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A NOTE ON COMMUTATIVITY OF SKEW DERIVATIONS IN PRIME RINGS

Abstract. Let R be a prime ring and set $(x \circ_1 y) = (x \circ y) = xy + yx$ for all $x, y \in R$ and inductively $(x \circ_k y) = ((x \circ_{k-1} y) \circ y)$ for $k > 1$. In this paper, we apply the theory of generalized polynomial identities with automorphism and skew derivations to investigate the commutativity of R satisfying certain properties on some appropriate subset of R.

1. Introduction, Notation and Statements of the Results

Let R be a prime ring, $Z(R)$ the center of R, and Q the right Martindale quotient ring of R. The center of Q, denoted by C, is called the extended centroid of R. For any $x, y \in R$, the symbol $[x, y]$ and $x \circ y$ stands for the commutator $xy - yx$ and anti-commutator $xy + yx$, respectively. Given $x, y \in R$, we set $x \circ_0 y = x$, $x \circ_1 y = x \circ y = xy + yx$ and inductively $x \circ_m y = (x \circ_{m-1} y) \circ y$ for $m > 1$. Recall that a ring R is prime if $xRy = \{0\}$ implies either $x = 0$ or $y = 0$, and R is semiprime if $xRx = \{0\}$ implies $x = 0$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + yd(x)$ holds for all $x, y \in R$. In particular d is an inner derivation induced by an element $q \in R$, if $d(x) = [q, x]$ holds for all $x \in R$. If R is a ring and $S \subseteq R$, a mapping $f : S \to R$ is called strong commutativity-preserving (scp) on S if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$. In [5], Brešar introduced the notion of generalized derivation: an additive mapping $F : R \longrightarrow R$ is called a generalized derivation if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Hence, the concept of generalized derivations covers both the concepts of a derivation and of a left multiplier (i.e., an additive mapping $f(xy) = f(x)y$ for all $x, y \in R$).

Given any automorphism φ of R, an additive mapping $\delta : R \to R$ satisfying $\delta(xy) = \delta(x)y + \varphi(x)\delta(y)$ for all $x, y \in R$ is called a φ-derivation of R, or a skew derivation of R with respect to φ, denoted by (δ, φ). It is easy to see if $\varphi = 1_R$, the identity map of R, then a φ-derivation is merely an ordinary derivation. And if $\varphi \neq 1_R$, then $\varphi - 1_R$ is a skew derivation. Thus the concept of skew derivations can be regard as a generalization of both derivations and automorphism. Any skew derivation (δ, φ) extends uniquely to a skew derivation of Q [15] via extensions of each map to Q. Thus, we may assume that any skew derivation of R is the restriction of a skew derivation of Q. When $\delta(x) = \varphi(x)b - bx$ for some $b \in Q$, then (δ, φ) is called an inner skew derivation, and otherwise it is outer. Recall that φ is called an inner automorphism if when acting on Q, $\varphi(q) = uqu^{-1}$ for some invertible $u \in Q$. When φ is not inner, then it is called an outer automorphism. The skew derivations have been extensively studied by many researchers from various views (see for instance [9], [15] and [20] where further references can be found).

Let $Q_{*C}C\{X\}$ be the free product of Q and the free algebra $C\{X\}$ over C on an infinite set X, of indeterminate. Elements of $Q_{*C}C\{X\}$ are called generalized polynomials and a typical element in $Q_{*C}C\{X\}$ is a finite sum of monomials of the form $\alpha a_{i_0}x_{j_1}a_{i_1}x_{j_2}\cdots x_{j_n}a_{i_n}$ where $\alpha \in C$, $a_{ik} \in Q$ and $x_{jk} \in X$. We say that R satisfies a nontrivial generalized polynomial identity (abbreviated as GPI) if there exists a nonzero polynomial $\phi(x_i) \in Q_{*C}C\{X\}$ such that $\phi(r_i) = 0$ for all $r_i \in R$. By a generalized polynomial identity with automorphisms and skew derivations, we mean an identity of R expressed as the form $\phi(\varphi_j(x_i), \delta_k(x_i))$, where each φ_j is an automorphism, each δ_k is a skew derivation of R and $\phi(y_{ij}, z_{ik})$ is a generalized polynomial in distinct indeterminates y_{ij}, z_{ik}.

We need some well-known facts which will be used in the sequel.

FACT 1 ([9]). Let R be a prime ring, δ is a nonzero skew derivation of R and

$$\Phi\left(x_1, x_2\cdots, x_n, \delta(x_1), \cdots, \delta(x_n)\right)$$

is a skew differential identity of R, then one of the following statements holds:

1. either δ is inner;
2. or R satisfies the generalized polynomial identity

$$\Phi\left(x_1, x_2\cdots, x_n, y_1, \cdots, y_n\right).$$

FACT 2 ([9, Theorem 1]). Let R be a prime ring with an automorphism φ. Suppose that (δ, φ) is a Q-outer derivation of R. Then any generalized polynomial identity of R in the form $\phi(x_i, \delta(x_i)) = 0$ yields the generalized polynomial identity $\phi(x_i, y_i) = 0$ of R, where x_i, y_i are distinct indeterminates.

