac{\partial}{\partial x^i}$ (defined on U). The functions N_i^j are known as the *coefficients of the nonlinear connection* defined by HTM.

Hence, for any (local) vector field $X = X^i(x)\frac{\partial}{\partial x^i}$ on M, the horizontal and the vertical lifts at $\mathfrak{u}$ are given by:

$$X^H_{\mathfrak{u}} = X^i(x) \left.\frac{\delta}{\delta x^i}\right|_{\mathfrak{u}}, \; X^V_{\mathfrak{u}} = X^i(x) \left.\frac{\partial}{\partial y^i}\right|_{\mathfrak{u}}. \tag{10}$$

Note that the classical situation is obtained when $N_i^j(x,y) = \Gamma^j_{ik}(x)y^k$, where $\Gamma^j_{ik}(x)$ are the coefficients of the Levi-Civita connection of g.

One can define on $T(M)$ a Riemannian metric g_{s}, of Sasaki type, as follows

$$g_{\mathsf{s}}(X^V, Y^V) = g(X,Y) \circ \pi, \; g_{\mathsf{s}}(X^H, Y^V) = 0, \; g_{\mathsf{s}}(X^H, Y^H) = g(X,Y) \circ \pi. \tag{11}$$

It follows that π is a Riemannian submersion.

Denote by $\sigma(\pi)$ the second fundamental form of the projection π, that is $\sigma(\pi) = \nabla d\pi$. We immediately have

$$\begin{cases} \sigma(\pi)\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \frac{1}{2} g^{hk}\left(\frac{\partial g_{ij}}{\partial x^k} - g_{il}B\Gamma^l_{jk} - g_{jl}B\Gamma^l_{ik}\right)\frac{\partial}{\partial x^h}, \\ \sigma(\pi)\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = 0, \end{cases}$$

where $B\Gamma^l_{jk}(x,y) = \frac{\partial N^l_k}{\partial y^j}$ represent the coefficients of the Berwald connection. See e.g. [6].

For an arbitrary nonlinear connection, the fibers of the projection π are not, in general, totally geodesic. Yet, in the classical case when $N_i^j(x,y) = \Gamma^j_{ik}(x)y^k$, the projection π

has all its fibers totally geodesic. However, the projection $\pi : (T(M), g_{\mathrm{s}}) \longrightarrow (M, g)$ is harmonic if and only if π has minimal fibers. See e.g. [19].

Interesting results concerning the geometry of the tangent bundle $T(M)$ equipped as before, may be obtained when the functions $N_k^l(x,y)$ are polynomials in variables y^i, for $i = 1, \ldots, n$. Therefore, let $N_k^l(x,y)$ be a polynomial of degree 2 in y of the following form:

$$N_k^l(x,y) = R_{skh}^l(x) y^s y^h + \Gamma_{ks}^l(x) y^s + T_{ks}^l(x) y^s + \Psi_k^l(x), \tag{12}$$

where R_{skh}^l (respectively T_{ks}^l and Ψ_k^l) are the coefficients of a $(1,3)$ (respectively $(1,2)$ and $(1,1)$) tensor field on M.

Compute the tension field of π. Since $\tau(\pi) = \mathrm{trace}_{g_{\mathrm{s}}}(\sigma(\pi))$ we have

$$\tau(\pi) = g^{ij} \sigma(\pi) \left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = \frac{1}{2}\, g^{kh} \left[T_{kl}^l(x) - \left(R_{lks}^l(x) + R_{skl}^l(x) \right) y^s \right] \frac{\partial}{\partial x^h}.$$

Let now ξ be a global vector field on $T(M)$, F a magnetic field on M whose Lorentz force is ϕ and $q \neq 0$ a real number. Then π is a magnetic map if and only if the magnetic equation (with strength q) is satisfied, that is $\tau(\pi) = q\, \phi(\pi_* \xi)$. Of course if ξ is vertical then π is magnetic if and only if it is harmonic. Take $\xi|_{\mathfrak{u}} = y^k \frac{\delta}{\delta x^k}|_{\mathfrak{u}}$, where $\mathfrak{u} = y^k \frac{\partial}{\partial x^k}|_p$, $p = \pi(\mathfrak{u})$. Then, the magnetic equation becomes

$$g^{kh} \left[T_{kl}^l(x) - \left(R_{lks}^l(x) + R_{skl}^l(x) \right) y^s \right] = 2q \phi_s^h(x) y^s,$$

which is equivalent to

$$T_{kl}^l(x) - \left(R_{lks}^l(x) + R_{skl}^l(x) \right) y^s = 2q\, F_{sk}(x) y^s,$$

for all $\mathfrak{u} \in T(M)$. Here ϕ_s^h (respectively F_{sk}) are coefficients of the Lorentz force ϕ (respectively of the magnetic field F) in the local chart on M.

It follows that

$$\begin{cases} T_{kl}^l(x) = 0, \\ R_{lks}^l(x) + R_{skl}^l(x) = 2q\, F_{sk}(x). \end{cases} \tag{13}$$

We can formulate the following result.

THEOREM 2. *Let $\pi : (T(M), g_{\mathrm{s}}) \longrightarrow (M, g)$ defined as before, where the nonlinear connection is defined by three tensor fields on M, namely $R \in \mathcal{T}_3^1(M)$, $T \in \mathcal{T}_2^1(M)$ and $\Psi \in \mathcal{T}_1^1(M)$, as in* (12). *Let ξ be the canonical horizontal vector field on $T(M)$ and F a magnetic field on M. Then π is a magnetic map with strength q with respect to ξ and F if and only if*

(i) *the 1-form* $\mathrm{trace}_g\big(\bullet \mapsto T(X, \bullet)\big)$ *vanishes;*

(ii) $2qF(X,Y) = \mathrm{trace}_g\big(\bullet \mapsto R(X,Y)\bullet\big) + \mathrm{trace}_g\big(\bullet \mapsto R(X,\bullet)Y\big)$.

REMARK 3. We have put $T(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}) = T^l_{ij}(x)\frac{\partial}{\partial x^l}$ and $R(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})\frac{\partial}{\partial x^k} = R^l{}_{kij}(x)\frac{\partial}{\partial x^l}$.

EXAMPLE 1. If J is a skew symmetric $(1,1)$ tensor field on M and a is a 1-form on M, then $T = a \otimes J$, defined by $T(X,Y) = a(X)JY$ fulfills condition (i) of Theorem 2.

EXAMPLE 2. Suppose that R is a curvature-like tensor on M, namely it has all the symmetries as the Riemannian curvature tensor (including the first Bianchi identity). Suppose that its Ricci tensor $\rho(R)$ is skew symmetric. Then, the condition (ii) is satisfied if and only if $\rho(R)$ is a closed 2-form and $F = \frac{1}{2q}\rho(R)$.

EXAMPLE 3. We give two other situations when the condition (ii) of Theorem 2 is fulfilled:

(a) $R = 2q\, g \otimes \phi$, that is $N^l_k(x,y) = g_{kh}(x)\phi^l_s(x)y^h y^s + \Gamma^l_{ks}(x)y^s + T^l_{ks}(x)y^s + \Psi^l_k(x)$,

(b) $R = \frac{2q}{n+1}\, F \otimes I$, that is $N^l_k(x,y) = F_{kh}(x)y^h y^l + \Gamma^l_{ks}(x)y^s + T^l_{ks}(x)y^s + \Psi^l_k(x)$,

where I is the identity tensor on $T(M)$. Here F is the magnetic field on M and ϕ is the corresponding Lorentz force.

EXAMPLE 4. If V is a Killing vector field on M, consider $R = g \otimes \nabla V$.
We have $R^h_{kij} = g_{ij}\nabla_k V^h$. Therefore $R^l_{lks} = g_{ks}\nabla_l V^l = 0$ (since V is divergence free) and $R^l_{skl} = \nabla_s V_k$, where $V_k = g_{kl}V^k$. As V is Killing, we obtain that F_0 is a 2-form, where $F_0(X,Y) = g(\nabla_X V, Y)$. In case when F_0 is closed, condition (ii) is satisfied with $F = \frac{1}{2q}\, F_0$. This happens in several situations, for example when:

- M is the Euclidean n-space $\mathbb{E}^n(x^1,\dots,x^n)$ and $V = \frac{\partial}{\partial x^j}$ (which is parallel);
- M is the Euclidean 3-space $\mathbb{E}^3(x,y,z)$ and $V = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$;
- M is a Sasakian manifold and V is the Reeb vector field; in such a case F_0 is the contact 2-form.

3. When a vector field is a magnetic map?

Let (M,g) be a compact, orientable Riemannian manifold of dimension n and $(T(M),g_S)$ its tangent bundle equipped with the Sasaki metric. On $T(M)$ we can also define an almost complex structure J_S by

$$J_S X^H = X^V,\; J_S X^V = -X^H, \text{ for all } X \in \mathfrak{X}(M). \tag{14}$$

It is known that $(T(M),g_S,J_S)$ is an almost Käherian manifold. See [8]. Hence, the Kähler 2-form $\Omega_S = g_S(J_S\,\cdot,\,\cdot)$ may be considered as a magnetic field on $T(M)$.

Let $\xi \in \mathfrak{X}(M)$ be thought as a map from (M,g) to $(T(M),g_S,J_S)$. We can compute the differential of this map, that is $\xi_{*,p} : T_pM \longrightarrow T_{(p,\xi(p))}T(M)$. If X is tangent to M we have

$$\xi_{*,p}X(p) = X^H_{\xi(p)} + (\nabla_X \xi)^V_{\xi(p)}. \tag{15}$$

We easily find the well known result: *The map* $\xi:(M,g)\longrightarrow(T(M),g_S)$ *is an isometric immersion if and only if* $\nabla\xi=0$. See e.g. [9, Proposition 2.1].

The aim of this section is to find conditions under which the vector field $\xi:M\longrightarrow T(M)$ is a magnetic map with respect to ξ itself and the magnetic field Ω_S on $T(M)$. To do this, we first compute the Hilbert-Schmidt norm $||d\xi||_g$. In a point $p\in M$, take an orthonormal frame $\{e_k\}_{k=1,\dots,n}$. We have

$$||d\xi||_g^2(p)=\sum_{k=1}^{n}\left[g_S(e_k^H,e_k^H)+g_S((\nabla_{e_k}\xi)^V,(\nabla_{e_k}\xi)^V)\right]_{\xi(p)}=n+||\nabla\xi||^2.$$

Then we find the Dirichlet energy of ξ on M, that is

$$E(\xi)=\frac{n}{2}\,\mathsf{vol}(M)+\frac{1}{2}\int_M||\nabla\xi||_g^2dv_g, \tag{16}$$

where dv_g is the volume form on M and $\mathsf{vol}(M)$ is the volume of M. The number

$$\mathcal{B}(\xi)=\int_M||\nabla\xi||_g^2dv_g \tag{17}$$

is called the *total bending* of the vector field ξ. We know that $\xi:(M,g)\longrightarrow(T(M),g_S)$ is harmonic if and only if ξ is parallel. In such a case it is an absolute minimum of the energy functional $E(\xi)$. See e.g. [14]. However, if M is not compact, tension field of ξ must be computed. In the book of Dragomir and Perrone [9], the authors write the following formula

$$\tau(\xi)=-\left\{(\mathrm{trace}_gR(\nabla_\bullet\xi,\xi)\bullet)^H+(\Delta_g\xi)^V\right\}\circ\xi. \tag{18}$$

Here Δ_g denotes the rough Laplacian on vector fields, defined by

$$\Delta_gX=-\sum_{k=1}^{n}\left[\nabla_{e_k}\nabla_{e_k}X-\nabla_{\nabla_{e_k}e_k}X\right],$$

where $\{e_k\}_{k=1,\dots,n}$ is an orthonormal frame on M.

The magnetic equation (3) writes as

$$\tau(\xi)=q\,J_S(\xi_*\xi),\ q\in\mathbb{R}. \tag{19}$$

Using (14) and (15) we get

$$J_S(\xi_*\xi)=\xi^V-(\nabla_\xi\xi)^H.$$

Now we plug this expression into (19), use (18) and then identify the horizontal and the vertical parts. We may state the following.

THEOREM 3. *Let* (M,g) *be a Riemannian manifold and* $(T(M),g_S,J_S)$ *its tangent bundle endowed with the usual almost Kählerian structure. Let* ξ *be a vector field on* M. *Then* ξ *is a magnetic map with strength* q *associated to* ξ *itself and the Kähler magnetic field* Ω_S *if and only if the following conditions hold:*

$$\mathrm{trace}_gR(\nabla_\bullet\xi,\xi)\bullet)=q\,\nabla_\xi\xi, \tag{20}$$

$$\Delta_g\xi=-q\,\xi. \tag{21}$$

COROLLARY 1. *Let $\xi : M \longrightarrow T(M)$ be a non-zero, non-harmonic magnetic map. If M is compact and oriented, then the strength q is strictly negative.*

Proof. The operator Δ_g satisfies the following identity

$$g(\Delta_g \xi, \xi) = \frac{1}{2}\Delta(||\xi||^2) + ||\nabla\xi||^2,$$

where Δ is the Beltrami Laplace operator on functions. Using (21) we obtain

$$\frac{1}{2}\Delta(||\xi||^2) + q||\xi||^2 + ||\nabla\xi||^2 = 0.$$

If $q \geq 0$ we observe that $\Delta(||\xi||^2) \leq 0$ and hence $||\xi||^2$ is a harmonic function. Then $q = 0$ and $\nabla\xi = 0$. Hence ξ is a harmonic map, which is false. Therefore, $q < 0$. $\square$

Interesting results may be obtained in the case where the curvature tensor has a certain expression. In what follows, we will describe two such situations.

1. Suppose that the manifold M is of constant sectional curvature c. Then its curvature tensor writes as

$$R(X,Y)Z = c\big(g(Y,Z)X - g(X,Z)Y\big), \text{ for all } X,Y,Z \in \mathfrak{X}(M).$$

We successively have

$$\begin{aligned} \mathrm{trace}_g R(\nabla_\bullet \xi, \xi)\bullet &= \sum_{k=1}^{n} R(\nabla_{e_k}\xi, \xi)e_k = c\sum_{k=1}^{n}\left[g(\xi, e_k)\nabla_{e_k}\xi - g(e_k, \nabla_{e_k}\xi)\xi\right] \\ &= c\left[\nabla_\xi \xi - (\mathrm{div}\ \xi)\xi\right). \end{aligned}$$

Here div ξ states for the divergence of the vector field ξ.
Consequently, the equation (20) becomes

$$(c-q)\nabla_\xi\xi - c(\mathrm{div}\ \xi)\xi = 0. \tag{22}$$

Let us observe the following:

(i) If $c = 0$, that is M is flat (not necessarily compact) it follows that ξ is self-parallel.

(ii) If $c \neq 0$, then we have
$$\left(1 - \frac{q}{c}\right)\nabla_\xi\xi = (\mathrm{div}\ \xi)\xi.$$
Hence, for $q = c$, the vector field ξ is divergence free.

2. Suppose now that M is a Sasakian space form. We briefly explain this structure. See for more details the Blair's book [4].

A (φ, ξ, η)-*structure* on a manifold M is defined by a field φ of endomorphisms of tangent spaces, a vector field ξ and a 1-form η satisfying

$$\eta(\xi) = 1, \ \ \varphi^2 = -\mathrm{I} + \eta\otimes\xi, \ \ \varphi\xi = 0, \ \ \eta\circ\varphi = 0.$$

If (M,φ,ξ,η) admits a compatible Riemannian metric g, namely

$$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y),\ \text{for all}\ X,Y\in\mathfrak{X}(M),$$

then M is said to have an *almost contact metric structure*, and (M,φ,ξ,η,g) is called an *almost contact metric manifold*. Consequently, ξ is unitary and $\eta(X)=g(\xi,X)$, for any $X\in\mathfrak{X}(M)$. Denoting by ∇ the Levi Civita connection associated to g, the Sasakian manifold (M,φ,ξ,η,g) is characterized by

$$(\nabla_X\varphi)Y=-g(X,Y)\xi+\eta(Y)X,\ \ \text{for any}\ \ X,Y\in\mathfrak{X}(M).$$

As a consequence, we have

$$\nabla_X\xi=\varphi X,\ \forall X\in\mathfrak{X}(M). \tag{23}$$

A plane section Π at $p\in M^{2n+1}$ is called a φ-section if it is invariant under φ_p. The sectional curvature $k(\Pi)$ of a φ-section is called the φ-*sectional curvature* of M^{2n+1} at p. A Sasakian manifold $(M^{2n+1},\varphi,\xi,\eta,g)$ is said to be a *Sasakian space form* and denote this by $M^{2n+1}(c)$, if it has constant φ-sectional curvature c. In such a case, the curvature tensor is given by

$$\begin{aligned}R(X,Y)Z=&\tfrac{c+3}{4}\big(g(Y,Z)X-g(X,Z)Y\big)\\&+\tfrac{c-1}{4}\Big(\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\\&+g(Z,\varphi Y)\varphi X-g(Z,\varphi X)\varphi Y+2g(X,\varphi Y)\varphi Z\Big).\end{aligned}\tag{24}$$

Our aim is to study the conditions when the Reeb vector field ξ is magnetic, that is it satisfies the condition in Theorem 3.
In order to compute the left side in (20) we consider an adapted orthonormal frame $\{e_i,\varphi e_i,\xi\ :\ i=1,\ldots,n\}$, where $e_i\in\ker\eta$ for all $i=1,\ldots,n$. From (24) we obtain

$$R(X,\xi)Z=\eta(Z)X-g(X,Z)\xi.$$

Now, since $\nabla_\bullet\xi=\varphi\,\bullet$ we get

$$R(\nabla_{e_i}\xi,\xi)e_i=-g(\varphi e_i,e_i)\xi=0,\ \text{for all}\ i=1,\ldots,n.$$

Hence the equation (20) is automatically satisfied in the light of the relation $\nabla_\xi\xi=0$.
Concerning (21) we should compute $\Delta_g\xi$. We have

$$\nabla_X\nabla_Y\xi-\nabla_{\nabla_XY}\xi=\nabla_X(\varphi Y)-\varphi\nabla_XY=(\nabla_X\varphi)Y=-g(X,Y)\xi+\eta(Y)X.$$

Setting successively $X=Y=e_i$, $X=Y=\varphi e_i$ and $X=Y=\xi$ we get

$$\Delta_g\xi=2n\xi.$$

Observe that we should take $q=-2n$.

4. Magnetic maps between tangent bundles

Let (M,g) be a Riemannian manifold and $T(M)$ its tangent bundle. A $(1,1)$-tensor field L on a Riemannian manifold (M,g) defines a map $L : T(M) \longrightarrow T(M)$, by $(p,\mathfrak{u}) \mapsto (p,L_p\mathfrak{u})$, for any $p \in M$ and $\mathfrak{u} \in T_pM$. An interesting problem is to determine conditions under which L is a magnetic map with respect to the geodesic flow (resp. the Liouville vector field) and the usual magnetic field Ω_S on $T(M)$. See [11] for the study of harmonicity of endomorphism fields on a pseudo-Riemannian manifold, when the complete lift metric on $T(M)$ is considered either on the source or on the target manifold.

Let us first consider $L = I$, the identity map of $T(M)$. More precisely, we study when the map $I : (T(M),G) \longrightarrow (T(M),g_S,J_S)$ is magnetic, for some metrics G on $T(M)$.

It is well known that the identity is a harmonic map when the same metric is considered either on the source, or on the target manifold. Therefore, the case $G = g_S$ is not interesting. Let us see what happens when G is the Cheeger-Gromoll metric g_{CG}.

We compute the second fundamental form $\sigma(I)$ on the two distributions VTM and HTM, respectively, using the formulas (5) and (7) to get

$$\begin{cases} \sigma(I)\left(\dot{\partial}_i,\dot{\partial}_j\right) = -\frac{1}{1+2t}\left(g_{ij}C - g_{i0}\dot{\partial}_j - g_{j0}\dot{\partial}_i\right) - \frac{1}{(1+2t)^2}\left(g_{ij} + g_{i0}g_{j0}\right)C, \\ \sigma(I)\left(\delta_i,\delta_j\right) = 0. \end{cases}$$

Then, we obtain the tension field $\tau(I)$, computing the *trace* of $\sigma(I)$, that is

$$\tau(I) = \left((1+2t)g^{ij} - y^i y^j\right)\sigma(I)\left(\dot{\partial}_i,\dot{\partial}_j\right) = \frac{2(1-n)(1+t)}{1+2t}\,C.$$

So I is a magnetic map (with strength $q = 1-n$) with respect to the vector field $\frac{2+g(\mathfrak{u},\mathfrak{u})}{1+g(\mathfrak{u},\mathfrak{u})}\xi$ and the magnetic field Ω_S. Here $\xi_{\mathfrak{u}} = y^i\delta_i|_{\mathfrak{u}}$ denotes (as above) the geodesic flow on $T(M)$.

Suppose now that we have a nonlinear connection on the tangent bundle of a Riemannian manifold M with the coefficients $N_k^l(x,y)$ as described in Section 2. Let us write down the (local) expressions of the Levi-Civita connection ${}^{\mathrm{s}}\nabla$ of the Sasaki type metric g_{s}:

$$\begin{cases} {}^{\mathrm{s}}\nabla_{\dot{\partial}_i}\dot{\partial}_j = -\frac{1}{2}g^{hk}\left(\frac{\partial g_{ij}}{\partial x^k} - g_{il}\frac{\partial N_k^l}{\partial y^j} - g_{jl}\frac{\partial N_k^l}{\partial y^i}\right)\bar{\delta}_h, \\ {}^{\mathrm{s}}\nabla_{\dot{\partial}_i}\bar{\delta}_j = -\frac{1}{2}R_{kj}^l g_{li}g^{kh}\bar{\delta}_h + \frac{1}{2}g^{hk}\left(\frac{\partial g_{ik}}{\partial x^j} - g_{il}\frac{\partial N_j^l}{\partial y^k} - g_{lk}\frac{\partial N_j^l}{\partial y^i}\right)\dot{\partial}_h, \\ {}^{\mathrm{s}}\nabla_{\bar{\delta}_i}\dot{\partial}_j = -\frac{1}{2}R_{ki}^l g_{lj}g^{kh}\bar{\delta}_h + \frac{1}{2}g^{hk}\left(\frac{\partial g_{jk}}{\partial x^i} - g_{jl}\frac{\partial N_i^l}{\partial y^k} + g_{lk}\frac{\partial N_i^l}{\partial y^j}\right)\dot{\partial}_h, \\ {}^{\mathrm{s}}\nabla_{\bar{\delta}_i}\bar{\delta}_j = \Gamma_{ij}^h\bar{\delta}_h + \frac{1}{2}R_{ji}^h\dot{\partial}_h. \end{cases}$$

See also [18]. Here we made the following notations:

$$\dot{\partial}_i = \left(\frac{\partial}{\partial x^i}\right)^V = \frac{\partial}{\partial y^i}\ ,\bar{\delta}_i = \left(\frac{\partial}{\partial x^i}\right)^H = \frac{\partial}{\partial x^i} - N_i^j(x,y)\frac{\partial}{\partial y^j}\ \text{and}\ R_{ij}^h = \bar{\delta}_i(N_j^h) - \bar{\delta}_j(N_i^h).$$

Looking back to (12), let us consider the coefficients of the nonlinear connection be given in the following way

$$N_k^l(x,y) = \Gamma_{ks}^l(x)y^s + \Psi_k^l(x),$$

where Ψ_k^i are the coefficients of a $(1,1)$ tensor field Ψ on M. Consequently we have

$$\bar{\delta}_i = \delta_i - \Psi_i^k(x)\dot{\partial}_k.$$

Compute the second fundamental form $\sigma(I)$ restricted to the two distributions VTM and HTM, respectively. Here the horizontal distribution HTM is that corresponding to the nonlinear connection on the source manifold $(T(M), g_{\mathrm{s}})$.
We find

$$\sigma(I)\left(\dot{\partial}_i, \dot{\partial}_j\right) = 0,$$

$$\sigma(I)\left(\bar{\delta}_i, \bar{\delta}_j\right) = -\tfrac{1}{2}\left(\Psi_i^k R_{j0k}^h + \Psi_j^k R_{i0k}^h\right)\delta_h - \tfrac{1}{2}\left(R_{0ji}^h + \nabla_j \Psi_i^h + \nabla_i \Psi_j^h\right)\dot{\partial}_h.$$

Therefore, the tension field $\tau(I)$ is given by

$$\tau(I) = \mathrm{trace}_{g_{\mathrm{s}}}\sigma(I) = -g^{ij}\Psi_i^k R_{j0k}^h \delta_h - g^{ij}\nabla_i \Psi_j^h \dot{\partial}_h.$$

If $\xi = A^h(x,y)\delta_h + B^h(x,y)\dot{\partial}_h$ is a vector field on $T(M)$ then I satisfies the magnetic equation $\tau(I) = J_S\xi$ if and only if

$$A^h = -g^{ij}\nabla_i \Psi_j^h \text{ and } B^h = g^{ij}\Psi_i^k R_{j0k}^h.$$

Hence I is a magnetic map with respect to ξ and Ω_S (with strength $q = 1$) if and only if ξ is given by

$$\xi = -\Big(\mathrm{trace}_g(\nabla_\bullet \Psi)\bullet\Big)^H + \Big(\mathrm{trace}_g R(\mathfrak{u}, \Psi\bullet)\bullet - \Psi \mathrm{trace}_g(\nabla_\bullet \Psi)\bullet\Big)^V.$$

Of course, the horizontal lift is considered with respect to the nonlinear connection.
As particular case we consider $\Psi_k^l = \delta_k^l$. Then I is a magnetic map if and only if ξ is given by

$$\xi = (Q\mathfrak{u})^V.$$

Here Q is the Ricci operator on M defined by $g(QX,Y) = Ric(X,Y)$, where Ric is the usual Ricci tensor on M.

The study of the magnetic equation for an arbitrary endomorphism $L : T(M) \longrightarrow T(M)$, when the Sasaki metric (on both source and target) is considered, will be done in a subsequent paper.

Let us consider now an arbitrary smooth map $f : M \longrightarrow N$ between two Riemannian manifolds (M,g) and (N,h). Let $F = df : T(M) \longrightarrow T(N)$ be the differential of f, defined by

$$F(p,\mathfrak{u}) = (f(p), f_{*,p}\mathfrak{u}), \text{ for every } p \in M \text{ and } \mathfrak{u} \in T_pM.$$

On $T(M)$ (respectively on $T(N)$) set the Sasakian metric g_S (respectively h_S). The tension field of F was computed by Sanini in [22]. In order to fix certain notations, let us briefly sketch some computations.

Let (U,x) be a local chart on M and $(\pi_M^{-1}(U),x,y)$ be the induced chart on $T(M)$, where $\pi_M : T(M) \longrightarrow M$ is the canonical projection. In the same manner define local charts on N and $T(N)$, respectively. From now on the indices i,j,k range from 1 to $m = \dim(M)$, while indices α,β,γ range from 1 to $n = \dim(N)$ and so, we distinguish geometric objects defined on M from those defined on N. We have

$$\begin{cases} F_*\dot{\partial}_i = f_i^\alpha \dot{\partial}_\alpha \\ F_*\delta_i = f_i^\alpha \delta_\alpha + \sigma(f)^\gamma(\partial_i,\partial_j) y^j \dot{\partial}_\gamma, \end{cases}$$

where $\sigma(f)$ is the second fundamental form of f, that is

$$\sigma(f)(\partial_i,\partial_j) = \left(f_{ij}^\gamma + {}^h\Gamma_{\alpha\beta}^\gamma(f(x)) f_i^\alpha f_j^\beta - {}^g\Gamma_{ij}^k(x) f_k^\gamma \right) \partial_\gamma.$$

Here we set $f_i^\alpha := \frac{\partial f^\alpha}{\partial x^i}$ and $f_{ij}^\alpha := \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j}$. Note that $\sigma(f)(X,Y)$ is a section in the induced bundle $f^{-1}T(N)$ (over M), for all X, Y tangent to M and it is symmetric in X and Y.

Computing the tension field $\tau(F)$ we find

$$\tau(F) = \Big[\tau(f) + \mathrm{trace}_g\, {}^hR\big(df(\mathfrak{u}), \sigma(f)(\,\bullet\,,\mathfrak{u})\big) df(\,\bullet\,) \Big]^H + \big[\big(\mathrm{div}\, \sigma(f)\big)(\mathfrak{u}) \big]^V, \tag{25}$$

where

$$\big(\mathrm{div}\, \sigma(f)\big)(X) = \mathrm{trace}_g\, \big(\bar{\nabla}_\bullet \sigma(f)\big)(\,\bullet\,,X).$$

Here $\bar{\nabla}$ is the induced connection in the vector bundle $S_2(T^*(M)) \otimes f^{-1}T(N)$ and it is defined by

$$\big(\bar{\nabla}_X \sigma\big)(Y,Z) = {}'\nabla_X \sigma(Y,Z) - \sigma({}^g\nabla_X Y, Z) - \sigma(Y, {}^g\nabla_X Z),$$

where ${}'\nabla$ is the induced connection in the induced bundle $f^{-1}T(N)$.

Observe that div $\sigma(f)$ is a 1-form on M with values in the induced bundle $f^{-1}T(N)$.

Now we consider on $T(N)$ the magnetic field Ω_S (with the corresponding Lorentz force J_S) and let q be a real number.

If C_M denotes the Liouville vector field on $T(M)$, then the magnetic equation (for F) with respect to C_M and Ω_S may be written as $\tau(F) = qJ_S F_* C_M$. Identifying the vertical and the horizontal parts respectively, we obtain

$$\begin{cases} \big(\mathrm{div}\, \sigma(f)\big)(\mathfrak{u}) = 0, \\ \tau(f) + \mathrm{trace}_g\, {}^hR\big(df(\mathfrak{u}), \sigma(f)(\,\bullet\,,\mathfrak{u})\big) df(\,\bullet\,) - q\, df(\mathfrak{u}) = 0. \end{cases} \tag{26}$$

The left side of the second equation represents a polynomial of second order in y^i, hence all the coefficients vanish. Therefore, we get:

THEOREM 4. *Under the previous hypothesis, df is magnetic with respect to C_M and Ω_S if and only if the following conditions are satisfied:*

f is harmonic;

$q = 0$ that is F is harmonic, or f is a constant map;

$\mathrm{trace}_g\, {}^hR\big(df(X), \sigma(f)(\,\bullet\,, X)\big)df(\,\bullet\,) = 0$*, for all X tangent to M;*

$\mathrm{div}\ \sigma(f) = 0.$

If ξ_M denotes the geodesic spray on $T(M)$, then the magnetic equation (for F) with respect to ξ_M and Ω_S yields the following equations for f:

$$\left\{\begin{array}{l} \tau(f) + \mathrm{trace}_g\, {}^hR\big(df(\mathfrak{u}), \sigma(f)(\,\bullet\,, \mathfrak{u})\big)df(\,\bullet\,) + q\,\sigma(f)(\mathfrak{u}, \mathfrak{u}) = 0 \\ \big(\mathrm{div}\ \sigma(f)\big)(\mathfrak{u}) = q\, df(\mathfrak{u}). \end{array}\right. \tag{27}$$

With the same argument as before, we obtain:

THEOREM 5. *Under the hypothesis above, df is magnetic with respect to ξ_M and Ω_S if and only if the following conditions are satisfied:*

f is harmonic;

$\mathrm{trace}_g\, {}^hR\big(df(X), \sigma(f)(\,\bullet\,, X)\big)df(\,\bullet\,) + q\,\sigma(f)(X,X) = 0$*, for all X tangent to M;*

$\mathrm{div}\ \sigma(f) = q\, df.$

Let us consider the case when f is an isometric immersion, namely when M is an immersed submanifold in N.

REMARK 4. If M is totally geodesic in N, that is $\sigma(f) = 0$ and $m \geq 1$, then the condition that df is magnetic with respect to ξ_M and Ω_S implies that df is harmonic.

REMARK 5. If $f : M \longrightarrow N$ is a minimal hypersurface in the real space form N, then df is magnetic with respect to ξ_M and Ω_S implies df is harmonic.

Proof. We use the Codazzi equation $(\bar\nabla_X\sigma)(Y,Z) - (\bar\nabla_Y\sigma)(X,Z) = \big({}^hR(X,Y)Z\big)^\perp$. Here $\bar\nabla$ is known as the Van der Waerden-Bortolotti connection; see e.g. [21, Chapter 3, §9]. Since N is a real space form, the right side of the above equation vanishes. When M is a hypersurface, the normal bundle is 1-dimensional; choose ν unitary and normal to M. Set $X = Z = e_i$, where $\{e_i\}_{i=1,\dots,m}$ is an orthonormal basis on M. We get

$$\big(\bar\nabla_{e_i}\sigma\big)(Y, e_i) - \big(\bar\nabla_Y\sigma\big)(e_i, e_i) = 0.$$

Summing up on $i = 1, \dots, m$ we find $\mathrm{div}\ \sigma(f)(Y) - \big[\nabla^\perp_Y(m\vec{H}) - 2\sigma({}^g\nabla_Y e_i, e_i)\big] = 0.$ We can choose the orthonormal basis on M consisting in eigenvectors of the shape operator. As M is minimal, i.e. the mean curvature vector $\vec{H}$ vanishes, we obtain $\mathrm{div}\ \sigma(f) = 0$.

It follows that $q = 0$ and hence df is harmonic. □

THEOREM 6. *If M is a submanifold in N such that* div $\sigma(f) = q\, df$, *then either f is constant or $q = 0$.*

Proof. For X tangent to M we know that div $\sigma(f)(X)$ is a normal vector. Thus div $\sigma(f) = 0$ and $qdf = 0$. Hence the conclusion. □

THEOREM 7. *If M is a submanifold in the real space form $N(c)$ such that* $\mathrm{trace}_g\, {}^hR\big(df(X), \sigma(f)(\bullet, X)\big)df(\bullet) + q\,\sigma(f)(X,X) = 0$, *then either f is totally geodesic or $q = c$.*

Proof. Since $df(X)$ is tangent and $\sigma(Y,Z)$ is normal to f we compute

$$\sum_{k=1}^{m} {}^hR\big(df(X), \sigma(f)(e_k, X)\big)df(e_k) = -c\sum_{k=1}^{m}\sigma(f)\big(g(X,e_k)e_k, X\big) = -c\sigma(f)(X,X).$$

As $\sigma(f)$ is symmetric we immediately get the conclusion. □

As consequence of these remarks we should ask:

Can we determine hypersurfaces in a space form $N(c)$ such that df is harmonic?

Doing similar computations as before we obtain either $c = 0$ or M is totally geodesic in N. Therefore, we formulate another question, that is:

Can we determine hypersurfaces in space forms $N(c)$ satisfying div $\sigma(f) = 0$?

Straightforward computations as in Remark 5 lead to div $\sigma(f) = m\nabla^{\perp}\vec{H}$. So, $\sigma(f)$ is divergence free if and only if M has parallel mean curvature vector. In case when M is a hypersurface in N, the condition div $\sigma(f) = 0$ implies that M is a CMC hypersurface in N.

Acknowledgement. The second author would like to express his sincere gratitude to Prof. Laura Falcitelli, for kind hospitality during his visits at the *Università degli Studi Aldo Moro* di Bari, in particular for the invitation at the wonderful conference organized in June 2015.

References

[1] ABBASSI M. T. K. AND CALVARUSO G., *g-natural contact metrics on unit tangent sphere bundles*, Monatsh. Math. **151** 2 (2007), 89–109.

[2] ABBASSI M. T. K. AND CALVARUSO G., *Harmonic maps having tangent bundles with g-natural metrics as source or target*, Rend. Sem. Mat. Univ. Politec. Torino **68** 1 (2010), 37–56.

[3] ABBASSI M.T.K. AND SARIH M., *On some hereditary properties of Riemannian g-natural metrics on tangent bundles of Riemannian manifolds*, Differential Geom. Appl. **22** 1 (2005), 19–47.

[4] BLAIR D.E., *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Math. 203, Birkhäuser, Boston-Basel-Berlin 2002.

[5] BOECKX E. AND VANHECKE L., *Harmonic and minimal vector fields on tangent and unit tangent bundles*, Differential Geom. Appl. **13** (2000), 77–93.

[6] BUCATARU I. AND MIRON R., *Finsler-Lagrange geometry. Applications to dynamical systems*, Editura Academiei Romane, 2007.

[7] CHEEGER J. AND GROMOLL D., *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. **96** (1972), 413–443.

[8] DOMBROWSKI P., *On the Geometry of the Tangent Bundle*, J. Reine Angew. Math. **210** (1962), 73–88.

[9] DRAGOMIR S. AND PERRONE D., *Harmonic vector fields. Variational principles and differential geometry*, Elsevier 2012.

[10] EELLS J. AND LEMAIRE L., *Selected Topics in Harmonic Maps*, Conf. Board Math. Sci 50, American Mathematical Society, Providence, RI 1983.

[11] GARCIA-RÍO E., VANHECKE L. AND VÁSQUEZ-ABAL M.E., *Harmonic endomorphism fields*, Illinois J. Math. **41** (1997), 23–30.

[12] GUDMUNDSSON S. AND KAPPOS E., *On the Geometry of the tangent bundle with the Cheeger-Gromoll metric*, Tokyo J. Math. **25** (2002), 75–83.

[13] INOGUCHI J. AND MUNTEANU M.I., *Magnetic maps*, Int. J. Geom. Methods Mod. Phys. **11** 6 (2014), art. 1450058, (22 pages).

[14] ISHIHARA T., *Harmonic sections of tangent bundles*, J. Math. Tokushima Univ. **13** (1979), 23–27.

[15] KOWALSKI O., *Curvature of the Induced Riemannian Metric on the Tangent Bundle of a Riemannian Manifold*, J. Reine Angew. Math. **250** (1971), 124–129.

[16] KOWALSKI O. AND SEKIZAWA M., *Natural transformations of Riemannian metrics on manifolds to metrics on tangent bundles. A classification*, Bull. Tokyo Gakugei Univ. (4) **40** (1988), 1–29.

[17] MUSSO E. AND TRICERRI F., *Riemannian metrics on tangent bundles*, Ann. Mat. Pura Appl. **150** (1988), 1–19.

[18] ONICIUC C., *The Nonlinear Connections and Harmonicity*, Gen. Math. **5** (1997), 287–296.

[19] ONICIUC C., *Nonlinear connections on tangent bundle and harmonicity*, Ital. J. Pure Appl. Math. **6** (1999), 109–122.

[20] OPROIU V., *Some new geometric structures on the tangent bundle*, Publ. Math. Debrecen **55** 3–4 (1999), 261–281.

[21] POSTNIKOV M.M., *Geometry VI: Riemannian Geometry*, Encyclopaedia of Mathematical Sciences vol. 91, Springer, New-York–Berlin 2001.

[22] SANINI A., *Applicazioni armoniche tra i fibrati tangenti di varietà riemanniane*, Boll. U.M.I. **6** 2A (1983), 55–63.

[23] SASAKI S., *On the differential geometry of tangent bundles of Riemannian manifolds*, Tohoku Math. J. **10** (1958), 338–354.

[24] SEKIZAWA M., *Curvatures of tangent bundles with Cheeger-Gromoll metric*, Tokyo J. Math. **14** (1991), 407–417.

[25] URAKAWA H., *Calculus of Variations and Harmonic Maps*, Translations of Mathematical Monographs Vol. 132, American Mathematical Society, Providence, RI, 1993.

2010 AMS Subject Classification: 53C07, 53C15, 53C43, 53D25, 58E20

Jun-ichi Inoguchi,
Institute of Mathematics, University of Tsukuba,
1-1-1 Tennodai Tsukuba, Ibaraki,
305-8571, Japan
e-mail: `inoguchi@math.tsukuba.ac.jp`

Marian Ioan Munteanu
University 'Al. I. Cuza' of Iasi,
Faculty of Mathematics, Bd. Carol I, no. 11,
700506 Iasi, Romania
e-mail: `marian.ioan.munteanu@gmail.com`

Lavoro pervenuto in redazione il 26.11.2015 e accettato il 19.01.2016.

Geom. Struc. on Riem. Man.-Bari
Vol. 73/1, 3–4 (2015), 117 – 125

A. Lotta[1]

RICCI NILSOLITONS ASSOCIATED TO GRAPHS AND EDGE-COLOURING

Abstract. We provide a method to attach to every simple graph a 2-step nilpotent Ricci nilsoliton.

1. Introduction

The purpose of this note is to provide a new method to attach to every simple graph a 2-step nilpotent *Ricci nilsoliton*, namely a simply connected 2-step nilpotent Lie group $(\mathcal{N}, g)$ endowed with a left-invariant metric g whose Ricci operator $Q : \mathfrak{n} \to \mathfrak{n}$ satisfies

$$Q = cI + D \text{ for some } c \in \mathbb{R} \text{ and } D \in Der(\mathfrak{n}), \tag{1}$$

where $\mathfrak{n}$ is the Lie algebra of $\mathcal{N}$.

Lauret proved that there is at most one Ricci soliton metric on a nilpotent Lie group, up to isometry and scaling [8, Theorem 3.5]. Ricci nilsolitons are noteworthy in connection with the problem of classifying the *homogeneous Einstein* Riemannian manifolds; indeed, the metric nilpotent Lie algebras $(\mathfrak{n}, <,>)$ satisfying (1) are exactly the nilradicals $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ of the standard metric Einstein solvable Lie algebras $\mathfrak{s}$ (endowed with the induced metric), see [8, Theorem 3.7]. For this reason, they are also called *Einstein nilradicals*. For more information and a list of recent relevant results concerning the classification of Einstein nilradicals see [9].

In [10] it is showed that the natural 2-step nilpotent Lie algebra $\mathfrak{n}_G$ attached to a simple graph G admits a Ricci soliton metric if and only if the graph is *positive*, i.e., all the entries of the vector

$$(3I + \mathrm{Adj}(L(G)))^{-1}(1, \ldots, 1)$$

are positive; here $\mathrm{Adj}(L(G))$ is the adjacency matrix of the line graph of G. For instance, regular graphs are positive. This criterion has been exploited in [11] in order to prove that for every pair (p,q) of integers with $q \geq 21$ and $q - 1 \leq p \leq \frac{1}{2}q^2 - \frac{5}{2}q + 9$, there do exist indecomposable 2-step nilpotent Lie algebras $\mathfrak{n}$ of type (p,q) which are *not* Einstein nilradicals, where type (p,q) means that $\dim \mathfrak{n} = p + q$ and $\dim [\mathfrak{n}, \mathfrak{n}] = p$. We recall that $\mathfrak{n}_G$ is defined in a very simple way as follows:

$$\mathfrak{n}_G = V_0 \oplus W_0,$$

where V_0 is the real vector space of dimension n = order of G spanned by the vertices $v_1, \ldots, v_n$ of G and W_0 is the vector space of dimension l = size of G, spanned by the

[1]This paper is dedicated to Professor Anna Maria Pastore on the occasion of her 70th birthday.

edges of G; the Lie bracket $V_0 \times V_0 \to W_0$ is determined declaring

$$[v_i, v_j] := \{v_i, v_j\}$$

provided the vertices v_i and v_j are adjacent and $i < j$, and setting $[v_i, v_j] = 0$ whenever v_i and v_j are not adjacent (cf. [2]). It should be remarked that, in general, the standard basis of $\mathfrak{n}_G$ made up by the vertices and the edges of a positive graph G is orthogonal but not orthonormal with respect to the Ricci soliton metric on the corresponding group $\mathcal{N}_G$ (cf. [4]).