FACT 3 ([9, Theorem 1]). Let R be a prime ring with an automorphism φ. Suppose that (δ, φ) is a Q-outer derivation of R. Then any generalized polynomial identity of R in the form $\phi(x_i, \varphi(x_i), \delta(x_i)) = 0$ yields the generalized polynomial identity $\phi(x_i, y_i, z_i) = 0$ of R, where x_i, y_i, z_i are distinct indeterminates.

FACT 4 ([17, Proposition]). Let R be a prime algebra over an infinite field k and let K be a field extension over k. Then R and $R \otimes_k K$ satisfy the same generalized polynomial identities with coefficients in R.

The next result is a slight generalization of [16, Lemma 2] and can be obtained directly by the proof of [16, Lemma 2] and Fact 4.

FACT 5. Let R be a non-commutative simple algebra, finite dimensional over its center $Z(R)$. Then $R \subseteq M_n(F)$ with $n > 1$ for some field F and R and $M_n(F)$ satisfy the same generalized polynomial identities with coefficients in R.

In 2002 Ashraf and Rehman [2], prove that if R is a prime ring, I is a nonzero ideal of R and d is a nonzero derivation of R such that $d(x\circ y) = x\circ y$ for all $x,y\in I$, then R is commutative. Recently, Argaç and Inceboz [1], generalized the above result as following: Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer, if R admits a nonzero derivation d with the property $(d(x\circ y))^n = x\circ y$ for all $x,y\in I$, then R is commutative. In [19, Theorem 2.3], Quadri et al., discussed the commutativity of prime rings with generalized derivations. More precisely, Quadri et al., prove that if R is a prime ring, I is a nonzero ideal of R and F is a generalized derivation associated with a nonzero derivation d such that $F(x\circ y) = x\circ y$ for all $x,y\in I$, then R is commutative. In 2012 Huang [12], generalized the result obtained by Quadri et al., and he proved that if R is a prime ring, I a nonzero ideal of R, n a fixed positive integer and F a generalized derivation associated with a nonzero derivation d such that $(F(x\circ y))^n = x\circ y$ for all $x,y\in I$, then R is commutative.

In 1994 Bell and Daif [4], initiated the study of strong commutativity-preserving (scp) maps and prove that a nonzero right ideal I of a semiprime ring is central, if R admits a derivation which is scp on I. In [2], Ashraf and Rehman prove that if R is a 2-torsion free prime ring, I is a nonzero ideal of R and d is a nonzero derivation of R such that $d(x)\circ d(y) = x\circ y$ for all $x,y\in I$, then R is commutative.

Here we will continue the study of analogue problems on ideals of a prime ring involving skew derivations. The goal of this paper is to extend Ashraf and Rehman theorem [2], and Huang theorem [12], by using the theory of generalized polynomial identities with automorphisms.

Explicitly we shall prove the following theorems.

THEOREM 6. *Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer. Suppose that (δ,φ) is a skew derivation of R such that $\delta(x\circ y) = (x\circ y)^n$ for all $x,y\in I$, then R is commutative.*

THEOREM 7. *Let R be a prime ring with $char(R)\neq 2$, I a nonzero ideal of R and m,n are fixed positive integers. Suppose that (δ,φ) is a skew derivation of R such that $\delta(x)\circ_m \delta(y) = x\circ_n y$ for all $x,y\in I$, then R is commutative.*

2. Proof of Theorem 6.

If $\delta = 0$, then $(x\circ y)^n = 0$ for all $x,y\in I$, which can be rewritten as $(xy+yx)^n = 0$ for all $x,y\in I$. It is obvious that, if $char(R)\neq 2$, then $(2x^2)^n = 0$ for all $x\in I$. This is a contradiction by Xu [21]. And if $charR = 2$, then $(xy+yx)^n = 0 = [x,y]^n$ for all $x,y\in I$. By Herstein [11, Theorem 2], we have $I\subseteq Z(R)$, and so R is commutative by Mayne [18, Lemma 3].

Now we assume that $\delta\neq 0$ and $\delta(x\circ y) = (x\circ y)^n$ for all $x,y\in I$, which can be rewritten as

$$\delta(x)y+\varphi(x)\delta(y)+\delta(y)x+\varphi(y)\delta(x)=(xy+yx)^n. \quad (1)$$

In the light of Kharchenko's theory [14], we split the proof into two cases:

Case 1. Let δ is Q-inner, then $\delta(x)=\varphi(x)q-qx$ for all $x\in R, q\in Q$. From (1) and also for all $x,y\in Q$ (see [9, Theorem 2]), we have

$$\begin{aligned}(\varphi(x)q-qx)y+\varphi(x)(\varphi(y)q-qy)+ \quad & (\varphi(y)q-qy)x+\varphi(y)(\varphi(x)q-qx)\\ &=(xy+yx)^n \text{ for all } x,y\in Q.\end{aligned} \quad (2)$$