Our construction provides instead, for *every* graph G, a bigger 2-step nilpotent Einstein nilradical $\mathfrak{n}$ containing $\mathfrak{n}_G$ as a *totally geodesic* subalgebra (see Theorem 1). This Lie algebra is constructed as

$$\mathfrak{n} = V \oplus W_G, \quad V := V_0 \otimes \mathbb{R}^{2N}, \tag{2}$$

where $N \in \mathbb{N}^* \cup \{\frac{1}{2}\}$ is uniquely determined by the degree sequence of the graph and W_G is a suitable subspace of $\mathfrak{so}(V)$, whose dimension s depends both on the degree sequence and the size of the graph. The Lie bracket $V \times V \to W_G$ of $\mathfrak{n}$ is defined in the canonical way by

$$< [X,Y], J > := < J(X), Y >, \qquad X, Y \in V, \quad J \in W_G, \tag{3}$$

where $<,>$ denotes both the inner product on V with respect to which the basis $\{v_i \otimes e_k\}$ is orthonormal and the inner product $< F, G > = -\mathrm{tr}(F \circ G)$ on $\mathfrak{so}(V)$.

In general, a 2-step nilpotent Lie algebra $\mathfrak{n} = V \oplus W$ constructed in this way starting from an Euclidean vector space $(V, <,>)$ of dimension q and a p-dimensional subspace $W \subset \mathfrak{so}(V)$, is usually called *standard of type* (p,q). Hence, using this terminology, our $\mathfrak{n}$ is of type $(s, 2Nn)$.

Requiring that $W = V^\perp$ and keeping on V and $\mathfrak{so}(V)$ the inner products $<,>$, a standard 2-step nilpotent Lie algebra $\mathfrak{n}$ is turned into a metric Lie algebra in a natural way, which we shall call a *standard metric* 2-step nilpotent Lie algebra. Certainly the left-invariant metric on the corresponding Lie group $\mathcal{N}$ is a Ricci soliton provided its Ricci tensor is *optimal*, in the sense that the restrictions of Q to V and to W are both scalar operators. This happens if and only if W admits a basis $U = \{J_i\}$ such that

$$\sum J_i^2 = -\rho Id, \quad < J_i, J_j > = \lambda \delta^i_j, \tag{4}$$

where λ and ρ are positive constants (cf. [3] or [6], where such a W is called a *uniform* subspace of $\mathfrak{so}(V)$). Indeed, it is known that $< QX, J > = 0$ for every $X \in V$ and $J \in W$, and moreover

$$Q|_V = \frac{1}{2} \sum F_i^2, \qquad Q|_W = \frac{1}{4} Id_W,$$

where $\{F_i\}$ is an arbitrary orthonormal basis of W (cf. e.g. [3, Prop. 3.1]).

In our case, we provide such a basis U_G of W_G parametrizing it by the edges of the graph G and a certain set of vertices; in building the corresponding operators J_i we make use of a fixed edge-colouring of the complete graph K_{2N} with $2N - 1$ colours.

We remark that the construction of $\mathfrak{n}$ is performed in such a way that $\mathfrak{n} = \mathfrak{n}_G$ if and only if the graph G is regular.

In the last section we also propose a method for attaching to G a standard Einstein nilradical $\mathfrak{n}_{\mathbb{A}}$ having the same properties, constructed using a Cayley-Dickson algebra $\mathbb{A}$ of dimension 2^M, where M is the smallest non negative integer such that $2^{M-1} \geq N$. The type of $\mathfrak{n}_{\mathbb{A}}$ is $(s, 2^M n)$.

2. Preliminary remarks

Given a simple graph G, we shall denote by $V(G)$ its vertex set and by $E(G)$ its edge set; recall that $V(G)$ is a finite set and $E(G)$ is a set of subsets of $V(G)$ each having cardinality 2.

We shall denote by K_n the *complete graph* with n vertices, namely $V(K_n) = \{1, \dots, n\}$ and $E(K_n)$ is the set of all subsets of $\{1, \dots, n\}$ having two elements.

We recall that an *edge colouring* of a graph G is a way of assigning a colour to each edge, in such a way that adjacent edges have different colours. In other words, an edge-colouring is a map

$$C_\mu : E(G) \to \{1, \dots, \mu\},$$

where μ is a positive integer, such that $C_\mu(\delta) \neq C_\mu(\delta')$ whenever $\delta \cap \delta' \neq \emptyset$. The *chromatic index* $\mathcal{X}'(G)$ of G is the minimum integer μ for which such a colouring C_μ exists.

We shall use the following basic result: $\mathcal{X}'(K_{2N}) = 2N - 1$ *for every positive integer* N. See e.g. [5]. Moreover, given a fixed colouring $C = C_{2N-1}$ of K_{2N}, for each colour $t \in \{1, \dots, 2N-1\}$ we shall denote by C_t the set of ordered pairs $(k, k') \in \{1, \dots, 2N\}^2$ such that

$$C(\{k, k'\}) = t \;\text{ and } k < k'.$$

Then, by definition

$$t \neq t' \Rightarrow C_t \cap C_{t'} = \emptyset. \tag{5}$$

Moreover, it easily established that each colour class $C^{-1}(t)$ has cardinality N, so that

$$|C_t| = N \qquad \text{for all } t \in \{1, \dots, 2N-1\}. \tag{6}$$

3. Einstein nilradicals attached to graphs

Let $G = (V(G), E(G))$ be a simple graph of order $n := |V(G)|$ and size $l := |E(G)|$. We shall denote by d_v the degree of a vertex $v \in V(G)$ and by $\Delta(G)$ the maximum degree.

Let $V^o(G)$ the set of vertices having odd degree, and let $V'(G)$ be the set of vertices whose degree is different from $\Delta(G)$. We put

$$2m := |V^o(G)|, \qquad r := |V'(G)|.$$

Denote by $v_1,\dots,v_n$ the vertices of G. We shall label the vertices of $V^o(G)$ and those in $V'(G)$ as follows:

$$V^o(G)=\{v_{q_1},\dots,v_{q_{2m}}\},\qquad V'(G)=\{v_{p_1}\dots,v_{p_r}\}.$$

Now we define a pair (N,s) of numbers uniquely determined by the graph, determining the type of the standard metric 2-step nilpotent Einstein nilradical we are going to attach to G. First we set

$$h_i:=\left[\frac{\Delta(G)-d_{v_{p_i}}+\varepsilon}{2}\right]\qquad \text{for each } v_{p_i}\in V'(G),$$

where [ ] means the integer part, $\varepsilon\in\{0,1\}$ and $\varepsilon=1$ iff $\Delta(G)$ is odd.
Next let h be the greatest of the h_i and set $\bar h:=\sum_{i=1}^r h_i$.
Hence, we define

$$s:=\begin{cases} l+\bar h+m & \text{if } 0<2m<n\\ l+\bar h & \text{otherwise}\end{cases}$$

and

$$N:=\text{smallest number in } \mathbb{N}^*\cup\{\frac{1}{2}\} \text{ such that } \begin{cases} 2N>h+1 & \text{if } 0<2m<n\\ 2N>h & \text{otherwise.}\end{cases}$$

We remark that the graph is regular iff $h=0$ and this is the only case where $N=\frac{1}{2}$.
As an example, consider the case of a non regular subcubic graph, i.e., $\Delta(G)=3$; then we have $(N,s)=(1,l+r)$ in the case $V^o(G)=V(G)$ and $(N,s)=(2,l+m+r)$ otherwise.

Keeping this notation, we prove the following result.

THEOREM 1. *Let G be a simple graph of order n. Let $\mathfrak{n}_G$ be the natural nilpotent Lie algebra attached to G. Then there exists a standard metric 2-step nilpotent Einstein nilradical $\mathfrak{n}$ of type $(s,2Nn)$ containing $\mathfrak{n}_G$ as a totally geodesic subalgebra.*

Proof. We consider the vector space V_0 spanned by the vertices of G and $V:=V_0\otimes\mathbb{R}^{2N}$ with its standard basis $\{v_i\otimes e_k\}$. We shall order the elements of $\mathfrak{B}$ according to

$$v_i\otimes e_k<v_j\otimes e_{k'} \text{ iff } i<j \text{ or } i=j \text{ and } k<k'$$

and we shall denote by $<,>$ the inner product on V with respect to which $\mathfrak{B}$ is an orthonormal basis. Given a pair (u,w) of elements of $\mathfrak{B}$, with $u<w$, we shall denote by F^u_w the skew-symmetric endomorphism of V such that

$$u\mapsto w,\quad w\mapsto -u,$$

and whose kernel contains all other vectors in $\mathfrak{B}$ different from u and w. Of course with respect to the standard inner product $<F,G>:=-\mathrm{tr}(F\circ G)$, the F^u_w make up an orthogonal basis of $\mathfrak{so}(V)$; observe that $||F^u_w||^2=2$.

Now we construct an s-dimensional subspace W_G of $\mathfrak{so}(V)$ attached to the graph G. To this aim, we also fix an edge colouring C of the complete graph K_{2N} with $2N-1$ colours. In the case $N=\frac{1}{2}$ we understand the empty colouring. Hence we build a subset $U_G \subset \mathfrak{so}(V)$ according to the following recipes A), B) and eventually C) in the case $0<2m<n$.

A) For each edge $\delta=\{v_p,v_q\},\ p<q$ we consider the following operator J_δ:

$$J_\delta := \sum_{k=1}^{2N} F_{v_q\otimes e_k}^{v_p\otimes e_k}.$$

B) For each vertex $v=v_{p_i}\in V'(G)$ we define the following h_i operators $J_{v,t}$:

$$J_{v,t} := \sqrt{2} \sum_{(k,k')\in C_t} F_{v\otimes e_{k'}}^{v\otimes e_k} \qquad t=1,\dots,h_i.$$

C) In the case where $0<2m<n$, we also define, for each vertex $w=v_{q_j}$ in $V^o(G)$ with $j=1,\dots,m$, an operator J_w by

$$J_w := \sum_{(k,k')\in C_{h+1}} F_{w\otimes e_{k'}}^{w\otimes e_k} + \sum_{(k,k')\in C_{h+1}} F_{w'\otimes e_{k'}}^{w'\otimes e_k}, \qquad \text{where } w' := v_{q_{j+m}}.$$

Let U_G be the set consisting of the s operators of type J_δ, $J_{v,t}$ and J_w. We claim that U_G satisfies (4). Observe first that

$$||J||^2 = 4N \qquad \text{for all } J\in U_G.$$

This is clear for the operators J_δ; as regards the operators $J_{t,v}$ and the J_w, this follows from (6). Moreover, (5) also guarantees that the elements of U_G are pairwise orthogonal. Finally, observe that

$$\sum_{\delta\in E(G)} J_\delta^2 \equiv -\mathrm{diag}(u,u,\dots,u), \quad u=(d_{v_1},\dots,d_{v_n})$$

where $\equiv$ means that the right-hand side square matrix of order $2Nn$ represents the operator on the left with respect to $\mathfrak{B}$. On the other hand, denoting by $\{\bar{e}_i\}$ the canonical basis of $\mathbb{R}^n$, for each $v=v_{p_i}\in V'(G)$ we have:

$$J_{v,t}^2 \equiv -\mathrm{diag}(2\bar{e}_{p_i}, 2\bar{e}_{p_i},\dots,2\bar{e}_{p_i})$$

and finally for each of the vertices $w=v_{q_j}$, $j=1,\dots,m$ of $V^o(G)$ we have

$$J_w^2 \equiv -\mathrm{diag}(\bar{e}_{q_j}+\bar{e}_{q_{j+m}}, \bar{e}_{q_j}+\bar{e}_{q_{j+m}},\dots,\bar{e}_{q_j}+\bar{e}_{q_{j+m}}).$$

According to the definition of the h_i, we conclude that

$$\sum_{J\in U_G} J^2 = -(\Delta(G)+\nu)\, Id_V,$$

where $\nu \in \{0,1\}$ and $\nu = 1$ iff $0 < 2m < n$ and $\Delta(G)$ is odd.
We have thus proved that the standard metric 2-step nilpotent Lie algebra $\mathfrak{n} = V \oplus W_G$, where $W_G = \mathrm{span}(U_G)$ is an Einstein nilradical.
Concerning the last claim, denoting by V_1 the subspace of V spanned by $\{v_1 \otimes e_1, \ldots, v_n \otimes e_1\}$ and by W_E the subspace of W_G spanned by the operators J_δ, $\delta \in E(G)$, then the linear isomorphism

$$\mathfrak{n}_G \to V_1 \oplus W_E$$

determined by

$$v_i \mapsto v_i \otimes e_1, \qquad \delta \mapsto J_\delta$$

is a Lie algebra isomorphism between $\mathfrak{n}_G$ and the Lie subalgebra $\mathfrak{n}_1 := V_1 \oplus W_E$ of $\mathfrak{n}$. Finally, it is readily verified that for every $X, Y \in \mathfrak{n}_1$ and $Z \in \mathfrak{n}_1^\perp$ we have

$$< [X,Z], Y >= 0$$

yielding that $\mathfrak{n}_1$ is a totally geodesic subalgebra of $\mathfrak{n}$. $\square$

Observe that the construction performed in the proof actually works also for every integer $N' \geq N$, providing a standard Einstein nilradical of type $(s, 2N'n)$; our choice aims at providing a totally geodesic embedding $\mathcal{N}_G \hookrightarrow \mathcal{N}$ with minimum codimension.

4. A construction involving Cayley-Dickson algebras

In this section we provide an alternative description of an Einstein nilradical $\mathfrak{n}$ attached to a graph, satisfying the conditions of Theorem 1. We recall that the Cayley-Dickson algebras constitute an infinite sequence of real algebras

$$\mathbb{A}_0 \subset \mathbb{A}_1 \subset \mathbb{A}_2 \subset \ldots \subset \mathbb{A}_M \subset \ldots$$

where $\mathbb{A}_0 = \mathbb{R}$, $\mathbb{A}_1 = \mathbb{C}$, $\mathbb{A}_2 = \mathbb{H}$ (quaternions), $\mathbb{A}_3 = \mathbb{O}$ (octonions), etc. Their construction can be described recursively as follows; set $\mathbb{A}_0 = \mathbb{R}$ and define $\mathbb{A}_{n+1} = \mathbb{A}_n \times \mathbb{A}_n$. Given $(a,b),(c,d)$ in $\mathbb{A}_{n+1}$, their product is defined by

$$(a,b)(c,d) = (ac - db^*, a^*d + cb),$$

where the *conjugate* of (a,b) is also defined recursively by $(a,b)^* := (a^*, -b)$. We have $\dim_{\mathbb{R}} \mathbb{A}_M = 2^M$; moreover, $\mathbb{A}_M$ admits a standard basis

$$i_0 := 1, i_1, \ldots, i_{2^M-1}$$

such that $i_k^* = -i_k$ for $k > 0$, and the corresponding multiplication table is

$$i_p i_q = \gamma(p,q)\, i_{p \oplus q}, \tag{7}$$

where, for each $p, q \in \mathcal{G}_M := \{0, \ldots, 2^M - 1\}$, $\gamma(p,q) = \pm 1$, and $\oplus$ is the group operation in $\mathcal{G}_M$ defined by $p \oplus q$:=bit-wise *exclusive* OR of the binary representations of

p and q; the group $(\mathcal{G}_M, \oplus)$ is canonically isomorphic to $\mathbb{Z}_2^M$. In particular, $i_k^2 = -1$ for each $k \in \mathcal{G}_M$, $k > 0$. For more information and a detailed study of the function $\gamma\colon \mathcal{G}_M \times \mathcal{G}_M \to \{-1,1\}$ we refer the reader to [1].
For each $x \in \mathbb{A}$, its *real part* is defined as usual by $Re(x) = x_0$, provided $x = \sum_{p\in\mathcal{G}_M} x_p i_p$. With respect to the inner product $< x,y >= Re(x^* y)$, the algebra $\mathbb{A}_M$ is known to satisfy the *adjoint properties*:

$$< xy,z >=< y,x^*z >, \qquad < x,yz >=< xz^*,y > .$$

Moreover, we remark that using the multiplication of $\mathbb{A}$ one can define an edge-colouring $C_{\mathbb{A}}$ of the complete graph K_{2^M} with vertex set $\{0,\dots,2^M-1\}$ as follows:

$$C_{\mathbb{A}}(\{k,k'\}) = t \quad \text{ iff } \; i_t i_{k'} = \pm i_k$$

for every $t \in \{1,\dots,2^M-1\}$ and $k,k' \in \{0,\dots,2^M-1\}$ with $k \neq k'$.

Consider now a graph G and the pair (N,s) defined above, and take the smallest non negative integer M such that $2^{M-1} \geq N$. In this case we consider a standard metric 2-step nilpotent Lie algebra

$$\mathfrak{n}_{\mathbb{A}} := \mathbb{A}^n \oplus W_G,$$

where $\mathbb{A} := \mathbb{A}_M$ and $\mathbb{A}^n$ is considered as a $n2^M$-dimensional Euclidean vector space in a natural way, declaring the natural basis $\{i_k\bar{e}_i\}$ to be orthonormal (here $\{\bar{e}_i\}$ is the canonical basis of $\mathbb{R}^n$ and $i_k\bar{e}_i$ is the vector of $\mathbb{A}^n$ whose only non null entry, indexed i, is equal to i_k).
In this case, the s operators making up the basis U_G of W_G will be all of type

$$< J_a(X),Y >:= Re({}^tX^*(aY)) \tag{8}$$

where a is a $n \times n$ matrix with entries in $\mathbb{A}$.
The s matrices are defined as follows:

A) For each edge $\delta = \{v_p, v_q\}$, $p < q$, we consider the matrix $a_\delta := E_{pq} - E_{qp}$ whose entries indexed by (p,q) and (q,p) are respectively 1 and -1 and all the other entries are equal to zero.

B) For each vertex $v = v_{p_i}$ in $V'(G)$, $i = 1,\dots,r$ we define the h_i matrices:

$$a_{v,t} := \mathrm{diag}(0,\dots,0,\sqrt{2}\,i_t,0,\dots,0), \qquad t = 1,\dots,h_i$$

each having exactly one non null entry indexed p_i.

C) In the case where $0 < 2m < n$, for each vertex $w = v_{q_j}$, $j = 1,\dots,m$ we introduce the matrix

$$a_w := \mathrm{diag}(0,\dots,0,i_{h+1},0,\dots,i_{h+1},0,\dots,0),$$

where the two non null entries are indexed respectively q_j and q_{j+m}.

Observe that for each of these matrices, the fact that the real bilinear form on $\mathbb{A}^n$ on the right hand side of (8) is skew-symmetric is guaranteed by the adjoint properties of $\mathbb{A}$. Taking into account the isomorphism $\mathbb{A}^n \cong V_0 \otimes \mathbb{R}^{2^M}$ such that

$$i_k \bar{e}_i \mapsto v_i \otimes e_{k+1},$$

the corresponding operators differ slightly from those defined in the proof of Theorem 1, since the twist function γ comes into play; namely one gets:

$$\begin{aligned}
J_\delta &= \sum_{k=1}^{2^M} F^{v_p \otimes e_k}_{v_q \otimes e_k} \\
J_{v,t} &= \sqrt{2} \sum_{(k,k') \in C_t} \gamma(t,k') F^{v \otimes e_k}_{v \otimes e_{k'}} \\
J_w &= \sum_{(k,k') \in C_{h+1}} \gamma(h+1,k') F^{w \otimes e_k}_{w \otimes e_{k'}} + \sum_{(k,k') \in C_{h+1}} \gamma(h+1,k') F^{w' \otimes e_k}_{w' \otimes e_{k'}}
\end{aligned}$$

where $C = C_{\mathbb{A}}$ is the colouring of K_{2^M} considered above. Again W_G is a uniform subspace of $\mathfrak{so}(\mathbb{A}^n)$, yielding that $\mathfrak{n}_{\mathbb{A}}$ is an Einstein nilradical.

EXAMPLE 1. Let G be the graph obtained from the complete graph K_6 by removing all the edges $\{i,j\}$ with i,j both in $\{1,2,3,4\}$. It can be checked directly that this graph is not positive (cf. also the classification in [7, Table 1]), so that $\mathfrak{n}_G$ is not an Einstein nilradical. In this case $(N,s) = (2,18)$ so that $\mathbb{A} = \mathbb{H}$ and $\mathfrak{n}_{\mathbb{H}} = \mathbb{H}^6 \oplus W_G$, where W_G is determined via (8) by the following 18 square matrices of order 6 with entries in $\mathbb{H}$:

$$\begin{aligned}
&E_{56} - E_{65},\, E_{i5} - E_{5i},\, E_{i6} - E_{6i}, \quad i = 1,\dots,4, \\
&\mathrm{diag}(\sqrt{2}i,0,0,0,0,0),\ \mathrm{diag}(\sqrt{2}j,0,0,0,0,0), \\
&\mathrm{diag}(0,\sqrt{2}i,0,0,0,0),\ \mathrm{diag}(0,\sqrt{2}j,0,0,0,0), \\
&\mathrm{diag}(0,0,\sqrt{2}i,0,0,0),\ \mathrm{diag}(0,0,\sqrt{2}j,0,0,0), \\
&\mathrm{diag}(0,0,0,\sqrt{2}i,0,0),\ \mathrm{diag}(0,0,0,\sqrt{2}j,0,0), \\
&\qquad\qquad\qquad\qquad\qquad\quad \mathrm{diag}(0,0,0,0,k,k).
\end{aligned}$$

References

[1] BALES J.W., *A tree for computing the Cayley-Dickson twist*, Missouri Journal of Mathematical Sciences **21** 2 (2009), 83–93.

[2] DANI S.G. AND MAINKAR M.G, *Anosov automorphisms on compact nil-manifolds associated with graphs*, Trans. A.M.S. **357** (2005), 2235–2251.

[3] EBERLEIN P., *Riemannian 2-step nilmanifolds with prescribed Ricci tensor*, Geometric and probabilistic structures in dynamics, Contemp. Math. **469** (2008), 167–195.

[4] FANAÏ H.-R., *Two-step nilpotent Lie algebras attached to graphs*, Int. Math. Forum **4** 41-44 (2009), 2143–2148.

[5] FIORINI S. AND WILSON R. J., *Edge-colourings of graphs*, Research Notes in Mathematics No. 16, Pitman, London, 1977.

[6] GORDON C. S. AND KERR M. M., *New homogeneous Einstein metrics of negative Ricci curvature*, Ann. Global Anal. Geom. **19** 1 (2001) , 75–101.

[7] LAFUENTE R. A., *Solvsolitons associated with graphs*, Adv. Geom. **13** 2 (2013), 255–275.

[8] LAURET J., *Ricci soliton homogeneous nilmanifolds*, Math. Ann. **319** (2001), 715–733.

[9] LAURET J., *Einstein solvmanifolds and nilsolitons*, New developments in Lie theory and geometry, Contemp. Math. **491** (2009), 1–35.

[10] LAURET J. AND WILL C., *Einstein solvmanifolds: existence and non-existence questions*, Math. Ann. **350** 1 (2011), 199–225.

[11] OSCARI D., *On the existence of nilsolitons on 2-step nilpotent Lie groups*, Adv. Geom. **14** 3 (2014), 483–497.

AMS Subject Classification: 53C25, 22E25

Antonio LOTTA,
Dipartimento di Matematica, Università degli Studi di Bari Aldo Moro
Via E. Orabona 4, 70125 Bari, Italy
e-mail: `antonio.lotta@uniba.it`

Lavoro pervenuto in redazione il 16.11.2015 e accettato il 16.02.2016.

Geom. Struc. on Riem. Man.-Bari
Vol. 73/1, 3–4 (2015), 127 – 150

C. Medori - A. Spiro[1]

STRUCTURE EQUATIONS OF LEVI DEGENERATE CR HYPERSURFACES OF UNIFORM TYPE

Abstract. We explicitly determine the structure equations of 5-dimensional Levi 2-non-degenerate CR hypersurfaces, using our recently constructed canonical Cartan connection for this class of CR manifolds. We also give an outline of the basic properties of absolute parallelisms and Cartan connections, together with a brief discussion of the absolute parallelisms for such CR manifolds existing in the literature.

1. Introduction

Let M be a $(2n+1)$-dimensional CR hypersurface, that is a manifold endowed with a pair $(\mathcal{D}, J)$ formed by

a) a distribution $\mathcal{D} \subset TM$ of codimension 1,

b) a smooth family J of complex structures $J_x : \mathcal{D}_x \longrightarrow \mathcal{D}_x$, satisfying the integrability condition, i.e. the complex distribution $\mathcal{D}^{10} \subset T^{\mathbb{C}}M$ of the $(+i)$-eigenspaces of the J_x is involutive.

We recall that the Levi form of M is defined as follows. For $x \in M$, let ϑ be a 1-form on a neighbourhood $\mathcal{U}$ of x with $\ker \vartheta_y = \mathcal{D}_y$ at each point y of $\mathcal{U}$. The *Levi form at x* is the symmetric bilinear map

$$\mathcal{L}_x : \mathcal{D}_x \times \mathcal{D}_x \to \mathbb{R}, \qquad \mathcal{L}_x(v, w) := d\vartheta_x(v, Jw),$$

which is well known to be J_x-invariant and independent on the choice ϑ, up to a scalar multiple. If the dimension of $\ker \mathcal{L}_x$ is constant over M, we call the CR hypersurface *of uniform type*. The case $\dim\ker \mathcal{L}_x = 0$ occurs if and only if the distribution $\mathcal{D}$ is contact and in this case $(M, \mathcal{D}, J)$ is called *Levi-nondegenerate*. If $\mathcal{D}$ is of uniform type with $\dim\ker \mathcal{L}_x > 0$ at all points, we call it *uniformly Levi-degenerate*.

The simplest examples of uniformly Levi degenerate CR hypersurfaces are given by the cartesian products $\overline{M} \times S$ of a Levi-nondegenerate CR hypersurface $(\overline{M}, \overline{\mathcal{D}}, \overline{J})$ and an m-dimensional complex manifold (S, J^S). The natural CR structure of $\overline{M} \times S$ is the pair $(\mathcal{D}, J)$ defined by

$$\mathcal{D}_x := \overline{\mathcal{D}}_{\bar{x}} + T_s S, \qquad J_x := \bar{J}_{\bar{x}} \times J^S_s \text{ for all } x = (\bar{x}, s) \in \overline{M} \times S.$$

If a CR hypersurface is locally CR equivalent with a cartesian product of this kind around any point, we say that *it admits local CR straightenings*.

[1] *Acknowledgments*. This research was partially supported by the Project MIUR “Real and Complex Manifolds: Geometry, Topology and Harmonic Analysis” and by GNSAGA of INdAM.

Under appropriate uniformity assumptions on the CR structure, any uniformly Levi degenerate CR hypersurface $(M,\mathcal{D},J)$ is equipped with a nested sequence of complex distributions

(1) $$\ldots\subset\mathcal{F}_2\subset\mathcal{F}_1\subset\mathcal{F}_0\subset\mathcal{F}_{-1}\subset\mathcal{F}_{-2}=T^{\mathbb{C}}M,$$

in which $\mathcal{F}_{-1}:=\mathcal{D}^{10}$ and all other subdistribution $\mathcal{F}_i$, $i\geq 0$, are inductively defined in a special way that implies that

$$[\mathcal{F}_i,\mathcal{F}_j]\subset\mathcal{F}_{i+j}\qquad\text{for each } i,j\geq -2$$

(here, we assume that $\mathcal{F}_{i+j}:=T^{\mathbb{C}}M$ if $i+j\leq -2$). This nested sequence of distributions necessarily stabilises after a finite number of steps and, by a result of Freeman ([11]), it has the following crucial property: $(M,\mathcal{D},J)$ *admits local CR straightenings if and only if the first stabilising distribution* $\mathcal{F}_k$*, that is such that* $\mathcal{F}_{k+\ell}=\mathcal{F}_k$ *for all* $\ell\geq 0$*, is non trivial.*

The uniformly Levi degenerate CR hypersurfaces with trivial stabilising distribution $\mathcal{F}_k$ (hence, with no CR straightenings) are called *Levi* $(k+1)$*-nondegenerate*. This notion extends the concept of Levi nondegeneracy, since the Levi 1-nondegenerate hypersurfaces are precisely the Levi nondegenerate hypersurfaces in the usual sense.

The smallest dimension for a CR hypersurface to be uniformly Levi degenerate and with no CR straightenings is 5. By dimension counting, any such 5-dimensional CR hypersurfaces is *2-nondegenerate*. For conciseness, we call the CR manifolds of this kind *girdled CR manifolds*.

The class of girdled CR manifolds and the associated equivalence problem has been the main object of investigation in several recent papers. In particular, in [14] we proved the existence of a canonical Cartan connection for any girdled CR manifold, obtaining in this way a solution to the equivalence problem and a complete set of invariants for this class of CR manifolds. Independently and with preprints posted almost at the same time, Isaev and Zaitsev presented in [13] an alternative solution, hence another set of invariants, for the same equivalence problem. Isaev and Zaitsev's solution is however not corresponding to a Cartan connection. Shortly after, a third solution and another set of invariants has been given by Pocchiola in [16].

Due to this, in several occasions, the following question has been posed: *Is there a way to compare one to the other such solutions to the equivalence problem of girdled CR manifolds?*

Having this question in mind, in this paper we newly present our solution to the equivalence problem for girdled CR manifolds, in a way that allows an immediate comparison with the other existing solutions. More precisely, we first provide a quick review of the notions of equivalence problems, absolute parallelisms and Cartan connections. The intention of such overview is twofold: to fix unambiguously the meaning of all terms of our discussion and to clarify the main reasons of interests for solutions to equivalence problems coming from canonical Cartan connections. We then describe in detail the canonical Cartan connections of girdled CR manifolds introduced in [14], giving the explicit expressions of the corresponding structure equations and making manifest all curvature restrictions that characterise such connections.

2. Equivalence problems and Cartan connections

Let G be a class of geometric structures, that is of pairs $(M, \mathcal{S})$ formed by a manifold M with some geometric datum $\mathcal{S}$ of fixed type (as, for instance, a Riemannian metric g, a distribution $\mathcal{D}$, a CR structure $(\mathcal{D}, J)$, etc.). Given two geometric structures $(M, \mathcal{S})$, $(M', \mathcal{S}')$ in G, the *local equivalences around points $x \in M$ and $x' \in M'$* are the local diffeomorphisms $f : \mathcal{U} \to \mathcal{U}'$ between neighbourhoods $\mathcal{U}$, $\mathcal{U}'$ of x, x', transforming $\mathcal{S}|_{\mathcal{U}}$ into $\mathcal{S}'|_{\mathcal{U}'}$. The *equivalence problem for the class* G it is the query for an algorithm that establishes when, given two points, there exists a local equivalence around such two points.

A standard approach to such problem consists in looking for constructions that give for each $(M, \mathcal{S})$ in G a unique triple $(P, (X_i), \widetilde{(\cdot)})$ made of:

i) a bundle $\pi : P \to M$ over the manifold M;

ii) an absolute parallelism (X_i) on P, i.e. an ordered N-tuples of vector fields $(X_1, \ldots, X_N)$ that gives a frame at each tangent space $T_u P$;

iii) an operator $\widetilde{(\cdot)}$ which maps each local equivalence $f : \mathcal{U} \to M'$ into a bundle diffeomorphism $\widetilde{f} : \mathcal{V} \subset P \to P'$ that projects onto f,

such that the following holds: *a local diffeomorphism $F : \mathcal{V} \subset P \to P'$ between the bundles P, P' of two structures $(M, \mathcal{S})$, $(M', \mathcal{S}')$ in* G *maps the associated parallelisms (X_i), (X_i') one into the other if and only if $F = \widetilde{f}$ for some local equivalence f.*

Triples $(P, (X_i), \widetilde{(\cdot)})$ with this property are called *canonical absolute parallelisms for the class* G and any algorithm that provides canonical absolute parallelisms solves the equivalence problem for G in the following sense.

Any absolute parallelism (X_i) is uniquely determined by the N-tuple of its dual 1-forms $(\omega^1, \ldots, \omega^N)$ and a local diffeomorphism transforms one absolute parallelism into another if and only it it transforms the corresponding dual coframe fields one into the other. We now observe that the differentials $d\omega^i$ admit unique expansions of the form $d\omega^i = \sum_{j<k} c^i_{jk} \omega^j \wedge \omega^k$. These are the so-called *structure equations of the parallelism* (X_i) and the functions c^i_{jk} are the associated *first order invariants*. Note that the invariants c^i_{jk} can be explicitly determined from the vector fields X_i by recalling that

$$c^i_{jk} = d\omega^i(X_j, X_k) = -\omega^i([X_j, X_k]). \tag{1}$$

Their differentials have the form $dc^i_{jk} = c^i_{jk|\ell_1} \omega^{\ell_1}$ and the functions $c^i_{jk|\ell_1}$ are called *second order invariants*. Their differentials $dc^i_{jk|\ell_1} = c^i_{jk|\ell_1\ell_2} \omega^{\ell_2}$ define the *third order invariants* $c^i_{jk|\ell_1\ell_2}$ and so on. By a fundamental result of Cartan and Sternberg, if appropriate constant rank conditions hold, there is an m_o such that all invariants of order $r \leq m_o + 1$ give a map $F^{(m_o)} := (c^i_{jk},\, c^i_{jk|\ell_1},\, c^i_{jk|\ell_1\ell_2},\, \cdots, c^i_{jk|\ell_1 \ldots \ell_{m_o}}) : P \to \mathbb{R}^{N_o}$, which completely characterises the pair $(P, (X_i))$ up local equivalences ([24], Thm. VII.4.1; see also [12, 17, 23]). So, any question on existence of equivalences between canonical absolute parallelisms of G is in principle completely solvable by studying the invariants of the parallelisms up to some finite order. This is the reason why any algorithm that provides canonical absolute parallelisms for G is considered as a solution to the equivalence problem for this class.

Solutions of this type to the equivalence problems are usually not unique. For instance, the so-called *G-structures of finite type* admit canonical absolute parallelisms, determined via a finite number steps, each of them based on choices of certain normalising conditions ([24, 12, 25, 15, 1, 17]). Different choices lead to non-equivalent canonical absolute parallelisms, hence to distinct solutions to the same equivalence problem. Other examples are provided by the celebrated absolute parallelisms of Chern and Moser ([9]) and of Tanaka ([26, 27]) for the Levi-nondegenerate CR hypersurfaces, whose first order invariants are actually constrained by non-equivalent sets of linear equations. There exists also three distinct solutions to the equivalence problems for the elliptic and hyperbolic CR manifolds of codimension two, which have been determined in [10, 18, 20].

Amongst all canonical absolute parallelisms that one might associate with the structures of a given class, there are sometimes some special ones that correspond to Cartan connections. As we will shortly see, parallelisms of this kind have several very important additional features.

We recall that a *Cartan connection on a manifold M, modelled on a homogeneous space G/H,* is a pair (P,ϖ), formed by a principal H-bundle $\pi : P \to M$ and a $\mathfrak{g}$-valued 1-form $\varpi : TP \to \mathfrak{g} = Lie(G)$ such that:

(a) for each $y \in P$, the map $\varpi_y : T_yP \to \mathfrak{g}$ is a linear isomorphism and $(\varpi_y)^{-1}|_{\mathfrak{h}} : \mathfrak{h} \to T^V_y P$ is the standard isomorphism, given by the right action of H on P, between $\mathfrak{h} = Lie(H)$ and the tangent space $T^V_y P$ of the fiber,

(b) $R^*_h\varpi = \mathrm{Ad}_{h^{-1}} \circ \varpi$ for any $h \in H$.

Given a class of geometric structures G, a correspondence between the structures in G and Cartan connections on the underlying manifolds, is called *canonical* if there is an associated bijection between the local equivalences $f : \mathcal{U} \to \mathcal{U}'$ between manifolds M, M' of G and the local diffeomorphisms $\widetilde{f} : P|_{\mathcal{U}} \to P'|_{\mathcal{U}'}$ between the bundles of the associated Cartan connections (P,ϖ), (P',ϖ'), that satisfy the condition $\widetilde{f}^*\varpi' = \varpi$.

Note that if there is a canonical Cartan connection (P,ω) for any manifold M of G, each basis (E^o_i) for $\mathfrak{g} = Lie(G)$ determines a canonical absolute parallelism $(P,(E_i),\widetilde{(\cdot)})$, formed by the bundle P and the absolute parallelism (E_i) given by the vector fields

$$E_i|_u := \varpi_u^{-1}(E^o_i), \qquad u \in P. \tag{2}$$

Hence, *any construction of canonical Cartan connections for a class* G *automatically provides a solution to the corresponding equivalence problem.*

However, the interest for Cartan connections is by far much wider than their uses for equivalence problems. For an introduction to the variety of possible applications, see e.g. [12, 22, 6, 7, 8, 4, 19, 2, 28] and references therein.

One of the most basic reasons of interest for Cartan connections is given by the following fact: *If* (P,ϖ) *is a Cartan connection on* M *modelled on* G/H*, the associated* $\mathfrak{g}$*-valued curvature* 2*-form* $= d\varpi + \frac{1}{2}[\varpi,\varpi]$ *on* P *vanishes identically if and only if* P *is locally equivalent to the Lie group* G *and* M *is locally equivalent to the homogeneous model* G/H. This means that if the elements of a class G of geometric structures admit canonical Cartan connections modelled on a given homogeneous spaces, for each of them there exists a very informative indicator (namely, the curvature) of how it locally deviates from the homogeneous model.

From this and other facts on Cartan connections, one has also that geometric structures admitting canonical Cartan connections are equipped with distinguished families of appropriate curves or submanifolds of higher dimension, which are invariant under local equivalences and play the same role of geodesics and chains in Riemannian geometry and in geometry of Levi non-degenerate hypersurfaces, respectively (see e.g. [3, 18]). Such distinguished curves and submanifolds can be also combined and determine systems of normal coordinates, which allow to reduce several questions to geometric properties of the homogeneous models (see, for instance, [18, 19]).

At the best of our knowledge, the first methodical study on the possibilities of constructing canonical Cartan connections was done by Tanaka in [27]. There he proved the existence of canonical Cartan connections for an important family of classes of geometric structures, modelled on homogeneous spaces G/H of (semi)simple Lie groups and with parabolic isotropy subgroups $H \subset G$. His results were later extended in various senses by T. Morimoto in [15] and Čap and Schichl in [5]. For a concise review of Tanaka's results, see [1].

We conclude this short discussion of Cartan connections recalling that in [1], Alekseevsky and the second author proved that Tanaka's method of construction of Cartan connections can be considered as a derivation of a more general method of construction of absolute parallelisms, also invented by Tanaka ([25]). This second method applies to a wider range of geometric structures, called *Tanaka's structures of finite type*, and produces canonical parallelisms $(P,(X_i),\widetilde{(\cdot)})$, formed by bundles $\pi : P \to M$ that *in general are not principal bundles* and by parallelisms (X_i) that *in general are not determined by a* $\mathfrak{g}$*-valued* 1*-form* ϖ satisfying the properties of Cartan connections. Nonetheless, for a special class of Tanaka structures, modelled on homogeneous spaces G/H of a semisimple G and parabolic subgroup $H \subset G$, the general construction can be performed in such a way that it produces a bundle $\pi : P \to M$, which *is a principal* H*-bundle*, and an absolute parallelism (X_i) on P, which *is determined by a Cartan connection* ϖ (see [1] for details).

3. Cartan connections of girdled CR manifolds and corresponding structure equations

Now we focus on girdled CR manifolds $(M,\mathcal{D},J)$, i.e. on 5-dimensional CR hypersurfaces of uniform type, which are Levi 2-nondegenerate. As we already mentioned in the introduction, any such CR hypersurface is Levi degenerate and yet admits no local straightenings. The name *girdled* has been chosen to allude to such lack of straightenings.

One of the most important examples of girdled CR manifolds and, as we will shortly see, a model for these geometric structures is given by the following homogeneous manifold. Consider the bilinear form $(\cdot,\cdot)$ and the pseudo-Hermitian form $<\cdot,\cdot>$ on $\mathbb{C}^5$ defined by

$$(1) \qquad (t,s) = t^T I_{3,2} s, \qquad < t,s >= (\bar{t},s), \qquad I_{3,2} = \left(\begin{array}{c|c} I_3 & 0 \\ \hline 0 & -I_2 \end{array} \right),$$

and the corresponding semi-algebraic subset $M_o \subset \mathbb{C}P^4$ defined by

$$(2) \qquad \begin{cases} (t,t) = (t^0)^2 + (t^1)^2 + (t^2)^2 - (t^3)^2 - (t^4)^2 = 0, \\ < t,t >= |t^0|^2 + |t^1|^2 + |t^2|^2 - |t^3|^2 - |t^4|^2 = 0, \\ \Im\left(t^3 \overline{t^4}\right) > 0. \end{cases}$$

It is known (see e.g. [21]) that M_o is a $\mathrm{SO}^o_{3,2}$-homogeneous, 5-dimensional CR submanifold of $\mathbb{C}P^4$ (here, $\mathrm{SO}^o_{3,2}$ is the identity component of $\mathrm{SO}_{3,2}$) and contains $\mathsf{T}_o = M_o \cap \{\Im(t^3\overline{(t^0+t^4)}) > 0\}$ as open dense subset, which is CR equivalent to the so called *tube over the future light cone in* $\mathbb{C}^3$, i.e. to the real hypersurface

$$(3) \qquad \mathsf{T} = \{ (z^1,z^2,z^3) \in \mathbb{C}^3 : (x^1)^2 + (x^2)^2 - (x^3)^2 = 0, x^3 > 0 \}.$$

It turns out that M_o is girdled and its group of CR automorphisms coincides with $\mathrm{Aut}\,(M_o) = \mathrm{SO}^o_{3,2}$. Hence, if we denote by $H \subset \mathrm{SO}^o_{3,2}$ the isotropy subgroup of $\mathrm{Aut}\,(M_o)$ at some point, M_o is CR equivalent to the homogeneous space $\mathrm{SO}^o_{3,2}/H$, equipped with an appropriate invariant girdled CR structure.
The homogeneous CR manifold $M_o = \mathrm{SO}^o_{3,2}/H$ is a modelling space, of which any girdled CR manifold can be considered as a local deformation. This is a consequence of the main theorem of our paper [14], namely

THEOREM 1. *For any 5-dimensional girdled CR manifold $(M,\mathcal{D},J)$, there exists a canonical Cartan connection (Q,ϖ), modelled on the homogeneous CR manifold $M_o = \mathrm{SO}^o_{3,2}/H$ described above.*

The proof of this theorem is constructive and provides an explicit description of the bundle $\pi : Q \longrightarrow M$ and of the $\mathfrak{so}_{3,2}$-valued 1-form ϖ (more precisely, of a collection of vector fields, by which ϖ is uniquely determined). Our construction is based on a modification of Tanaka's general scheme for building up absolute parallelisms. The fact that our collection of vector fields actually defines a Cartan connection is a consequence of an appropriate tuning of each step of the construction.
As it is shown in [1] (see also above, end of §2), even the classical Tanaka's method can be used to produce Cartan connections, provided that appropriate algebraic conditions are satisfied. Such conditions certainly occur for the *parabolic geometries* [5], i.e. the

geometric structures modelled on homogeneous spaces G/H of semisimple Lie groups G with parabolic H. Since the girdled CR manifolds are modelled on a homogeneous space G/H of the semisimple Lie group $G = \mathrm{SO}^o_{3,2}$ with a *non parabolic* H, our result shows that the above conditions might occur for a wider and interesting class of homogeneous models.

As pointed out in §2, the absolute parallelism, that is determined by the canonical Cartan connection (Q, ϖ) and a basis of $\mathfrak{so}_{3,2}$, provides a solution to the equivalence problem for girdled CR manifolds. At the best of our knowledge, at the moment there are two other absolute parallelisms for girdled CR manifolds, hence two other solutions to the same problem ([13, 16]), but none of them corresponds to a Cartan connection.

In the next sections, we select a special basis for $\mathfrak{so}_{3,2}$ and we write explicitly the structure equations of the absolute parallelism corresponding to such special basis. Such explicit expressions also allow immediate comparisons with the structure equations of the parallelisms provided by the other solutions to the equivalence problem for girdled CR manifolds.

3.1. A convenient basis for $\mathfrak{so}_{3,2}$

The Lie algebra $\mathfrak{g} = \mathfrak{so}_{3,2}$ has a natural structure of graded Lie algebra, which can be explicitly described as follows. Consider a system of projective coordinates on $\mathbb{C}P^4$, in which the scalar product $(\cdot,\cdot)$ defined in (1) assumes the form

$$(t,s) = t^T I\, s \qquad \text{with} \qquad I = \left(\begin{array}{cc|c|cc} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right). \tag{4}$$

By means of these new coordinates, the Lie algebra $\mathfrak{so}_{3,2}$ of the isometries of $(\cdot,\cdot)$ can be identified with the Lie algebra of real matrices A such that $A^T I + IA = 0$, i.e., of the form

$$A = \left(\begin{array}{cc|c|cc} a_1 & a_2 & a_5 & a_7 & 0 \\ a_3 & a_4 & a_6 & 0 & -a_7 \\ \hline a_8 & a_9 & 0 & -a_6 & -a_5 \\ \hline a_{10} & 0 & -a_9 & -a_4 & -a_2 \\ 0 & -a_{10} & -a_8 & -a_3 & -a_1 \end{array}\right), \qquad \text{for some } a_i \in \mathbb{R}.$$

This shows that $\mathfrak{so}_{3,2}$ is the direct sum of the vector subspaces

$$\begin{gathered} \mathfrak{g}_{-2} = < e^o_{-2} >,\ \mathfrak{g}_{-1} = < e^o_{-1|1}, e^o_{-1|2} >,\ \mathfrak{g}_0 = < e^o_{0|1}, e^o_{0|2}, E^o_{0|1}, E^o_{0|2} >, \\ \mathfrak{g}_1 = < E^o_{1|1}, E^o_{1|2} >,\ \mathfrak{g}_2 = < E^o_2 >, \end{gathered} \tag{5}$$

spanned by the matrices

$$
\begin{aligned}
&e^o_{-2}=\left(\begin{array}{cc|c|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{array}\right),\ e^o_{-1|1}=\left(\begin{array}{cc|c|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{array}\right),\ e^o_{-1|2}=\left(\begin{array}{cc|c|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),\\
&e^o_{0|1}=\left(\begin{array}{cc|c|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right),\ e^o_{0|2}=\left(\begin{array}{cc|c|cc} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right),\\
(6)\qquad &E^o_{0|1}=\left(\begin{array}{cc|c|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right),\ E^o_{0|2}=\left(\begin{array}{cc|c|cc} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right),\\
&E^o_{1|1}=\left(\begin{array}{cc|c|cc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),\ E^o_{1|2}=\left(\begin{array}{cc|c|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),\ E^o_{2}=\left(\begin{array}{cc|c|cc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).
\end{aligned}
$$

We will refer to the collection $\mathcal{B}^o$ of these matrices as *standard basis* of $\mathfrak{so}_{3,2}$.

Note that the each $\mathfrak{g}_k$ in (5) is the eigenspace of the adjoint action of the *grading element* $Z := E^o_{0|1}$ with eigenvalue k, so that

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \qquad \text{for all } i, j,$$

where, by convention, we assume $\mathfrak{g}_k = \{0\}$ for any $k \notin \{-2,-1,0,2,2\}$. In other words, $\mathfrak{so}_{3,2}$ *has a natural structure of graded Lie algebra*.

We note that also the Lie algebra $\mathfrak{h} = Lie(H)$ of the isotropy subgroup $H \subset \mathrm{SO}^o_{3,2}$ at $x_o = [1 : i : 0 : 0 : 0]$ is natural graded. Indeed, it decomposes into the direct sum $\mathfrak{h} = \mathfrak{h}_0 + \mathfrak{g}_1 + \mathfrak{g}_2$ with $\mathfrak{h}_0 := \langle E_{0|1}, E_{0|2} \rangle$.

We conclude this section introducing another convenient basis for $\mathfrak{so}_{3,2}$, which is a technical modification of $\mathcal{B}^o$, more suitable for several arguments concerning the CR structure of the model space $\mathrm{SO}^o_{3,2} \cdot x_o \simeq \mathrm{SO}^o_{3,2}/H$. Indeed, in many places it is more appropriate to consider instead of the four elements $E^o_{\ell|j}$, $e^o_{-\ell|j}$, $\ell = 0, 1$, $j = 1, 2$, the four complex matrices in $(\mathfrak{so}_{3,2})^{\mathbb{C}}$:

$$
(7)\qquad \begin{array}{ll} E^o_{\ell(10)} = \frac{1}{2}\left(E^o_{\ell|1} - iE_{\ell|2}\right), & E^o_{\ell(01)} = \overline{E^o_{\ell(10)}}, \\ e^o_{-\ell(10)} = \frac{1}{2}\left(e^o_{-\ell|1} - ie^o_{-\ell|2}\right), & e^o_{-\ell(01)} = \overline{e^o_{-\ell(10)}}, \end{array} \qquad \ell = 0, 1.
$$

So, in the following, instead of expanding the elements of $\mathfrak{so}_{3,2}$ in terms of the standard basis $\mathcal{B}^o$, we often expand the same elements in terms of the *standard CR basis*

$$
(8)\qquad \mathcal{B}^{CR} = \left(e^o_{-2}, e^o_{-1(10)}, e^o_{-1(01)}, e^o_{0(10)}, e^o_{0(01)}, E^o_{0(10)}, E^o_{0(01)}, E^o_{1(10)}, E^o_{1(01)}, E^o_{2}\right).
$$

Since the elements $X \in \mathfrak{so}_{3,2}$ are real matrices, their expansion in the standard CR basis has the form $X = \sum^A e^o_A + \sum \mu^A E^o_A$, with coefficients satisfying the reality conditions

$$\lambda^{-2}, \mu^{-2} \in \mathbb{R} \qquad \text{and} \qquad \lambda^{-\ell(01)} = \overline{\lambda^{-\ell(10)}},\ \mu^{\ell(01)} = \overline{\mu^{\ell(10)}} \text{ for } \ell = 0, 1.$$

A table of all Lie brackets between elements in $\mathcal{B}^{CR}$ can be found in [14].

3.2. The absolute parallelism associated with the standard basis

Consider now a girdled CR manifold $(M, \mathcal{D}, J)$ and its canonical Cartan connection (Q, ϖ) modelled on $M_o = \mathrm{SO}^o_{3,2}/H$. As we discussed in §2, the relation (2) associates with each element e^o_A or E^o_B of the standard basis $\mathcal{B}^o$ of $\mathfrak{so}_{3,2}$ a vector field that we denote by e_A or E_B, respectively. The ordered 10-tuple (e_A, E_B) is the *absolute parallelism corresponding to the basis* $\mathcal{B}^o$.

As we observed, in place of these (real) vector fields, it is often more convenient to consider the collection of (real and complex) vector fields

$$(9) \qquad \left(e_{-2}, e_{-1(10)}, e_{-1(01)}, e_{0(10)}, e_{0(01)}, E_{0(10)}, E_{0(01)}, E_{1(10)}, E_{1(01)}, E_2\right),$$

with $e_{-\ell(10)}, e_{-\ell(01)}, E_{\ell(10)}, E_{\ell(01)}$, $\ell = 0, 1$, defined by

$$(10) \qquad \begin{aligned} E_{\ell(10)} &:= \tfrac{1}{2}\left(E_{\ell|1} - iE_{\ell|2}\right), & E_{\ell(01)} &:= \overline{E_{\ell(10)}}, \\ e_{-\ell(10)} &:= \tfrac{1}{2}\left(e_{-\ell|1} - ie_{-\ell|2}\right), & e_{-\ell(01)} &:= \overline{e_{-\ell(10)}}, \end{aligned} \qquad \ell = 0, 1.$$

This is the collection of complex vector fields that corresponds to the elements of the standard CR basis $\mathcal{B}^{CR}$ by means of (2). From now, we will use the notation e_A and E_B to indicate just these vector fields.

The vector fields e_A, E_B uniquely determine their dual (real and complex) 1-forms ϑ^A, ω^B, defined by

$$(11) \qquad \vartheta^A(e_C) = \delta^A_C, \quad \vartheta^A(E_D) = 0, \quad \omega^B(e_A) = 0, \quad \omega^B(E_D) = \delta^B_D.$$

Note that the $\mathfrak{g}$-valued 1-form ϖ can be written in terms of such 1-forms as

$$(12) \qquad \varpi = \sum_A e^o_A \otimes \vartheta^A + \sum_B E^o_B \otimes \omega^B.$$

The vector fields (e_A, E_B) and the dual 1-forms (ϑ^A, ω^B) have several geometric features, which derive from the special step-by-step construction of the Cartan connection ϖ given in [14]. Let us briefly recall them.

First of all, we remind that the girdled CR manifold $(M, \mathcal{D}, J)$ is naturally equipped with a J-invariant, 2-dimensional, involutive subdistribution $\mathcal{E}$ of the distribution $\mathcal{D}$, defined at each point $x \in M$ by (see e.g. [14], §2.1):

$$(13) \qquad \mathcal{E}_x := \big\{ v \in \mathcal{D}_x : \text{there is vector field } X \text{ in } \mathcal{D} \text{ such that } X_x = v \text{ and } [X, Y]_x \in \mathcal{D}_x \text{ for all vector fields } Y \text{ in } \mathcal{D} \big\}.$$

In other words, $\mathcal{E}$ is the J-invariant distribution of vector spaces, generated by the real vectors that are in the kernels of the Levi forms of $(\mathcal{D}, J)$.
In [14], the bundle $\pi : Q \to M$ is obtained as the last step of a tower of three principal bundles, one defined over the other, as in the diagram

$$Q = P^2 \xrightarrow{\pi^2} P^1 \xrightarrow{\pi^1} P^0 \xrightarrow{\pi^0} M, \qquad \text{with } \pi := \pi^0 \circ \pi^1 \circ \pi^2.$$

In turn, each bundle P^i is defined as a quotient $P^i = P^i_\sharp / N^i_\sharp$ by the action of a special group of matrices $N^i_\sharp$, of an appropriate principal bundle $P^i_\sharp$ of linear frames of the lower order bundle

$$\begin{tikzcd}
\mathcal{F}r(P^1) & \mathcal{F}r(P^0) & \mathcal{F}r(M) & \\
\cup & \cup & \cup & \\
P^2_\sharp \arrow[ddr] & P^1_\sharp \arrow[ddr] & P^0_\sharp \arrow[ddr] & \\
& & & \\
P^2 \arrow[r] & P^1 \arrow[r] & P^0 \arrow[r] & M \\
\| & \| & \| & \\
P^2_\sharp / N^2_\sharp & P^1_\sharp / N^1_\sharp & P^0_\sharp / N_\sharp &
\end{tikzcd}$$

The absolute parallelism (e_A, E_B) on $Q = P^2$ is defined as the unique frame field that takes values in a very special trivial subbundle $P^3_\sharp$ of the linear frame bundle $\mathcal{F}r(P^2)$ of P^2.

Each bundle of linear frames $P^i_\sharp \subset \mathcal{F}r(P^{i-1})$, $0 \leq i \leq 3$ (here, we set $M = P^{-1}$), is determined by all linear frames of P^{i-1} that are adapted to the natural distributions of P^{i-1} (for instance, when $P^{i-1} = M$, the frames are adapted to the J-invariant distributions $\mathcal{E}$ and $\mathcal{D}$) and satisfy three sets of conditions:

a) if the base point of the frame is $u = [(f_i)] \in P^{i-1} = P^{i-1}_\sharp / N^{i-1}$, the first vectors of a frame in $P^i_\sharp|_u$ are constrained to project onto one of the linear frames (f_i) in $P^{i-1}_\sharp$, which belong to the equivalence class $u = [(f_i)]$;

b) the other vectors of a frame in $P^i_\sharp|_u$ must be vertical with respect to the projection $\pi^i : P^{i-1} \to P^{i-2}$;

c) any linear frame in $P^i_\sharp|_u$ is constrained by an appropriate set of normalising conditions; such conditions depend on which bundle $P^i_\sharp$ of linear frames we are considering – we refer to [14] for the explicit formulation of such normalising conditions.

Due to (a), the frames in $P^i_\sharp$ not only satisfy the normalising conditions quoted in (c), but also all conditions that are residuals of the three types of conditions for the frames in $P^{i-1}_\sharp$, in $P^{i-2}_\sharp$, etc. In particular, the frame field in $P^3_\sharp$ that gives the absolute parallelism (e_A, E_B) on $Q = P^2$ satisfies a set of conditions that inherits from the three types of constraints on the linear frames of the previous steps. Amongst such conditions one has that

1) (the real and imaginary parts of) the vector fields E_A are the infinitesimal transformations associated with (the real and imaginary parts of) the elements E^o_A, determined by the right action of H on Q; in particular, they are generators of the *vertical distribution* $\mathcal{V} \subset TQ$, i.e. the distribution of the tangent spaces of the fibres of $\pi : Q \to M$;

2) the distribution $\mathcal{H} \subset TQ$, generated by (the real and imaginary parts of) the vector fields e_A, is such that for any $u \in Q$ the projection $\pi_*|_u : T_uQ \to T_xM$, $x = \pi(u)$, gives a linear isomorphism $\pi_* : \mathcal{H}_u \to T_xM$ between $\mathcal{H}_u$ and T_xM;

3) for any $u \in Q$, the complex subspaces of $\mathcal{H}^{\mathbb{C}}_u$

$$\mathcal{D}^{10(\mathcal{H})}_u := \langle e_{-1(10)}|_u, e_{0(10)}|_u \rangle, \quad \mathcal{E}^{10(\mathcal{H})}_u := \langle e_{0(10)}|_u \rangle, \tag{14}$$

$$\mathcal{D}^{01(\mathcal{H})}_u := \langle e_{-1(01)}|_u, e_{0(01)}|_u \rangle, \quad \mathcal{E}^{01(\mathcal{H})}_u := \langle e_{0(01)}|_u \rangle, \tag{15}$$

project isomorphically onto the holomorphic spaces $\mathcal{D}^{10}_x \subset \mathcal{D}^{\mathbb{C}}_x$, $\mathcal{E}^{10}_x \subset \mathcal{E}^{\mathbb{C}}_x$, $x = \pi(u)$, and the antiholomorphic spaces $\mathcal{D}^{01}_x = \overline{\mathcal{D}^{10}_x}$, $\mathcal{E}^{01}_x = \overline{\mathcal{E}^{10}_x}$, respectively.

The other conditions correspond to constraints on the curvature 2-form κ of the Cartan connection and will be discussed in the next section.

3.3. The curvature constraints on the Cartan connection

Consider now the *curvature* 2*-form* κ of the Cartan connection (Q, ϖ), that is the $\mathfrak{so}_{3,2}$-valued 2-form on Q, defined by

$$\kappa := d\varpi + \tfrac{1}{2}[\varpi, \varpi].$$

From basic properties of Cartan connections and the fact that the vector fields E_A are infinitesimal transformations on Q, corresponding to the elements $E^o_A \in \mathfrak{h} = Lie(H)$, the expansion of κ in terms of the pointwise linearly independent 2-forms $(\vartheta^A \wedge \vartheta^B, \vartheta^A \wedge \omega^C, \omega^C \wedge \omega^D)$, determined by the dual coframe (11), has necessarily the form

$$\kappa = d\varpi + \tfrac{1}{2}[\varpi, \varpi] = \sum T^A_{BC} e^o_A \otimes \vartheta^B \wedge \vartheta^C + \sum R^D_{BC} E^o_D \otimes \vartheta^B \wedge \vartheta^C, \tag{16}$$

for appropriate (real and complex) functions T^A_{BC} and R^D_{BC}.

The curvature components T^A_{BC} and R^D_{BC} are determined by the Lie brackets of pairs of vector fields e_A as follows. Let us denote by $(\cdot)^{e^o_A} : \mathfrak{so}_{3,2} \to \langle e^o_A \rangle$ and $(\cdot)^{E^o_B} : \mathfrak{so}_{3,2} \longrightarrow \langle e^o_B \rangle$ the standard projections of $\mathfrak{so}_{3,2}$ along the vectors of the basis $\mathcal{B}^{CR}$ and by $\mathfrak{c}^A_{BC} := ([e^o_B, e^o_C])^{e^o_A}$ and $\mathfrak{d}^D_{BC} := ([e^o_B, e^o_C])^{E^o_D}$ the structure constants of $\mathfrak{so}_{3,2}$ in the basis $\mathcal{B}^{CR}$. Then, by Koszul formula for exterior derivatives and the definition of κ, we have

$$T^A_{BC} = d\vartheta^A(e_B, e_C) + ([e^o_B, e^o_C])^{e^o_A} = -\vartheta^A([e_B, e_C]) + \mathfrak{c}^A_{BC}, \tag{17}$$

(18) $$R^D_{BC} = d\omega^D(e_B, e_C) + ([e^o_B, e^o_C])^{E^o_D} = -\omega^D([e_B, e_C]) + \mathfrak{d}^D_{BC}.$$

By comparison with (1), we immediately see that, modulo the structures constants of $\mathfrak{so}_{3,2}$, *the curvature components T^A_{BC} and R^D_{BC} of the curvature 2-form κ are nothing but the structure functions of the absolute parallelism (e_A, E_B)*. In fact, this is a well known general fact on Cartan connections.

As mentioned above, besides the conditions (1) – (3) of §3.2, the absolute parallelism (e_A, E_B) is constrained by other normalising conditions. They are conditions on the Lie brackets between the vectors e_A and, through (17) and (18), they can all be expressed in terms of the curvature components T^A_{BC} and R^D_{BC}. For reader's convenience, we give here the complete list of such constraints and we refer to [14] for further details.

Integrability of the complex structure and involutivity of $\mathcal{E}^{\mathbb{C}}$.
From (17) and the fact that the complex distributions $\mathcal{E}^{10(\mathcal{H})} + \mathcal{E}^{01(\mathcal{H})}$ and $\mathcal{D}^{10(\mathcal{H})}$, defined in (14) and (15), project onto the involutive distributions $\mathcal{E}^{\mathbb{C}}$ and $\mathcal{D}^{10}$ of M, one has

(19) $$\begin{aligned} &T^A_{0(10)\,0(01)} = 0 \text{ for } A \in \{-2, -1(10), -1(01)\}, \\ &T^{A'}_{i(10)\,j(10)} = \overline{T^{A'}_{i(10)\,j(10)}} = 0 \text{ for } i, j \in \{-1, 0\},\ A' \in \{-2, -1(01), 0(01)\}. \end{aligned}$$

The distribution $\mathcal{E}^{10}$ is in the kernel of Levi forms.
From (17) and the fact that the spaces $\mathcal{E}^{10(\mathcal{H})}_u$, $\mathcal{E}^{01(\mathcal{H})}_u$ project into the kernel of the Levi form, one has

(20) $$T^{-2}_{-1(01)\,0(10)} = T^{-2}_{-1(10)\,0(01)} = 0.$$

Normalising conditions on the frames in $P^0_\sharp$.
From (17) and condition (5.2) in [14], one has

(21) $$T^{-2}_{-1(10)\,-1(01)} = T^{-1(10)}_{-1(01)\,0(10)} = T^{-1(01)}_{-1(10)\,0(01)} = 0.$$

Normalising conditions on the frames of $P^1_\sharp$.
From (17) and the normalising conditions in [14], given by formula (6.7) and condition $\beta_K = 0$ after Lemma 6.5 of that paper, one has

(22) $$\begin{aligned} &T^{-1(10)}_{-1(10)\,0(10)} = T^{-1(10)}_{-1(10)\,0(01)} = T^{-1(01)}_{-1(01)\,0(01)} = T^{-1(01)}_{-1(01)\,0(10)} = 0, \\ &T^{0(10)}_{-1(01)\,0(01)} = T^{0(01)}_{-1(10)\,0(10)} = 0. \end{aligned}$$

Property of the strongly adapted frames in $P^1_\sharp$.
From (17) and Lemma 6.6 (ii) in [14] one has

(23) $$T^{-2}_{-2\,0(10)} = T^{-2}_{-2\,0(01)} = 0.$$

Normalising conditions on the strongly adapted frames in $P^2_\sharp$.
From (17) and the normalising conditions (7.2) on γ_K in [14], one has

$$T^{0(10)}_{-1(10)\,0(10)} = T^{0(01)}_{-1(01)\,0(01)} = T^{0(10)}_{-1(10)\,0(01)} = T^{0(01)}_{-1(01)\,0(10)} = 0. \tag{24}$$

Normalising conditions on the strongly adapted frames in $P^3_\sharp$.
From (18) and the normalising conditions (8.2) on ε_K in [14], one has

$$R^{0(10)}_{-2\,0(10)} = R^{0(01)}_{-2\,0(01)} = R^{0(01)}_{-2\,0(10)} = R^{0(10)}_{-2\,0(01)} = 0. \tag{25}$$

Besides (19) – (25), the absolute parallelism is subjected to three further conditions of cohomological nature. They are

a) the condition given in (6.21) of [14], which is equivalent to a system of linear equations on $T^{-2}_{-2\,-1(10)}$, $T^{-2}_{-2\,-1(01)}$ and $T^{-1(10)}_{-1(10)\;\;-1(01)}$;

b) the condition given in (7.4) in [14], which is equivalent to a system of linear equations on $T^{-1(10)}_{-2\,-1(10)}$, $T^{-1(10)}_{-2\,-1(01)}$, $T^{0(10)}_{-1(10)\,-1(01)}$, $R^{0(10)}_{-1(10)\,-1(01)}$ and their complex conjugates;

c) the condition given in (8.4) in [14], which is equivalent to a system of linear equations on $T^{0(10)}_{-2\,-1(10)}$, $T^{0(10)}_{-2\,-1(01)}$, $R^{0(10)}_{-2(10)\,-1(01)}$, $R^{0(10)}_{-2\,-1(01)}$, $R^{1(10)}_{-1(10)\,-1(01)}$ and their complex conjugates.

The explicit expressions for the linear systems corresponding to the constraints (a), (b), (c) can be determined with straightforward computations. An exposition of such computations, which uses only elementary tools, can be found in the appendix. The result is that the constraints (a), (b) and (c) are equivalent to the linear equations

$$\text{(a)} \quad T^{-2}_{-2\,-1(10)} = T^{-2}_{-2\,-1(01)} = T^{-1(10)}_{-1(10)\,-1(01)} = 0, \tag{26}$$

$$\begin{aligned}\text{(b)} \quad & T^{-1(10)}_{-2\,-1(10)} = T^{-1(10)}_{-2\,-1(01)} = T^{0(10)}_{-1(10)\,-1(01)} = T^{-1(01)}_{-2\,-1(01)} = T^{-1(01)}_{-2\,-1(10)} = \\ & \qquad = T^{0(01)}_{-1(10)\,-1(01)} = R^{0(10)}_{-1(10)\,-1(01)} = R^{0(01)}_{-1(10)\,-1(01)} = 0,\end{aligned} \tag{27}$$

$$\begin{aligned}\text{(c)} \quad & \overline{R^{0(10)}_{-2\,-1(10)}} = -\frac{1}{2}T^{0(10)}_{-2\,-1(10)} - \frac{1}{2}R^{0(10)}_{-2\,-1(01)}, \\ & R^{1(10)}_{-1(10)\,-1(01)} = \frac{i}{2}T^{0(10)}_{-2\,-1(10)} - \frac{i}{2}R^{0(10)}_{-2\,-1(01)}\end{aligned} \tag{28}$$

and to the equations that follows from (28) by complex conjugation.

3.4. The structure equations of a girdled CR manifold

The projections of the values of the curvature κ along each element of the basis $\mathcal{B}^{CR}$ give explicit expressions for the exterior differentials $d\vartheta^A$ and $d\omega^B$ in terms of the pointwise linearly independent 2-forms $(\vartheta^A \wedge \vartheta^B, \vartheta^A \wedge \omega^C, \omega^C \wedge \omega^B)$, i.e. the *structure equations of the absolute parallelism* (e_A, E_B) (see §2). Here is the complete list of these structure equations, where we set equal to 0 all terms T^A_{BC} that are bound to vanish by the curvature constraints in §3.3.

$$d\vartheta^{-2} + \frac{i}{2}\vartheta^{-1(10)} \wedge \vartheta^{-1(01)} - \left(\omega^{0(10)} + \omega^{0(01)}\right) \wedge \vartheta^{-2} = 0, \tag{29}$$

$$\begin{aligned} d\vartheta^{-1(10)} - \vartheta^{0(10)} \wedge \vartheta^{-1(01)} - \omega^{0(10)} \wedge \vartheta^{-1(10)} + i\omega^{1(10)} \wedge \vartheta^{-2} = \\ = T^{-1(10)}_{-2\,0(10)} \vartheta^{-2} \wedge \vartheta^{0(10)} + T^{-1(10)}_{-2\,0(01)} \vartheta^{-2} \wedge \vartheta^{0(01)}, \end{aligned} \tag{30}$$

$$\begin{aligned} d\vartheta^{0(10)} - \left(\omega^{0(10)} - \omega^{0(01)}\right) \wedge \vartheta^{0(10)} + \frac{1}{2}\omega^{1(10)} \wedge \vartheta^{-1(10)} = \\ = T^{0(10)}_{-2\,-1(10)} \vartheta^{-2} \wedge \vartheta^{-1(10)} + T^{0(10)}_{-2\,-1(01)} \vartheta^{-2} \wedge \vartheta^{-1(01)} + \\ + T^{0(10)}_{-2\,0(10)} \vartheta^{-2} \wedge \vartheta^{0(10)} + T^{0(10)}_{-2\,0(01)} \vartheta^{-2} \wedge \vartheta^{0(01)} + T^{0(10)}_{-1(01)\,0(10)} \vartheta^{-1(01)} \wedge \vartheta^{0(10)} + \\ + T^{0(10)}_{0(10)\,0(01)} \vartheta^{0(10)} \wedge \vartheta^{0(01)}, \end{aligned} \tag{31}$$

$$\begin{aligned} d\omega^{0(10)} - \vartheta^{0(10)} \wedge \vartheta^{0(01)} + \frac{1}{2}\omega^{1(01)} \wedge \vartheta^{-1(10)} + \omega^2 \wedge \vartheta^{-2} = \\ = \underbrace{R^{0(10)}_{-2\,-1(10)}}_{\textbf{constrained by } (28)} \vartheta^{-2} \wedge \vartheta^{-1(10)} + R^{0(10)}_{-2\,-1(01)} \vartheta^{-2} \wedge \vartheta^{-1(01)} + \\ + R^{0(10)}_{-1(10)\,0(10)} \vartheta^{-1(10)} \wedge \vartheta^{0(10)} + R^{0(10)}_{-1(10)\,0(01)} \vartheta^{-1(10)} \wedge \vartheta^{0(01)} + \\ + R^{0(10)}_{-1(01)\,0(10)} \vartheta^{-1(01)} \wedge \vartheta^{0(10)} + R^{0(10)}_{-1(01)\,0(01)} \vartheta^{-1(01)} \wedge \vartheta^{0(01)} + \\ + R^{0(10)}_{0(10)\,0(01)} \vartheta^{0(10)} \wedge \vartheta^{0(01)}, \end{aligned} \tag{32}$$

$$\begin{aligned} d\omega^{1(10)} - \omega^{1(01)} \wedge \vartheta^{0(10)} - \omega^{1(10)} \wedge \omega^{0(01)} + i\omega^2 \wedge \vartheta^{-1(10)} = \\ = R^{1(10)}_{-2\,-1(10)} \vartheta^{-2} \wedge \vartheta^{-1(10)} + R^{1(10)}_{-2\,-1(01)} \vartheta^{-2} \wedge \vartheta^{-1(01)} + \\ + R^{1(10)}_{-2\,0(10)} \vartheta^{-2} \wedge \vartheta^{0(10)} + R^{1(10)}_{-2\,0(01)} \vartheta^{-2} \wedge \vartheta^{0(01)} + \\ + \underbrace{R^{1(10)}_{-1(10)\,-1(01)}}_{\textbf{constrained by } (28)} \vartheta^{-1(10)} \wedge \vartheta^{-1(01)} + \\ + R^{1(10)}_{-1(10)\,0(10)} \vartheta^{-1(10)} \wedge \vartheta^{0(10)} + R^{1(10)}_{-1(10)\,0(01)} \vartheta^{-1(10)} \wedge \vartheta^{0(01)} + \end{aligned} \tag{33}$$

$$+R^{1(10)}_{-1(01)\,0(10)}\vartheta^{-1(01)}\wedge\vartheta^{0(10)}+R^{1(10)}_{-1(01)\,0(01)}\vartheta^{-1(01)}\wedge\vartheta^{0(01)}+$$
$$+R^{1(10)}_{0(10)\,0(01)}\vartheta^{0(10)}\wedge\vartheta^{0(01)},$$

$$d\omega^2-\frac{i}{2}\omega^{1(10)}\wedge\omega^{1(01)}+\left(\omega^{0(10)}+\omega^{0(01)}\right)\wedge\omega^2= \tag{34}$$
$$=R^2_{-2-1(10)}\vartheta^{-2}\wedge\vartheta^{-1(10)}+R^2_{-2-1(01)}\vartheta^{-2}\wedge\vartheta^{-1(01)}+$$
$$+R^2_{-2\,0(10)}\vartheta^{-2}\wedge\vartheta^{0(10)}+R^2_{-2\,0(01)}\vartheta^{-2}\wedge\vartheta^{0(01)}+$$
$$+R^2_{-1(10)-1(01)}\vartheta^{-1(10)}\wedge\vartheta^{-1(01)}+$$
$$+R^2_{-1(10)\,0(10)}\vartheta^{-1(10)}\wedge\vartheta^{0(10)}+R^2_{-1(10)\,0(01)}\vartheta^{-1(10)}\wedge\vartheta^{0(01)}+$$
$$+R^2_{-1(01)\,0(10)}\vartheta^{-1(01)}\wedge\vartheta^{0(10)}+R^2_{-1(01)\,0(01)}\vartheta^{-1(01)}\wedge\vartheta^{0(01)}+$$
$$+R^2_{0(10)\,0(01)}\vartheta^{0(10)}\wedge\vartheta^{0(01)}.$$

3.5. Comparison with other absolute parallelisms

As we already mentioned, other canonical absolute parallelisms for girdled CR manifolds, not associated with Cartan connections, have been recently given in [13, 16]. Note also that the absolute parallelism in [16] is defined only for the girdled CR manifolds admitting no local equivalence with the homogeneous girdled CR manifold M_o.

Let us now focus on the canonical absolute parallelism $(P,(X_i),\widetilde{(\cdot)})$, defined in [13] for an arbitrary girdled CR manifold $(M,\mathcal{D},J)$. There, the bundle $\pi : P\to M$ has 5-dimensional fibers, but it has no natural structure of principal bundle over M. The absolute parallelism $(X_i)_{i=1}^{10}$ on P is associated with a dual coframes field, given by the real and imaginary parts of ten $\mathbb{C}$-valued 1-forms, denoted by $(\omega,\omega^1,\overline{\omega^1},\varphi^2,\overline{\varphi^2},\theta^2,\overline{\theta^2},\varphi^1,\overline{\varphi^1},\psi)$ and with ω and ψ taking only imaginary values.

Since the bundle P and the principal bundle Q of our Cartan connection have the same dimension, if we consider them as mere bundles with no further structures, we may locally identify them. From the construction of P, we may also assume that, under this identification, the 1-form ω is equal to $\omega=-2i\vartheta^{-2}$, where ϑ^{-2} is the 1-form of our parallelism, defined in §3.2.

We now recall that the 1-forms of the absolute parallelism in [13] are characterised by the fact that they satisfy an appropriate set of structure equations. The first two of this set are

$$d\omega=-\omega^1\wedge\omega^{\bar 1}-\omega\wedge(\varphi^2+\varphi^{\bar 2}), \tag{35}$$
$$d\omega^1=\theta^2\wedge\omega^{\bar 1}-\omega^1\wedge\varphi^2-\omega\wedge\varphi^1. \tag{36}$$

Comparing them with our structure equations (29) and (30) and through a tedious but straightforward computation, one can check that the equations (35) and (36) are satisfied by the 1-forms on $P \simeq Q$, defined by

$$
\begin{aligned}
&\omega := -2i\vartheta^{-2},\\
&\omega^1 := \vartheta^{-1(10)} - \overline{T^{-1(10)}_{-20(10)}}\vartheta^{-2},\\
&\varphi^2 := \omega^{0(10)} + \frac{i}{2}\overline{T^{-1(10)}_{-20(10)}}\vartheta^{-1(01)},\\
&\theta^2 := \vartheta^{0(10)} + i\overline{T^{-1(10)}_{-20(10)}}\left(\vartheta^{-1(10)} - \overline{T^{-1(10)}_{-20(10)}}\vartheta^{-2}\right),\\
&\varphi^1 := \frac{1}{2}\omega^{1(10)} - \frac{i}{2}T^{-1(10)}_{-20(01)}\vartheta^{0(01)} - \frac{1}{2}\overline{T^{-1(10)}_{-20(10)}}T^{-1(10)}_{-20(10)}\vartheta^{-1(10)} +\\
&\qquad + \frac{1}{4}\overline{T^{-1(10)}_{-20(10)}T^{-1(10)}_{-20(10)}}\vartheta^{-1(01)} - \frac{i}{2}\overline{T^{-1(10)}_{-20(10)}}\omega^{0(01)} - \frac{i}{2}d\overline{T^{-1(10)}_{-20(10)}}.
\end{aligned}
\tag{37}
$$

Now, we expect that if the 1-forms (37) are appropriately modified with additional terms involving $\omega^{1(10)}, \omega^{1(01)}$ and ω^2, they will satisfy not only the first two structure equations of the absolute parallelism in [13], but also all other structure equations of that parallelism.

On the basis of this expectation, the construction in [13] seems to start diverging from ours precisely when the absolute parallelism is required to satisfy (36). In fact, this is a constraint that amounts to impose that the curvature components $T^{-1(10)}_{-20(10)}$ and $T^{-1(10)}_{-20(01)}$ are absorbed into the definition of the vector fields of the absolute parallelism. Since these curvature components are not invariant under the right action of the structure group of $\pi : Q \to M$, the constraint given by (36) is plausibly one of the main reasons for the fact that the constructive process in [13] does not produce a Cartan connection.

An analogous comparison between our canonical Cartan connection and the parallelism in [16] might be done following the same line of arguments. We leave this task to the interested reader.

Appendix

A.1. The Cartan-Killing form of $\mathfrak{so}_{3,2}$

For the following computations, it turns out that the standard basis $\mathcal{B}^o$ of $\mathfrak{so}_{3,2}$, defined in (6), is not very convenient. In place of that basis, it is by far more useful to consider a new basis $\mathcal{B} = (f_\alpha)_{\alpha=1,\dots,10}$, with elements

$$
\begin{aligned}
& f_1 := \frac{1}{\sqrt{6}} e_{-2}, \quad f_2 := \frac{1}{\sqrt{6}}(e_{-1(10)} + e_{-1(01)}), \quad f_3 := \frac{i}{\sqrt{6}}(e_{-1(10)} - e_{-1(01)}),\\
& f_4 := \frac{1}{\sqrt{12}}(e_{0(10)} + e_{0(01)}), \quad f_5 := \frac{i}{\sqrt{12}}(e_{0(10)} - e_{0(01)}),\\
& f_6 := \frac{1}{\sqrt{12}}(E_{0(10)} + E_{0(01)}), \quad f_7 := \frac{i}{\sqrt{12}}(E_{0(10)} - E_{0(01)}),\\
& f_8 := \frac{1}{\sqrt{6}}(E_{1(10)} + E_{1(01)}), \quad f_9 := \frac{i}{\sqrt{6}}(E_{1(10)} - E_{1(01)}), \quad f_{10} := \frac{1}{\sqrt{6}} E^2.
\end{aligned}
\tag{A.1}
$$

The main motivation for considering such new basis comes from the fact the entries of the Cartan-Killing form $\langle \cdot, \cdot \rangle$ of $\mathfrak{so}_{3,2}$ in this basis are equal to ± 1 or 0. More precisely, using Table 1 in [14], one can check that the components of $\langle \cdot, \cdot \rangle$ in the basis $\mathcal{B}$ are

$$
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&1&0\\
0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&-1&0&0&0\\
0&1&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0
\end{pmatrix}.
\tag{A.2}
$$

A.2. The space $\ker \partial^*|_{C^2_1(\mathfrak{m},\mathfrak{g})}$

We now want to show that the action of the codifferential ∂^* of $\mathfrak{so}_{3,2}$ on the bilinear maps of shifting degree $+1$ in $\mathrm{Hom}(Ł^2\mathfrak{m}_-,\mathfrak{g})$, $\mathfrak{m}_- := \mathfrak{g}_{-2} + \mathfrak{g}_{-1}$, has trivial kernel. As above, we denote by $\langle \cdot, \cdot \rangle$ the Cartan-Killing form of $\mathfrak{so}_{3,2}$, by $\mathcal{B} = (f_\alpha)$ the basis defined in (A.1) and by $\mathcal{B}^* = (f^\alpha)$ its dual basis. Finally, for each element $f_\alpha \in \mathcal{B}$, we denote by $\widehat{f_\alpha}$ the unique element in $\mathcal{B}$ such that $f^\alpha = \pm\langle \widehat{f_\alpha}, \cdot \rangle$ and by $\widehat{f}^\alpha$ the corresponding dual element in $\mathcal{B}^*$. The rest of the notation is taken from [14].
Consider a bilinear map $\tau \in \mathrm{Hom}(Ł^2\mathfrak{m}_-,\mathfrak{g})$ of shifting degree $+1$:

$$
\tau = \tau^1_{12} f_1 \otimes (f^1 \wedge f^2) + \tau^1_{13} f_1 \otimes (f^1 \wedge f^3) + \tau^2_{23} f_2 \otimes (f^2 \wedge f^3) + \tau^3_{23} f_3 \otimes (f^2 \wedge f^3).
\tag{A.3}
$$

By definition of ∂^*, this tensor is in $\ker \partial^*$ if and only if

$$
\langle \partial^* \tau, A \rangle = -\langle \tau, \partial A \rangle = 0
\tag{A.4}
$$

for any $A = A^\alpha_\beta f_\alpha \otimes f^\beta \in \mathrm{Hom}(\mathfrak{h},\mathfrak{g})$. From (A.3), equation (A.4) is equivalent to a linear equation on the τ^i_{jk} whose non trivial coefficients are

$$
\widehat{f}^1(\partial A(\widehat{f}_1,\widehat{f}_2)), \quad \widehat{f}^1(\partial A(\widehat{f}_1,\widehat{f}_3)), \quad \widehat{f}^2(\partial A(\widehat{f}_2,\widehat{f}_3)), \quad \widehat{f}^3(\partial A(\widehat{f}_2,\widehat{f}_3)).
$$

The computation of $\widehat{f}^1(\partial A(\widehat{f}_1,\widehat{f}_2))$ is straightforward and gives

$$\begin{aligned}\widehat{f}^1(\partial A(\widehat{f}_1,\widehat{f}_2)) &= f^{10}(\partial A(f_{10},f_8)) =\\ &= f^{10}\Big(f_{10}\cdot A(f_8)-f_8\cdot A(f_{10})-A([f_{10},f_8])\Big) =\\ &= A^\alpha_8 f^{10}(\mathrm{ad}_{f_{10}}(f_\alpha))-A^\alpha_{10}f^{10}(\mathrm{ad}_{f_8}(f_\alpha))-f^{10}(A([f_{10},f_8]))=\\ &= -\frac{1}{\sqrt{3}}A^6_8+\frac{1}{\sqrt{6}}A^9_{10}.\end{aligned}$$

Similar computations give all other coefficients of the equations and (A.4) reduces to

$$\begin{aligned}&\tau^1_{12}\big(-\frac{1}{\sqrt{3}}A^6_8+\frac{1}{\sqrt{6}}A^9_{10}\big)+\tau^1_{13}\big(-\frac{1}{\sqrt{3}}A^6_9-\frac{1}{\sqrt{6}}A^8_{10}\big)+\\ &+\tau^2_{23}\big(-\frac{1}{2\sqrt{3}}A^6_9-\frac{1}{2\sqrt{3}}A^4_9+\frac{1}{2\sqrt{3}}A^7_8+\frac{1}{2\sqrt{3}}A^5_8+\frac{1}{\sqrt{6}}A^8_{10}\big)+\\ &+\tau^3_{23}\big(\frac{1}{2\sqrt{3}}A^7_9-\frac{1}{2\sqrt{3}}A^5_9+\frac{1}{2\sqrt{3}}A^6_8-\frac{1}{2\sqrt{3}}A^4_8+\frac{1}{\sqrt{6}}A^9_{10}\big)=0.\end{aligned}\tag{A.5}$$

By arbitrariness of A, it follows that $\tau\in\ker\partial^*$ if and only if $\tau^1_{12}=\tau^1_{13}=\tau^2_{23}=\tau^3_{23}=0$, meaning that $\ker\partial^*|_{C^2_1(\mathfrak{m},\mathfrak{g})}=0$, as claimed.

A.3. The space $(\partial\mathfrak{l}^1)^\perp$

We recall that, according to Lemma 6.5 in [14] and the definition of the (abelian) group L^1, the abelian Lie algebra $\mathfrak{l}^1=Lie(L^1)$ can be identified with the real vector space generated by the linear maps

$$\begin{aligned}&B_1:=e_{-1(10)}\otimes e^{-2}+e_{-1(01)}\otimes e^{-2},\\ &B_2:=ie_{-1(10)}\otimes e^{-2}-ie_{-1(01)}\otimes e^{-2},\\ &B_3:=\big(e_{0(10)}-E_{0(01)}\big)\otimes e^{-1(10)}+\big(e_{0(01)}-E_{0(10)}\big)\otimes e^{-1(01)},\\ &B_4:=i\big(e_{0(10)}-E_{0(01)}\big)\otimes e^{-1(10)}-i\big(e_{0(01)}-E_{0(10)}\big)\otimes e^{-1(01)},\\ &B_5:=i\big(e_{0(10)}+E_{0(01)}\big)\otimes e^{-1(10)}-i\big(e_{0(01)}+E_{0(10)}\big)\otimes e^{-1(01)},\\ &B_6:=\big(e_{0(10)}+E_{0(01)}\big)\otimes e^{-1(10)}-\big(e_{0(01)}+E_{0(10)}\big)\otimes e^{-1(01)},\\ &B_7:=\big(E_{0(10)}+E_{0(01)}\big)\otimes e^{-1(10)}+\big(E_{0(10)}+E_{0(01)}\big)\otimes e^{-1(01)},\\ &B_8:=i\big(E_{0(10)}+E_{0(01)}\big)\otimes e^{-1(10)}-i\big(E_{0(10)}+E_{0(01)}\big)\otimes e^{-1(01)}.\end{aligned}\tag{A.6}$$

We also recall that the elements $\tau^1\in Tor^1(\mathfrak{m})$ have the form

$$\begin{aligned}\tau^1=\tau^{-2}_{-2-1(10)}e_{-2}\otimes e^{-2}\wedge e^{-1(10)}+\overline{\tau^{-2}_{-2-1(10)}}e_{-2}\otimes e^{-2}\wedge e^{-1(01)}+\\ +\tau^{-1(10)}_{-1(10)-1(01)}e_{-1(10)}\otimes e^{-1(10)*}\wedge e^{-1(01)*}+\\ +\overline{\tau^{-1(01)}_{-1(10)-1(01)}}e_{-1(01)}\otimes e^{-1(10)*}\wedge e^{-1(01)*}.\end{aligned}$$

Now, let us choose as $\mathrm{ad}_{E^0_2}$-invariant inner product on the space $Tor^1(\mathfrak{m})$ the sum of an arbitrary inner product on $\mathfrak{m}_{-2}$ and the standard Hermitian inner product of $\mathbb{C} \simeq \mathfrak{m}_{-1}$. This implies that, in order to determine the subspace $(\partial\mathfrak{l}^1)^{\perp} \subset Tor^1(\mathfrak{m})$, the only relevant components of the generators ∂B_i of $\partial\mathfrak{l}^1$ are the components

$$(\partial B_i)^{-2}_{-2-1(10)} := e^{-2}\left(\partial B_i(e_{-2}, e_{-1(10)})\right),$$
$$(\partial B_i)^{-1(10)}_{-1(10)-1(01)} := e^{-1(10)}\left(\partial B_i(e_{-1(10)}, e_{-1(01)})\right).$$

We now observe that

$$(\partial B_3)^{-2}_{-2-1(10)} = e^{-2}\left([e_{-2}, B_3(e_{-1(10)})] - [e_{-1(10)}, B_3(e_{-2})]\right) = -1,$$
$$(\partial B_4)^{-2}_{-2-1(10)} = e^{-2}\left([e_{-2}, B_4(e_{-1(10)})] - [e_{-1(10)}, B_4(e_{-2})]\right) = -i,$$

meaning that $\partial\mathfrak{l}^1$ contains the 2-dimensional real subspace generated by

$$e_{-2} \otimes e^{-2} \wedge e^{-1(10)} + \overline{e_{-2} \otimes e^{-2} \wedge e^{-1(10)}},$$
$$ie_{-2} \otimes e^{-2} \wedge e^{-1(10)} - \overline{ie_{-2} \otimes e^{-2} \wedge e^{-1(10)}}.$$

This yields that if $\tau^1 \in (\partial\mathfrak{l}^1)^{\perp}$, then $\tau^{-2}_{-2-1(10)} = \overline{\tau^{-2}_{-2-1(10)}} = 0$. Similar computations show that

$$(\partial B_1)^{-1(10)}_{-1(10)-1(01)} = -\frac{i}{2}, \qquad (\partial B_2)^{-1(01)}_{-1(10)-1(01)} = -\frac{1}{2},$$

hence that $\partial\mathfrak{l}^1$ contains the 2-dimensional real subspace generated by

$$e_{-1(10)} \otimes e^{-1(10)} \wedge e^{-1(01)} + \overline{e_{-1(10)} \otimes e^{-1(10)} \wedge e^{-1(01)}},$$
$$ie_{-1(10)} \otimes e^{-1(10)} \wedge e^{-1(01)} - \overline{ie_{-1(10)} \otimes e^{-1(10)} \wedge e^{-1(01)}},$$

and therefore that if $\tau^1 \in (\partial\mathfrak{l}^1)^{\perp}$, then $\tau^{-1(10)}_{-1(10)-1(01)} = \overline{\tau^{-1(10)}_{-1(10)-1(01)}} = 0$. We therefore conclude that $(\partial\mathfrak{l}^1)^{\perp} = 0$.

Since $\ker\partial^*|_{C^2_1(\mathfrak{m},\mathfrak{g})}$ is trivial as well (see §A.2), condition (6.21) in [14] is equivalent to requiring that *the c-torsion c^1_K is identically equal to* 0.

A.4. The space $\ker\partial^*|_{C^2_2(\mathfrak{m}_-,\mathfrak{g})}$

Here, we want show that the space of the bilinear maps in $\mathrm{Hom}(Ł^2\mathfrak{m}_-, \mathfrak{g})$ of shifting degree $+2$ that are in $\ker\partial^*|_{C^2_2(\mathfrak{m},\mathfrak{g})}$, is trivial. This amount to say that condition (7.4) of [14] reduces to $c^2_K = 0$.

A bilinear map $\tau \in \mathrm{Hom}(Ł^2\mathfrak{m}_-, \mathfrak{g})$ of shifting degree $+2$ has the form

$$\tau = \tau^2_{12} f_2 \otimes (f^1 \wedge f^2) + \tau^3_{12} f_3 \otimes (f^1 \wedge f^2) + \tau^2_{13} f_2 \otimes (f^1 \wedge f^3) + \tau^3_{13} f_3 \otimes (f^1 \wedge f^3) + \\ + \tau^4_{23} f_4 \otimes (f^2 \wedge f^3) + \tau^5_{23} f_5 \otimes (f^2 \wedge f^3) + \tau^6_{23} f_6 \otimes (f^2 \wedge f^3) + \tau^7_{23} f_7 \otimes (f^2 \wedge f^3). \tag{A.7}$$

As in §A.2, this tensor is in $\ker \partial^*$ if and only if $\langle \partial^* \tau, A\rangle = -\langle \tau, \partial A\rangle = 0$ for any element $A = A^{\alpha}_{\beta} f_\alpha \otimes f^\beta \in \mathrm{Hom}(\mathfrak{h}, \mathfrak{g})$. By (A.7), this corresponds to a linear equation on the components τ^i_{jk} with coefficients

$$\begin{aligned}
&\widehat{f}^2(\partial A(\widehat{f}_1, \widehat{f}_2)), \quad \widehat{f}^3(\partial A(\widehat{f}_1, \widehat{f}_2)), \quad \widehat{f}^2(\partial A(\widehat{f}_1, \widehat{f}_3)), \quad \widehat{f}^3(\partial A(\widehat{f}_1, \widehat{f}_3)),\\
&\widehat{f}^4(\partial A(\widehat{f}_2, \widehat{f}_3)), \quad \widehat{f}^5(\partial A(\widehat{f}_2, \widehat{f}_3)), \quad \widehat{f}^6(\partial A(\widehat{f}_2, \widehat{f}_3)), \quad \widehat{f}^7(\partial A(\widehat{f}_2, \widehat{f}_3)).
\end{aligned}$$

With the same computations of §A.2, we compute all these coefficients and get that the equation $\langle \tau, \partial A\rangle = 0$ has the explicit expression

$$\begin{aligned}
&\tau^2_{12}\left(-\frac{1}{\sqrt{6}}A^3_8 + \frac{1}{2\sqrt{3}}A^4_{10} + \frac{1}{2\sqrt{3}}A^6_{10}\right) + \tau^3_{12}\left(\frac{1}{\sqrt{6}}A^2_8 + \frac{1}{2\sqrt{3}}A^5_{10} - \frac{1}{2\sqrt{3}}A^7_{10}\right) +\\
&+\tau^2_{13}\left(-\frac{1}{\sqrt{6}}A^3_9 + \frac{1}{2\sqrt{3}}A^5_{10} + \frac{1}{2\sqrt{3}}A^7_{10}\right) + \tau^3_{13}\left(\frac{1}{\sqrt{6}}A^2_9 - \frac{1}{2\sqrt{3}}A^4_{10} + \frac{1}{2\sqrt{3}}A^6_{10}\right) +\\
&+\tau^4_{23}\left(\frac{1}{2\sqrt{3}}A^2_9 + \frac{1}{2\sqrt{3}}A^3_8 + \frac{1}{\sqrt{6}}A^4_{10}\right) + \tau^5_{23}\left(\frac{1}{2\sqrt{3}}A^3_9 - \frac{1}{2\sqrt{3}}A^2_8 + \frac{1}{\sqrt{6}}A^5_{10}\right) +\\
&+\tau^6_{23}\left(\frac{1}{2\sqrt{3}}A^2_9 - \frac{1}{2\sqrt{3}}A^3_8 + \frac{1}{\sqrt{6}}A^6_{10}\right) + \tau^7_{23}\left(-\frac{1}{2\sqrt{3}}A^3_9 - \frac{1}{2\sqrt{3}}A^2_8 - \frac{1}{\sqrt{6}}A^7_{10}\right) = 0.
\end{aligned} \tag{A.8}$$

Since this has to be satisfied for each A, factoring the components of A we get that $\tau \in \ker \partial^*|_{C^2_2(\mathfrak{m},\mathfrak{g})}$ if and only if its components satisfy the system

$$\begin{aligned}
&\tau^3_{12} - \frac{1}{\sqrt{2}}\tau^5_{23} - \frac{1}{\sqrt{2}}\tau^7_{23} = 0, \quad \tau^2_{12} - \frac{1}{\sqrt{2}}\tau^4_{23} + \frac{1}{\sqrt{2}}\tau^6_{23} = 0,\\
&\tau^3_{13} + \frac{1}{\sqrt{2}}\tau^4_{23} + \frac{1}{\sqrt{2}}\tau^6_{23} = 0, \quad \tau^2_{13} - \frac{1}{\sqrt{2}}\tau^5_{23} + \frac{1}{\sqrt{2}}\tau^7_{23} = 0,\\
&\tau^4_{23} + \frac{1}{\sqrt{2}}\tau^2_{12} - \frac{1}{\sqrt{2}}\tau^3_{13} = 0, \quad \tau^5_{23} + \frac{1}{\sqrt{2}}\tau^3_{12} + \frac{1}{\sqrt{2}}\tau^2_{13} = 0,\\
&\tau^6_{23} + \frac{1}{\sqrt{2}}\tau^2_{12} + \frac{1}{\sqrt{2}}\tau^3_{13} = 0, \quad \tau^7_{23} + \frac{1}{\sqrt{2}}\tau^3_{12} - \frac{1}{\sqrt{2}}\tau^2_{13} = 0.
\end{aligned}$$

A simple check shows that this system has just the trivial solution. This means that $\ker \partial^*|_{C^2_2(\mathfrak{m},\mathfrak{g})} = 0$ and that (7.4) of [14] is equivalent to $c^2_K = 0$.

A.5. The space $\ker \partial^*|_{C^2_3(\mathfrak{m}_-,\mathfrak{g})}$

In this section we determine explicitly the bilinear maps in $\mathrm{Hom}(Ł^2\mathfrak{m}_-, \mathfrak{g})$ of shifting degree $+3$ that are in $\ker \partial^*$.

A bilinear map $\tau \in \mathrm{Hom}(Ł^2\mathfrak{m}_-, \mathfrak{g})$ of shifting degree $+3$ has the form

$$\begin{aligned}
\tau = {}&\tau^4_{12} f_4 \otimes (f^1 \wedge f^2) + \tau^5_{12} f_5 \otimes (f^1 \wedge f^2) + \tau^6_{12} f_6 \otimes (f^1 \wedge f^2) + \tau^7_{12} f_7 \otimes (f^1 \wedge f^2) +\\
&+\tau^4_{13} f_4 \otimes (f^1 \wedge f^3) + \tau^5_{13} f_5 \otimes (f^1 \wedge f^3) + \tau^6_{13} f_6 \otimes (f^1 \wedge f^3) + \tau^7_{13} f_7 \otimes (f^1 \wedge f^3) +\\
&+\tau^8_{23} f_8 \otimes (f^2 \wedge f^3) + \tau^9_{23} f_9 \otimes (f^2 \wedge f^3).
\end{aligned} \tag{A.9}$$

As in the previous sections, this is in $\ker \partial^*$ if and only if $\langle \partial^* \tau, A\rangle = -\langle \tau, \partial A\rangle = 0$ for any $A = A^{\alpha}_{\beta} f_\alpha \otimes f^\beta \in \mathrm{Hom}(\mathfrak{h}, \mathfrak{g})$. From (A.9), this condition is a linear equation in the components τ^i_{jk} with coefficients

$$\begin{aligned}
&\widehat{f}^4(\partial A(\widehat{f}_1,\widehat{f}_2)), \quad \widehat{f}^5(\partial A(\widehat{f}_1,\widehat{f}_2)), \quad \widehat{f}^6(\partial A(\widehat{f}_1,\widehat{f}_2)), \quad \widehat{f}^7(\partial A(\widehat{f}_1,\widehat{f}_2)),\\
&\widehat{f}^4(\partial A(\widehat{f}_1,\widehat{f}_3)), \quad \widehat{f}^5(\partial A(\widehat{f}_1,\widehat{f}_3)), \quad \widehat{f}^6(\partial A(\widehat{f}_1,\widehat{f}_3)), \quad \widehat{f}^7(\partial A(\widehat{f}_1,\widehat{f}_3)),\\
&\widehat{f}^8(\partial A(\widehat{f}_2,\widehat{f}_3)), \quad \widehat{f}^9(\partial A(\widehat{f}_2,\widehat{f}_3)).
\end{aligned}$$

We explicitly compute these coefficients with the same standard computations indicated in §A.2. Then we obtain that the condition $\langle \tau, \partial A\rangle = 0$ has the explicit form

$$\begin{aligned}
&\tau^4_{12}\left(-\frac{1}{2\sqrt{3}}A^2_{10}\right) + \tau^5_{12}\left(-\frac{1}{2\sqrt{3}}A^3_{10}\right) + \tau^6_{12}\left(\frac{1}{\sqrt{3}}A^1_8 - \frac{1}{2\sqrt{3}}A^2_{10}\right) +\\
&+ \tau^7_{12}\left(\frac{1}{2\sqrt{3}}A^3_{10}\right) + \tau^4_{13}\left(\frac{1}{2\sqrt{3}}A^3_{10}\right) + \tau^5_{13}\left(-\frac{1}{2\sqrt{3}}A^2_{10}\right) +\\
&+ \tau^6_{13}\left(\frac{1}{\sqrt{3}}A^1_9 - \frac{1}{2\sqrt{3}}A^3_{10}\right) + \tau^7_{13}\left(-\frac{1}{2\sqrt{3}}A^2_{10}\right) +\\
&+ \tau^8_{23}\left(\frac{1}{\sqrt{6}}A^1_8 - \frac{1}{\sqrt{6}}A^2_{10}\right) + \tau^9_{23}\left(\frac{1}{\sqrt{6}}A^1_9 - \frac{1}{\sqrt{6}}A^3_{10}\right) = 0.
\end{aligned} \tag{A.10}$$

Since this needs to hold for each A, factoring the components of A we get that $\tau \in \ker \partial^*|_{C^2_3(\mathfrak{m},\mathfrak{g})}$ if and only if its components satisfy the system for

$$\begin{aligned}
&\frac{1}{2\sqrt{3}}\tau^4_{12} + \frac{1}{2\sqrt{3}}\tau^6_{12} + \frac{1}{2\sqrt{3}}\tau^5_{13} + \frac{1}{2\sqrt{3}}\tau^7_{13} - \frac{1}{\sqrt{6}}\tau^8_{23} = 0,\\
&-\frac{1}{2\sqrt{3}}\tau^5_{12} + \frac{1}{2\sqrt{3}}\tau^7_{12} + \frac{1}{2\sqrt{3}}\tau^4_{13} - \frac{1}{2\sqrt{3}}\tau^6_{13} + \frac{1}{\sqrt{6}}\tau^9_{23} = 0,\\
&\frac{1}{\sqrt{3}}\tau^6_{12} + \frac{1}{\sqrt{6}}\tau^8_{23} = 0, \qquad \frac{1}{\sqrt{3}}\tau^6_{13} + \frac{1}{\sqrt{6}}\tau^9_{23} = 0.
\end{aligned}$$

Using the last two equations to simplify the first two, the system reduces to

$$\begin{aligned}
&\tau^4_{12} + \tau^5_{13} = -3\tau^6_{12} - \tau^7_{13}, \qquad \tau^4_{13} - \tau^5_{12} = 3\tau^6_{13} - \tau^7_{12},\\
&\tau^8_{23} = -\sqrt{2}\,\tau^6_{12}, \qquad \tau^9_{23} = -\sqrt{2}\,\tau^6_{13}.
\end{aligned} \tag{A.11}$$

This means that the space $\ker \partial^*|_{C^2_3(\mathfrak{m},\mathfrak{g})}$ is 6-dimensional and that condition (8.4) of [14] correspond to a system of linear equations on the curvature components

$$T^{0(10)}_{-2\,-1(10)} \; T^{0(10)}_{-2\,-1(01)}, \; R^{0(10)}_{-2\,-1(10)}, \; R^{0(10)}_{-2\,-1(01)}, \; R^{1(10)}_{-1(10)\,-1(01)}.$$

In order to make explicit these equations, we have to convert the system (A.11) on the components of τ in the basis $\mathcal{B}$ into a system on the components of τ in the standard CR basis $\mathcal{B}^{CR}$. For this purpose, we recall that

$$\begin{aligned}
&e_{-2} = \sqrt{6} f_1, \quad e_{-1(10)} = \frac{\sqrt{6}}{2}(f_2 - i f_3), \quad e_{-1(01)} = \frac{\sqrt{6}}{2}(f_2 + i f_3),\\
&e_{0(10)} = \frac{\sqrt{12}}{2}(f_4 - i f_5), \quad e_{0(01)} = \frac{\sqrt{12}}{2}(f_4 + i f_5), \quad E_{0(10)} = \frac{\sqrt{12}}{2}(f_6 - i f_7),\\
&e_{0(01)} = \frac{\sqrt{12}}{2}(f_6 + i f_7), \quad E_{1(10)} = \frac{\sqrt{6}}{2}(f_8 - i f_9), \quad E_{1(01)} = \frac{\sqrt{6}}{2}(f_8 + i f_9)
\end{aligned}$$

and that, for the dual vectors,

$$e^{0(10)} = \frac{1}{\sqrt{12}}(f^4 + if^5), \quad e^{0(01)} = \frac{1}{\sqrt{12}}(f^4 - if^5), \quad E^{0(10)} = \frac{1}{\sqrt{12}}(f^6 + if^7),$$
$$E^{0(01)} = \frac{1}{\sqrt{12}}(f^6 - if^7), \quad E^{1(10)} = \frac{1}{\sqrt{6}}(f^8 + if^9), \quad E^{1(01)} = \frac{1}{\sqrt{6}}(f^8 - if^9).$$

From this we get that

$$\text{(A.12)} \qquad T^{0(10)}_{-2\,-1(10)} = \frac{\sqrt{3}}{2}(f^4 + if^5)(\tau(f_1, f_2) - i\tau(f_1, f_3)) = \\ = \frac{\sqrt{3}}{2}(\tau^4_{12} + \tau^5_{13}) + i\frac{\sqrt{3}}{2}(-\tau^4_{13} + \tau^5_{12}),$$

$$\text{(A.13)} \qquad T^{0(10)}_{-2\,-1(01)} = \frac{\sqrt{3}}{2}(f^4 + if^5)(\tau(f_1, f_2) + i\tau(f_1, f_3)) = \\ = \frac{\sqrt{3}}{2}(\tau^4_{12} - \tau^5_{13}) + i\frac{\sqrt{3}}{2}(\tau^4_{13} + \tau^5_{12}),$$

$$\text{(A.14)} \qquad R^{0(10)}_{-2\,-1(10)} = \frac{\sqrt{3}}{2}(f^6 + if^7)(\tau(f_1, f_2) - i\tau(f_1, f_3)) = \\ = \frac{\sqrt{3}}{2}(\tau^6_{12} + \tau^7_{13}) + i\frac{\sqrt{3}}{2}(-\tau^6_{13} + \tau^7_{12}),$$

$$\text{(A.15)} \qquad R^{0(10)}_{-2\,-1(01)} = \frac{\sqrt{3}}{2}(f^6 + if^7)(\tau(f_1, f_2) + i\tau(f_1, f_3)) = \\ = \frac{\sqrt{3}}{2}(\tau^6_{12} - \tau^7_{13}) + i\frac{\sqrt{3}}{2}(\tau^6_{13} + \tau^7_{12}),$$

$$\text{(A.16)} \qquad R^{1(10)}_{-1(10)\,-1(01)} = \frac{\sqrt{3}}{\sqrt{2}}(f^8 + if^9)(i\tau(f_2, f_3)) = \sqrt{\frac{3}{2}}(\tau^8_{23} + i\tau^9_{23}).$$

From this we see that

$$\frac{1}{\sqrt{3}}(\overline{R^{0(10)}_{-2\,-1(10)}} + R^{0(10)}_{-2\,-1(01)}) = \tau^6_{12} + i\tau^6_{13}, \; -i\frac{\sqrt{2}}{\sqrt{3}}R^{1(10)}_{-1(10)\,-1(01)} = \tau^8_{23} + i\tau^9_{23},$$

which yields that the last equation in (A.11) is equivalent to

$$\text{(A.17)} \qquad R^{1(10)}_{-1(10)\,-1(01)} = -i\overline{R^{0(10)}_{-2\,-1(10)}} - iR^{0(10)}_{-2\,-1(01)}.$$

On the other hand, since

$$\frac{2}{\sqrt{3}}T^{0(10)}_{-2\,-1(10)} = (\tau^4_{12} + \tau^5_{13}) - i(\tau^4_{13} - \tau^5_{12}),$$
$$\frac{2}{\sqrt{3}}\left(\overline{R^{0(10)}_{-2\,-1(10)}} + R^{0(10)}_{-2\,-1(01)}\right) + \frac{2}{\sqrt{3}}\overline{R^{0(10)}_{-2\,-1(10)}} = 3\tau^6_{12} + \tau^7_{13} + i(3\tau^6_{13} - \tau^7_{12}),$$

we immediately see that the first two equations in (A.11) are equivalent to

$$T^{0(10)}_{-2\,-1(10)} = -2\overline{R^{0(10)}_{-2\,-1(10)}} - R^{0(10)}_{-2\,-1(01)}. \tag{A.18}$$

Rearranging in an appropriate way the equations (A.17) and (A.18), we conclude that condition (8.4) in [14] is equivalent to the following equalities:

$$\begin{aligned} \overline{R^{0(10)}_{-2\,-1(10)}} &= -\frac{1}{2}T^{0(10)}_{-2\,-1(10)} - \frac{1}{2}R^{0(10)}_{-2\,-1(01)}, \\ R^{1(10)}_{-1(10)\,-1(01)} &= \frac{i}{2}T^{0(10)}_{-2\,-1(10)} - \frac{i}{2}R^{0(10)}_{-2\,-1(01)}. \end{aligned} \tag{A.19}$$

References

[1] D. V. ALEKSEEVSKY AND A. SPIRO, *Prolongations of Tanaka structures and regular CR structures* in "Selected topics in Cauchy-Riemann geometry", Quad. Mat., 9, Dept. Math., Seconda Univ. Napoli, Caserta 2001.

[2] V. BELOSHAPKA, V. EZHOV AND G. SCHMALZ, *Canonical Cartan connection and holomorphic invariants on Engel CR manifolds*, Russ. J. Math. Phys. **14** (2007), 121–133.

[3] D. BURNS JR, K. DIEDERICH AND S. SHNIDER, *Distinguished curves in pseudoconvex boundaries*, Duke Math. J. **44** (1977), 407–431.

[4] A. ČAP AND A. R. GOVER, *Tractor Calculi for Parabolic Geometries*, Trans. Amer. Math. Soc. **354** (2002), 1511–1548.

[5] A. ČAP AND H. SCHICHL, *Parabolic Geometries and Canonical Cartan Connections*, Hokkaido Math. J. **29** (2000), 453–505.

[6] A. ČAP, J. SLOVAK AND V. SOUČEK, *Invariant operators on manifolds with almost Hermitian symmetric structures, I. Invariant differentiation*, Acta Math. Univ. Commenianae **66** (1997), 33–69.

[7] A.ČAP, J. SLOVAK AND V. SOUČEK, *Invariant operators on manifolds with almost Hermitian symmetric structures, II. Normal Cartan connections*, Acta Math. Univ. Commenianae **66** (1997), 203–220.

[8] A. ČAP, J. SLOVAK AND V. SOUČEK, *Invariant operators on manifolds with almost Hermitian symmetric structures, III. Standard operators*, Diff. Geom. Appl. **12** (2000), 51–84.

[9] S.S. CHERN AND J. MOSER, *Real Hypersurfaces in Complex Manifolds*, Acta Math. **133** (1974), 219–271; Erratum in Acta Math. **150** (1983), 297.

[10] V. EZHOV, A. ISAEV AND G. SCHMALZ, *Invariants of elliptic and hyperbolic CR-structures of codimension 2*, Internat. J. Math. **10** (1999), 1–52.

[11] M. FREEMAN, *Local biholomorphic straightening of real submanifolds*, Ann. of Math. **106** (1977), 319–352.

[12] S. KOBAYASHI, *Transformation Groups in Differential Geometry*, Springer, New York 1972.

[13] A. ISAEV AND D. ZAITSEV, *Reduction of Five-Dimensional Uniformly Levi Degenerate CR Structures to Absolute Parallelisms*, J. Geom. Anal. **23** (2013), 1571–1605.

[14] C. MEDORI AND A. SPIRO, *The equivalence problem for five-dimensional Levi degenerate CR manifolds*, Int. Math. Res. Not., IMRN **20** (2014), 5602–5647.

[15] T. MORIMOTO, *Geometric structures on filtered manifolds*, Hokkaido Math. J. **22** (1993), 263–347.

[16] S. POCCHIOLA, *Explicit absolute parallelism for* 2*-nondegenerate real hypersurfaces $M^5 \subset \mathbb{C}^3$ of constant Levi rank* 1, preprint, arXiv:1312.6400 (2013).

[17] F. PODESTÀ AND A. SPIRO, *Introduzione ai Gruppi di Trasformazione*, Preprint Series of the Department V. Volterra of the University of Ancona, Ancona 1996.

[18] G. SCHMALZ AND J. SLOVÁK, *The Geometry of Hyperbolic and Elliptic CR manifolds of codimension two*, Asian J. Math. **4** (2000), 565–598.

[19] G. SCHMALZ AND A. SPIRO, *Normal systems of coordinates on manifolds of Chern-Moser type*, J. Korean Math. Soc. **40** (2003), 461–486.

[20] G. SCHMALZ AND A. SPIRO, *Explicit construction of a Chern-Moser connection for CR manifolds of codimension two*, Ann. Mat. Pura Appl. (4) **185** (2006), 337–379.

[21] A. G. SERGEEV AND V. S. VLADIMIROV, *Complex Analysis in the Future Tube*, in "Several Complex Variables II (eds. G. M. Khenkin, A. G. Vitushkin)", Encyclopaedia of Mathematical Sciences, Springer, Berlin 1994.

[22] R. W. SHARPE, *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program*, Springer, New York 1997.

[23] A. SPIRO, *The structure equations of a complex Finsler manifold*, Asian J. Math. **5** (2001), 291–326.

[24] S. STERNBERG, *Lectures on Differential Geometry*, Prentice-Hall, Englewood Cliff NJ 1964.

[25] N. TANAKA, *On differential systems, graded Lie algebras and pseudo-groups*, J. Math. Kyoto Univ. **10** (1970), 1–82.

[26] N. TANAKA, *On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections*, Japan J. Math. **2** (1976), 131–190.

[27] N. TANAKA, *On the equivalence problems associated with simple graded Lie algebras*, Hokkaido Math. J. **8** (1979), 23–84.

[28] D. K. WISE, *MacDowell-Mansouri gravity and Cartan geometry*, Classical Quantum Gravity **27** (2010), 155010, 26pp.

AMS Subject Classification: 32V05, 32V35, 53C10, 53C05

Costantino MEDORI,
Dipartimento di Matematica e Informatica,
Università di Parma,
Parma, ITALY
e-mail: `costantino.medori@unipr.it`

Andrea SPIRO,
Scuola di Scienze e Tecnologie,
Università di Camerino,
Camerino, ITALY
e-mail: `andrea.spiro@unicam.it`

Lavoro pervenuto in redazione il 26.10.2015 e accettato il 27.01.2016.

Geom. Struc. on Riem. Man.-Bari
Vol. 73/1, 3–4 (2015), 151 – 159

P. Mutlu and Z. Şentürk

ON WALKER TYPE IDENTITIES LOCALLY CONFORMAL KAEHLER SPACE FORMS

Abstract. The notion of locally conformal Kaehler manifold (l.c.K-manifold) in Hermitian Geometry has been introduced by I. Vaisman in 1976. In this work, we present results on l.c.K-space forms satisfying curvature identities named Walker type identities.

1. Introduction

Let (M,g,J) be a real $m = 2n$-dimensional Hermitian manifold with the structure (J, g), where J is the almost complex structure and g is the Hermitian metric. Then

$$J^2 = -Id, \qquad g(JX,JY) = g(X,Y),$$

for any vector fields X and Y tangent to M. The fundamental 2-form Ω is defined by

$$\Omega(X,Y) = g(JX,Y) = -\Omega(Y,X).$$

The manifold M is called a *locally conformal Kaehler manifold (an l.c.K-manifold)* if each point x in M has an open neighborhood U with a positive differentiable function $\rho : U \to \mathbb{R}$ such that

$$g^* = e^{-2\rho} g \mid_{\mathrm{U}}$$

is a Kaehlerian metric on U. Especially, if we can take $U = M$, then the manifold M is said to be *globally conformal Kaehler*.

A Hermitian manifold whose metric is locally conformal to a Kaehler metric is called an l.c.K-manifold. I. Vaisman gives its characterization as follows [10] :

A Hermitian manifold M is an l.c.K-manifold if and only if there exists on M a global closed 1-form α such that

$$d\Omega = 2\alpha \wedge \Omega \, ,$$

where α is called the *Lee form*.

A Hermitian manifold (M,g,J) is an l.c.K-manifold if and only if

$$\text{(A.1)} \qquad \nabla_k J_{ij} = -\beta_i g_{kj} + \beta_j g_{ki} - \alpha_i J_{kj} + \alpha_j J_{ki} \, ,$$

where

$$\beta_j = -\alpha_r J_j^r.$$

An l.c.K-manifold M is called *an l.c.K-space form* if the holomorphic sectional curvature of the section $\{X,JX\}$ at each point of M has a constant value. Let M(c) be an l.c.K-space form with constant holomorphic sectional curvature c, then the Riemannian curvature tensor R_{ijhk} with respect to g_{ij} has the form [8]

$$\begin{aligned} R_{hijk} &= \frac{c}{4}(g_{hk}g_{ij} - g_{hj}g_{ik} + J_{hk}J_{ij} - J_{hj}J_{ik} - 2J_{hi}J_{jk}) \\ &+ \frac{3}{4}(P_{hk}g_{ij} + P_{ij}g_{hk} - P_{hj}g_{ik} - P_{ik}g_{hj}) \\ &- \frac{1}{4}(\widetilde{P}_{hk}J_{ij} + \widetilde{P}_{ij}J_{hk} - \widetilde{P}_{hj}J_{ik} - \widetilde{P}_{ik}J_{hj} - 2\widetilde{P}_{hi}J_{jk} - 2\widetilde{P}_{jk}J_{hi}), \end{aligned} \tag{A.2}$$

where $\widetilde{P}_{ij} = -P_{ir}J^r_j$,

$$P_{ij} = -\nabla_i\alpha_j - \alpha_i\alpha_j + \frac{\|\alpha\|^2}{2}g_{ij} \tag{A.3}$$

is hybrid, *i.e.*,$P_{ir}J^r_j + P_{jr}J^r_i = 0$, $P_{ij} = P_{ji}$ and $\|\alpha\|$ denotes the length of Lee form. Contracting (A.2) with g^{hk}, we have

$$S_{ij} = \frac{1}{4}\{(m+2)c + 3\,trP\}g_{ij} + \frac{3}{4}(m-4)P_{ij}\ , \tag{A.4}$$

where S denotes the Ricci tensor with respect to g.

PROPOSITION 1. *[9] If the tensor field P is hybrid and the trace of the tensor field P is constant in a 4-dimensional l.c.K-space form M(c), then M(c) is Einstein.*

THEOREM 1. *[9] A real m-dimensional ($m \neq 4$) l.c.K-space form $M(c)$ in which the tensor field P is hybrid and the trace of the tensor field P is constant is Einstein if and only if the tensor field P is proportional to g.*

2. Preliminaries

Let (M,g) be an n-dimensional, $n \geq 3$, semi-Riemannian connected manifold of class C^∞ with Levi-Civita connection ∇. The Ricci operator $\mathcal{S}$ is defined by $g(\mathcal{S}X,Y) = S(X,Y)$, where X, Y $\in \Xi(M)$, $\Xi(M)$ being the Lie algebra of vector fields on M.
We define the endomorphisms $X \wedge_A Y$, $\mathcal{R}(X,Y)Z$ and $\mathcal{C}(X,Y)$ of $\Xi(M)$ by

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y\ , \tag{A.5}$$

$$\mathcal{R}(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\ , \tag{A.6}$$

$$\begin{aligned} \mathcal{C}(X,Y)Z &= \mathcal{R}(X,Y)Z \\ &- \frac{1}{n-2}(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1}X \wedge_g Y)Z\ , \end{aligned} \tag{A.7}$$

respectively, where X, Y, Z $\in \Xi(M)$, A is a symmetric (0,2)-tensor, κ the scalar curvature and $[X,Y]$ is the Lie bracket of vector fields X and Y. In particular we have $(X \wedge_g Y) = X \wedge Y$.

The Riemannian-Christoffel curvature tensor R, the Weyl conformal curvature tensor C and the (0,4)-tensor G of (M,g) are defined by

$$\begin{aligned} R(X_1,X_2,X_3,X_4) &= g(\mathcal{R}(X_1,X_2)X_3,X_4)\,, \\ C(X_1,X_2,X_3,X_4) &= g(\mathcal{C}(X_1,X_2)X_3,X_4)\,, \\ G(X_1,X_2,X_3,X_4) &= g((X_1\wedge_g X_2)X_3,X_4)\,, \end{aligned} \tag{A.8}$$

respectively, where $X_1,X_2,X_3,X_4 \in \Xi(M)$.
A tensor $\mathcal{B}$ of type (1,3) on M is said to be a *generalized curvature tensor* if

$$\begin{aligned} \sum_{X_1,X_2,X_3} \mathcal{B}(X_1,X_2)X_3 &= 0\,, \\ \mathcal{B}(X_1,X_2)+\mathcal{B}(X_2,X_1) &= 0\,, \\ B(X_1,X_2,X_3,X_4) &= B(X_3,X_4,X_1,X_2)\,, \end{aligned} \tag{A.9}$$

where $B(X_1,X_2,X_3,X_4) = g(\mathcal{B}(X_1,X_2)X_3,X_4)$.
For symmetric (0,2)-tensor E and F we define their Kulkarni-Nomizu product $E\wedge F$ by

$$\begin{aligned} (E\wedge F)(X_1,X_2,X_3,X_4) &= E(X_1,X_4)F(X_2,X_3)+E(X_2,X_3)F(X_1,X_4) \\ &- E(X_1,X_3)F(X_2,X_4)-E(X_2,X_4)F(X_1,X_3). \end{aligned}$$

For a symmetric (0,2)-tensor E and a (0,k)-tensor T, $k\geq 2$, we define their Kulkarni-Nomizu product $E\wedge T$ by [3]

$$\begin{aligned} (E\wedge T)(X_1,X_2,X_3,X_4;Y_3,...,Y_k) &= E(X_1,X_4)T(X_2,X_3,Y_3,...,Y_k) \\ &+ E(X_2,X_3)T(X_1,X_4,Y_3,...,Y_k) \\ &- E(X_1,X_3)T(X_2,X_4,Y_3,...,Y_k) \\ &- E(X_2,X_4)T(X_1,X_3,Y_3,...,Y_k). \end{aligned}$$

For symmetric (0, 2)-tensor E and F we have [6]

$$Q(E,E\wedge F) = -Q(F,\bar{E}), \tag{A.10}$$

where $\bar{E} = \frac{1}{2}E\wedge E$. We also have [7]

$$E\wedge Q(E,F) = -Q(F,\bar{E}). \tag{A.11}$$

For a (0,k)-tensor field T, $k\geq 1$, a (0,2)-tensor field A and a generalized curvature tensor $\mathcal{B}$ on (M,g), we define the tensors $B\cdot T$ and $Q(A,T)$ by

$$\begin{aligned} (B\cdot T)(X_1,...,X_k;X,Y) &= -T(\mathcal{B}(X,Y)X_1,X_2,...,X_k) \\ &- ...-T(X_1,X_2,...,X_{k-1},\mathcal{B}(X,Y)X_k), \end{aligned} \tag{A.12}$$

$$\begin{aligned} Q(A,T)(X_1,...,X_k;X,Y) &= -T((X\wedge_A Y)X_1,X_2,...,X_k) \\ &- ...-T(X_1,X_2,...,X_{k-1},(X\wedge_A Y)X_k), \end{aligned} \tag{A.13}$$

respectively, where $X, Y, X_1, X_2, ..., X_k \in \Xi(M)$.
Putting in the above formulas $\mathcal{B} = \mathcal{R}$ or $\mathcal{B} = \mathcal{C}$, $T = R$ or $T = C$ or $T = S$, $A = g$ or $A = S$, we obtain the tensors $R \cdot R$, $R \cdot C$, $R \cdot S$, $C \cdot S$, Q (g, R), Q (S, R), Q (g, C), Q (g, S) and Q (S, C) respectively.
Let (M, g) be covered by a system of charts $\{W; x^k\}$. We define by g_{ij}, R_{hijk}, S_{ij}, $G_{hijk} = g_{hk}g_{ij} - g_{hj}g_{ik}$ and

$$\begin{aligned} C_{hijk} &= R_{hijk} - \frac{1}{n-2}(g_{hk}S_{ij} - g_{hj}S_{ik} + g_{ij}S_{hk} - g_{ik}S_{hj}) \\ &+ \frac{\kappa}{(n-1)(n-2)} G_{hijk}\,, \end{aligned} \tag{A.14}$$

the local components of the metric tensor g, the Riemannian-Christoffel curvature tensor R, the Ricci tensor S, the tensor G and the Weyl tensor C, respectively. Further, we denote by $S_{ij} = S_{ir}g^r_j$ and $S^j_i = g^{jr}S_{ir}$.
The local components of the (0,6)-tensors $R \cdot T$ and Q (g, T) on M are the following:

$$(R \cdot T)_{hijklm} = g^{rs}(T_{rijk}R_{shlm} + T_{hrjk}R_{silm} + T_{hirk}R_{sjlm} + T_{hijr}R_{sklm}), \tag{A.15}$$

$$\begin{aligned} Q(g,T)_{hijklm} &= -g_{mh}T_{lijk} - g_{mi}T_{hljk} - g_{mj}T_{hilk} - g_{mk}T_{hijl} \\ &+ g_{lh}T_{mijk} + g_{li}T_{hmjk} + g_{lj}T_{himk} + g_{lk}T_{hijm}\,, \end{aligned} \tag{A.16}$$

where T_{hijk} are the local components of the tensor T.
In this part we present some considerations leading to the definition of Deszcz Symmetric (Pseudosymmetric in the sense of Deszcz) and Ricci-pseudosymmetric manifolds.
A semi-Riemannian manifold (M, g) satisfying the condition $\nabla R = 0$ is said to be *locally symmetric*. Locally symmetric manifolds form a subclass of the class of manifolds characterized by the condition

$$R \cdot R = 0. \tag{A.17}$$

Semi-Riemannian manifolds fulfilling (A.17) are called *semisymmetric*. Here $R \cdot R$ is a (0,6)-tensor with components

$$\begin{aligned} (R \cdot R)_{hijklm} &= \nabla_m \nabla_l R_{hijk} - \nabla_l \nabla_m R_{hijk} \\ &= g^{rs}(R_{rijk}R_{shlm} + R_{hrjk}R_{silm} + R_{hirk}R_{sjlm} + R_{hijr}R_{sklm}). \end{aligned} \tag{A.18}$$

A more general class of manifolds than the class of semisymmetric manifolds is the class of Deszcz Symmetric manifolds.
A semi-Riemannian manifold (M, g) is said to be *Deszcz Symmetric* [2] if at every point of M the condition

$$R \cdot R = L_R Q(g, R) \tag{A.19}$$

holds on the set $\mathcal{U}_R = \{x \in M \mid R - \frac{\kappa}{n(n-1)} G \neq 0 \text{ at } x\}$, where L_R is some function on $\mathcal{U}_R$. There exist various examples of Deszcz Symmetric manifolds which are not semisymmetric.

A semi-Riemannian manifold is said to be *Ricci-semisymmetric* if we have $R \cdot S = 0$ on M.

A semi-Riemannian manifold (M, g) is said to be *Ricci-pseudosymmetric* ([2], [5]) if at every point of M the condition

$$R \cdot S = L_S Q(g, S) \tag{A.20}$$

holds on the set $\mathcal{U}_S = \{x \in M \mid S - \frac{\kappa}{n} g \neq 0 \; at \; x\}$, where L_S is some function on $\mathcal{U}_S$. The class of Ricci-pseudosymmetric manifolds is an extension of the class of Ricci-semisymmetric manifolds as well as of the class of pseudosymmetric manifolds. Every pseudosymmetric manifold is Ricci-pseudosymmetric. The converse statement is not true. Evidently, every Ricci-semisymmetric $(R \cdot S = 0)$ is Ricci-pseudosymmetric. There exist various examples of Ricci-pseudosymmetric manifolds which are not pseudosymmetric.

3. On Walker Type Identities Locally Conformal Kaehler Space Forms

In this section, we present results on l.c.K-space forms satisfying curvature identities named Walker type identities.

LEMMA 1. *[1] For a symmetric (0,2)-tensor A and a generalized curvature tensor $\mathcal{B}$ on a semi-Riemannian manifold (M,g)*, $n \geq 3$, *we have*

$$Q(A, \mathcal{B})_{hijklm} + Q(A, \mathcal{B})_{jklmhi} + Q(A, \mathcal{B})_{lmhijk} = 0. \tag{A.21}$$

It is well-known that the following identity

$$(R \cdot R)_{hijklm} + (R \cdot R)_{jklmhi} + (R \cdot R)_{lmhijk} = 0 \tag{A.22}$$

holds on any semi-Riemannian manifold. The equation (A.22) is called *the Walker type identity*.

On any semi-Riemannian manifold (M, g), $n \geq 4$, the following three identities are equivalent to each other [4]:

$$(R \cdot C)_{hijklm} + (R \cdot C)_{jklmhi} + (R \cdot C)_{lmhijk} = 0, \tag{A.23}$$

$$(C \cdot R)_{hijklm} + (C \cdot R)_{jklmhi} + (C \cdot R)_{lmhijk} = 0 \tag{A.24}$$

and

$$(R \cdot C - C \cdot R)_{hijklm} + (R \cdot C - C \cdot R)_{jklmhi} + (R \cdot C - C \cdot R)_{lmhijk} = 0. \tag{A.25}$$

The equations (A.23) - (A.25) are named *the Walker type identities*. We also can consider the following Walker type identity

$$(C \cdot C)_{hijklm} + (C \cdot C)_{jklmhi} + (C \cdot C)_{lmhijk} = 0. \tag{A.26}$$

THEOREM 2. *Let M(c) be a 4-dimensional l.c.K-space form, such that the tensor field P is hybrid and the trace of the tensor field P is constant. Then the Walker type identities (A.23) - (A.25) and (A.26) hold on M(c).*

Proof. In view of (A.15) , we have

$$(R\cdot C)_{hijklm} = g^{rs}(C_{rijk}R_{shlm} + C_{hrjk}R_{silm} + C_{hirk}R_{sjlm} + C_{hijr}R_{sklm}), \tag{A.27}$$

$$(C\cdot R)_{hijklm} = g^{rs}(R_{rijk}C_{shlm} + R_{hrjk}C_{silm} + R_{hirk}C_{sjlm} + R_{hijr}C_{sklm}). \tag{A.28}$$

Using (A.14) in (A.27) we obtain

$$\begin{aligned}
(R\cdot C)_{hijklm} &= (R\cdot R)_{hijklm} - \frac{1}{(m-2)}\Big[R_{hklm}S_{ij} - R_{jhlm}S_{ik} + R_{jilm}S_{hk} \\
&- R_{kilm}S_{hj} - R_{hjlm}S_{ik} + R_{ijlm}S_{hk} + R_{khlm}S_{ij} - R_{iklm}S_{hj} \\
&+ g_{ij}S^s_kR_{shlm} + g_{hk}S^s_jR_{silm} + g_{hk}S^s_iR_{sjlm} + g_{ij}S^s_hR_{sklm} \\
&- g_{ik}S^s_jR_{shlm} - g_{hj}S^s_kR_{silm} - g_{ik}S^s_hR_{sjlm} - g_{hj}S^s_iR_{sklm}\Big] \\
&+ \frac{\kappa}{(m-1)(m-2)}\Big[R_{hklm}g_{ij} - R_{jhlm}g_{ik} + R_{jilm}g_{hk} \\
&- R_{kilm}g_{hj} + R_{ijlm}g_{hk} - R_{hjlm}g_{ik} + R_{hklm}g_{ij} - R_{iklm}g_{hj}\Big] \\
&= (R\cdot R)_{hijklm} - \frac{1}{(m-2)}\Big[g_{ij}(A_{khlm} + A_{hklm}) + g_{hk}(A_{jilm} + A_{ijlm}) \\
&- g_{ik}(A_{jhlm} + A_{hjlm}) - g_{hj}(A_{kilm} + A_{iklm})\Big],
\end{aligned} \tag{A.29}$$

where

$$A_{mijk} = S^s_mR_{sijk}. \tag{A.30}$$

Applying, in the same way, (A.14) in (A.28) we get

$$\begin{aligned}
(C\cdot R)_{hijklm} &= (R\cdot R)_{hijklm} - \frac{1}{(m-2)}Q(S,R)_{hijklm} \\
&+ \frac{\kappa}{(m-1)(m-2)}Q(g,R)_{hijklm} \\
&- \frac{1}{(m-2)}(g_{lh}A_{mijk} - g_{mh}A_{lijk} - g_{li}A_{mhjk} + g_{mi}A_{lhjk} \\
&+ g_{lj}A_{mkhi} - g_{mj}A_{lkhi} - g_{kl}A_{mjhi} + g_{km}A_{ljhi}).
\end{aligned} \tag{A.31}$$

Substituting (A.4) into (A.29) and (A.31) we get

$$(R\cdot C) = (R\cdot R) - \frac{\beta}{(m-2)}T \tag{A.32}$$

and

$$
\begin{aligned}
(C\cdot R) &= (R\cdot R) - \frac{(m-2)\alpha - \beta\, trP}{(m-2)(m-1)} Q(g,R) - \frac{\beta}{(m-2)} Q(P,R) \\
&- \frac{\beta}{(m-2)} \acute{T}, \qquad \text{(A.33)}
\end{aligned}
$$

where T and $\acute{T}$ are the (0,6)-tensor fields whose local components are given by

$$
\begin{aligned}
T_{hijklm} &= g_{ij}(E_{khlm} + E_{hklm}) + g_{hk}(E_{jilm} + E_{ijlm}) \\
&- g_{ik}(E_{jhlm} + E_{hjlm}) - g_{hj}(E_{kilm} + E_{iklm}), \qquad \text{(A.34)}
\end{aligned}
$$

$$
\begin{aligned}
\acute{T}_{hijklm} &= \Big(g_{lh}E_{mijk} - g_{mh}E_{lijk} - g_{li}E_{mhjk} + g_{mi}E_{lhjk} \\
&+ g_{lj}E_{mkhi} - g_{mj}E_{lkhi} - g_{kl}E_{mjhi} + g_{km}E_{ljhi} \Big), \qquad \text{(A.35)}
\end{aligned}
$$

$E_{hijk} = P^s_h R_{sijk}$, $\alpha = \frac{1}{4}\{(m+2)c + 3\, trP\}$ and $\beta = \frac{3}{4}(m-4)$.
Then by direct computation, one obtains:

$$
\sum_{(X_1,X_2),(X_3,X_4),(X,Y)} (T + \acute{T})(X_1,X_2,X_3,X_4,X,Y) = 0.
$$

Hence (A.23) (equivalently (A.24), (A.25)) holds if and only if

$$
\frac{\beta}{(m-2)} \sum_{(X_1,X_2),(X_3,X_4),(X,Y)} T(X_1,X_2,X_3,X_4,X,Y) = 0.
$$

In particular, if $m = 4$, then $\beta = 0$, so (A.23) holds.
Further, we note that (A.14) turns into $C = R - \frac{2c+trP}{4} G$. This gives

$$
\begin{aligned}
C\cdot C &= C\cdot(R - \frac{2c+trP}{4} G) = C\cdot R \\
&= (R - \frac{2c+trP}{4} G)\cdot R = R\cdot R - \frac{2c+trP}{4} Q(g,R). \qquad \text{(A.36)}
\end{aligned}
$$

Now using (A.21) and (A.22) we complete the proof. □

THEOREM 3. *Let M(c) be an m-dimensional* $(m > 4)$ *l.c.K-space form. If the tensor field P is proportional to g and the trace of the tensor field P is constant, then the Walker type identities (A.23) - (A.25) and (A.26) hold on M(c).*

Proof. In view of Theorem 1., the m-dimensional l.c.K-space form $M(c)$ is Einstein, so (A.23) - (A.25) hold on $M(c)$. Substituting $S = \frac{1}{4}\Big\{(m+2)c + \frac{6(m-2)}{m} trP\Big\} g$ into (A.14), we have

$$
C = R - \frac{1}{4(m-1)} \left\{(m+2)c + \frac{6(m-2)}{m} trP\right\} G.
$$

This gives

$$\begin{aligned} C\cdot C &= C\cdot\Big(R-\frac{1}{4(m-1)}\Big\{(m+2)c+\frac{6(m-2)}{m}trP\Big\}\,G\Big)\\ &= C\cdot R\\ &= \Big(R-\frac{1}{4(m-1)}\Big\{(m+2)c+\frac{6(m-2)}{m}trP\Big\}\,G\Big)\cdot R\\ &= R\cdot R-\frac{1}{4(m-1)}\Big\{(m+2)c+\frac{6(m-2)}{m}trP\Big\}\,Q(g,R). \end{aligned}$$

Using (A.21) and (A.22), we get the result. □

LEMMA 2. *Let M(c) be an m-dimensional* $(m>4)$ *l.c.K-space form such that the tensor field P is hybrid. Then, we have*

$$(m-2)\Big((R\cdot C)_{hijklm}+(R\cdot C)_{jklmhi}+(R\cdot C)_{lmhijk}\Big)$$

$$=-\beta\Big((g\wedge(R\cdot P))_{hijklm}+(g\wedge(R\cdot P))_{jklmhi}+(g\wedge(R\cdot P))_{lmhijk}\Big). \quad \text{(A.37)}$$

Proof. Substituting (A.4) into (A.14), we obtain

$$C=R-\frac{\beta}{m-2}(g\wedge P)-\frac{\alpha(m-2)-\beta\,trP}{(m-1)(m-2)}G, \quad \text{(A.38)}$$

where $\alpha=\frac{1}{4}\{(m+2)c+3\,trP\}$ and $\beta=\frac{3}{4}(m-4)$.
From (A.38), we get

$$R\cdot C=R\cdot R-\frac{\beta}{m-2}g\wedge(R\cdot P). \quad \text{(A.39)}$$

Using (A.22) the proof is completed. □

LEMMA 3. *If one of the Walker type identities (A.23) - (A.25) holds on an m-dimensional* $(m>4)$ *l.c.K-space form M(c) and the tensor field P is hybrid, then on M(c) we have*

$$(g\wedge(R\cdot P))_{hijklm}+(g\wedge(R\cdot P))_{jklmhi}+(g\wedge(R\cdot P))_{lmhijk}=0. \quad \text{(A.40)}$$

Proof. Lemma 2. completes the proof. □

THEOREM 4. *On any Ricci-pseudosymmetric l.c.K-space form M(c),* $(m>4)$ *such that the tensor field P is hybrid, the Walker type identities (A.23) - (A.25) hold on* $\mathcal{U}_S\subset M$.

Proof. In view of (A.20) and (A.4), m-dimensional Ricci-pseudosymmetric $(m>4)$ l.c.K-space forms satisfy

$$R\cdot P=L_S Q(g,P). \quad \text{(A.41)}$$

Using (A.41) and (A.37), we obtain the following identity on $\mathcal{U}_S$

$$(m-2)\Big((R\cdot C)_{hijklm}+(R\cdot C)_{jklmhi}+(R\cdot C)_{lmhijk}\Big)$$

$$=-\beta L_S\Big((g\wedge Q(g,P))_{hijklm}+(g\wedge Q(g,P))_{jklmhi}+(g\wedge Q(g,P))_{lmhijk}\Big).$$

Making use of (A.11) and (A.21), we obtain on $\mathcal{U}_S$

$$(m-2)\Big((R\cdot C)_{hijklm}+(R\cdot C)_{jklmhi}+(R\cdot C)_{lmhijk}\Big)$$

$$=-\beta L_S\Big(Q(P,G)_{hijklm}+Q(P,G)_{jklmhi}+Q(P,G)_{lmhijk}\Big)=0.$$

Hence (A.23) (equivalently (A.24), (A.25)) holds on M(c). □

Acknowledgements. The authors would like to thank the referee for his/her valuable comments.

References

[1] DEPREZ J., DESZCZ R., VERSTRAELEN L., *Examples of pseudo-symmetric conformally flat warped products*, Chinese J. Math. **17** (1989), 51–65.

[2] DESZCZ R., *On pseudosymmetric spaces*, Bull. Soc. Math. Belg. Sér. **A 44** (1992), 1–34.

[3] DESZCZ R. AND GŁOGOWSKA M., *Some nonsemisymmetric Ricci-semisymmetric warped product hypersurfaces*, Publ. Inst. Math.(Beograd) (N.S.) **72** 86 (2002), 81–93.

[4] DESZCZ R., GŁOGOWSKA M., VERSTRAELEN L., *On some generalized Einstein metric conditions on hypersurfaces in semi-Riemannian space forms*, Colloq. Math. **96** (2003), 149–166.

[5] DESZCZ R. AND HOTLOS M., *Remarks on Riemannian manifolds satisfying a certain curvature condition imposed on the Ricci tensor*, Prace. Nauk. Pol. Szczec. **11** (1989), 23–34.

[6] DESZCZ R., VERSTRAELEN L., VRANCKEN L., *On the symmetry of warped product spacetimes*, Gen. Relativity Gravitation **23** (1991), 671–681.

[7] DESZCZ R. AND VERSTRAELEN L., *Hypersurfaces of semi-Riemannian conformally flat manifolds*, Geometry and Topology of Submanifolds, III, World Sci., River Edge, NJ 1991, 131–147.

[8] KASHIWADA T., *Some properties of locally conformal Kaehler manifolds*, Hokkaido Math. J. **8** (1978), 191–198.

[9] MATSUMOTO K., *Locally conformal Kaehler manifolds and their submanifolds*, Mem. Secţ. Ştiinţ. Acad. Română Ser. IV **14** 2 (1991), 7–49.

[10] VAISMAN I., *On locally conformal almost Kaehler manifolds*, Israel J. of Math. **24** (1976), 338–351.

AMS Subject Classification: 53B20, 53B25, 53B30, 53B50, 53C25

Pegah MUTLU, Zerrin ŞENTÜRK ,
Mathematics Engineering Department, Istanbul Technical University
Maslak, TR-34469, Istanbul, TURKEY
e-mail: `sariaslani@itu.edu.tr, senturk@itu.edu.tr`

Lavoro pervenuto in redazione il 13.11.2015 e accettato il 16.02.2016.

Geom. Struc. on Riem. Man.-Bari
Vol. 73/1, 3–4 (2015), 161 – 170

A. I. Nistor

MOTION OF CHARGED PARTICLES IN A KILLING MAGNETIC FIELD IN $\mathbb{H}^2 \times \mathbb{R}$

Dedicated to Prof. Anna Maria Pastore on the occasion of her 70th birthday

Abstract. In this paper we study the equations of motion for charged particles in a magnetic field generated by the Killing vector fields of $\mathbb{H}^2 \times \mathbb{R}$. Furthermore, we obtain some explicit parametrizations of such magnetic curves.

1. Introduction

The importance of the study of the motion of charged particles in certain magnetic or electric fields is given by the good physical understanding of some dynamical processes. Usually, these fields are known as functions depending on position and time. In the following we will consider only the action of time-independent magnetic fields and vanishing electric fields, keeping the problems in the so-called magnetostatics theory.

From the point of view of Physics, according to [1], if we consider a particle of charge $\mathbf{e}$, under the action of the Lorentz force $\mathbf{F}$ generated by the magnetic field $\mathbf{B}$, the equation of motion (Lorentz equation) can be written as:

$$(*) \qquad \frac{d\mathbf{p}}{dt} = \mathbf{F} = \mathbf{e}(\mathbf{v} \times \mathbf{B}),$$

where $\mathbf{p}$ denotes the momentum of the particle and $\mathbf{v}$ is its velocity vector. We know that the momentum $\mathbf{p}$ is collinear to $\mathbf{v}$, more precisely we have the relation:

$$(**) \qquad \mathbf{p} = m\,\mathbf{v} \Big/ \sqrt{1 - \frac{v^2}{c^2}},$$

where $v = |\mathbf{v}|$ and c is the light speed. Because of equations $(*)$ and $(**)$ we observe that $\frac{d\mathbf{p}}{dt}$ is orthogonal to $\mathbf{p}$ and therefore $|\mathbf{p}|$ is constant. Consequently, the velocity v and the energy $\varepsilon = mc^2 / \sqrt{1 - \frac{v^2}{c^2}}$ are both constant. Inserting $(**)$ in $(*)$, the equation of motion becomes $\dfrac{d\mathbf{v}}{dt} = \phi(\mathbf{v}(t))$, where ϕ is a skew-symmetric operator such that $\phi(\mathbf{v})$ is orthogonal to $\mathbf{v}$.

This problem can be formulated in Riemannian geometry as follows. On a Riemannian manifold $(M, \tilde{g})$ a magnetic field is defined as a closed 2-form F. The corresponding Lorentz force ϕ is a skew-symmetric operator given by $\tilde{g}(\phi(X), Y) = F(X,Y)$, for all $X, Y \in \mathfrak{X}(M)$. Notice that F and ϕ are metrically equivalent, and moreover, they are physically equivalent, because ϕ is obtained from F raising its second index [1]. A magnetic trajectory is defined as a smooth curve γ on M which satisfies the Lorentz equation $\tilde{\nabla}_{\dot\gamma}\dot\gamma = \phi(\dot\gamma)$, where $\tilde{\nabla}$ is the Levi-Civita connection on M. If we deal with a null Lorentz force, the magnetic trajectories are the geodesics of the ambient space. Hence, the geodesics can be regarded as magnetic trajectories described by particles moving freely, only under the action of gravity.

The aim of this paper is to investigate the equations of motion for charged particles under the action of Killing magnetic fields in the homogeneous 3-space $\mathbb{H}^2\times\mathbb{R}$, in order to complete the study started for such magnetic fields in the Euclidean 3-space [3] and in the product space $\mathbb{S}^2\times\mathbb{R}$ [7].

2. Preliminaries

Let $\gamma : I\subset\mathbb{R}\to\mathbb{H}^2\times\mathbb{R}$ be a smooth curve in the product space $\mathbb{H}^2\times\mathbb{R}$, where $\mathbb{H}^2$ denotes the hyperbolic plane and $\mathbb{R}$ denotes the real line. In the following we work with the the upper half-plane model of the hyperbolic plane, that is we consider

$$\mathbb{H}^2 = \{(x,y)\in\mathbb{R}^2, y>0\},$$

endowed with the metric $g_{\mathbb{H}} = \frac{dx^2+dy^2}{y^2}$, of constant Gaussian curvature -1. It is well known that the geodesics in $\mathbb{H}^2$ are half-circles perpendicular to the x-axis and straight vertical half-lines with the end on the x-axis. In these notations, the ambient space is given as the Riemannian product of the hyperbolic plane $(\mathbb{H}^2(-1), g_{\mathbb{H}})$ and the one dimensional Euclidean space endowed with the usual metric. The metric on the ambient space is therefore given by $\widetilde{g} = g_{\mathbb{H}} + dt^2$, where t is the global coordinate on $\mathbb{R}$. Denote by $\widetilde{\nabla}$ its corresponding Levi-Civita connection, whose non-vanishing Christoffel symbols are:

$$\widetilde{\Gamma}^1_{12} = \widetilde{\Gamma}^1_{21} = \widetilde{\Gamma}^2_{22} = -1/y, \quad \widetilde{\Gamma}^2_{11} = 1/y. \tag{A.1}$$

Setting s the arc-length parameter on γ, the curve is parametrized, in local coordinates (x,y,t), as

$$\gamma(s) = (x(s), y(s), t(s)), \tag{A.2}$$

such that

$$\frac{\dot x^2+\dot y^2}{y^2} + \dot t^2 = 1. \tag{A.3}$$

Recall that the curve γ is a geodesic in $(\mathbb{H}^2\times\mathbb{R}, \widetilde{g}, \widetilde{\nabla})$ if its velocity vector field is parallel along γ, namely

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma = 0. \tag{A.4}$$

Using the expressions of the Christoffel symbols from (A.1) we get that the components of γ satisfy the following system of ordinary differential equations:

$$\text{(A.5)} \qquad \begin{cases} \ddot{x} - \frac{2\dot{x}\dot{y}}{y} = 0, \\ \ddot{y} + \frac{\dot{x}^2 - \dot{y}^2}{y} = 0, \\ \ddot{t} = 0, \end{cases}$$

with the initial conditions $x(0) = x_0,\ y(0) = y_0,\ t(0) = t_0,\ \dot{x}(0) = u_0,\ \dot{y}(0) = v_0,$ $\dot{t}(0) = \zeta_0$.
Solving this Cauchy problem, the explicit parametrizations of geodesics in $\mathbb{H}^2 \times \mathbb{R}$ are obtained as follows.

THEOREM 1. *The geodesics of the product space $\mathbb{H}^2 \times \mathbb{R}$ can be explicitly parametrized as:*

(a) $\gamma(s) = (x_0,\ y_0 e^{s\sqrt{1-\zeta_0^2}},\ \zeta_0 s + t_0)$,

(b) $\gamma(s) = (-d + r\cos\lambda(s),\ r\sin\lambda(s),\ \zeta_0 s + t_0)$,

where $r^2 := \frac{1-\zeta_0^2}{c^2}$, *with* $c \in \mathbb{R} \setminus \{0\}$, $\lambda(s) = 2\arctan(\delta e^{s\sqrt{1-\zeta_0^2}})$, *with* $\delta = \tan\frac{\lambda_0}{2}$. *The numbers λ_0 and d satisfy the initial conditions* $y_0 = r\sin\lambda_0$ *and* $d = y_0\cot\lambda_0 - x_0$.

Proof. From the last equation of (A.5) we immediately get

$$\text{(A.6)} \qquad t(s) = \zeta_0 s + t_0,\ \ \zeta_0, t_0 \in \mathbb{R}.$$

In order to determine the other two coordinate functions, that is x and y, we distinguish two cases.
Case 1. $\dot{x} = 0$. Thus, $x(s) = x_0$, $x_0 \in \mathbb{R}$. Replacing x in the second equation we find that $\frac{\ddot{y}y - \dot{y}^2}{y}$ vanishes or, equivalently, $\frac{\dot{y}}{y}$ is constant. Now, taking into account that s is the arc-length parameter, and using (A.6), we obtain that $y(s)$ satisfies the ordinary differential equation $\frac{\dot{y}}{y} = \pm\sqrt{1-\zeta_0^2}$. It follows that $y(s) = y_0 e^{s\sqrt{1-\zeta_0^2}}$, concluding the proof of statement (a) in the theorem.
Case 2. $\dot{x} \neq 0$. From the first equation of (A.5) we get

$$\text{(A.7)} \qquad \dot{x} = cy^2,\ c \in \mathbb{R}^*.$$

Combining this relation with the second equation of (A.5) we get

$$\text{(A.8)} \qquad \frac{d}{ds}\left(\frac{\dot{y}}{y}\right) = -c\dot{x}.$$

Integrating once with respect to s, we have $\frac{\dot{y}}{y} = -c(x+d)$, $d \in \mathbb{R}$. Computing the square of this equality and taking into account the arc-length parametrization condition, as well as equations (A.6) and (A.7), we obtain a relation between the x and y coordinates, as follows

$$y^2 + (x+d)^2 = r^2,\ \ r^2 := \frac{1-\zeta_0^2}{c^2}.$$

Hence, there exists a function $\lambda(s)$ such that

$$x(s) = -d + r\cos\lambda(s), \quad y(s) = r\sin\lambda(s). \tag{A.9}$$

Again, as s is the arc-length, we get, up to the orientation of γ, that

$$\lambda(s) = 2\arctan(\delta e^{s\sqrt{1-\zeta_0^2}}), \tag{A.10}$$

where $\delta = \tan\frac{\lambda_0}{2}$ and λ_0 satisfies the initial conditions $x_0 = -d + r\cos\lambda_0,\ y_0 = r\sin\lambda_0$. Moreover, from these relations we get $d = y_0\cot\lambda_0 - x_0$. Hence, item (b) of the theorem is proved. $\square$

The geodesics of $\mathbb{H}^2 \times \mathbb{R}$, thought as a particular case of Bianchi-Cartan-Vrănceanu spaces, were described in [8]. It is important to observe that the hyperbolic plane $\mathbb{H}^2$ was modelled by the Poincaré disk.

3. Magnetic curves in $\mathbb{H}^2 \times \mathbb{R}$

Recall that a *magnetic field* may be regarded as a closed 2-form F and the Lorentz force corresponding to it in the ambient space $(\mathbb{H}^2 \times \mathbb{R}, \widetilde{g}, \widetilde{\nabla})$ is a $(1,1)-$type tensor field ϕ given by:

$$\widetilde{g}(\phi(X), Y) = F(X, Y), \tag{A.11}$$

where X, Y are vector fields in $\mathbb{H}^2 \times \mathbb{R}$. Moreover, the *magnetic trajectories* of the magnetic flow generated by the magnetic field F are given by those curves γ which are the solutions of the *Lorentz equation*:

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \phi(\dot\gamma). \tag{A.12}$$

As $\mathbb{H}^2 \times \mathbb{R}$ is a 3-dimensional manifold, 2-forms may be identified with the corresponding vector fields using the Hodge operator and the volume form. Hence, magnetic fields can be thought as *divergence free* vector fields. An important class is obtained when a Killing vector field generates the magnetic field. In this situation, the magnetic trajectories are known as *Killing magnetic curves*.

Let us fix a Killing vector field V. If F_V is the magnetic field corresponding to V, then the Lorentz force of the magnetic background $(\mathbb{H}^2 \times \mathbb{R}, \widetilde{g}, F_V)$ may be expressed in terms of the cross product $\wedge$ on $\mathbb{H}^2 \times \mathbb{R}$,

$$\phi(X) = V \wedge X \text{ for any } X \in \mathfrak{X}(\mathbb{H}^2 \times \mathbb{R}), \tag{A.13}$$

and the Lorentz equation (A.12) becomes

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma = V \wedge \dot\gamma. \tag{A.14}$$

In order to define a cross product on $\mathbb{H}^2 \times \mathbb{R}$, from the parametrization (A.2) we compute first

$$\dot\gamma = \dot x\frac{\partial}{\partial x} + \dot y\frac{\partial}{\partial y} + \dot t\frac{\partial}{\partial t},$$

and since $\frac{\partial}{\partial x} \perp \frac{\partial}{\partial y}$, we can choose an orthonormal basis $\left\{ e_1 = y\frac{\partial}{\partial x},\ e_2 = y\frac{\partial}{\partial y},\ \frac{\partial}{\partial t} \right\}$ such that the cross product $\wedge$ is defined as follows:

$$e_1 \wedge e_2 := \frac{\partial}{\partial t},\quad \frac{\partial}{\partial t} \wedge e_1 := e_2,\quad \frac{\partial}{\partial t} \wedge e_2 := -e_1. \tag{A.15}$$

Hence, using the definition (A.15), we may easily find:

$$\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} = \frac{1}{y^2}\frac{\partial}{\partial t},\quad \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial x} = \frac{\partial}{\partial y},\quad \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial y} = -\frac{\partial}{\partial x}. \tag{A.16}$$

In this section we explicitly determine the normal magnetic curves associated to Killing vector fields in $\mathbb{H}^2 \times \mathbb{R}$, where we use the upper half-plane model of $\mathbb{H}^2$. Recall that a basis of Killing vector fields in $\mathbb{H}^2 \times \mathbb{R}$ [6] is given by:

$$\left\{ \frac{\partial}{\partial t},\ \frac{\partial}{\partial x},\ \frac{x^2 - y^2 + 1}{2}\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y},\ x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \right\}.$$

In the next two subsections we will consider the translational Killing vector fields, $V_1 = \frac{\partial}{\partial t}$ and $V_2 = \frac{\partial}{\partial x}$ respectively, in order to classify the magnetic curves corresponding to the magnetic fields defined by V_1 and V_2.

A.1. Magnetic curves associated with the Killing vector field $V_1 = \frac{\partial}{\partial t}$

The Killing magnetic field F_1 corresponding to the Killing vector field $V_1 = \frac{\partial}{\partial t}$ can be expressed as

$$F_1\left(\frac{\partial}{\partial t}, -\right) = 0 \quad \text{and} \quad F_1\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \frac{1}{y^2}.$$

This is a consequence of (A.11), (A.13) and (A.16).

In the following we give the explicit parametrization of Killing magnetic curves generated by the magnetic field F_1.

THEOREM 2. *Let* $\gamma : I \subset \mathbb{R} \to \mathbb{H}^2 \times \mathbb{R}$ *be a smooth curve parametrized by arc-length. The curve* γ *is the Killing magnetic curve corresponding to the magnetic field defined by the Killing vector field* $V_1 = \frac{\partial}{\partial t}$ *if and only if it is parametrized as:*

(a) a geodesic line $\gamma(s) = (x_0, y_0, t_0 \pm s)$ *in a point* $(x_0, y_0) \in \mathbb{H}^2$*,*

(b) an Euclidean line $\gamma(s) = (x_0 \pm s y_0, y_0, t_0)$ *parallel to* $Ox-$*axis,*

(c) a helix $\gamma(s) = \left(x_0 + \frac{y_0(1+\sin\theta)\tan\theta\sin(s\cos\theta)}{1+\sin\theta\cos(s\cos\theta)},\ \frac{y_0(1+\sin\theta)}{1+\sin\theta\cos(s\cos\theta)},\ s\cos\theta + t_0 \right)$*,*

(d) a hyperbolic circle $\gamma(s) = \left(x_0 - \frac{s y_0}{s^2+1}, \frac{y_0}{s^2+1}, t_0 \right)$*, which is also an Euclidean circle centered at* $(x_0, \frac{y_0}{2})$ *and of radius* $\frac{y_0}{2}$*,*

where $x_0, t_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}_+^*$ and θ represents the constant angle made by the curve γ with the real axis $\mathbb{R}$.

Proof. Replacing $V = \partial_t$ in the expression of the Lorentz equation (A.14), we get

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \partial_t \wedge \dot\gamma. \tag{A.17}$$

We have already computed the left side of (A.17), namely

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \left(\ddot{x} - \frac{2\dot{x}\dot{y}}{y}\right)\partial_x + \left(\ddot{y} + \frac{\dot{x}^2 - \dot{y}^2}{y}\right)\partial_y + \ddot{t}\partial_t \ .$$

Now, using the formulas (A.16), we find the right side of (A.17), that is

$$\partial_t \wedge \dot\gamma = -\dot{y}\partial_x + \dot{x}\partial_y \ .$$

It follows that the magnetic curve $\gamma = (x, y, t)$ satisfies the following system of ordinary differential equations:

$$\begin{cases} \ddot{x} - \frac{2\dot{x}\dot{y}}{y} = -\dot{y}, \\ \ddot{y} + \frac{\dot{x}^2 - \dot{y}^2}{y} = \dot{x}, \\ \ddot{t} = 0. \end{cases} \tag{A.18}$$

Our purpose is to solve this system.

The condition (A.3) yields that there exists a function $\theta(s) \in [0, \pi]$ such that $\dot{t} = \cos\theta(s)$ and $\frac{\dot{x}^2 + \dot{y}^2}{y^2} = \sin^2\theta(s)$. From the last equation in (A.18) we see that $\dot{t}$ is constant; thus, the function θ is constant. Hence, the expression of the third coordinate of γ is given by

$$t(s) = s\cos\theta + t_0, \quad t_0 \in \mathbb{R}. \tag{A.19}$$

Notice that θ represents the constant angle made by the unit tangent to the curve $\dot\gamma$ and the real line ∂_t.

The first two coordinates of γ are related by the condition

$$\dot{x}^2 + \dot{y}^2 = y^2 \sin^2\theta. \tag{A.20}$$

In this equation we easily observe that there exists a smooth function $\alpha(s) \in [0, 2\pi)$ such that

$$\dot{x}(s) = y(s)\sin\theta\cos\alpha(s), \tag{A.21a}$$

$$\dot{y}(s) = y(s)\sin\theta\sin\alpha(s). \tag{A.21b}$$

Taking the second derivative in (A.21a) and (A.21b) and replacing all these expressions in the first two equations of (A.18), we get:

$$\sin\theta\sin\alpha(s)\dot\alpha(s) + \sin^2\theta\sin\alpha(s)\cos\alpha(s) = \sin\theta\sin\alpha(s), \tag{A.22a}$$

$$\sin\theta\cos\alpha(s)\dot\alpha(s) + \sin^2\theta\cos^2\alpha(s) = \sin\theta\cos\alpha(s). \tag{A.22b}$$

At this point we distinguish several cases for the angle θ and the function $\alpha(s)$.

(a) $\sin\theta = 0$. From (A.21a) and (A.21b) we get that $\dot{x} = \dot{y} = 0$, hence $x(s) = x_0$, $y(s) = y_0$, where $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}^*_+$. From (A.19) it follows that the last component of γ is given by
$t(s) = \pm s + t_0$, proving item (a) of the theorem.

(b) $\cos\theta \neq 0$, $\sin\theta \neq 0$ and $\sin\alpha(s) = 0$, which implies that $\cos\alpha(s)\dot{\alpha}(s) = 0$, and from (A.22b) we have that $\sin\theta = \cos\alpha(s) = 1$, thus $\cos\theta = 0$. Hence, from (A.19) we have $t(s) = t_0$, $t_0 \in \mathbb{R}$, from (A.21b) $y(s) = y_0$, $y_0 \in \mathbb{R}^+$, and from (A.21a) $x(s) = sy_0 + x_0$, $x_0 \in \mathbb{R}$, proving item (b).

Notice that the situation $\sin\theta \neq 0,\ \cos\alpha(s) = 0$ yields a contradiction.

(c) In the general situation $\cos\theta \neq 0$, $\sin\theta \neq 0$, $\sin\alpha(s) \neq 0$ and $\cos\alpha(s) \neq 0$ we get that α is a solution for the ordinary differential equation:

$$\dot{\alpha}(s) = 1 - \sin\theta\cos\alpha(s). \tag{A.23}$$

Integrating (A.23) and for an appropriate choice of the integration constant, we obtain the expression of the function α:

$$\alpha(s) = 2\arctan\left(\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\tan\frac{s\cos\theta}{2}\right). \tag{A.24}$$

Replacing the expression (A.24) of α in (A.21b) and solving the ordinary differential equation, we get the coordinate function $y(s)$ of γ as:

$$y(s) = \frac{y_0(1+\sin\theta)}{1+\sin\theta\cos(s\cos\theta)}. \tag{A.25}$$

Next, computing the expression of $\cos\alpha$ using (A.24) and replacing it together with the expression (A.25) of y in (A.21a) and solving the obtained ordinary differential equation in x, we get (when $\cos\theta \neq 0$)

$$x(s) = x_0 + \frac{y_0(1+\sin\theta)\tan\theta\sin(s\cos\theta)}{1+\sin\theta\cos(s\cos\theta)}. \tag{A.26}$$

Thus, combining (A.19), (A.25) and (A.26), case (c) of the theorem is proved.

(d) Finally, we study the remained case $\cos\theta = 0$. Consequently, we get $t = t_0$ with $t_0 \in \mathbb{R}$ and $\dot{x}^2 + \dot{y}^2 = y^2$. As before, there exists a function α such that

$$\dot{x}(s) = y(s)\cos\alpha(s), \tag{A.27a}$$
$$\dot{y}(s) = y(s)\sin\alpha(s). \tag{A.27b}$$

In a similar manner as in the previous situation, we get that α satisfies the ordinary differential equation

$$\dot{\alpha}(s) = 2\sin^2\left(\frac{\alpha(s)}{2}\right). \tag{A.28}$$

If $\sin\alpha = 0$, then item (b) of the theorem is obtained again.
When $\sin\alpha \neq 0$, we find the solution of (A.28), $\alpha(s) = -2\arctan\left(\frac{1}{s-s_0}\right)$, where s_0 is such that $\tan\frac{\alpha(0)}{2} = \frac{1}{s_0}$. Solving the ordinary differential equations (A.27a) and (A.27b), one gets the parametrization of γ from item (d), concluding the direct implication.
The converse part is a simple verification of the fact that the parametrizations from items (a)-(d) define magnetic curves, namely they are solutions of the Lorentz equation (A.17).

□

These magnetic curves may be characterized in the following manner:

THEOREM 3. *The non-geodesic Killing magnetic curves in the magnetic background* $(\mathbb{H}^2\times\mathbb{R},\widetilde{g},F_1)$ *are the geodesics of the cylindrical surface* $C\times\mathbb{R}$, *where the curve* C *represents a hyperbolic circle of curvature* $\bar{\kappa} = \pm\frac{1}{\sin\theta}$ *in* $\mathbb{H}^2$.

For similar results see e.g. [2, 4, 5].

A.2. Magnetic curves associated with the Killing vector field $V_2 = \frac{\partial}{\partial x}$

The Killing magnetic field F_2 corresponding to the Killing vector field $V_2 = \frac{\partial}{\partial x}$ is given by

$$F_2\left(\frac{\partial}{\partial x},-\right) = 0 \quad \text{and} \quad F_2\left(\frac{\partial}{\partial y},\frac{\partial}{\partial t}\right) = \frac{1}{y^2}\ .$$

The equations of motion of charged particles evolving in the magnetic background $(\mathbb{H}^2\times\mathbb{R},\tilde{g},F_2)$ corresponding to Killing magnetic vector field V_2 are given in the following.

THEOREM 4. *Let* $\gamma : I\subset\mathbb{R}\to\mathbb{H}^2\times\mathbb{R}$ *be a smooth curve parametrized by arc-length. The curve* γ *is Killing magnetic curve corresponding to the magnetic field defined by the Killing vector field* $V_2 = \frac{\partial}{\partial x}$ *if it satisfies the following equations:*

$$\text{(A.29)}\qquad \begin{cases} \dot{x}(s) = \beta y^2(s), \\ \dot{t}(s) = -\frac{1}{y(s)} + \alpha, \\ \dot{y}^2(s)+\beta^2y^4(s)+(\alpha^2-1)y^2(s)-2\alpha y(s)+1 = 0, \quad \alpha,\beta\in\mathbb{R}. \end{cases}$$

Proof. We proceed in the same manner as in the previous section. Computing the cross product $\frac{\partial}{\partial x}\wedge\dot{\gamma}$ we obtain

$$V_2\wedge\dot{\gamma} = \frac{\dot{y}}{y^2}\frac{\partial}{\partial t} - \dot{t}\frac{\partial}{\partial y}\ .$$

Then, the Lorentz equation

$$\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = V_2\wedge\dot{\gamma},$$

becomes

$$\text{(A.30)} \qquad \begin{cases} \ddot{x} - \frac{2\dot{x}\dot{y}}{y} = 0, \\ \ddot{y} + \frac{\dot{x}^2 - \dot{y}^2}{y} + \dot{t} = 0, \\ \ddot{t} - \frac{\dot{y}}{y^2} = 0. \end{cases}$$

Combining (A.30) with the arc-length parametrization condition (A.3) we get the formulas (A.29).

□

Final remarks. In order to determine explicitly the expression of the y-coordinate, we have to make use of elliptic functions, in the same manner as in [7].

To close this section, let us discuss some particular cases involving the integration constants α and β.

Case A. $\beta = 0$, or equivalently $\dot{x} = 0$. Hence $x(s) = x_0$, $x_0 \in \mathbb{R}$. The coordinate y satisfies the following ordinary differential equation:

$$\text{(A.31)} \qquad \dot{y}^2(s) + (\alpha^2 - 1)y^2(s) - 2\alpha y(s) + 1 = 0.$$

It is clear that for $\alpha \le -1$ there is no solution (since $y > 0$). For $\alpha > -1$ we distinguish three situations, namely

- if $|\alpha| < 1$, then $y(s) = \frac{1}{1-\alpha^2}\left(\cosh(s\sqrt{1-\alpha^2}) - \alpha\right)$; hence $y(s) \ge \frac{1}{1+\alpha}$; we find $t(s) = \alpha\, s - 2\arctan\left(\sqrt{\frac{1+\alpha}{1-\alpha}}\tanh\frac{s\sqrt{1-\alpha^2}}{2}\right)$;
- if $\alpha = 1$, then $y(s) = \frac{1}{2}\left(1 + (s-s_0)^2\right)$ and $t(s) = s - \sqrt{2}\arctan\left(\sqrt{2}(s-s_0)\right)$, $s_0 \in \mathbb{R}$;
- if $\alpha > 1$, then $y(s) = \frac{1}{\alpha^2-1}\left(\alpha - \cos(s\sqrt{\alpha^2-1})\right)$; hence y is bounded, namely $y(s) \in \left(\frac{1}{\alpha+1}\,,\frac{1}{\alpha-1}\right)$ for all s; then $t(s) = \alpha\, s - 2\arctan\left(\sqrt{\frac{\alpha+1}{\alpha-1}}\tan\frac{s\sqrt{\alpha^2-1}}{2}\right)$.

Case B. $\beta \neq 0$, $\alpha = 0$. The function y satisfies the equation

$$\text{(A.32)} \qquad \dot{y}^2(s) + \beta^2 y^4(s) - y^2(s) + 1 = 0.$$

Obviously, if $\beta^2 > \frac{1}{4}$ there is no solution. If $\beta^2 = \frac{1}{4}$ then $y(s) = \sqrt{2}$. The other two components are $t(s) = t_0 - \frac{s}{\sqrt{2}}$ and $x(s) = x_0 \pm s$. When $0 < \beta^2 < \frac{1}{4}$ the solution involves elliptic functions.

Case C. Let us look for horizontal magnetic curves, that is for which $t = t_0$, constant. We immediately deduce that $y(s) = \frac{1}{\alpha}$ and hence we should have $\alpha > 0$. Using the first equation in (A.29) we find the expression of x, that is $x(s) = x_0 + \frac{\beta}{\alpha^2}\, s$. The last equations lead to a relation between α and β, namely $\alpha^2 = \beta^2$.

Acknowledgement. This work was supported by a grant of the Romanian Ministry of Education, CNCS–UEFISCDI, project number PN-II-RU-PD-2012-3-0387.

References

[1] BARROS M., ROMERO A., CABRERIZO J.L. AND FERNÁNDEZ M., *The Gauss-Landau-Hall problem on Riemannian surfaces*, J. Math. Phys. **46** (2005), art. 112905.

[2] CABRERIZO J.L., FERNÁNDEZ M. AND GÓMEZ J.S., *The contact magnetic flow in* 3*D Sasakian manifolds*, J. Phys. A **42** (2009), art. 195201.

[3] DRUŢĂ-ROMANIUC S.L. AND MUNTEANU M.I., *Magnetic curves corresponding to Killing magnetic fields in* E^3, J. Math. Phys. **52** (2011) 11, art. 113506.

[4] DRUŢĂ-ROMANIUC S.L., INOGUCHI J., MUNTEANU M.I. AND NISTOR A.I., *Magnetic curves in Sasakian manifolds*, J. Nonlinear Math. Phys. **22** (2015), 428–447.

[5] DRUŢĂ-ROMANIUC S.L., INOGUCHI J., MUNTEANU M.I. AND NISTOR A.I., *Magnetic curves in cosymplectic manifolds*, to appear in Rep. Math. Phys., **78** (2016) 1, 33–48.

[6] MONTALDO S. AND ONNIS I.I., *Geodesics on an invariant surface*, J. Geom. Phys. **61** (2011) 8, 1385–1395.

[7] MUNTEANU M.I. AND NISTOR A.I., *The classification of Killing magnetic curves in* $\mathbb{S}^2 \times \mathbb{R}$, J. Geom. Phys. **62** (2012), 170–182.

[8] PIU P. AND PROFIR M.M., *On the three-dimensional homogenous* $SO(2)$*-isotropic Riemannian manifolds*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **57** (2011) 2, 361–376.

AMS Subject Classification: 53B25

Ana Irina NISTOR
'Gh. Asachi' Technical University of Iaşi
Department of Mathematics and Informatics
Bd. Carol I, no.11A
700506 Iaşi, ROMANIA
e-mail: ana.irina.nistor@gmail.com

Lavoro pervenuto in redazione il 13.11.2015 e accettato il 26.01.16.

Geom. Struc. on Riem. Man.-Bari
Vol. 73/1, 3–4 (2015), 171 – 182

A. Perrone

MAGNETIC CURVES OF THE REEB VECTOR FIELD OF A NORMAL ALMOST PARACONTACT THREE-MANIFOLD

Abstract. In this paper we first show that in any (oriented) Lorentzian three-manifold (M,g), magnetic fields F of positive constant length are in correspondence with the almost paracontact structures compatible with the Lorentzian metric g and with divergence-free Reeb vector field. Then, we report the results of [5], that is, we consider a normal almost paracontact metric structure, with divergence-free Reeb vector field, and describe the magnetic curves of the corresponding magnetic field.

Key words and phrases. Almost paracontact metric structures, normal structures, Killing vector fields, magnetic curves.

1. Introduction

Let (M,g) be a pseudo-Riemannian manifold and ∇ its Levi-Civita connection. A *magnetic field* on (M,g) is a closed two-form F on M. The *Lorentz force* ϕ corresponding to F, is a skew-symmetric $(1,1)$-tensor field uniquely determined by $g(\phi X,Y) = F(X,Y)$, for any vector fields X,Y on M. A smooth curve on M is called a *magnetic curve*, or *trajectory* for the magnetic field F if it is a solution of the *Lorentz equation* $\nabla_{\dot\gamma}\dot\gamma = \phi(\dot\gamma)$. As such, they are a natural generalization of geodesics of M, that satisfy the Lorentz equation in the absence of any magnetic field. However, it is relevant to note that the magnetic curves never reduce to geodesics [1, Prop. 2.1].

In the last decade, magnetic curves have been widely studied (see, for example, [1, 2], [6]–[12] and references therein). Such curves, under a geometric point of view, are often related to slant curves with respect to some natural Killing vector fields (see, for example, Proposition 1 and [15]). While, under a physical point of view, magnetic curves shape the magnetic flow in a background magnetic field, and describe the motion of charged particles in several physical scenarios (see, for example, [2]).

In particular, the three-dimensional case has been investigated, because it shows some special behaviors [4, 6, 8, 9, 11]. On an oriented three-dimensional pseudo-Riemannian manifold (M,g), the Lorentz force is provided via the cross product, and magnetic fields are in a one-to-one correspondence with divergence-free vector fields. As Killing vector fields are divergence-free, one may define a special class of magnetic fields called *Killing magnetic fields*, namely, the ones corresponding to Killing vector fields. This leads to investigate magnetic curves related to some Killing vector fields which appear naturally in the geometry of the three-dimensional pseudo-Riemannian manifold itself.

In the contact Riemannian case, Cabrerizo et al. in [6] have studied the magnetic fields determined by the Reeb vector field of any Sasakian three-manifold, determining the magnetic trajectories and proving that they are helices with axis the Reeb vector field itself. In dimension three, paracontact structures are the Lorentzian counterpart to contact Riemannian structures.

This paper is based on the Chapter 5 of [13] and on the joint paper [5]. More precisely, in Section 2 we report some basic information about magnetic curves, in particular in dimension three. In Section 3 we show that in any (oriented) Lorentzian three-manifold (M,g) of signature $(++-)$, magnetic fields F_V with $g(V,V) = constant > 0$ are in one-to-one correspondence with the almost paracontact structures compatible with the Lorentzian metric g and with divergence-free Reeb vector field. Moreover, the magnetic field F_V is a Killing magnetic field if and only if the corresponding almost paracontact structure is normal. These results motivate the study of the magnetic curves in almost paracontact metric three-manifolds. Then, in Section 4 we report the results of [5], that is, we describe the magnetic curves corresponding to the magnetic field $F = qF_\xi, q \in \mathbb{R} \setminus \{0\}$, associated with the Reeb vector field ξ of a normal almost paracontact metric three-manifold with ξ divergence-free. We observe that the class of normal almost paracontact metric manifolds with divergence-free Reeb vector field is very large. In particular, it includes paraSasakian and paracosymplectic manifolds. Moreover, with respect to the contact metric case, the study here is at the same time more complex and interesting, because a metric compatible with an almost paracontact three-structure is Lorentzian and so, the vectors $\dot\gamma$ and $\nabla_{\dot\gamma}\dot\gamma$ can have any causal character. Next, as an application, explicit descriptions are given for the magnetic curves of the standard left-invariant paraSasakian structure of the hyperbolic Heisenberg group and of a model of paracosymplectic three-manifold.

2. Magnetic curves

The magnetic curves on a pseudo-Riemannian manifold (M,g) are trajectories of charged particles, moving on M under the action of a magnetic field. A *magnetic field* on (M,g) is a closed two-form F on M, to which one may associate a skew-symmetric $(1,1)$-tensor field ϕ on M, called the *Lorentz force*, uniquely determined by $g(\phi X,Y) = F(X,Y)$, for any vector fields X,Y on M. Let us remark that ϕ is metrically equivalent to F, so that no information is lost when ϕ is considered instead of F, and ϕ and F are then said to be physically equivalent.

A smooth parametrized curve $\gamma(t)$ in M is called *magnetic curve* of the magnetic field F if it satisfies the *Lorentz equation*

$$\nabla_{\dot\gamma}\dot\gamma = \phi(\dot\gamma), \tag{A.1}$$

where ∇ is the Levi-Civita connection of (M,g). It is well known that for any point $p \in M$ and for any vector $X_p \in T_pM$ there exists a unique geodesic $\gamma(t)$ such that $\gamma(0) = p$ and $\dot\gamma = X_p$. When $F \neq 0$, the same existence and uniqueness property can be stated for magnetic curves.

We observe that a magnetic curve γ has constant speed:

$$\frac{1}{2}\frac{d}{ds}g(\dot\gamma,\dot\gamma) = g(\nabla_{\dot\gamma}\dot\gamma,\dot\gamma) = g(\phi\dot\gamma,\dot\gamma) = F(\dot\gamma,\dot\gamma) = 0.$$

On a three-dimensional pseudo-Riemannian manifold (M,g) oriented by the volume form Ω, one may define a Lorentz cross product via Ω on M, putting

$$g(X \wedge Y, Z) = \Omega(X,Y,Z),$$

for any vector fields X,Y,Z on M. Moreover, in dimension three, the two-forms F are in a one-to-one correspondence with the vector fields V by

$$F(X,Y) = \Omega(V,X,Y),$$

for any vector fields X,Y on M. Since $F_V(X,Y) := \Omega(V,X,Y)$ satisfies

$$dF_V = \mathcal{L}_V \Omega = (\mathrm{div}\ V)\Omega,$$

then F_V is closed if and only if V is divergence-free. Therefore, magnetic fields are in a one-to-one correspondence with divergence-free vector fields. Besides, for any vector field V, with $\mathrm{div}V = 0$, from $g(\phi X,Y) = F_V(X,Y) = \Omega(V,X,Y) = g(V \wedge X, Y)$, we get that the Lorentz force may be expressed as

$$\phi X = V \wedge X,$$

and so, Equation (A.1) becomes

$$\nabla_{\dot\gamma}\dot\gamma = V \wedge \dot\gamma. \tag{A.2}$$

Therefore, magnetic curves associated to V correspond to solutions of the ordinary differential equation (A.2). In particular, a large class of Lorentz forces may be obtained from Killing vector fields on M (since V Killing implies V divergence-free). We now state the following.

PROPOSITION 1. *A divergence-free vector field V on a pseudo-Riemannian three-manifold (M,g) is Killing if and only if $g(V,\dot\gamma)$ is constant for any magnetic curve $\gamma(s)$ of F_V.*

Proof. It is well known that V is a Killing vector field if and only if it satisfies

$$g(\nabla_X V, Y) + g(X, \nabla_Y V) = 0,$$

for any tangent vector fields X,Y. Therefore, if V is a Killing vector field and γ is a magnetic curve, using Equation (A.2) we get that $g(V,\dot\gamma)$ is constant. In fact,

$$\frac{d}{ds}g(V,\dot\gamma) = g(\nabla_{\dot\gamma}V,\dot\gamma) + g(V,\nabla_{\dot\gamma}\dot\gamma) = g(\nabla_{\dot\gamma}V,\dot\gamma) + g(V, V\wedge\dot\gamma) = g(\nabla_{\dot\gamma}V,\dot\gamma) = 0.$$

Conversely, for any vector field X, consider the magnetic curve γ such that $\dot\gamma(0) = X_p$, where $\gamma(0) = p$. Since $g(V,\dot\gamma)$ is constant we have

$$0 = \frac{d}{ds}g(V,\dot\gamma) = g(\nabla_{\dot\gamma}V,\dot\gamma) + g(V,\nabla_{\dot\gamma}\dot\gamma) = g(\nabla_{\dot\gamma}V,\dot\gamma).$$

Hence, $g(\nabla_X V, X)(p) = g(\nabla_{X_p}V, X_p) = 0$ for any $p \in M$. Consequently, we obtain

$$0 = g(\nabla_{X+Y}V, X+Y) = g(\nabla_X V, Y) + g(\nabla_Y V, X)$$

and so V is Killing. □

In the Riemannian case Proposition 1 corresponds to Lemma 4.1 of [6]. Under a geometric point of view, this proposition implies that magnetic curves are related to slant curves with respect to some natural Killing vector fields (see, for example, [15]).

3. Almost paracontact structures and magnetic fields

A.1. Almost paracontact metric structures

An *almost paracontact structure* on a $(2n+1)$-dimensional (connected) smooth manifold M is a triple (φ,ξ,η), where φ is a tensor of type $(1,1)$, ξ a vector field (called the *Reeb vector field* or the *characteristic vector field*) and η a one-form, satisfying

(i) $$\eta(\xi)=1, \qquad \varphi^2 = I_d - \eta\otimes\xi$$

(ii) the tensor field φ induces an almost paracomplex structure on the distribution $\mathcal{D}=\ker\eta$, that is, the eigendistributions $\mathcal{D}^+$ and $\mathcal{D}^-$, corresponding to the eigenvalues 1, -1 of φ respectively, have equal dimension n.

As a consequence of (i), one gets $\varphi\xi=0$, $\eta\circ\varphi=0$. M is said to be an *almost paracontact manifold* if it is equipped with an almost paracontact structure. M is called an *almost paracontact metric manifold* if it is equipped with an almost paracontact metric structure, that is, if it admits a pseudo-Riemannian metric g (called *compatible metric*), such that

$$g(\varphi X,\varphi Y) = -g(X,Y)+\eta(X)\eta(Y). \tag{A.3}$$

Any compatible metric g with a given almost paracontact structure is of signature $(n+1,n)$. We note that condition (i) and Equation (A.3) imply that $\eta(X)=g(\xi,X)$ and condition (ii).

For an almost paracontact metric manifold, we can define the *fundamental two-form* Φ by $\Phi(X,Y)=g(X,\varphi Y)$, for all tangent vector fields X,Y. If $\Phi=d\eta$, then the manifold is called a *paracontact metric manifold* and g the *associated metric*.

An almost paracontact metric structure (φ,ξ,η,g) is said to be *normal* if it satisfies the condition $N^{(1)} := [\varphi,\varphi]-2d\eta\otimes\xi=0$, where $[\varphi,\varphi]$ is the Nijenhuis torsion tensor of φ. In dimension three, normal almost paracontact metric structures are characterized by condition (see [14],[15]):

$$(\nabla_X\varphi)Y = \alpha(g(\varphi X,Y)\xi-\eta(Y)\varphi X)+\beta(g(X,Y)\xi-\eta(Y)X), \tag{A.4}$$

or equivalently,

$$\nabla_X\xi = \alpha(X-\eta(X)\xi)+\beta\varphi X, \tag{A.5}$$

where $2\alpha=\mathrm{tr}\nabla\xi=\mathrm{div}\xi$ and $2\beta=\mathrm{tr}(\varphi\nabla\xi)$. A normal almost paracontact metric structure is called

- *quasi-paraSasakian* if $\alpha=0\neq\beta$ (in particular, β-paraSasakian when β is a constant, and when $\beta=-1$ the structure is paraSasakian);

- *paracosymplectic* if $\nabla\eta = \nabla\Phi = 0$. In the three-dimensional case, this is equivalent to the condition $\nabla\xi = 0$, that is, the structure is normal with $\alpha = \beta = 0$.

A.2. Paracontact magnetic fields

Let $(M, \varphi, \xi, \eta, g)$ be an almost paracontact metric three-manifold with $\mathrm{div}\xi = 0$. Then, the volume form is given by $\Omega = 3\eta \wedge \Phi$ and the magnetic field $F_\xi = \Phi$, as

$$F_\xi(X,Y) := \Omega(\xi, X, Y) = 3(\eta \wedge \Phi)(\xi, X, Y) = \Phi(X,Y).$$

So, the almost paracontact metric structure (φ, ξ, η, g), with $\mathrm{div}\xi = 0$, determines the set $\{F' = qF_\xi,\, q \in \mathbb{R} \setminus \{0\}\}$ of magnetic fields on the Lorentzian manifold (M, g). Vice versa, consider now on an oriented Lorentzian three-manifold (M, g), that we suppose of signature $(+ + -)$, a magnetic field F with corresponding divergence-free vector field V and so, with Lorentz force ϕ expressed by the Lorentz cross product $\phi X = V \wedge X$. This Lorentz cross product satisfies (see, for example, [8])

$$\begin{aligned} X \wedge (Y \wedge Z) &= g(X,Y)Z - g(X,Z)Y, \\ g(Z \wedge X, Z \wedge Y) &= g(Z,X)g(Z,Y) - g(Z,Z)g(X,Y). \end{aligned}$$

Then, by using these properties, the vector field V, the Lorentz force ϕ and the one-form $\eta_V = g(V, \cdot)$ satisfy:

$$\begin{aligned} \phi^2 X = V \wedge (V \wedge X) &= g(V,V)X - g(V,X)V = g(V,V)X - \eta_V(X)V, \\ g(\phi X, \phi Y) = g(V \wedge X, V \wedge Y) &= g(V,X)g(V,Y) - g(V,V)g(X,Y) \\ &= \eta_V(X)\eta_V(Y) - g(V,V)g(X,Y). \end{aligned}$$

If $g(V,V) = q^2 > 0$, for some smooth function $q \neq 0$, then the tensors

$$\Big(\varphi = -\frac{1}{q}\phi,\ \xi = \frac{1}{q}V,\ \eta = \frac{1}{q}\eta_V,\ g\Big)$$

satisfy condition (i) and Equation (A.3), and such conditions define an almost paracontact metric structure. Besides, if $q \neq 0$ is a real constant, then ξ is divergence-free, and the magnetic field F is given by

$$F(X,Y) = g(\phi X, Y) = -g(q\varphi X, Y) = q\Phi(X,Y) = qF_\xi.$$

We note that, given the magnetic field $F = F_V$, $g(V,V) = \mathit{constant} = q^2 > 0$, if we consider the magnetic field $F' = \lambda F = \lambda F_V$, $\lambda \in \mathbb{R} \setminus \{0\}$, then F' determines the same almost paracontact structure (φ, ξ, η) compatible with g and with $\mathrm{div}\xi = 0$. In fact,

$$\begin{aligned} &F' = \lambda F_V = \lambda\Omega(V, \cdot, \cdot) = F_{V'}, && V' = \lambda V, && g(V', V') = q'^2 = \lambda^2 q^2, \\ &\phi' X = V' \wedge X = \lambda V \wedge X = \lambda\phi X, && \varphi' = -\tfrac{1}{q'}\phi' = -\tfrac{1}{\lambda q}\lambda\phi = \varphi, \\ &\xi' = \tfrac{1}{q'}V' = \xi, && \eta' = \tfrac{1}{q'}\eta_{V'} = \tfrac{1}{\lambda q}g(V', \cdot) = \tfrac{1}{q}g(V, \cdot) = \eta. \end{aligned}$$

Moreover, if the almost paracontact structure (φ,ξ,η) compatible with g, and with ξ divergence-free, is normal then from Equation (A.5) we get $\nabla\xi = \beta\varphi$. This gives that ξ is Killing, and thus F_V is a Killing magnetic field. Vice versa, suppose that the magnetic field F_V, $g(V,V) = constant = q^2 > 0$, is Killing, then also the corresponding Reeb vector field ξ is Killing. Besides, it is well known that a unit Killing vector field is geodesic, and hence ξ satisfies $\nabla_\xi\xi = 0$. Then, by using (4) of Proposition 3.1 of [3] we have that the tensor $h := (1/2)\mathcal{L}_\xi\varphi$ vanishes. On the other hand, in dimension three, if $h = 0$ then the almost paracontact structure is normal. In fact, first of all, we note that the distributions $\mathcal{D}^+$ and $\mathcal{D}^-$ are involutive, because they have dimension one. Then, for any $X,Y \in \Gamma(\mathcal{D}^+)$, or $X,Y \in \Gamma(\mathcal{D}^-)$, we have $d\eta(X,Y) = 0$ and so

$$N^{(1)}(X,Y) = [\varphi,\varphi](X,Y) = \varphi^2[X,Y] + [\varphi X,\varphi Y] - \varphi[\varphi X,Y] - \varphi[X,\varphi Y] = 0.$$

Moreover, for $X \in \Gamma(\mathcal{D}^+)$ and $Y \in \Gamma(\mathcal{D}^-)$, we get $2d\eta(X,Y) = -\eta[X,Y]$ and so

$$N^{(1)}(X,Y) = \varphi^2[X,Y] + [\varphi X,\varphi Y] - \varphi[\varphi X,Y] - \varphi[X,\varphi Y] + \eta[X,Y]\xi = 0.$$

Similarly for $X \in \Gamma(\mathcal{D}^-)$ and $Y \in \Gamma(\mathcal{D}^+)$. Finally, for $X \in \Gamma(\mathcal{D}^\pm)$

$$\begin{aligned} N^{(1)}(X,\xi) &= \varphi^2[X,\xi] - \varphi[\pm X,\xi] - 2d\eta(X,\xi)\xi = [X,\xi] \mp \varphi[X,\xi] \\ &= \mp(\mathcal{L}_\xi\varphi)X = \mp 2h(X) = 0. \end{aligned}$$

Analogously $N^{(1)}(\xi,X) = 0$.

Summarizing, we proved the following

THEOREM 1. *In any (oriented) Lorentzian three-manifold (M,g) of signature $(++-)$, magnetic fields F_V with $g(V,V) = constant > 0$, defined up to a constant of proportionality, are in one-to-one correspondence with the almost paracontact structures compatible with the Lorentzian metric g and with divergence-free Reeb vector field. Moreover, the magnetic field F_V is a Killing magnetic field if and only if the corresponding almost paracontact structure is normal.*

REMARK 1. By above proof we get that if an (oriented) Lorentzian three-manifold (M,g) of signature $(++-)$ admits a magnetic fields F_V, $g(V,V) = q^2 > 0$, for some smooth function $q \neq 0$, then it admits an almost paracontact structure (φ,ξ,η) compatible with the Lorentzian metric g.

Theorem 1 motivates the study of the magnetic curves in almost paracontact metric three-manifolds.

Let (M,φ,ξ,η,g) be an almost paracontact metric three-manifold with $\mathrm{div}\xi = 0$. For the magnetic field F with *strength* q, corresponding to the given almost paracontact metric structure, that is, for $F = qF_\xi$, $q \in \mathbb{R}\setminus\{0\}$, we have

$$g(\phi X,Y) = F(X,Y) = q\Phi(X,Y) = qg(X,\varphi Y) = -qg(\varphi X,Y),$$

for all tangent vector fields X,Y. Therefore, we get $\phi = -q\varphi$ and the Lorentz equation (A.1) becomes

$$\nabla_{\dot\gamma}\dot\gamma = -q\varphi\dot\gamma. \tag{A.6}$$

Furthermore, since $g(\xi \wedge X,Y) = F_\xi(X,Y) = \Phi(X,Y) = -g(\varphi X,Y)$, we have $\varphi X = -\xi \wedge X$ and so, the Lorentz force is expressed by

$$\phi X = q\xi \wedge X.$$

If we suppose that $\gamma(t)$ is non-geodesic, then from Equation (A.6) we have that $\gamma(t)$ can not be an integral curve of ξ (for the integral curves of ξ we have $\varphi\dot\gamma = \varphi\xi_{\gamma(t)} = 0$).

4. Magnetic curves in a normal almost paracontact metric 3D-manifold

The results of this section were obtained in the joint paper [5]. We shall only report the essential steps, for more details see the same paper [5].

Let (M,φ,ξ,η,g) be a normal almost paracontact metric three-manifold. Suppose that the two-form F_ξ is a magnetic field, that is, $\mathrm{div}\xi = 0$. Then, from Theorem 1 it follows that ξ is Killing. We now study the magnetic curves corresponding to the Killing magnetic field $F = qF_\xi$, $q \in \mathbb{R}\setminus\{0\}$.

Let γ denote such a (non-geodesic) magnetic curve. We know that γ has constant speed and, as ξ is a Killing vector field, by Proposition 1, we get that

$$a_0 := \eta(\dot\gamma) = g(\xi,\dot\gamma) \text{ is a constant.}$$

We now shall treat separately three cases, depending on the causality of γ.

Case I: γ is light-like.
First, we analyze the acceleration vector field. By Equation (A.6), we find

$$g(\nabla_{\dot\gamma}\dot\gamma,\nabla_{\dot\gamma}\dot\gamma) = q^2 g(-\varphi\dot\gamma,-\varphi\dot\gamma) = q^2(-g(\dot\gamma,\dot\gamma)+\eta(\dot\gamma)^2) = q^2\eta(\dot\gamma)^2 = q^2a_0^2. \tag{A.7}$$

If $\eta(\dot\gamma) = 0$, then γ is a *Legendre curve*. In this case, we find that $\nabla_{\dot\gamma}\dot\gamma$ is light-like and by Theorem 4.1 in [14] we conclude that γ is a geodesic. Therefore, from now on we shall assume that $a_0 = \eta(\dot\gamma) \neq 0$. From Equation (A.7) we now get $g(\nabla_{\dot\gamma}\dot\gamma,\nabla_{\dot\gamma}\dot\gamma) = q^2a_0^2 > 0$ and so, $\nabla_{\dot\gamma}\dot\gamma$ is space-like. Thus, reparametrizing γ, we can reduce to the case where the curvature satisfies $\kappa = \|\nabla_{\dot\gamma}\dot\gamma\| = 1$ (*pseudo-arc length parameter*), that is, $q^2a_0^2 = 1$. In fact, after an affine reparametrization, γ still remains a magnetic curve but with a different strength.

The normal vector field $N = \nabla_{\dot\gamma}\dot\gamma$ is space-like, $T = \dot\gamma$ is light-like and $g(N,T) = 0$. Hence, there exists a unique light-like vector field B, such that $g(B,N) = 0$ and $g(B,T) = 1$, so that $\{T,N,B\}$ is a null basis along the curve γ. We recall that the torsion τ of γ is given by $\tau := g(\nabla_{\dot\gamma}N, B)$. Writing $\nabla_{\dot\gamma}N = aT + bN + cB$, for some smooth real functions a, b, c, we get the second Frenet formula

$$\nabla_{\dot\gamma}N = \tau T - B. \tag{A.8}$$

We now determine τ. As $N = -q\varphi\dot\gamma$, we have

$$\nabla_{\dot\gamma}N = -q\nabla_{\dot\gamma}(\varphi\dot\gamma) = -q((\nabla_{\dot\gamma}\varphi)\dot\gamma + \varphi\nabla_{\dot\gamma}\dot\gamma) = -q((\nabla_{\dot\gamma}\varphi)\dot\gamma + \varphi N).$$

Using Equations (A.8) and (A.4) (with $\alpha = 0$), we find

$$\tau T - B = -\beta q\big(g(\dot\gamma,\dot\gamma)\xi - \eta(\dot\gamma)\dot\gamma\big) - q\varphi N = \beta q a_0 T - q\varphi N.$$

Thus, $q\varphi N = (-\tau + \beta q a_0)T + B$ and so, $q^2 g(\varphi N, \varphi N) = -2(\tau - \beta q a_0)g(T, B)$, that is, using Equation (A.3), $q^2 g(N,N) = 2(\tau - \beta q a_0)$, which yields

$$\tau = \frac{q^2}{2} + \beta q a_0 = \frac{q^2}{2} \pm \beta.$$

Then, τ is constant (and so, γ is a helix) if and only if β is constant, that is, when M is either β-paraSasakian or paracosymplectic.
Finally, expressing ξ with respect the basis $\{T, N, B\}$, using Equation (A.8) we find $\xi = \frac{1}{2a_0}\, T + a_0 B$.

Case II: γ is time-like.
Let γ be parametrized by arc length, so that $g(\dot\gamma,\dot\gamma) = -1$. By Equation (A.6), we have

$$g(\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma) = q^2 g(\varphi\dot\gamma, \varphi\dot\gamma) = q^2(-g(\dot\gamma,\dot\gamma) + \eta(\dot\gamma)^2) = q^2(1 + \eta(\dot\gamma)^2) = q^2(1 + a_0^2).$$

It follows that $\nabla_{\dot\gamma}\dot\gamma$ is space-like. Moreover, there exists a real constant θ, such that $a_0 = \sinh\theta$, where θ is the hyperbolic angle between the space-like vector ξ and the time-like vector $\dot\gamma$.
The Frenet frame on γ is defined by the time-like vector $T = \dot\gamma$, the space-like vector $\nabla_{\dot\gamma}\dot\gamma = \kappa N$, and the space-like vector $B = T \wedge N$, where the positive function κ denotes the curvature of γ, and N is the unitary normal.
We get $\kappa = |q|\cosh\theta$. Hence, the curvature κ is a nonzero constant, and in particular $\kappa^2 = q^2(1 + a_0^2)$. Besides, we find $\tau = \beta - a_0 q$ and $\xi = -a_0 T + \frac{\kappa}{q} B$.

Case III: γ is space-like. Let γ be parametrized by arc length, that is, without loss of generality, we assume that $g(\dot\gamma,\dot\gamma) = 1$. Taking into account Equation (A.6), we get

$$g(\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma) = q^2 g(-\varphi\dot\gamma, -\varphi\dot\gamma) = q^2(-g(\dot\gamma,\dot\gamma) + \eta(\dot\gamma)^2) = q^2(-1 + \eta(\dot\gamma)^2) = q^2(a_0^2 - 1).$$

We shall distinguish three cases, depending on whether $a_0^2 > 1$ (that is, $\nabla_{\dot\gamma}\dot\gamma$ is space-like), $0 \le a_0^2 < 1$ ($\nabla_{\dot\gamma}\dot\gamma$ is time-like) or $a_0^2 = 1$ ($\nabla_{\dot\gamma}\dot\gamma$ is light-like).

Case III.a: $\nabla_{\dot\gamma}\dot\gamma$ is space-like.

Since $a_0^2 > 1$, there exists some constant θ, such that $a_0 = \varepsilon_0 \cosh\theta$. Besides, both the tangent $T = \dot\gamma$ and the normal $N = \frac{1}{\kappa}\nabla_{\dot\gamma}\dot\gamma$ are space-like vector fields and we get $\kappa = |q\sinh\theta|$. The binormal B is a time-like vector field defined by $B = T \wedge N$ and hence $\{T,N,B\}$ is the pseudo-orthonormal Frenet frame along the curve γ. Finally, we get that the torsion of γ is $\tau := g(\nabla_{\dot\gamma}N,B) = -(\beta + a_0 q)$ and $\xi = a_0 T - \frac{\kappa}{q}B$.

Case III.b: $\nabla_{\dot\gamma}\dot\gamma$ *is time-like.*
In such a case, $0 \le a_0^2 < 1$, so there exists $\theta \in]0,\pi[$ such that $a_0 = \cos\theta$. As before, we determine the Frenet frame $\{T,N,B\}$, where $T = \dot\gamma$ is space-like, $N = \frac{1}{\kappa}\nabla_{\dot\gamma}\dot\gamma$ is time-like, $B = T \wedge N$ is space-like and $\kappa = |q\sin\theta|$, the torsion $\tau = \beta + a_0 q$ and $\xi = a_0 T - \frac{\kappa}{q}B$.

Case III.c: $\nabla_{\dot\gamma}\dot\gamma$ *is light-like.*
As $a_0^2 = 1$, we can write $a_0 = \pm 1$. In this case, $T = \dot\gamma$ is space-like, $N = \nabla_{\dot\gamma}\dot\gamma$ is light-like, the curvature κ is not defined and there exists a unique light-like vector field B, such that $g(B,T) = 0$ and $g(B,N) = 1$. So, $\{T,N,B\}$ is a null basis along the curve γ. Finally, we obtain that the torsion τ is given by $\tau = \mp(\beta + a_0 q)$ and $\xi = a_0 T \pm \frac{1}{q}B$.

Summarizing, we obtain the following

THEOREM 2. *Let (M,φ,ξ,η,g) be a normal almost paracontact metric three-manifold with ξ divergence-free and $\gamma(s) : I \longrightarrow M$ be a non-geodesic magnetic curve associated to ξ with strength q. We put $2\beta = \mathrm{tr}(\varphi\nabla\xi)$, then we have the following:*

- *if $\gamma(s)$ is a light-like curve, parametrized by pseudo-arc length, then the acceleration is space-like, $\xi = \frac{1}{2a_0}T + a_0 B$, $\kappa = 1$ and $\tau = \frac{q^2}{2} \pm \beta$, where $a_0^2 q^2 = 1$;*

- *if $\gamma(s)$ is a unit speed time-like curve, then the acceleration is space-like, $\xi = -a_0 T + \frac{\kappa}{q}B$, $\kappa^2 = (1 + a_0^2)q^2$ and $\tau = \beta - a_0 q$;*

- *if $\gamma(s)$ is a unit speed space-like curve with a space-like acceleration, then $\xi = a_0 T - \frac{\kappa}{q}B$, $\kappa^2 = (a_0^2 - 1)q^2$ and $\tau = -(\beta + a_0 q)$;*

- *if $\gamma(s)$ is a unit speed space-like curve with a time-like acceleration, then $\xi = a_0 T - \frac{\kappa}{q}B$, $\kappa^2 = (1 - a_0^2)q^2$ and $\tau = \beta + a_0 q$;*

- *if $\gamma(s)$ is a unit speed space-like curve with a light-like acceleration, then $\xi = a_0 T \pm \frac{1}{q}N$, κ is not defined, $\tau = \mp(\beta + a_0 q)$ and $a_0 = \pm 1$;*

where κ and τ are, respectively, the curvature and the torsion of the curve γ, $a_0 := \eta(\dot\gamma)$ is constant and (T,N,B) is the corresponding Frenet frame along γ.

Moreover, in all cases the curve $\gamma(s)$ is a helix if and only if β is constant, that is, M is either β-paraSasakian ($\beta \neq 0$) or paracosymplectic ($\beta = 0$).

Magnetic curves of the hyperbolic Heisenberg group H_h^3
Consider the Lie group $\mathbb{R}^3$ with the following group law

(A.9) $$(x,y,z)\cdot(x',y',z') := (x + x', y + y', z + z' - xy' + x'y).$$

A basis of left-invariant vector fields is given by $\xi = 2\frac{\partial}{\partial z}$, $U = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z}$, $V = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial z}$. Next, consider the standard left-invariant paraSasakian structure (φ, ξ, η, g) defined by

$$\eta = \frac{1}{2}dz + (xdy - ydx), \qquad g = dx \otimes dx - dy \otimes dy + \eta \otimes \eta,$$

$$\varphi(\xi) = 0, \quad \varphi(U) = V, \quad \varphi(V) = U.$$

The Lie group $\mathbb{R}^3$, with the law group (A.9), equipped with the pseudo-Riemannian metric g, is called *hyperbolic Heisenberg group*, and we shall denote it by H_h^3.

Let $\gamma(s) = (x(s), y(s), z(s))$ be a magnetic curve of H_h^3, that is, a solution of the Lorentz equation $\nabla_{\dot\gamma}\dot\gamma = -q\varphi\dot\gamma$. We put $\dot\gamma = f_1 U + f_2 V + f_3 \xi$ where f_1, f_2, f_3 are smooth functions on H_h^3. In particular, $f_3 = g(\dot\gamma(s), \xi) = \eta(\dot\gamma) =: a_0$ is a constant, because ξ is Killing. Next, with respect to the φ-basis (ξ, U, V), we get

$$(\dot f_1 - 2f_2 a_0)U + (\dot f_2 - 2f_1 a_0)V = \nabla_{\dot\gamma}\dot\gamma = -q\varphi\dot\gamma = -q(f_1 V + f_2 U).$$

Therefore, the curve $\gamma(s)$ is a magnetic curve if and only if f_1, f_2 satisfy the following system of differential equations: $\dot f_1 = \lambda f_2$, $\dot f_2 = \lambda f_1$, where $\lambda := 2a_0 - q =$ is a constant.

• If $\lambda = 0$, we find that $\gamma(s) = \big(x_0 + c_1 s,\ y_0 + c_2 s,\ z_0 + 2(a_0 + c_1 y_0 - c_2 x_0)s\big)$, for some real constants c_1, c_2, x_0, y_0, z_0, that is, it is a straight line that can be either space-like, time-like or light-like, as $g(\dot\gamma, \dot\gamma) = c_1^2 - c_2^2 + a_0^2$.

• If $\lambda \neq 0$, we find that $\gamma(s) = (x(s), y(s), z(s))$, where

$$\begin{cases} x(s) = x_0 + \frac{c_1}{\lambda}\sinh(\lambda s) + \frac{c_2}{\lambda}\cosh(\lambda s), \\ y(s) = y_0 + \frac{c_1}{\lambda}\cosh(\lambda s) + \frac{c_2}{\lambda}\sinh(\lambda s), \\ z(s) = z_0 + 2\big(a_0 + \frac{c_1^2 - c_2^2}{\lambda}\big) + \frac{2}{\lambda}(c_1 y_0 - c_2 x_0)\sinh(\lambda s) + \frac{2}{\lambda}(c_2 y_0 - c_1 x_0)\cosh(\lambda s), \end{cases}$$

for some real constants c_1, c_2, x_0, y_0, z_0, and it can be either space-like, time-like or light-like as $g(\dot\gamma, \dot\gamma) = c_1^2 - c_2^2 + a_0^2$.

Examples of magnetic curves in the paracosymplectic case

Let $M = \mathbb{R}_- \times \mathbb{R}^2$ with the standard Cartesian coordinates (x, y, z), we put $\partial_x = \frac{\partial}{\partial_x}$, $\partial_y = \frac{\partial}{\partial_y}$ and $\partial_z = \frac{\partial}{\partial_z}$. Consider on M the paracosymplectic structure (φ, ξ, η, g) defined by

$$\varphi\partial_x = \partial_y, \qquad \varphi\partial_y = \partial_x, \qquad \varphi\partial_z = 0, \qquad \partial_z = \xi,$$

$$\eta = dz, \qquad g = -x(dx \otimes dx - dy \otimes dy) + \eta \otimes \eta.$$

Now, let $\gamma(s) = \big(x(s), y(s), z(s)\big)$ be a magnetic curve of M, that is, a solution of $\nabla_{\dot\gamma}\dot\gamma = -q\varphi\dot\gamma$ with $x(s) < 0$. We put $\dot\gamma = \big(\dot x(s),\ \dot y(s),\ \dot z(s)\big) = \big(f_1(s),\ f_2(s),\ f_3(s)\big)$, where f_1, f_2, f_3 are smooth functions on M. In particular, as ξ is Killing, $\dot z(s) = g(\dot\gamma(s), \xi) = \eta(\dot\gamma) =: a_0$ is a constant. So, we get $z(s) = a_0 s + z_0$, for some constant z_0. Next, we explicitly compute the two members of the Lorentz equation with respect to the basis $(\partial_x, \partial_y, \partial_z)$. Since

$$\dot{f}_1\partial_x + \dot{f}_2\partial_y + f_1\nabla_{\dot\gamma}\partial_x + f_2\nabla_{\dot\gamma}\partial_y + a_0\nabla_{\dot\gamma}\xi = \nabla_{\dot\gamma}\dot\gamma = -q\varphi\dot\gamma = -q(f_2\partial_x + f_1\partial_y),$$

we have that $\gamma(s)$ is a magnetic curve if and only if f_1, f_2 satisfy the following system of differential equations:

$$\dot{f}_1 + \frac{f_1^2}{2x} + \frac{f_2^2}{2x} = -qf_2, \quad \dot{f}_2 + \frac{f_1 f_2}{x} = -qf_1.$$

We note that if $f_2 = 0$, then the second equation gives $f_1 = 0$ and so, γ is a geodesic. In order to find explicit solutions, we can consider special cases, and so, we find that examples of magnetic curves of M are:

- $\gamma(s) = \big(a_1,\ -2qa_1 s + a_2,\ a_0 s + z_0\big)$, for any real constants $a_1 < 0, a_0, a_2, z_0$, and it can be either light-like, time-like or space-like, as $g(\dot\gamma, \dot\gamma) = a_0^2 + 4q^2 a_1^3$.
- $\gamma(s) = \big(-\sqrt{k_2 - k_1 e^{-\varepsilon q s}},\ -\varepsilon\sqrt{k_2 - k_1 e^{-\varepsilon q s}},\ a_0 s + z_0\big)$, for any real constants $a_0, k_1 \neq 0$ and k_2, z_0, and it is space-like, as $g(\dot\gamma, \dot\gamma) = a_0^2$.

We close this paper with the following.

Open Problem: to describe the magnetic curves corresponding to the magnetic field $F = qF_\xi, q \in \mathbb{R} \setminus \{0\}$, associated with the Reeb vector field ξ, of an arbitrary almost paracontact metric manifold (with $\mathrm{div}\xi = 0$).

References

[1] BARROS M. AND ROMERO A., *The Gauss-Landau-Hall problem on Riemannian surfaces*, J. Math. Phys. **46** (2005), 112905, 15pp.

[2] BARROS M. AND ROMERO A., *Magnetic vortices*, EPL (Europhysics Letters) **77** (2007), 34002, 5pp.

[3] CALVARUSO G. AND PERRONE A., *Ricci solitons in three-dimensional paracontact geometry*, J. Geom. Phys. **98** (2015), 1–12.

[4] CALVARUSO G. AND MUNTEANU M.I., *Hopf magnetic curves in the anti-de Sitter space* $\mathbb{H}_1^3$, submitted.

[5] CALVARUSO G., MUNTEANU M.I., PERRONE A., *Killing magnetic curves in three-dimensional almost paracontact manifolds*, J. Math. Anal. Appl. **426** 1 (2015), 423–439.

[6] CABRERIZO J.L., FERNANDEZ M. , GÓMEZ J.S., *The contact magnetic flow in 3D Sasakian manifolds*, J. Phys. A: Math. Theor. **42** 19 (2009), 195201, 10pp.

[7] DRUŢĂ-ROMANIUC S.L., INOGUCHI J., MUNTEANU M.I., NISTOR A.I., *Magnetic curves in Sasakian manifolds*, J. Nonlinear Math. Phys. **22** 3 (2015), 428–447.

[8] DRUŢĂ-ROMANIUC S.L. AND MUNTEANU M.I., *Killing magnetic curves in a Minkowski 3-space*, Nonlinear Anal. Real World Appl. **14** 1 (2013), 383–396.

[9] DRUŢĂ-ROMANIUC S.L. AND MUNTEANU M.I., *Magnetic curves corresponding to Killing magnetic fields in* $\mathbb{E}^3$, J. Math. Phys. **52** 11 (2011), 113506, 11 pp.

[10] JLELI M., MUNTEANU M.I., NISTOR A.I., *Magnetic trajectories in an almost contact metric manifold* $\mathbb{R}^{2N+1}$, Results Math. **67** 1-2 (2015), 125-134.

[11] MUNTEANU M.I. AND NISTOR A.I., *The classification of Killing magnetic curves in $S^2 \times \mathbb{R}$*, J. Geom. Phys. **62** 2 (2012), 170–182.

[12] MUNTEANU M.I. AND NISTOR A.I., *A note on magnetic curves on $\mathbb{S}^{2n+1}$*, C. R. Acad. Sci. Paris **352** 5 (2014), 447–449.

[13] PERRONE A., *Geometry of paracontact metric manifolds*, Doctoral thesis, University of Salento, 2015.

[14] WELYCZKO J., *On Legendre Curves in 3-Dimensional Normal Almost Paracontact Metric Manifolds*, Results Math. **54** (2009), 377–387.

[15] WELYCZKO J., *Slant Curves in 3-Dimensional Normal Almost Paracontact Metric Manifolds*, Mediterr. J. Math. **11** 3 (2014), 965–978.

AMS Subject Classification: 53C15, 53C25, 53C80, 53D15.

Antonella PERRONE,
Dipartimento di Matematica e Fisica “Ennio De Giorgi”, Università del Salento
Prov. Lecce-Arnesano, 73100, Lecce, ITALY
e-mail: `perroneantonella@outlook.it`

Lavoro pervenuto in redazione il 06.10.2015 e accettato il 28.10.2015.

Geom. Struc. on Riem. Man.-Bari
Vol. 73/1, 3–4 (2015), 183 – 199

H. Tadano[1]

SOME MYERS TYPE THEOREMS AND HITCHIN–THORPE INEQUALITIES FOR SHRINKING RICCI SOLITONS

Abstract. In this expository paper, we shall give a brief survey of recent development on some compactness criteria and diameter estimates for complete shrinking Ricci solitons. As an application, we shall provide some new sufficient conditions for four-dimensional compact shrinking Ricci solitons to satisfy the Hitchin–Thorpe inequality.

1. Ricci Solitons

In this expository paper, we shall give a brief survey of recent development on some compactness criteria and diameter estimates for complete shrinking Ricci solitons. Ricci solitons were introduced by R. Hamilton [20] and are natural generalizations of Einstein manifolds. They correspond to self-similar solutions to the Ricci flow and often arise as singularity models [6, 21]. The importance of Ricci solitons was demonstrated in a series of three papers by G. Perelman [33, 34, 35], where Ricci solitons played crucial roles in the affirmative resolution of the Poincaré conjecture. Besides their geometric importance, Ricci solitons are also of great interest in theoretical physics and have been studied actively in relation to string theory [8, 15].

DEFINITION 1. A complete Riemannian manifold (M,g) is called a *Ricci soliton* [20] if there exists a vector field $V \in \mathfrak{X}(M)$ satisfying the equation

$$\mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_V g = \lambda g \tag{A.1}$$

for some constant $\lambda \in \mathbb{R}$, where Ric_g denotes the Ricci curvature of (M,g) and $\mathcal{L}_V$ is the Lie derivative in the direction of V. We say that the soliton (M,g) is *shrinking*, *steady* and *expanding* described as $\lambda > 0, \lambda = 0$ and $\lambda < 0$, respectively. A typical example of Ricci solitons is an Einstein manifold, where V is given by a Killing vector field. In such a case, we say that the soliton is *trivial*. When V is replaced with the gradient vector field ∇f for some smooth function $f : M \to \mathbb{R}$, the soliton (M,g) is called a *gradient Ricci soliton*. We refer to f as a *potential function*. Then (A.1) becomes

$$\mathrm{Ric}_g + \mathrm{Hess}\, f = \lambda g, \tag{A.2}$$

where $\mathrm{Hess}\, f$ denotes the Hessian of the potential function f.

EXAMPLE 1 (Cigar solitons). A typical example of gradient Ricci solitons is a *cigar soliton* $(\mathbb{R}^2, \frac{dx^2+dy^2}{1+x^2+y^2})$ discovered by R. Hamilton [20], where its potential function is

[1] This work was partially supported by Moriyasu Graduate Student Scholarship Foundation.

given by the function $f(x,y) = -\log(1+x^2+y^2)$. This is a complete non-compact steady Ricci soliton and is also known in the physics literature as *Witten's black hole*.

EXAMPLE 2 (Gaussian solitons). Another typical example of gradient Ricci solitons is a *Gaussian soliton* $(\mathbb{R}^n, g_0)$, where g_0 is the canonical flat metric on $\mathbb{R}^n$ and its potential function is given by the function $f(x) = \pm\frac{1}{4}|x|^2$. This is a complete non-compact shrinking (respectively, expanding) Ricci soliton.

Thanks to G. Perelman [33], any vector field on compact Ricci solitons appearing in (A.1) must be the sum of a gradient vector field and a Killing vector field. It is well-known now that compact steady and expanding Ricci solitons must be trivial, as well as compact shrinking Ricci solitons in dimension two and three [6]. Examples of compact non-trivial shrinking Kähler–Ricci solitons were constructed by N. Koiso [23], H.-D. Cao [5], X.-J. Wang and X. Zhu [44] and F. Podestà and A. Spiro [36]. Examples of non-compact non-trivial Kähler–Ricci solitons were given by M. Feldman, T. Ilmanen and D. Knopf [11] and A. S. Dancer and M. Y. Wang [10].

2. Some Myers type theorems

In this section, we shall introduce some Myers type theorems for complete shrinking Ricci solitons. Throughout this paper, we shall assume that all Riemannian manifolds are smooth, connected without boundary. To give nice compactness criteria for complete Riemannian manifolds is one of the most interesting problems in Riemannian geometry. The celebrated theorem of Myers guarantees the compactness of complete Riemannian manifolds under some positive lower bounds on the Ricci curvature.

THEOREM 1 (S. B. Myers [32]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $\lambda > 0$ such that the Ricci curvature satisfies $\mathrm{Ric}_g \geqslant \lambda g$. Then (M,g) must be compact with finite fundamental group. Moreover,*

$$\mathrm{diam}(M,g) \leqslant \pi\sqrt{\frac{n-1}{\lambda}}.$$

The Myers theorem above has been widely generalized by many authors [1, 4, 18, 19, 31]. The first generalization was given by W. Ambrose [1], where the positive lower bound on the Ricci curvature was replaced with an integral condition of the Ricci curvature along some geodesics.

THEOREM 2 (W. Ambrose [1]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some point $p \in M$ for which every geodesic $\gamma : [0,+\infty) \to M$ emanating from p satisfies*

$$\int_0^{+\infty} \mathrm{Ric}_g(\dot\gamma(s), \dot\gamma(s))\,ds = +\infty.$$

Then (M,g) must be compact.

On the other hand, motivated by relativistic cosmology, G. J. Galloway [18] proved the following compactness theorem by perturbing the constant lower bound on the Ricci curvature by the derivative in the radial direction of some bounded function:

THEOREM 3 (G. J. Galloway [18]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exist some constants $\lambda > 0$ and $L \geqslant 0$ such that for every pair of points in M and minimal geodesic γ joining those points, the Ricci curvature satisfies*

$$\mathrm{Ric}_g(\dot\gamma,\dot\gamma)|_{\gamma(s)} \geqslant \lambda + \frac{d\phi}{ds}(s),$$

where ϕ is some smooth function of the arc length satisfying $|\phi| \leqslant L$ along γ. Then (M,g) must be compact. Moreover,

$$\mathrm{diam}(M,g) \leqslant \frac{\pi}{\lambda}\left(L+\sqrt{L^2+(n-1)\lambda}\right).$$

DEFINITION 2. A *smooth metric measure space* is a complete Riemannian manifold (M,g) with the weighted volume form $d\mu := e^{-f}d\mathrm{vol}_g$, where $f : M \to \mathbb{R}$ is a smooth function on M and vol_g is the Riemannian density with respect to the metric g. For a smooth metric measure space (M,g) and a positive constant $k \in (0,+\infty)$, we put

$$\mathrm{Ric}_f := \mathrm{Ric}_g + \mathrm{Hess}\, f \quad \text{and} \quad \mathrm{Ric}_f^k := \mathrm{Ric}_g + \mathrm{Hess}\, f - \frac{1}{k}df\otimes df \tag{A.3}$$

and call them a *Bakry–Émery Ricci curvature* and a *k-Bakry–Émery Ricci curvature*, respectively. We call f a *potential function*. More generally, for a smooth vector field $V \in \mathfrak{X}(M)$ and a positive constant $k \in (0,+\infty)$, we define

$$\mathrm{Ric}_V := \mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_V g \quad \text{and} \quad \mathrm{Ric}_V^k := \mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_V g - \frac{1}{k}V^*\otimes V^*,$$

where V^* is the metric dual of V with respect to g. We call them a *modified Ricci curvature* and a *k-modified Ricci curvature*, respectively. Then, we put

$$\Delta_f := \Delta_g - \nabla f\cdot\nabla \quad \text{and} \quad \Delta_V := \Delta_g - V\cdot\nabla \tag{A.4}$$

and call them a *Witten–Laplacian* and a *V–Laplacian*, respectively. Here, Δ_g is the Laplacian with respect to g.

Note that, if $f : M \to \mathbb{R}$ is constant in (A.3) and (A.4), then the Bakry–Émery Ricci curvature and the Witten–Laplacian are reduced to the Ricci curvature and the Laplacian, respectively. As with the ordinary case, for any smooth functions u,v on M with compact support, we have

$$\int_M g(\nabla u,\nabla v)d\mu = -\int_M(\Delta_f u)v\,d\mu = -\int_M u(\Delta_f v)d\mu.$$

Moreover, D. Bakry and M. Émery [3] proved that, for any smooth function u on M,

$$\frac{1}{2}\Delta_f|\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \mathrm{Ric}_f(\nabla u,\nabla u) + g(\nabla\Delta_f u,\nabla u), \tag{A.5}$$

which may be regarded as a natural extension of the Bochner–Weitzenböck formula.

Since Ricci solitons are generalizations of Einstein manifolds, it is natural to ask whether classical theorems for Einstein manifolds with positive Ricci curvature remain valid in the case of Ricci solitons with positive modified Ricci curvature. However, a positive lower bound on the modified Ricci curvature does not imply the compactness of complete Ricci solitons. In fact, the shrinking Gaussian soliton is non-compact.

The compactness of complete shrinking Ricci solitons may be characterized by the boundedness of the norm of its vector fields.

THEOREM 4 (M. Fernández-López and E. García-Río [12]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $\lambda > 0$ such that the modified Ricci curvature satisfies $\mathrm{Ric}_V \geqslant \lambda g$. Then M is compact if and only if $|V|$ is bounded on M.*

Recently, the Bakry–Émery Ricci curvature and the Witten–Laplacian have received much attention in various areas of mathematics, since they are good substitutes for the Ricci curvature and the Laplacian respectively, allowing us to establish many interesting results in smooth metric measure spaces, such as eigenvalue estimates [16], Li–Yau Harnack inequalities [26] and comparison theorems [45]. In particular, G. Wei and W. Wylie [45] proved the following Myers type theorem via the Bakry–Émery Ricci curvature which extends Theorem 1 to the case of smooth metric measure spaces:

THEOREM 5 (G. Wei and W. Wylie [45]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $\lambda > 0$ such that the Bakry–Émery Ricci curvature satisfies $\mathrm{Ric}_f \geqslant \lambda g$. If the potential function satisfies $|f| \leqslant H$ for some non-negative constant $H \geqslant 0$, then (M,g) must be compact. Moreover,*

$$\mathrm{diam}(M,g) \leqslant \pi\sqrt{\frac{n-1}{\lambda}} + \frac{4H}{\sqrt{(n-1)\lambda}}. \tag{A.6}$$

REMARK 1. Under the same assumption as in Theorem 5, M. Limoncu [29] gave the upper diameter estimate

$$\mathrm{diam}(M,g) \leqslant \frac{\pi}{\sqrt{\lambda}}\sqrt{n-1+2\sqrt{2}H}, \tag{A.7}$$

which may be sharper than (A.6) if $H > \frac{(n-1)\pi}{8}(\sqrt{2}\pi - 4)$.

On the other hand, the author [39] gave the following diameter estimate under the same assumption as in Theorem 5:

THEOREM 6 ([39]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $\lambda > 0$ such that the Bakry–Émery*

Ricci curvature satisfies $\mathrm{Ric}_f \geqslant \lambda g$. *If the potential function satisfies* $|f| \leqslant H$ *for some non-negative constant* $H \geqslant 0$, *then* (M,g) *must be compact. Moreover,*

$$\mathrm{diam}(M,g) \leqslant \frac{\pi}{\sqrt{\lambda}}\sqrt{n-1+\frac{8H}{\pi}}. \tag{A.8}$$

REMARK 2. Since $\frac{8}{\pi} < 2\sqrt{2}$, the diameter estimate (A.8) above is sharper than (A.7) by M. Limoncu [29]. Moreover, we may easily see that (A.8) is also sharper than (A.6) by G. Wei and W. Wylie [45] without any assumptions on H.

Recall from Theorem 4 that the compactness of complete shrinking Ricci solitons may be characterized by the boundedness of the norm of its vector fields. However, no an upper diameter estimate was given in Theorem 4. By extending the proof of the Myers theorem, M. Limoncu [28] gave such a diameter estimate.

THEOREM 7 (M. Limoncu [28]). *Let* (M,g) *be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant* $\lambda > 0$ *such that the modified Ricci curvature satisfies* $\mathrm{Ric}_V \geqslant \lambda g$. *If the vector field satisfies* $|V| \leqslant K$ *for some non-negative constant* $K \geqslant 0$, *then* (M,g) *must be compact. Moreover,*

$$\mathrm{diam}(M,g) \leqslant \frac{\pi}{\lambda}\left(\frac{K}{\sqrt{2}}+\sqrt{\frac{K^2}{2}+(n-1)\lambda}\right). \tag{A.9}$$

On the other hand, the diameter estimate (A.9) above was improved by the author [40] under the same assumption as in Theorem 7.

THEOREM 8 ([40]). *Let* (M,g) *be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant* $\lambda > 0$ *such that the modified Ricci curvature satisfies* $\mathrm{Ric}_V \geqslant \lambda g$. *If the vector field satisfies* $|V| \leqslant K$ *for some non-negative constant* $K \geqslant 0$, *then* (M,g) *must be compact. Moreover,*

$$\mathrm{diam}(M,g) \leqslant \frac{1}{\lambda}\left(2K+\sqrt{4K^2+(n-1)\lambda\pi^2}\right).$$

An interesting problem in smooth metric measure spaces is to establish Ambrose and Galloway types theorems via the Bakry–Émery Ricci curvature. An Ambrose type theorem via the Bakry–Émery Ricci curvature was first established by S. Zhang [46] under the assumption that the potential function of the Bakry–Émery Ricci curvature has at most linear growth.

THEOREM 9 (S. Zhang [46]). *Let* (M,g) *be an n-dimensional complete Riemannian manifold. Suppose that there exists some point* $p \in M$ *for which every geodesic* $\gamma : [0,+\infty) \to M$ *emanating from p satisfies*

$$\int_0^{+\infty} \mathrm{Ric}_f(\dot\gamma(s),\dot\gamma(s))ds = +\infty$$

and the potential function satisfies $f(x) \leqslant \gamma(d(x,p)+\alpha)$ *for some constants* γ *and* α*, where* $d(x,p)$ *is the distance between* x *and* p*, then* (M,g) *must be compact.*

More generally, we may prove the following Ambrose type theorem via the modified Ricci curvature which may be considered as a generalization of Theorem 4:

THEOREM 10 ([41]). *Let* (M,g) *be an* n*-dimensional complete Riemannian manifold. Suppose that there exists some point* $p \in M$ *for which every geodesic* $\gamma : [0,+\infty) \to M$ *emanating from* p *satisfies*

$$\int_0^{+\infty} \mathrm{Ric}_V(\dot{\gamma}(s),\dot{\gamma}(s))ds = +\infty$$

and the vector field satisfies $|V| \leqslant K$ *for some non-negative constant* $K \geqslant 0$*, then* (M,g) *must be compact.*

M. P. Cavalcante, J. Q. Oliveira and M. S. Santos [7] established the following Galloway type theorem via the Bakry–Émery Ricci curvature:

THEOREM 11 (M. P. Cavalcante, J. Q. Oliveira and M. S. Santos [7]). *Let* (M,g) *be an* n*-dimensional complete Riemannian manifold. Suppose that there exist some constants* $\lambda > 0$ *and* $L \geqslant 0$ *such that for every pair of points in* M *and minimal geodesic* γ *joining those points, the Bakry–Émery Ricci curvature satisfies*

$$\mathrm{Ric}_f(\dot{\gamma},\dot{\gamma})|_{\gamma(s)} \geqslant \lambda + \frac{d\phi}{ds}(s),$$

where ϕ *is some smooth function of the arc length satisfying* $|\phi| \leqslant L$ *along* γ*. If the potential function satisfies* $|f| \leqslant H$ *for some non-negative constant* $H \geqslant 0$*, then* (M,g) *must be compact. Moreover,*

$$\mathrm{diam}(M,g) \leqslant \frac{\pi}{\lambda}\left(L + \sqrt{L^2 + \{(n-1)+2\sqrt{2}H\}\lambda}\right). \tag{A.10}$$

REMARK 3. Under the same assumption as in Theorem 11, the author [41] gave the upper diameter estimate

$$\mathrm{diam}(M,g) \leqslant \frac{1}{\lambda}\left(2L + \sqrt{4L^2 + \{(n-1)\pi + 8H\}\lambda\pi}\right), \tag{A.11}$$

which is slightly sharper than (A.10). Moreover, by taking $L = 0$, (A.11) is reduced to (A.8).

On the other hand, a Galloway type theorem via the modified Ricci curvature was first established by the author [41].

THEOREM 12 ([41]). *Let* (M,g) *be an* n*-dimensional complete Riemannian manifold. Suppose that there exist some constants* $\lambda > 0$ *and* $L \geqslant 0$ *such that for every pair of* *points in M and minimal geodesic γ joining those points, the modified Ricci curvature satisfies*

$$\mathrm{Ric}_V(\dot\gamma,\dot\gamma)|_{\gamma(s)} \geqslant \lambda + \frac{d\phi}{ds}(s),$$

where ϕ is some smooth function of the arc length satisfying $|\phi| \leqslant L$ along γ. If the vector field satisfies $|V| \leqslant K$ for some non-negative constant $K \geqslant 0$, then (M,g) must be compact. Moreover,

$$\mathrm{diam}(M,g) \leqslant \frac{1}{\lambda}\left(2(L+K)+\sqrt{4(L+K)^2+(n-1)\lambda\pi^2}\right).$$

REMARK 4. By taking $L = 0$, Theorem 12 above is reduced to Theorem 8.

M. Limoncu [28] established the following Myers type theorem via the k-modified Ricci curvature without making any assumptions on its vector field:

THEOREM 13 (M. Limoncu [28]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $\lambda > 0$ such that the k-modified Ricci curvature satisfies $\mathrm{Ric}_V^k \geqslant \lambda g$, where $k \in (0,+\infty)$. Then (M,g) must be compact. Moreover,*

$$\mathrm{diam}(M,g) \leqslant \frac{\pi}{\sqrt{\lambda}}\sqrt{n+k-1}. \tag{A.12}$$

REMARK 5. In the case that the vector field V is replaced with the gradient of some smooth function $f : M \to \mathbb{R}$ as $V = \nabla f$, Theorem 13 was already proved by Z. Qian [37].

On the other hand, M. P. Cavalcante, J. Q. Oliveira and M. S. Santos [7] proved the following Ambrose type theorem via the k-Bakry–Émery Ricci curvature:

THEOREM 14 (M. P. Cavalcante, J. Q. Oliveira and M. S. Santos [7]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some point $p \in M$ for which every geodesic $\gamma : [0,+\infty) \to M$ emanating from p satisfies*

$$\int_0^{+\infty} \mathrm{Ric}_f^k(\dot\gamma(s),\dot\gamma(s))ds = +\infty,$$

where $k \in (0,+\infty)$. Then (M,g) must be compact.

The key ingredient in the proof of Theorem 14 above is the Riccati inequality

$$\mathrm{Ric}_f^k(\partial_r,\partial_r) \leqslant -\dot m_f - \frac{(m_f)^2}{n+k-1},$$

which may be derived by applying the Bochner–Weitzenböck formula (A.5) to the distance function $r = r(x)$. Here $m_f := \Delta_f r$.

The Bochner–Weitzenböck formula (A.5) may be extended as follows:

LEMMA 1 (Y. Li [27]). *Let (M,g) be a Riemannian manifold. For any smooth vector field $V \in \mathfrak{X}(M)$ and smooth function $u : M \to \mathbb{R}$, we have*

$$\frac{1}{2}\Delta_V |\nabla u|^2 = |\operatorname{Hess} u|^2 + \operatorname{Ric}_V(\nabla u, \nabla u) + g(\nabla \Delta_V u, \nabla u). \tag{A.13}$$

REMARK 6. In Lemma 1 above, if the vector field V is replaced with the gradient of some smooth function $f : M \to \mathbb{R}$ as $V = \nabla f$, then (A.13) is reduced to the Bochner–Weitzenböck formula (A.5) via the Bakry–Émery Ricci curvature.

By applying the Bochner–Weitzenböck formula (A.13) to the distance function $r = r(x)$, we may obtain the Riccati inequality for the k-modified Ricci curvature

$$\operatorname{Ric}_V^k(\partial_r, \partial_r) \leqslant -\dot{m}_V - \frac{(m_V)^2}{n+k-1},$$

where $m_V := \Delta_V r$. By using this Riccati inequality, we may prove the following Ambrose type theorem via the k-modified Ricci curvature:

THEOREM 15 ([41]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some point $p \in M$ for which every geodesic $\gamma : [0,+\infty) \to M$ emanating from p satisfies*

$$\int_0^{+\infty} \operatorname{Ric}_V^k(\dot{\gamma}(s), \dot{\gamma}(s))ds = +\infty,$$

where $k \in (0,+\infty)$. Then (M,g) must be compact.

On the other hand, a Galloway type theorem via the k-modified Ricci curvature was first established by M. P. Cavalcante, J. Q. Oliveira and M. S. Santos [7].

THEOREM 16 (M. P. Cavalcante, J. Q. Oliveira and M. S. Santos [7]). *Let (M,g) be an n-dimensional complete Riemannian manifold. Suppose that there exist some constants $\lambda > 0$ and $L \geqslant 0$ such that for every pair of points in M and minimal geodesic γ joining those points, the k-modified Ricci curvature satisfies*

$$\operatorname{Ric}_V^k(\dot{\gamma}, \dot{\gamma})|_{\gamma(s)} \geqslant \lambda + \frac{d\phi}{ds}(s),$$

where ϕ is some smooth function of the arc length satisfying $|\phi| \leqslant L$ along γ and $k \in (0,+\infty)$. Then (M,g) must be compact. Moreover,

$$\operatorname{diam}(M,g) \leqslant \frac{\pi}{\lambda}\left(L + \sqrt{L^2 + (n+k-1)\lambda}\right). \tag{A.14}$$

REMARK 7. Under the same assumption as in Theorem 16, the author [41] gave the upper diameter estimate

$$\operatorname{diam}(M,g) \leqslant \frac{1}{\lambda}\left(2L + \sqrt{4L^2 + (n+k-1)\lambda\pi^2}\right), \tag{A.15}$$

which is slightly sharper than (A.14). Moreover, by taking $L = 0$, (A.15) is reduced to (A.12).

REMARK 8. In the case that the vector field V is replaced with the gradient of some smooth function $f : M \to \mathbb{R}$ as $V = \nabla f$, Theorem 16 was already proved by M. Rimoldi [38].

3. Diameter bounds for compact Ricci solitons

In this section, we shall introduce some recent development on lower and upper diameter bounds for compact shrinking Ricci solitons.

A.1. Lower diameter bounds

Diameter bounds for compact shrinking Ricci solitons have been recently investigated by many authors [2, 9, 13, 16, 17, 39, 40]. In particular, a lower diameter bound for compact non-trivial shrinking Ricci solitons was first investigated by M. Fernández-López and E. García-Río [13] in terms of the Ricci curvature and the range of the potential function.

THEOREM 17 (M. Fernández-López and E. García-Río [13]). *Let (M,g) be an n-dimensional compact non-trivial shrinking Ricci soliton satisfying* (A.2). *Then*

(A.16)
$$\operatorname{diam}(M,g) \geqslant \max\left\{\sqrt{\frac{2(f_{\max}-f_{\min})}{C-\lambda}}, \sqrt{\frac{2(f_{\max}-f_{\min})}{\lambda-c}}, 2\sqrt{\frac{2(f_{\max}-f_{\min})}{C-c}}\right\},$$

where $f_{\max}$ and $f_{\min}$ respectively denote the maximum and the minimum values of the potential function on the soliton.

In Theorem 17 above and throughout this paper, the numbers

$$C := \max_{v\in TM}\{\mathrm{Ric}_g(v,v) : |v| = 1\} \quad \text{and} \quad c := \min_{v\in TM}\{\mathrm{Ric}_g(v,v) : |v| = 1\}$$

respectively denote the maximum and the minimum values of the Ricci curvature on the unit sphere bundle over (M,g). Note that $cg \leqslant \mathrm{Ric} \leqslant Cg$.

When the soliton has positive Ricci curvature, the diameter bound (A.16) above may be written in terms of the range of the scalar curvature.

COROLLARY 1 (M. Fernández-López and E. García-Río [13]). *Let (M,g) be an n-dimensional compact non-trivial shrinking Ricci soliton satisfying* (A.2). *Suppose that the soliton has positive Ricci curvature. Then*

$$\operatorname{diam}(M,g) \geqslant \max\left\{\sqrt{\frac{R_{\max}-R_{\min}}{\lambda(C-\lambda)}}, \sqrt{\frac{R_{\max}-R_{\min}}{\lambda(\lambda-c)}}, 2\sqrt{\frac{R_{\max}-R_{\min}}{\lambda(C-c)}}\right\},$$

where $R_{\max}$ and $R_{\min}$ respectively denote the maximum and the minimum values of the scalar curvature on the soliton.

On the other hand, a universal lower diameter bound for compact non-trivial shrinking Ricci solitons was first given by A. Futaki and Y. Sano [17] in relation to study of the first non-zero eigenvalue of the Witten–Laplacian.

THEOREM 18 (A. Futaki and Y. Sano [17]). *Let (M,g) be an n-dimensional compact non-trivial shrinking Ricci soliton satisfying* (A.2). *Then*

$$\operatorname{diam}(M,g) \geqslant \frac{10\pi}{13\sqrt{\lambda}}. \tag{A.17}$$

The lower diameter bound (A.17) above was improved by many authors [2, 9, 16]. In particular, the following sharper diameter bound was given independently by Y. Chu and Z. Hu [9] and A. Futaki, H. Li and X.-D. Li [16]:

THEOREM 19 (Y. Chu and Z. Hu [9] and A. Futaki, H. Li and X.-D. Li [16]). *Let (M,g) be an n-dimensional compact non-trivial shrinking Ricci soliton satisfying* (A.2). *Then*

$$\operatorname{diam}(M,g) \geqslant \frac{2(\sqrt{2}-1)\pi}{\sqrt{\lambda}}. \tag{A.18}$$

REMARK 9. Theorem 19 above says that, if a compact shrinking Ricci soliton (M,g) does not satisfy the inequality (A.18), then the soliton (M,g) must be trivial. Hence, this theorem gives us a gap phenomenon between Einstein manifolds and non-trivial Ricci solitons. See [14, 25] for other gap theorems for gradient Ricci solitons and Kähler–Ricci solitons, respectively.

A.2. Upper diameter bounds

Myers type theorems via the Bakry–Émery and modified Ricci curvatures in the previous section are closely related to an upper diameter bound for compact shrinking Ricci solitons. We recall the following useful propositions:

PROPOSITION 1 (R. Hamilton [21]). *Let (M,g) be an n-dimensional gradient Ricci soliton satisfying* (A.2). *Then*

$$R + |\nabla f|^2 - 2\lambda f = C_0 \tag{A.19}$$

for some real constant C_0, where R denotes the scalar curvature on the soliton.

PROPOSITION 2 (M. Fernández-López and E. García-Río [13, 14]). *Let (M,g) be an n-dimensional compact shrinking Ricci soliton satisfying* (A.2). *Then*

$$|\nabla f|^2 \leqslant R_{\max} - R. \tag{A.20}$$

Moreover, if the soliton has positive Ricci curvature, then

$$2\lambda f_{\max} - 2\lambda f_{\min} = R_{\max} - R_{\min}. \tag{A.21}$$

Recall from Theorem 17 and Corollary 1 that a lower diameter bound for compact non-trivial shrinking Ricci solitons was given in terms of the range of the potential function, as well as in terms of the range of the scalar curvature. M. Fernández-López and E. García-Río [13] conjectured that an upper diameter bound for compact shrinking Ricci solitons would also be given in terms of the range of the potential function, as well as in terms of the range of the scalar curvature. By combining Theorem 8 and the gradient estimate (A.20), we may give the following upper diameter bound for compact shrinking Ricci solitons which may be considered as an answer to the conjecture by M. Fernández-López and E. García-Río:

THEOREM 20 ([40]). *Let (M,g) be an n-dimensional compact shrinking Ricci soliton satisfying* (A.2). *Then*

$$\operatorname{diam}(M,g) \leqslant \frac{1}{\lambda}\left(2\sqrt{R_{\max}-R_{\min}}+\sqrt{4(R_{\max}-R_{\min})+(n-1)\lambda\pi^2}\right). \tag{A.22}$$

REMARK 10. In Theorem 20 above, if the soliton has constant scalar curvature, then the soliton appears as an Einstein manifold and the diameter bound (A.22) above is reduced to the Myers diameter bound [32] for Einstein manifolds with positive Ricci curvature.

When the soliton has positive Ricci curvature, the diameter bound (A.22) above may be written in terms of the range of the potential function due to the relation (A.21).

COROLLARY 2 ([40]). *Let (M,g) be an n-dimensional compact shrinking Ricci soliton satisfying* (A.2). *Suppose that the soliton has positive Ricci curvature. Then*

$$\operatorname{diam}(M,g) \leqslant 2\sqrt{\frac{2(f_{\max}-f_{\min})}{\lambda}}+\sqrt{\frac{8(f_{\max}-f_{\min})+(n-1)\pi^2}{\lambda}}.$$

On the other hand, by making some normalization on the potential function, we may give another diameter bound for compact shrinking Ricci solitons. Recall from Proposition 1 that any potential function f on a gradient shrinking Ricci soliton satisfies (A.19). By adding some constant on f, we may normalize f such that

$$R+|\nabla f|^2 = 2\lambda f. \tag{A.23}$$

By combining Theorem 6 and the normalizing condition (A.23), we may give the following upper diameter bound for compact shrinking Ricci solitons in terms of the maximum value of the scalar curvature:

COROLLARY 3 ([39]). *Let (M,g) be an n-dimensional compact shrinking Ricci soliton satisfying* (A.2). *Suppose that the soliton is normalized in sense of* (A.23). *Then*

$$\operatorname{diam}(M,g) \leqslant \frac{\pi}{\sqrt{\lambda}}\sqrt{n-1+\frac{4R_{\max}}{\pi\lambda}}.$$

4. An application to the Hitchin–Thorpe inequality

In this section, we shall introduce some validity of the Hitchin–Thorpe inequality for four-dimensional compact shrinking Ricci solitons. J. Thorpe [43] and N. Hitchin [22] proved independently that, if a four-dimensional compact Riemannian manifold M admits an Einstein metric, then the Euler number $\chi(M)$ and the signature $\tau(M)$ must satisfy the inequality

$$2\chi(M) \geqslant 3|\tau(M)|.$$

This inequality is known as the *Hitchin–Thorpe inequality* and has various geometric implications. For instance, if a four-dimensional compact Riemannian manifold M does not satisfy the inequality, then M never admit any Einstein metric. On the other hand, C. LeBrun [24] proved that there are infinitely many four-dimensional compact simply connected Riemannian manifolds which do not admit any Einstein metric, but nevertheless satisfy the *strict Hitchin–Thorpe inequality*

$$2\chi(M) > 3|\tau(M)|.$$

Just as in Einstein manifolds, we may expect some topological obstructions to the existence of four-dimensional compact Ricci solitons. The validity of the Hitchin–Thorpe inequality for four-dimensional compact shrinking Ricci solitons was first investigated by L. Ma [30] assuming some upper bounds on the L^2-norm of the scalar curvature.

THEOREM 21 (L. Ma [30]). *Let (M,g) be a four-dimensional compact shrinking Ricci soliton satisfying* (A.2). *If the scalar curvature satisfies*

$$\int_M R^2 \leqslant 24\lambda^2 \mathrm{vol}(M,g),$$

then the soliton must satisfy the Hitchin–Thorpe inequality.

M. Fernández-López and E. García-Río [13] investigated the validity of the Hitchin–Thorpe inequality assuming some upper diameter bounds in terms of the Ricci curvature.

THEOREM 22 (M. Fernández-López and E. García-Río [13]). *Let (M,g) be a four-dimensional compact non-trivial shrinking Ricci soliton satisfying* (A.2). *If*

$$\mathrm{diam}(M,g) \leqslant \max\left\{\sqrt{\frac{2}{C-\lambda}}, \sqrt{\frac{2}{\lambda-c}}, 2\sqrt{\frac{2}{C-c}}\right\},$$

then the soliton must satisfy the Hitchin–Thorpe inequality.

The validity of the Hitchin–Thorpe inequality for four-dimensional compact shrinking Ricci solitons may also be obtained by assuming some lower diameter bounds in terms of the range of the scalar curvature.

COROLLARY 4 ([40]). *Let* (M,g) *be a four-dimensional compact shrinking Ricci soliton satisfying* (A.2). *If*

$$\left(2+\sqrt{4+\frac{3}{2}\pi^2}\right)\frac{\sqrt{R_{\max}-R_{\min}}}{\lambda}\leqslant \operatorname{diam}(M,g), \tag{A.24}$$

then the soliton must satisfy the Hitchin–Thorpe inequality.

REMARK 11. In Corollary 4 above, if the soliton has constant scalar curvature, then the soliton appears as an Einstein manifold and the assumption (A.24) above is trivially satisfied. Hence, Corollary 4 may be regarded as a natural generalization of the Hitchin–Thorpe inequality [22, 43] for Einstein manifolds with positive Ricci curvature.

On the other hand, if the soliton is normalized in sense of (A.23), we may give the following sufficient condition for four-dimensional compact shrinking Ricci solitons to satisfy the Hitchin–Thorpe inequality:

COROLLARY 5 ([39]). *Let* (M,g) *be a four-dimensional compact shrinking Ricci soliton satisfying* (A.2). *Suppose that the soliton is normalized in sense of* (A.23). *If*

$$\sqrt{\frac{R_{\max}}{\lambda^2}\left(4\pi+\frac{\pi^2}{2}\right)}\leqslant \operatorname{diam}(M,g),$$

then the soliton must satisfy the Hitchin–Thorpe inequality.

5. Addendum

After this paper was submitted, the author obtained the following lower diameter bound for compact shrinking Ricci solitons in terms of the scalar curvature:

THEOREM 23 ([42]). *Let* (M,g) *be an n-dimensional compact non-trivial shrinking Ricci soliton satisfying* (A.2). *Then*

$$\operatorname{diam}(M,g)\geqslant \frac{R_{\max}-n\lambda}{2\lambda\sqrt{R_{\max}-R_{\min}}}. \tag{A.25}$$

Note that, if the maximum value $R_{\max}$ of the scalar curvature is sufficiently large, then the diameter bound (A.25) above may be sharper than the diameter bound (A.18) obtained by Y. Chu and Z. Hu [9] and A. Futaki, H. Li and X.-D. Li [16].

Moreover, by combining Theorem 19 and Theorem 21, we may provide the following new sufficient condition for four-dimensional compact non-trivial shrinking Ricci solitons to satisfy the Hitchin–Thorpe inequality:

THEOREM 24 ([42]). *Let* (M,g) *be a four-dimensional compact non-trivial shrinking Ricci soliton satisfying* (A.2). *If*

$$\mathrm{diam}(M,g) \leqslant \frac{2\sqrt{2}(\sqrt{2}-1)\pi}{\sqrt{R_{\max}-R_{\min}}},$$

then the soliton must satisfy the Hitchin–Thorpe inequality.

Acknowledgements. The author would like to thank Professor Toshiki Mabuchi for his encouragement. This paper is related to the conference "Geometric structures on Riemannian manifolds" held in Bari, Italy. The author would like to thank the organizers of the conference for their warm hospitality. The author would also like to thank the referee for the careful reading of this paper and for pointing out the reference [10].

References

[1] W. AMBROSE, *A theorem of Myers*, Duke Math. J. **24**, no. 3, (1957), 345–348.

[2] B. ANDREWS AND L. NI, *Eigenvalue comparison on Bakry–Emery manifolds*, Comm. Partial Differential Equations **37**, no. 11, (2012), 2081–2092.

[3] D. BAKRY AND M. ÉMERY, *Diffusion hypercontractives*, Séminaire de probabilités, XIX, 1983/84, 177–206, Lecture Notes in Math., 1123, Springer, Berlin, 1985.

[4] E. CALABI, *On Ricci curvature and geodesics*, Duke Math. J. **34**, no. 4, (1967), 667–676.

[5] H.-D. CAO, *Existence of gradient Kähler–Ricci solitons*, Elliptic and parabolic methods in geometry (Minneapolis, MN, 1994), 1–16, A. K. Peters (ed.), Wellesley, MA, 1996.

[6] ______, *Recent progress on Ricci solitons*, Recent advances in geometric analysis, 1–38, Adv. Lect. Math. (ALM), 11, Int. Press, Somerville, MA, 2010.

[7] M. P. CAVALCANTE, J. Q. OLIVEIRA AND M. S. SANTOS, *Compactness in weighted manifolds and applications*, Results Math. **68**, no. 1, (2015), 143–156.

[8] T. CHAVE AND G. VALENT, *On a class of compact and non-compact quasi–Einstein metrics and their renormalizability properties*, Nuclear Phys. B **478**, no. 3, (1996), 758–778.

[9] Y. CHU AND Z. HU, *Lower bound estimates of the first eigenvalue for the f–Laplacian and their applications*, Q. J. Math. **64**, no. 4, (2013), 1023–1041.

[10] A. S. DANCER AND M. Y. WANG, *On Ricci solitons of cohomogeneity one*, Ann. Global Anal. Geom. **39**, no. 3, (2011), 259–292.

[11] M. FELDMAN, T. ILMANEN AND D. KNOPF, *Rotationally symmetric shrinking and expanding gradient Kähler–Ricci solitons*, J. Differential Geom. **65**, no. 2, (2003), 169–209.

[12] M. FERNÁNDEZ-LÓPEZ AND E. GARCÍA-RÍO, *A remark on compact Ricci solitons*, Math. Ann. **340**, no. 4, (2008), 893–896.

[13] ______, *Diameter bounds and Hitchin–Thorpe inequalities for compact Ricci solitons*, Q. J. Math. **61**, no. 3, (2010), 319–327.

[14] ______, *Some gap theorems for gradient Ricci solitons*, Internat. J. Math. **23**, no. 7, (2012), 1250072, 9pages.

[15] D. FRIEDAN, *Nonlinear models in* $2+\varepsilon$ *dimensions*, Ann. Phys. **163**, no. 2, (1985), 318–419.

[16] A. FUTAKI, H. LI AND X.-D. LI, *On the first eigenvalue of the Witten–Laplacian and the diameter of compact shrinking solitons*, Ann. Global Anal. Geom. **44**, no. 2, (2013), 105–114.

[17] A. FUTAKI AND Y. SANO, *Lower diameter bounds for compact shrinking Ricci solitons*, Asian J. Math. **17**, no. 1, (2013), 17–32.

[18] G. J. GALLOWAY, *A generalization of Myers theorem and an application to relativistic cosmology*, J. Differential Geom. **14**, no. 1, (1979), 105–116.

[19] ______, *Compactness criteria for Riemannian manifolds*, Proc. Amer. Math. Soc. **84**, no. 1, (1982), 106–110.

[20] R. HAMILTON, *The Ricci flow on surfaces*, Mathematics and general relativity (Santa Cruz, CA, 1986), 237–262, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988.

[21] ______, *The formation of singularities in the Ricci flow*, Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), 7–136, Int. Press, Cambridge, MA, 1995.

[22] N. HITCHIN, *Compact four-dimensional Einstein manifolds*, J. Differential Geom. **9**, no. 3, (1974), 435–442.

[23] N. KOISO, *On rotationally symmetric Hamilton's equation for Kähler–Einstein metrics*, Recent topics in differential and analytic geometry, 327–337, Adv. Stud. Pure Math., 18-I, Academic Press, Boston, MA, 1990.

[24] C. LEBRUN, *Four-manifolds without Einstein metrics*, Math. Res. Lett. **3**, no. 2, (1996), 133–147.

[25] H. LI, *Gap theorems for Kähler–Ricci solitons*, Arch. Math. (Basel) **91**, no. 2, (2008), 187–192.

[26] X.-D. LI, *Liouville theorems for symmetric diffusion operators on complete Riemannian manifolds*, J. Math. Pures Appl. **84**, no. 10, (2005), 1295–1361.

[27] Y. LI, *Li–Yau–Hamilton estimates and Bakry–Emery–Ricci curvature*, Nonlinear Anal. **113** (2015), 1–32.

[28] M. LIMONCU, *Modifications of the Ricci tensor and applications*, Arch. Math. (Basel) **95**, no. 2, (2010), 191–199.

[29] ______, *The Bakry–Emery Ricci tensor and its applications to some compactness theorems*, Math. Z. **271**, no. 3, (2012), 715–722.

[30] L. MA, *Remarks on compact Ricci solitons of dimension four*, C. R. Acad. Sci. Paris, Ser. I **351** no. 21–22, (2013), 817–823.

[31] P. MASTROLIA, M. RIMOLDI AND G. VERONELLI, *Myers–type theorems and some related oscillation results*, J. Geom. Anal. **22**, no. 3, (2012), 763–779.

[32] S. B. MYERS, *Riemannian manifolds with positive mean curvature*, Duke Math. J. **8**, no. 2, (1941), 401–404.

[33] G. PERELMAN, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math/0211159, Preprint, 2002.

[34] ______, *Ricci flow with surgery on three-manifolds*, arXiv:math/0303109, Preprint, 2003.

[35] ______, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv:math/0307245, Preprint, 2003.

[36] F. PODESTÀ AND A. SPIRO, *Kähler–Ricci solitons on homogeneous toric bundles*, J. Reine Angew. Math. **642** (2010), 109–127.

[37] Z. QIAN, *Estimates for weighted volumes and applications*, Quart. J. Math. Oxford Ser. (2) **48**, no. 2, (1997), 235–242.

[38] M. RIMOLDI, *A remark on Einstein warped products*, Pacific J. Math. **252**, no. 1, (2011), 207–218.

[39] H. TADANO, *Remark on a diameter bound for complete Riemannian manifolds with positive Bakry–Émery Ricci curvature*, Diff. Geom. Appl. **44** (2016), 136–143.

[40] ______, *An upper diameter bound for compact Ricci solitons with application to the Hitchin–Thorpe inequality*, J. Math. Phys. **58** (2017), 023503, 8 pages.

[41] ______, *Some Ambrose and Galloway types theorems via Bakry–Émery and modified Ricci curvatures*, Preprint, 2015.

[42] ______, *Remark on a lower diameter bound for compact shrinking Ricci solitons*, Preprint, 2016.

[43] J. A. THORPE, *Some remarks on the Gauss–Bonnet integral*, J. Math. Mech. **18**, no. 8, (1969), 779–786.

[44] X.-J. WANG AND X. ZHU, *Kähler–Ricci solitons on toric manifolds with positive first Chern class*, Adv. Math. **188**, no. 1, (2004), 87–103.

[45] G. WEI AND W. WYLIE, *Comparison geometry for the Bakry–Emery Ricci tensor*, J. Differential Geom. **83**, no. 2, (2009), 377–405.

[46] S. ZHANG, *A theorem of Ambrose for Bakry–Emery Ricci tensor*, Ann. Global Anal. Geom. **45**, no. 3, (2014), 233–238.

AMS Subject Classification: Primary 53C21, Secondary 53C20, 53C25

Homare TADANO,
Department of Mathematics, Graduate School of Science, Osaka University
1-1 Machikaneyama, Toyonaka, Osaka 560-0043, JAPAN
e-mail: `h-tadano@cr.math.sci.osaka-u.ac.jp`

Lavoro pervenuto in redazione il 19.01.2016 e accettato il 30.03.2016.

Geom. Struc. on Riem. Man.-Bari
Vol. 73/1, 3–4 (2015), 201 – 215

R. A. Wolak

ORBIFOLDS, GEOMETRIC STRUCTURES AND FOLIATIONS. APPLICATIONS TO HARMONIC MAPS

Dedicated to Anna Maria Pastore on her 70th birthday

Abstract. In recent years a lot of attention has been paid to topological spaces which are a bit more general than smooth manifolds - orbifolds. Orbifolds are intuitively speaking manifolds with some singularities. The formal definition is also modelled on that of manifolds, an orbifold is a topological space which locally is homeomorphic to the orbit space of a finite group acting on R^n. Orbifolds were defined by Satake, as V-manifolds, cf. [25], then studied by W. Thurston, cf. [29], who introduced the term "orbifold". Due to their importance in physics, and in particular in the string theory, orbifolds have been drawing more and more attention. In this paper we propose to show that the classical theory of geometrical structures, cf. [27, 15, 22], easily translates itself to the context of orbifolds and is closely related to the theory of foliated geometrical structures, cf. [32]. Finally, we propose a foliated approach to the study of harmonic maps between Riemannian orbifolds based on our previous research into transversely harmonic maps.

1. Orbifolds and their smooth complete maps

Let X be a topological space, and fix $n \geq 0$. An n-dimensional orbifold chart on X is given by a connected open subset $\tilde{U} \subset R^n$, a finite group Γ of smooth diffeomorphisms of $\tilde{U}$, and a map $\phi\colon \tilde{U} \to X$ so that ϕ is Γ-invariant and induces a homeomorphism of $\tilde{U}/\Gamma$ onto an open subset $U \subset X$.

An embedding $\lambda\colon (\tilde{U},\Gamma,\phi) \to (\tilde{V},\Delta,\psi)$ between two such charts is a smooth embedding $\lambda\colon \tilde{U} \to \tilde{V}$ with $\psi\lambda = \phi$. Then there exists a homomorphism $\alpha\colon \Gamma \to \Delta$ such that the mapping λ is α-equivariant.

An orbifold atlas on X is a family $\mathcal{U} = \{(\tilde{U},\Gamma,\phi)\}$ of such charts, which cover X and are locally compatible:

given any two charts $(\tilde{U},\Gamma,\phi)$ for $U = \phi(\tilde{U}) \subset X$ and $(\tilde{V},\Delta,\psi)$ for $V = \psi(\tilde{V}) \subset X$ and a point $x \in U \cap V$, there exists an open neighborhood $W \subset U \cap V$ of x and a chart $(\tilde{W},\Lambda,\tau)$ for $W = \tau(\tilde{W}) \subset X$ such that there are two embeddings $(\tilde{W},\Lambda,\tau) \to (\tilde{U},\Gamma,\phi)$ and $(\tilde{W},\Lambda,\tau) \to (\tilde{V},\Delta,\psi)$.

An atlas $\mathcal{U}$ is said to refine another atlas $\mathcal{V}$ if for every chart in $\mathcal{U}$ there exists an embedding into some chart of $\mathcal{V}$. Two orbifold atlases are said to be equivalent if they have a common refinement.

With these notions defined above we can formulate the definition of an orbifold

DEFINITION 1. *An effective orbifold X of dimension n is a paracompact Hausdorff space X equipped with an equivalence class $[\mathcal{U}]$ of n-dimensional orbifold atlases.*

In this paper we will consider only effective orbifolds without further mentioning it.

For any effective orbifold we can find an atlas as in the following definition due to Borzellino and Brunsden, cf. [2]

DEFINITION 2. *An n-dimensional smooth orbifold O consists of a paracompact, Hausdorff topological space X_O called the underlying space, with the following local structure. For each $x \in X_O$ and neighborhood U of x, there is a neighborhood $U_x \subset U$ of x, an open set $\tilde{U}_x$ diffeomorphic to R^n, a finite group Γ_x acting smoothly and effectively on $\tilde{U}_x$ which fixes $0 \in \tilde{U}_x$, and a homeomorphism $\phi_x : \tilde{U}_x/\Gamma_x \to U_x$ with $\phi_x(0) = x$. These actions are subject to the condition that for a neighborhood $U_z \subset U_x$ with the corresponding $\tilde{U}_z \hat{=} R^n$, group Γ_z and homeomorphism $\phi_z \colon \tilde{U}_z/\Gamma_z \to U_z$ there is a smooth embedding $\tilde{\psi}_{zx} \colon \tilde{U}_z \to \tilde{U}_x$ and an injective homomorphism $\theta_{zx} \colon \Gamma_z \to \Gamma_x$ so that $\tilde{\psi}_{zx}$ is equivariant with respect to θ_{zx}. This means that for any $\gamma \in \Gamma_z; \tilde{\psi}_{zx}(\gamma\tilde{y}) = \theta_{zx}(\gamma)\tilde{\psi}_{zx}(\tilde{y})$ for all $\tilde{y} \in \tilde{U}_z$ and the following diagram commutes:*

$$\begin{tikzcd}
\tilde{U}_z \arrow[r, "\tilde{\psi}_{zx}"] \arrow[d] & \tilde{U}_x \arrow[d] \\
\tilde{U}_z/\Gamma_z \arrow[r, "\psi_{zx}=\tilde{\psi}_{zx}/\Gamma_z"] \arrow[d, "\phi_z"] & \tilde{U}_x/\theta_{zx}(\Gamma_z) \arrow[d, "\phi_x"] \\
U_z \arrow[r, "\subset"] & U_x
\end{tikzcd}$$

For a given orbifold atlas $\mathcal{U}$, the underlying topological space X_O is homeomorphic to the topological space $X_{\mathcal{U}}$ defined as the quotient of the disjoint union of $\tilde{U}_i$ by the equivalence relation induced by the action of the groups Γ_i and the compatibility condition. When there is no ambiguity the underlying space of the orbifold X will be denoted by the same letter X.
We will refer to the neighborhood U_x or $(\tilde{U}_x, \Gamma_x)$ or $(\tilde{U}_x, \Gamma_x, \rho_x, \phi_x)$ as an orbifold chart. In the 4-tuple notation, we are making explicit the representation $\rho_x : \Gamma_x \to \mathrm{Diff}(\tilde{U}_x)$. The isotropy group of x is the group Γ_x. The definition of orbifold implies that the germ of the action of Γ_x in a neighborhood of the origin of R^n is unique, so that by shrinking $\tilde{U}_x$ if necessary, Γ_x is well-defined up to isomorphism. The singular set of O is the set of points $x \in X_O$ with $\Gamma_x \neq \{e\}$.

We present the definition of a complete orbifold map from [2].

DEFINITION 3. *A C^∞ complete orbifold map $(f, \{\tilde{f}_x\}, \{\Theta_{f,x}\})$ between smooth orbifolds O and $\mathcal{P}$ consists of the following:*

(1) a continuous map $f \colon X_O \to X_{\mathcal{P}}$ of the underlying topological spaces,

(2) for each $y \in X_O$, there is a group homomorphism $\Theta_{f,y} \colon \Gamma_y \to \Gamma_{f(y)}$,

(3) a smooth $\Theta_{f,y}$-equivariant lift $\tilde{f}_y \colon \tilde{U}_y \to \tilde{V}_{f(y)}$ where $(\tilde{U}_y, \Gamma_y)$ is an orbifold chart at y and $(\tilde{V}_{f(y)}, \Gamma_{f(y)})$ is an orbifold chart at $f(y)$. That is the following diagram commutes:

$$\begin{tikzcd}
\tilde{U}_y \arrow[r, "\tilde{f}_y"] \arrow[d] & \tilde{V}_{f(y)} \arrow[d] \\
\tilde{U}_y/\Gamma_y \arrow[r, "\tilde{f}_y/\Theta_{f,y}"] \arrow[dd] & \tilde{V}_{f(y)}/\Theta_{f,y}(\Gamma_y) \arrow[d] \\
 & \tilde{V}_{f(y)}/\Gamma_{f(y)} \arrow[d] \\
U_y \arrow[r, "f"] & V_{f(y)}
\end{tikzcd}$$

(4) (Equivalence) Two complete orbifold maps $(f,\{\tilde{f}_x\},\{\Theta_{f,x}\})$ and $(g,\{\tilde{g}_x\},\{\Theta_{g,x}\})$ are considered equivalent if for each $x \in X_O$ $\tilde{f}_x = \tilde{g}_x$ as germs at x and $\Theta_{f,x} = \Theta_{g,x}$ That is, there exists an orbifold chart $(\tilde{U}_x, \Gamma_x)$ at x such that $\tilde{f}_x|_{\tilde{U}_x} = \tilde{g}_x|_{\tilde{U}_x}$ and $\Theta_{f,x} = \Theta_{g,x}$. Note that this implies that $f = g$.

The set of smooth complete orbifold maps from $\mathcal{O}$ to $\mathcal{P}$ will be denoted by $C^\infty_{Orb}(\mathcal{O},\mathcal{P})$

In a very similar way to submanifolds one can define suborbifolds. The first definition of a suborbifold was given by W. Thurston in [29], but it seems to be too restrictive. We recall and use the definition formulated in [3]

DEFINITION 4. *An (embedded) suborbifold P of an orbifold O consists of the following:*

i) A subspace $X_P \subset X_O$ equipped with the subspace topology,

ii) For each $x \in X_P$ and neighbourhood W of x in X_O there is an orbifold chart $(\tilde{U}_x, \Gamma_x, \rho_x, \phi_x)$ about x in O with $U_x \subset W$, a subgroup $\Lambda_x \subset \Gamma_x$ of the isotropy group of x in O and a $\rho_x(\Lambda_x)$-invariant submanifold $\tilde{V}_x \subset \tilde{U}_x \cong R^n$, so that $(\tilde{V}_x, \Lambda_x/\Omega_x, \rho_x|_{\Lambda_x}, \Psi_x)$ is an orbifold chart for P where $\Omega_x = \{\gamma\Lambda_x \colon \rho_x(\gamma)|_{\tilde{V}_x} = id\}$. (In particular, the intrinsic isotropy subgroup at x),

iii) $V_x = \psi_x(\tilde{V}_x/\rho_x(\Lambda_x)) = U_x \cap X_P$ is an orbifold chart domain for $x \in P$.

Thurston's definition is a bit more restrictive:

DEFINITION 5. *A $Y \subset X$ is called a full suborbifold of X if Y is a suborbifold with $\Lambda_x = \Gamma_x$ for all points x of Y.*

For the differences in the definitions and the reasons for them see [3].

2. Fibre bundles over orbifolds

Let F be a smooth manifold.

DEFINITION 6. *An orbifold E is called an orbifold fibre bundle over the orbifold X with standard fibre F if*
i) there exists a smooth orbifold map $p\colon E \to X$,
ii) there exists an orbifold atlas $\mathcal{U}$ of X, i.e., for any $x \in X$ there exists an orbifold chart (U_i, Γ_i, ψ_i) of $\mathcal{U}$ such that $x \in U_i$ and $\tilde{U}_i$ an open subset of R^n, Γ_i is a finite group of diffeomorphisms of $\tilde{U}_i$, and $\tilde{U}_i/\Gamma_i$ is homeomorphic to U_i, ψ_i being the homeomorphism,
iii) let $V_i = p^{-1}(U_i)$ and $\tilde{V}_i = \tilde{U}_i \times F$, then there exist a group Λ_i of fibre preserving diffeomorphisms of $\tilde{V}_i$ and a homeomorphism $\phi_i\colon \tilde{V}_i/\Lambda_i \to V_i$ such that $\{(\tilde{V}_i, \Lambda_i)\}$ form an atlas of the orbifold E,
iv) and the following diagram is commutative

$$\begin{tikzcd}
\tilde{U}_i \times F \arrow[rr, "\tilde{p}=p\times id"] \arrow[d] & & \tilde{U}_i \arrow[d] \\
\tilde{V}_i/\Lambda_i \arrow[d, "\phi_i"] & & \tilde{U}_i/\Gamma_i \arrow[d, "\psi_i"] \\
V_i \arrow[rr, "p"] & & U_i
\end{tikzcd}$$

Remark We can assume that $\tilde{p}$ is (Λ_i, Γ_i)-equivariant. Obviously, p is a C^∞-complete orbifold map. In fact, we will consider only fibre budnles for which the groups Λ_i and Γ_i are isomorphic.

Examples

a) The **tangent bundle** TX of an orbifold X, $F = R^n$

We construct the tangent bundle as follows. Take any orbifold atlas $\mathcal{U} = \{(\tilde{U}_i, \Gamma_i, \phi_i)\}$. Then take $\tilde{V}_i = T\tilde{U}_i = \tilde{U}_i \times R^n$, as the group Σ_i local transformations take $\Sigma_i = \{d\gamma\colon \gamma \in \Gamma_i\}$ and as ψ_i the quotient map $\tilde{V}_i \to \tilde{V}_i/\Sigma_i$. The condition of local compatibility of these charts is obviously satisfied with the required embeddings provided by the differentials of the embeddings of the compatibility condition of the atlas $\mathcal{U}$. The topological space TX is defined as the quotient of the disjoint union of $T\tilde{U}_i$ by the equivalence relation induced by the action of the groups Σ_i and the embeddings from the definition of an orbifold atlas. Equivalent atlases of the orbifold X define equivalent atlases of TX.

A smooth complete orbifold mapping $f\colon X \to Y$ defines the mapping $df\colon TX \to TY$ of the orbifold tangent bundles which is called the differential of f. If $(\tilde{U}, \Gamma, \phi)$ is a chart of X and $(\tilde{W}, \Delta, \rho)$ is a chart of Y such that $f(U) \subset W$ and $\tilde{f}\colon \tilde{U} \to \tilde{W}$ covers f then the differential $d\tilde{f}$ satisfies the condition of a complete orbifold mapping for the just defined atlases of the orbifold tangent bundles TX and TY. It is a simple exercise to verify that df is a smooth complete map of the orbifolds TX and TY. Thus we have defined a functor from the category of (effective) orbifolds and their complete smooth mappings into the category of orbifold vector bundles and their complete smooth vector bundle mappings.

b) The **linear frame** bundle of an orbifold, $F = GL(n)$

The same construction as in the point a) using the frame bundle of open subsets of R^n instead of the tangent bundle defines the orbifold principal frame bundle of an orbifold. As the local groups are finite, and the fact that one can make them local isometries, the total space of the orbifold principal bundle is in fact a manifold. Any smooth complete mapping of orbifolds defines a smooth complete mapping of their frame bundles. The above correspondence is a functor from the category of (effective) orbifolds and their complete smooth mappings into the category of principal frame bundles and their smooth bundle mappings.

c) **Higher order frame and tangent** bundles

The same procedure can be applied to the functor of higher order frame and tangent bundles. In this way we define the functor from the category of smooth orbifolds and their smooth complete maps to the category of orbifold fibre bundles and their smooth complete fibre maps.

Similar constructions work well in the cases of fibre bundles listed below:

d) associated bundles to higher order frame bundles

e) natural bundles.

It is not difficult to verify that one can define the dual vector bundle E^* of any orbifold vector bundle E, and that E^* is an orbifold vector bundle. Moreover, any tensor product of orbifold vector bundles over a given orbifold is itselt an orbifold vector bundle. Therefore we can define the orbifold tensor algebra

$$\bigotimes TX = \bigoplus_{p,q} \bigotimes_{p}^{q} TX = \bigoplus_{p,q} (\otimes^p TX) \otimes (\otimes^q TX^*).$$

Likewise for any finite number of orbifold vector bundles over a given orbifold X we can define their skewsymmetric product which itself is an orbifold vector bundle over X.

Another classical construction for fibre bundles over smooth manifolds works well within the framework of orbifolds, the pullback. Let $f\colon X \to Y$ be a smooth complete orbifold mapping between two orbifolds X and Y. Let $p\colon E \to Y$ be a smooth orbifold bundle, then the pullback bundle

$$f^{-1}E = \{(x,w) \in X \times E\colon f(x) = p(w)\}$$

is a well-defined orbifold bundle over X with the same standard fibre as the bundle E.

Remark The theory of natural bundles is well presented in [21], see also [28].

A.1. Sections of fibre bundles over an orbifold

We shall consider sections of orbifold fibre bundles which are smooth complete orbifold mappings.

If $p\colon E \to X$ is a smooth orbifold bundle, $Sect_{comp}(X,E)$ denotes the space of smooth complete sections of E, i.e., the set of all smooth complete orbifold mapping $s\colon X \to E$ such that $ps = id_X$.

Let $\mathcal{U}$ be an orbifold atlas of the orbifold X. The existence of a section s of the fibre bundle $s\colon X \to E$ is equivalent to the existence on each U_i of a section s_i of $\tilde{U}_i \times F$ which is (Γ_i, Λ_i)-equivariant and satisfy the compatibility condition.

Riemannian metrics

A Riemannian metric on an orbifold X is given by a family of Riemannian metrics g_i on $\tilde{U}_i$ which are Γ_i invariant, i.e. elements of Γ_i are isometries of the Riemannian metrics g_i.

PROPOSITION 1. *On any orbifold X there exists a Riemannian metric g_X.*

As the metrics g_i on $\tilde{U}_i$ are compatible, so are the associated Levi-Civita connections ∇^i. The induced object on the orbifold X we call the Levi-Civita connection of the Riemannian metric g. The operator ∇ can be characterized as a two-linear mapping

$$\mathcal{X}_{comp}(X) \times \mathcal{X}_{comp}(X) \to \mathcal{X}_{comp}(X)$$

satisfying the standard condtions for connections. The operator ∇ can be extended to the orbifold tensor algebra $\bigotimes TX$ in the same way as in the case of manifolds. For a given complete vector field Z on the orbifold X and a complete (p,q) tensor field T, $\nabla_Z T$ is also a complete (p,q) tensor field, or ∇T a complete $(p,q+1)$ tensor field.

A complete section of the orbifold tensor bundle $\otimes^p_q TX$ is called a (p,q)-orbi-tensor field on the orbifold X. Let $T_0 \in (\otimes^p R^n) \otimes (\otimes^q R^{n*})$ be a (p,q)-tensor of R^n. A tensor field T is called 0-deformable if for any point $x \in X$ there exists a frame p at that point such that $p^* T_x = T_0$ for some tensor T_0.

A.2. Geometrical structures on orbifolds

In the case of manifolds a very general definition defines a geometrical structure as a submanifold or a subbundle of a natural bundle over the chosen manifold. The same approach can be extended to the case of orbifolds.

A suborbifold H of the total space of an orbifold natural bundle K which is at the same time an orbifold natural bundle is called a orbifold (fibre) subbundle.

DEFINITION 7. *A geometrical structure on an orbifold X is an orbifold subbundle of a natural orbifold bundle H over X.*

In particular, one can develop the theory of classical orbifold G-structures for any classical Lie group $G \subset GL(n)$. For a fixed Lie group G orbifold G-structures on an orbifold of dimension n is an orbifold subbundle $B(X,G)$ of the linear frame bundle $L(X)$. As in the case of manifolds any 0-deformable orbi-tensor field T determines an orbifold G_T-structure. In this case the corresponding G-structure is defined as, for any point $x \in X$,

$$B(T)_x = \{p \in L(X)_x \colon p^* T_x = T_0\}$$

and the structure group is the Lie subgroup of $GL(n)$:

$$G(T_0) = \{A \in GL(n) \colon A^* T_0 = T_0\}.$$

Remark Many geometrical objects or procedures associated to G-structures over manifolds have their orbifold counterparts. It is not difficult to see that for an orbifold G-structure we can define the structure tensor in the same way as for manifolds, cf. [15, 27]. The structure tensor c is a smooth function with values in

$$Hom(R^n \wedge R^n, R^n)/\partial Hom(R^n, Lie(G)).$$

The vanishing of the structure tensor of an orbifold G-structure is equivalent to the existence of a torsionless orbifold connection in this orbifold G-structure. Likewise, one can prove that the 1st prolongation of an orbifold G-structure is an orbifold bundle. So the theory of prolongations of G-structures works well for orbifolds, cf. [15, 27].

Like in the classical case of manifolds we can define many geometrical structures in the dual way, via orbi-tensor fields or via orbifold reductions of the orbifold frame bundle.

Examples

a) **Riemannian structure (Riemannian metric)** In the previous subsection we have defined an orbifold Riemannian metric as a section of the orbifold bundle $\bigwedge^2 T^*X$. It is equivalent to a choice of an orbifold $O(n)$ reduction of the orbifold linear frame bundle LX.

b) **Symplectic structure** A symplectic structure on the orbifold X can be introduced by choosing a symplectic form on X, or equivalently an orbifold $Sp(2m)$-reduction of the orbifold linear frame bundle $L(X)$, cf. [31, 1].

c) **Kähler structure** A $U(n)$-reduction of the orbifold linear frame bundle LX of the orbifold X would give us an almost Kähler structure on X. The vanishing of its structure tensor is equivalent to the existence of a $U(n)$ torsionless connection, thus to the integrability of the almost-complex structure.

d) In a similar way we can define hyperKähler or quaternionic structures on orbifolds, cf. [4].

Remark Having said that we realize that with no difficulty we can define on orbifolds contact structures, K-contact and Sasakian structures as well as 3-Sasakian or K-structures. The only difference is that all the objects considered have to be orbifold tensor fields.

A.3. Harmonic mappings

Any smooth complete orbifold mapping $f\colon X \to Y$ defines a smooth complete orbifold mapping $df\colon TX \to TY$ called the differential of f. The differential df is a smooth complete section of the bundle

$$TX \otimes f^{-1}TY.$$

If the orbifolds X and Y are Riemannian, then the associated Levi-Civita connections ∇^X and ∇^Y, respectively, define a connection D in the vector bundle $TX \otimes f^{-1}TY$. Therefore it makes sense to consider

$$Ddf$$

and its trace

$$\tau(f)$$

$\tau(f)$ is a complete section of the bundle $f^{-1}TY \to X$. We call it the tension field of the complete mapping f between the Riemannian orbifolds (X, g_X) and (Y, g_Y).

DEFINITION 8. *A complete orbifold mapping $f\colon X \to Y$ between two Riemannian orbifolds (X, g_X) and (Y, g_Y) is harmonic if its tension field vanishes.*

Remark A different approach to the study of harmonic maps between orbifolds was proposed by Y.-J. Chiang in [7]. The same approach can be applied to the study of bi-harmonic, f-harmonic, etc. mappings as well as harmonic morphisms between orbifolds using the theory developed by Y.-J. Chiang and the author, cf. [8, 9, 10, 11, 12].

3. Orbifolds and foliations

At the begining of the seventies foliations with all leaves compact were an object of very interesting studies. In the case of compact manifolds it was proved that the local boundedness of the volume of leaves is equivalent to the leaf space being Hausdorff, or to the fact that the holonomy group of any leaf is finite. That in turn means that the leaf space of such a foliation is an orbifold, cf. [14, 22, 13]. In this case the foliation is Riemannian and taut, cf. [23]. A few years later A. Haefliger et al. cf. [16], remarked that any orbifold can be realized as the leaf space of a Riemannian foliation with compact leaves. It is a simple remark, having noticed that an orbifold admits a Riemannian metric, they take the associated orthonormal frame bundle, which in this case is a smooth manifold. If we take a compact orbifold, its orthonormal frame bundle is a compact manifold foliated by a foliation with all leaves compact, the fibres of this bundle. Its leaf space can be identified with the initial orbifold. Therefore Riemannian foliations with all leaves compact, or more general Riemannian foliations with leaves of finite holonomy, can be considered as natural desingularization objects of orbifolds. For a given orbifold X we will denote this foliated manifold by $(O_X, \mathcal{F}_X)$.

In the study of the geometry of a single orbifold this duality between the orbifold and its orthogonal frame bundle foliated by the compact fibres, between orbifold objects and foliated objects, works well, cf. e.g. [1].

The theory of foliated G-structures was developed mainly by Pierre Molino, cf. [23] and his numerous papers. The most general theory was proposed by the author in [32]. It is based on the theory of natural bundles.

Let $(M, g, \mathcal{F})$ be a compact Riemannian foliated manifold with compact leaves. In fact, we can assume less: it is sufficient to have a complete bundle-like Riemannian metric and leaves of finite holonomy. Then according to the Reeb stability theorem, cf. [24, 5], each leaf L admits a foliated (saturated by leaves) tubular neighbourhood U. Any fibre of such a tubular neighbourhood is a transverse submanifold intersecting all leaves contained in the neighbourhood. The holonomy of the zero section, a leaf of $\mathcal{F}$, defines a finite subgroup Γ of diffeomorphisms of the fibre D. The space $U/\mathcal{F}$ of leaves contained in the tubular neighbourhood U is identified with the the orbit space D/Γ. Let U and V be two such tubular neighbourhoods of leaves L_1 and L_2, respectively. If the intersection is not empty, it is an open saturated subset. For any leaf L in $U \cap V$ we can find an open saturated tubular neighbourhood $W \subset U \cap V$. The choice of a fibre of W defines two embeddings of $W/\mathcal{F}$ into $U/\mathcal{F}$ and $V/\mathcal{F}$, respectively. Thus the orbifold charts of the leaf space $M/\mathcal{F}$ defined using these tubular neighbourhoods are compatible, hence we have an orbifold atlas on $M/\mathcal{F}$. Different choices of tubular neighbourhoods and of different fibres in them produce different but equivalent orbifold atlases. Thus we have defined an orbifold structure on $M/\mathcal{F}$.

In the language of foliations, the orbifold associated to a foliations with compact leaves and finite holonomy contains the following data: the underlying topological space is the space of leaves of the foliation, the orbifold atlas corresponds to the choice of a complete transverse manifold, and the local finite groups are the elements of the holonomy pseudogroup induced on this transverse manifold. The embeddings from the definition of the orbifold atlas are also induced by elements of the holonomy pseudogroup, those which map open subsets of one connected component of the transverse manifold into another. The choices described in the previous paragraph lead to different but equivalent pseudogroups, cf. [17].

In [32], the author demonstrated that foliated geometrical structures are in one-to-one correspondence with holonomy invariant geometrical structures on the transverse manifold. In the case of foliated manifolds with foliations with compact leaves and finite holonomy, this result can be reformulated saying that foliated geometrical structures are in one-to-one correspondence with orbifold geometrical structures on the leaf space equiped with the induced orbifold structure. In particular, any bundle-like Riemannian metric of the foliated manifold $(M,\mathcal{F})$ induces an orbifold Riemannian metric on the leaf space $M/\mathcal{F}$ and vice versa.

Let $f\colon (M_1,\mathcal{F}_1) \to (M_2,\mathcal{F}_2)$ be a foliated smooth mapping between two foliated manifolds. We can choose two open coverings $\mathcal{U}_1$ and $\mathcal{U}_2$ by tubular neighbourhoods of leaves of $(M_1,\mathcal{F}_1)$ and $(M_2,\mathcal{F}_2)$, respectively, such that for any $U \in \mathcal{U}_1$ there exists $V \in \mathcal{U}_2$ such that $f(U)\subset V$. This choice ensures that the foliated mapping f induces a holonomy invariant mapping $\bar{f}$ of the associated transverse manifolds, cf. [18]. Therefore the induced mapping $\tilde{f}\colon M_1/\mathcal{F}_1 \to M_2/\mathcal{F}_2$ between the leaf spaces, considered as orbifolds, is a complete orbifold mapping. The remark can be understood as an orbifold version of the considerations of Section 3.2 of [18] about the mapping induced on the transverse manifolds by a foliated mapping.

Let us recall the precise definition of the mapping $\bar{f}$ presented in [18].

Let $f\colon (M_1,\mathcal{F}_1) \to (M_2,\mathcal{F}_2)$ be a foliated smooth mapping between two foliated manifolds. We suppose that $\mathcal{U} = \{(U_i,\phi_i,g_{ij})\}_I$ is a cocycle defining the foliation $\mathcal{F}_1$, and denote by $\mathcal{V} = \{(V_\alpha,\psi_\alpha,h_{\alpha\beta})\}_A$ a cocycle defining the foliation $\mathcal{F}_2$ such that, for any $i \in I$, there exists $\alpha(i) \in A$ such that $f(U_i)\subset V_{\alpha(i)}$. Let $\bar{U}_i = \phi_i(U_i)$ and $\bar{V}_\alpha = \psi_\alpha(V_\alpha)$. Then the manifold $N_1 = \coprod \bar{U}_i$ is a transverse manifold of the foliation $\mathcal{F}_1$, and $N_2 = \coprod \bar{V}_\alpha$ is a transverse manifold of the foliation $\mathcal{F}_2$. The transformations g_{ij} generate a pseudogroup $\mathcal{H}_1$, which is called the holonomy pseudogroup of $\mathcal{F}_1$ associated with the cocycle $\mathcal{U}$, and the transformations $h_{\alpha\beta}$ generate a pseudogroup $\mathcal{H}_2$, which is called the holonomy pseudogroup of $\mathcal{F}_1$ associated with the cocycle $\mathcal{V}$. On the level of transverse manifolds, the map f induces a smooth map $\bar{f}$, as, for any $i \in I$, the following diagram is commutative:

$$\begin{tikzcd}
U_i \arrow[r, "f|U_i"] \arrow[d, "\tilde{\phi}_i"] & \tilde{U}_x \arrow[d, "\tilde{\psi}_{\alpha(i)}"] \\
\bar{U}_i \arrow[r, "\bar{f}_{\alpha(i)i}"] & \bar{V}_{\alpha(i)}
\end{tikzcd}$$

The map $\bar{f}\colon N_1 \to N_2$ is defined as follows:

$$\bar{f}|\bar{U}_i = \bar{f}_{\alpha(i)i}.$$

Taken open subsets U_i, U_j with $U_i \cap U_j \neq \emptyset$, then $f(U_i \cap U_j)\subset V_{\alpha(i)} \cap V_{\alpha(j)}$. This intersection covers the open subsets $\bar{U}_{ji}\subset\bar{U}_i$ and $\bar{U}_{ij}\subset\bar{U}_j$. Likewise $V_{\alpha(i)} \cap V_{\alpha(j)}$ covers $\bar{V}_{\alpha(j)\alpha(i)}\subset\bar{V}_{\alpha(i)}$ and $\bar{V}_{\alpha(i)\alpha(j)}\subset\bar{V}_{\alpha(j)}$. Moreover, the map $g_{ji}\colon \bar{U}_{ji} \to \bar{U}_{ij}$ is a diffeomorphism as well as the map $h_{\alpha(j)\alpha(i)}\colon \bar{V}_{\alpha(j)\alpha(i)} \to \bar{V}_{\alpha(i)\alpha(j)}$. Then

$$h_{\alpha(j)\alpha(i)}\bar{f}_{\alpha(i)i}|\bar{U}_{ji} = \bar{f}_{\alpha(j)j}g_{ji}|\bar{U}_{ji}.$$

Let X and Y be two orbifolds, and $O(X)$ and $O(Y)$ their respective foliated desingularizations, i.e., foliated manifolds whose leaf space is X and Y, respectively. It would have been very nice if any smooth complete orbifold mapping $f\colon X \to Y$ could be lifted to a smooth foliated mapping $O(f)\colon O(X) \to O(Y)$. In general, this is not the case as the differential df does not send orthonormal frames into orthonormal frames. It is the case only if f is an isometry of (X, g_X) into (Y, g_Y). To rectify this we can consider linear frame bundles $L(X)$ and $L(Y)$ of the orbifolds X and Y, respectively. Any smooth complete orbifold local diffeomorphism $f\colon X \to Y$ lifts to a smooth mapping $L(f)\colon L(X) \to L(Y)$ which is a foliated mapping for the natural foliations $\mathcal{F}_X$ and $\mathcal{F}_Y$ by fibres of these two orbifold fibre bundles$L(X)$ and $L(Y)$, respectively. These foliations are Riemannian for the natural liftings of the Riemannian metrics of orbifolds. The holonomy groups of their leaves are finite.

4. Transversely harmonic maps

Let $\mathcal{F}$ be a foliation on a Riemannian n-manifold (M, g). Then $\mathcal{F}$ is defined by a cocycle $\mathcal{U} = \{U_i, f_i, g_{ij}\}_{i,j\in I}$ modeled on a q-manifold N_0 such that
(1) $\{U_i\}_{i\in I}$ is an open covering of M,
(2) $f_i : U_i \to N_0$ are submersions with connected fibres,
(3) $g_{ij} : N_0 \to N_0$ are local diffeomorphisms of N_0 with $f_i = g_{ij} f_j$ on $U_i \cap U_j$.
The connected components of the trace of any leaf of $\mathcal{F}$ on U_i consist of the fibres of f_i. The open subsets $N_i = f_i(U_i) \subset N_0$ form a q-manifold $N = \coprod N_i$, which can be considered as a transverse manifold of the foliation $\mathcal{F}$. The pseudogroup $\mathcal{H}_N$ of local diffeomorphisms of N generated by g_{ij} is called the holonomy pseudogroup of the foliated manifold $(M, \mathcal{F})$ defined by the cocycle $\mathcal{U}$. If the foliation $\mathcal{F}$ is Riemannian for the Riemannian metric g, then it induces a Riemannian metric $\bar{g}$ on N such that the submersions f_i are Riemannian submersions and the elements of the holonomy group are local isometries.
Let $\phi : U \to R^p \times R^q$, $\phi = (\phi^1, \phi^2) = (x_1, ..., x_p, y_1, ..., y_q)$ be an adapted chart on a foliated manifold $(M, \mathcal{F})$. Then on U the vector fields $\frac{\partial}{\partial x_1}, \dots \frac{\partial}{\partial x_p}$ span the bundle $T\mathcal{F}$ tangent to the leaves of the foliation $\mathcal{F}$, the equivalence classes denoted by $\frac{\bar{\partial}}{\partial y_1}, \dots \frac{\bar{\partial}}{\partial y_q}$ of $\frac{\partial}{\partial y_1}, \dots \frac{\partial}{\partial y_q}$ span the normal bundle $N(M, \mathcal{F}) = TM/T\mathcal{F}$ which is isomorphic to the subbundle $T\mathcal{F}^{\perp}$.
Suppose that $(M, \mathcal{F}, g)$ is a Riemannian foliation. The sheaf $\Gamma_b(T\mathcal{F}^{\perp})$ of foliated sections of the vector bundle $T\mathcal{F}^{\perp} \to M$ may be described as follows: If U is an open subset of M, then $X \in \Gamma_b(U, T\mathcal{F}^{\perp})$ if and only if for each local Riemannian submersion $\phi : U \to \bar{U}$ defining $\mathcal{F}$, the restriction of X to U is projectable via the map ϕ on a vector field $\bar{X}$ on $\bar{U}$.
The vector bundle $N(M, \mathcal{F})$ is a foliated vector bundle over $(M, \mathcal{F})$, i.e., it admits a foliation $\mathcal{F}_N$ of the same dimension as $\mathcal{F}$ whose leaves are covering spaces of leaves of $\mathcal{F}$. The vector bundle $\mathcal{F}^{\perp}$ is also foliated as it is isomorphic to the normal bundle. As $\mathcal{F}^{\perp}$ is foliated, so is its dual bundle $\mathcal{F}^{\perp^*}$ and any tensor product of these vector

bundles as well as any pull-back of a foliated vector bundle by a foliated mapping.

DEFINITION 9. *A basic partial connection* $(M, \mathcal{F}, g)$ *is a sheaf operator* $D : \Gamma_b(U, T\mathcal{F}^\perp) \times \Gamma_b(U, T\mathcal{F}^\perp) \to \Gamma_b(U, T\mathcal{F}^\perp)$ *such that*
1. $D_{fX+hY}Z = fD_XY + hD_YZ$,
2. D_X *is R-linear,*
3. $D_X fY = X(f)Y + fD_XY$ *(the transversal Leibniz rule),*
for any $X, Y, Z \in \Gamma_b(U, T\mathcal{F}^\perp)$ *and any basic functions* $f, h \in C_b^\infty(U)$*, where* U *is any open subset of* M*.*

Let ∇ be the Levi-Civita connection of g, then for any open subset U of M and $X, Y \in \Gamma_b(U, T\mathcal{F}^\perp)$ we define D as

$$D_XY = (\nabla_XY)^\perp$$

where $(\nabla_XY)^\perp$ is the horizontal component of ∇_XY. It is easy to check that D is a basic partial connection on $(M, \mathcal{F}, g)$. Let $\phi : U \to \bar{U}$ be a Riemannian submersion defining the foliation $\mathcal{F}$ on an open set U. Let us assume that $X, Y \in \Gamma_b(U, T\mathcal{F}^\perp)$, and $\bar{X}, \bar{Y}$ be the push forward vector fields via the map ϕ. Then

$$d\phi(D_XY) = \nabla^{\bar{g}}_{\bar{X}}\bar{Y}$$

where $\nabla^{\bar{g}}$ is the Levi-Civita connection of the metric $\bar{g}$ on the transverse manifold.

Let $f : (M_1, \mathcal{F}_1, g) \to (M_2, \mathcal{F}_2, h)$ be a foliated map between two foliated Riemannian manifolds. The covariant derivative $D(\Pi_2 dfi_1)$ is a global foliated section of the bundle

$$(T\mathcal{F}_1^\perp)^* \otimes (T\mathcal{F}_1^\perp)^* \otimes f^{-1}T\mathcal{F}_2^\perp \to M_1$$

where i_1 is the inclusion of $(T\mathcal{F}_1^\perp)^*$ into TM_1 and Π_2 is the orthogonal projection of TM_2 onto $T\mathcal{F}_2^\perp$. Its trace

$$\tau_b(f)$$

is a foliated section of the bundle $f^{-1}T\mathcal{F}_2^\perp \to M_1$. We call it the tension field of the foliated mapping f.

DEFINITION 10. *A foliated map* $f : (M_1, \mathcal{F}_1, g) \to (M_2, \mathcal{F}_2, h)$ *between two foliated Riemannian manifolds is called transversally harmonic if* $\tau_b(f) = 0$.

As we have shown any foliated map $f : (M_1, \mathcal{F}_1, g) \to (M_2, \mathcal{F}_2, h)$ induces a map $\bar{f}$ of the corresponding transverse manifolds. Since the tension tensor of a foliated map is itself a foliated section, it induces a holonomy invariant object on the transverse manifold. In fact, if p_i is a local submersion defining the foliation $\mathcal{F}_i$ for $i = 1,2$, then we have the following relation between the tension tensors of the mappings f and $\bar{f}$:

$$dp_2 \tau_b(f)_x = \tau(\bar{f})_{p_1(x)}.$$

Let $f : (M_1, \mathcal{F}_1, g) \to (M_2, \mathcal{F}_2, h)$ be a foliated mapping. The restriction of f to any leaf L of $\mathcal{F}_1$ is a smooth mapping of the manifold L into a leaf of the foliation $\mathcal{F}_2$. Both submanifolds can be provided with the restrictions of the Riemannian metrics g and h, respectively. If for each leaf L of the foliation $\mathcal{F}_1$ the mappings just described are harmonic, we say that the mapping f is *leaf-wise harmonic.*

Taking into account our previous considerations we can formulate the following theorems, whose proofs can be found in [18, 19]:

THEOREM 1. *Let $f : (M_1, \mathcal{F}_1, g) \to (M_2, \mathcal{F}_2, h)$ be a smooth foliated mapping between regular foliated Riemannian manifolds. Let $\mathcal{U}$ and $\mathcal{V}$ be two cocycles defining the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$, repectively. Then the map f is transversally harmonic if and only if the induced map $\bar{f} \colon N_{\mathcal{U}} \to N_{\mathcal{V}}$ between the associated transverse manifolds is harmonic.*

THEOREM 2. *Let $f : (M_1, \mathcal{F}_1, g) \to (M_2, \mathcal{F}_2, h)$ be a foliated harmonic map between two manifolds with Riemannian foliations. Moreover, assume that all leaves of the foliation $\mathcal{F}_1$ are minimal, the foliation $\mathcal{F}_2$ is totally geodesic, and f is horizontal, that is $df(T\mathcal{F}_1^{\perp}) \subset T\mathcal{F}_2^{\perp}$, then the map f is transversally harmonic.*

COROLLARY 1. *If the foliation $\mathcal{F}_1$ is minimal, and $\mathcal{F}_2$ totally geodesic, then the map f is harmonic if and only if f is transversally harmonic and leaf-wise harmonic.*

THEOREM 3. *Let $f : (M_1, \mathcal{F}_1, g) \to (M_2, \mathcal{F}_2, h)$ be a smooth foliated mapping between foliated Riemannian manifolds. If the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are totally geodesic and the horizontal distribution of $\mathcal{F}_2$ is integrable, then the map f is transversally harmonic if and only if the horizontal part of the tension tensor $\tau(f)$ is zero. In particular, the harmonicity of f implies its transverse harmonicity.*

The above results are "foliated" versions of Theorem 1.3.5 and its corollary of Smith's Ph.D. thesis, cf. [26], pp. 17 and 18, and the results of Xin, Theorem 6.4 and its corollary of [34], see also [33].

Finally, let us formulate the following two theorems for maps between Riemannian orbifolds which can serve as the starting point for the study of harmonic maps between orbifolds. The proofs are simple consequence of our previous considerations.

THEOREM 4. *Let $f : (M_1, \mathcal{F}_1, g) \to (M_2, \mathcal{F}_2, h)$ be a foliated map between two foliated Riemannian manifolds with compact leaves. Let $\bar{f}$ be the induced map between the leaf spaces with induced orbifold Riemannian metrics, $\bar{f} \colon M_1/\mathcal{F}_1, \to M_2/\mathcal{F}_2$. The map f is transversely harmonic iff the map $\bar{f}$ is harmonic.*

THEOREM 5. *Let $f\colon X \to Y$ be a smooth complete embedding of a Riemannian orbifold (X,g) into another Riemannian orbifold (Y,h). Then f is harmonic iff the induced (foliated) mapping $L(f)\colon (L(X),\mathcal{F}_X,g_L) \to (L(Y),\mathcal{F}_Y,h_L)$ is transversely harmonic.*

Remark The proposed approach to the study of harmonic maps between orbifolds has some deficiences. We can not lift any smooth map to a foliated map of its foliated desingularizations. However, if we just want to find harmonic maps, then the proposed methods can offer some solutions. The foliations $\mathcal{F}_X$ and $\mathcal{F}_Y$ are taut, so harmonic foliated maps between O_X and O_Y are not far from being transversally harmonic, and thus from inducing harmonic maps between the orbifolds. On the other hand, any compact Sasakian manifold fibers over a Kählerian orbifold, cf. [4], so the study of geometrical and cohomological properties of orbifolds can lead to the development of new obstructions to the existence of such structures.

References

[1] BĄK Ł. AND CZARNECKI A., *A remark on the Brylinski conjecture for orbifolds*, J. Austr. Math. Soc. **91** (2011), 1–12.

[2] BORZELLINO J.E. AND BRUNSDEN V., *Elementary orbifold differential topology*, Topology and its Applications **159** (2012), 3583–3589.

[3] BORZELLINO J.E. AND BRUNSDEN V., *On the notions of suborbifold and orbifold embedding*, to appear in Algebraic and Geometric Topology (2015)

[4] BOYER CH. AND GALICKI K., *Sasakian Geometry*, Oxford Un. Press 2008.

[5] CAMACHO C. AND LINS NETO A., *Geometric Theory of Foliations*, Birkhäuser, Boston 1985.

[6] CANDEL A. AND CONLON L., *Foliations I*, Graduate Studies in Mathematics, vol. 23, AMS, 2000.

[7] CHIANG Y.-J,, *Harmonic maps of V-manifolds*, Ann. Global Anal. Geom. **8** 3 (1990), 315–344.

[8] CHIANG Y.-J. AND WOLAK R., *Transversally biharmonic maps between foliated Riemannian manifolds*, Int. J. Math. **19** 8 (2008), 981–996.

[9] CHIANG Y.-J. AND WOLAK R., *Transversal biwave maps*, Archivum Mathematicum (Brno) **46** (2010), 211–226.

[10] CHIANG Y.-J. AND WOLAK R., *Transversal wave maps and transversal exponential wave maps*, Journal of Geometry **104** (2013), 443–459.

[11] CHIANG Y.-J. AND WOLAK R., *Remarks of transversally f-biharmonic maps*, Balkan Society of Geometers Proceeding **21** (2014), 38–49.

[12] CHIANG Y.-J. AND WOLAK R., *Transversally harmonic morphisms between foliated Riemannian manifolds*, to be published in Southeast Asian Bull. Math.

[13] EDWARDS R., MILLE K., SULLIVAN D., *Foliations with all leaves compact*, Topology **16** (1977), 13–32.

[14] EPSTEIN D.B.A., *Foliations with all leaves compact*, Annales de l'institut Fourier **26** 1 (1976), 265–282.

[15] FUJIMOTO A., *Theory of G-structures*, Publications of the Study Group of Geometry 1, Tokyo 1972.

[16] GIRBAU, J., HAEFLIGER, A. AND SUNDARARAMAN, D. *On deformations of transversely holomorphic foliations*. J. Reine Angew. Math. **345** (1983), 122–147.

[17] HAEFLIGER A., *Pseudogroups of local isometries*, in *Differential Geometry, Proceedings Vth International Colloquium on Differential Geometry, Santiago de Compostela 1984,* (Ed. L. A. Cordero), Pitman, London 1985.

[18] KONDERAK J.J. AND WOLAK R., *Transversally harmonic maps between manifolds with Riemannian foliations*, Quart. J. Math. (Oxford) **54** (2003), 335–354.

[19] KONDERAK J.J. AND WOLAK R., *Some remarks on transversally harmonic maps*, Glasgow Math. J. **50** (2008), 1–16,

[20] LEHMANN-LEJEUNE J., *Intégrabilité des G-structures définies par une 1-forme* 0*-déformable à valeurs dans le fibré tangent*, Annales de l'Institut Fourier **16** 2 (1966), 329-387.

[21] KOLAR I., MICHOR P., SLOVAK J., *Natural Operations in Differential Geometry*, Springer Verlag 1993.

[22] MILLET K., *Compact foliations*, in *Differential Topology and Geometry* , Lecture Notes in Mathematics 484, Springer Verlag, 1975, 277–287.

[23] MOLINO P., *Riemannian Foliations*, Progress in Mathematics, no. 73, Birkhäuser Boston, Inc., Boston, MA, 1988.

[24] REEB G., *Sur certaines propriétés toplogiques des variétés feuillétées*, Actualités Sci. Indust. 1183, Hermann, Paris 1952.

[25] SATAKE I., *On a generalization of the notion of manifold*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 359–363.

[26] SMITH R.T., *Harmonic mappings of spheres*, Ph.D. Thesis, Warwick, 1972.

[27] STERNBERG S., *Lectures on Differential Geometry*, Prentice-Hall, Englewood Cliffs, N. J. 1964.

[28] PALAIS R.S. AND TERNG C.-L., *Natural bundles have finite order*, Topology **16** 3 (1977), 271–277.

[29] THURSTON W., *The Geometry and Topology of Three-Manifolds*, Princeton University Press 1997.

[30] TONDEUR PH., *Geometry of Foliation*, Birkhäuser, Basel 1997.

[31] WANG Z.Z. AND ZAFFRAN D., *A remark on the hard Lefschetz theorem for Kähler orbifolds*, Proc. Amer. Math. Soc. **137** (2009), 2497–2501.

[32] WOLAK R., *Foliated and associated geometric structures on foliated manifolds*, Ann. Fac. Sci. Toulouse Math. (5) **10** (1989), 337–360.

[33] XIN Y.L., *Riemannian submersions and equivariant harmonic maps*, Proc. Sympo. Diff. Geom. in Honor of Professor Su Buchin on his 90-th Birthday, World Sci. Publ. Co. 1993, 272–287.

[34] XIN Y., *Geometry of Harmonic Maps*, Birkhäuser 1996.

AMS Subject Classification: 53C12

Robert A. WOLAK,
Wydzial Mathematyki i Informatyki, Uniwersytet Jagiellonski,
Lojasiewicza 6, 30-348 Krakow, Poland,
e-mail: `robert.wolak@im.uj.edu.pl`

Lavoro pervenuto in redazione il 09.11.2015 e accettato il 19.01.2016.

Special Issues and Proceedings published in the Rendiconti

Differential Geometry	(1992)
Numerical Methods in Astrophysics and Cosmology	(1993)
Partial Differential Equations, I-II	(1993–1994)
Problems in Algebraic Learning, I-II	(1994)
Number Theory, I-II	(1995)
Geometrical Structures for Physical Theories, I-II	(1996)
Jacobian Conjecture and Dynamical Systems	(1997)
Control Theory and its Applications	(1998)
Geometry, Continua and Microstructures, I-II	(2000)
Partial Differential Operators	(2000)
Liaison and Related Topics	(2001)
Turin Fortnight Lectures on Nonlinear Analysis	(2002)
Microlocal Analysis and Related Topics	(2003)
Splines, Radial Basis Functions and Applications	(2003)
Polynomial Interpolation and Projective Embeddings - Lecture Notes of the School	(2004)
Polynomial Interpolation and Projective Embeddings - Proceedings of the Workshop of the School	(2005)
Control Theory and Stabilization, I-II	(2005–2006)
Syzygy 2005	(2006)
Subalpine Rhapsody in Dynamics	(2007)
ISASUT Intensive Seminar on Non Linear Waves, Generalized Continua and Complex Structures	(2007)
Lezioni Lagrangiane 2007–2008	(2008)
Second Conference on Pseudo-Differential Operators and Related Topics: Invited Lectures	(2008)
Second Conference on Pseudo-Differential Operators and Related Topics	(2009)
In Memoriam Aristide Sanini	(2009)
Workshop on Hodge Theory and Algebraic Geometry	(2010)
School on Hodge Theory	(2011)
Generalized Functions, Linear and Nonlinear Problems. Proceedings of the International Conference GF 2011	(2011)
Forty years of Analysis in Turin. A conference in honour of Angelo Negro	(2012)
Proceedings of the School (and Workshop) on Invariant Theory and Projective Geometry	(2013)
Stochastic Analysis at the 8th Congress of Isaac	(2013)
Special issue dedicated to Alberto Conte on the occasion of his 70th birthday	(2013)
Rate-independent evolutions and hysteresis modelling	(2014)

Geometric structures
on Riemannian Manifolds-Bari

Volume 73/1, N. 3–4 (2015)

CONTENTS

ISSN 0373-1243

UNIVERSITÀ DEGLI STUDI DI TORINO
DIPARTIMENTO DI MATEMATICA