If φ is not Q-inner, then from (2), we have

$$\begin{aligned}(uq-qx)y+u(vq-qy)+&(vq-qy)x+v(uq-qx)\\ &=(xy+yx)^n \text{ for all } x,y,u,v\in Q.\end{aligned}$$

In particular $u=v=0$, then Q satisfied the following polynomial identity

$$-(qxy+qyx)=(xy+yx)^n, \text{ for all } x,y\in Q.$$

By Chuang [9, Theorem 1 and Theorem 2], shows that Q satisfies this polynomial identity and hence R as well. Note that this is a polynomial identity and hence there exist a field $\mathbb{F}$ such that $R\subseteq M_k(\mathbb{F})$, the ring of $k\times k$ matrices over a field $\mathbb{F}$, where $k\geq 1$. Moreover, R and $M_k(\mathbb{F})$ satisfy the same polynomial identity[6], that is $M_k(\mathbb{F})$ satisfy

$$-(qxy+qyx)=(xy+yx)^n.$$

Denote e_{ij} the usual matrix unit with 1 in (i,j)-entry and zero elsewhere. By choosing $x=e_{12}, y=e_{21}, q=e_{22}$, we see that

$$\begin{aligned}0=-q(x\circ y)-(x\circ y)^n&=(-e_{22}(e_{12}\circ e_{21}))-(e_{12}\circ e_{21})^n\\ &=-e_{11}-2e_{22}\neq 0, \text{ a contradiction.}\end{aligned}$$

Now consider, if φ is Q-inner, then there exist an invertible element $T\in Q$, $\varphi(x)=TxT^{-1}$ for all $x\in R$. From (2) we can write,

$$\begin{aligned}(TxT^{-1}q-qx)y+TxT^{-1}(TyT^{-1}q-qy)+&(TyT^{-1}q-qy)x+TyT^{-1}(TxT^{-1}q-qx)\\ &=(xy+yx)^n \text{ for all } x,y\in Q.\end{aligned}$$

We can see easily that if $T^{-1}q\in C$, then

$$\begin{aligned}\delta(x)=\varphi(x)q-qx&=TxT^{-1}q-qx=T(xT^{-1}q-T^{-1}qx)\\ &=T[x,T^{-1}q]=0, \text{ a contradiction.}\end{aligned}$$

Thus $T^{-1}q\notin C$. With this,

$$\begin{aligned}\phi(x,y) \;=\; &(TxT^{-1}q-qx)y+TxT^{-1}(TyT^{-1}q-qy)\\ &+(TyT^{-1}q-qy)x+TyT^{-1}(TxT^{-1}q-qx)-(xy+yx)^n.\end{aligned} \quad (3)$$

One can easily seen that $\phi(x,y)=0$ is a nontrivial generalized polynomial identity of $\mathcal{Q}$. Let $\mathcal{F}$ be the algebraic closure of C if C is infinite, otherwise let $\mathcal{F}$ be C. By Fact 4, $\phi(x,y)$ is also a generalized polynomial identity of $\mathcal{Q}\otimes_C\mathcal{F}$. Moreover, in view of [10, Theorem 3.5], $\mathcal{Q}\otimes_C\mathcal{F}$ is a prime ring with $\mathcal{F}$ as its extended centroid. Thus $\mathcal{Q}\otimes_C\mathcal{F}$ is a prime ring satisfies a nontrivial generalized polynomial identity and its extended centroid $\mathcal{F}$ is either an algebraically closed field or a finite field. Since both $\mathcal{Q}$ and $\mathcal{Q}\otimes_C\mathcal{F}$ are prime and centrally closed [10, Theorem 3.5], we may replace R by $\mathcal{Q}$ or $\mathcal{Q}\otimes_C\mathcal{F}$. Thus we may assume that R is centrally closed and the field $\mathcal{F}$ which is either algebraically closed or finite and R satisfies generalized polynomial identity (3). By Martindale's theorem [3, Corollary 6.1.7], R is a primitive ring having nonzero socle with the field $\mathcal{D}$ as its associated division ring. Moreover, by Jacobson theorem [13, p.75], R is isomorphic to a dense subring of the ring of linear transformations on a vector space $\mathcal{V}$ over $\mathcal{D}$ (or $End(\mathcal{V}_\mathcal{D})$ in brief), containing nonzero linear transformations of finite rank. If $\mathcal{V}$ is a finite dimensional over $\mathcal{D}$, then the density of R on $\mathcal{V}$ implies that $R\cong M_k(\mathcal{D})$, where $k=dim_\mathcal{D}\mathcal{V}$.

Assume first that $dim(\mathcal{V}_\mathcal{D})\geq 2$.

Step 1. We want to show that w and $T^{-1}qw$ are linearly $\mathcal{D}$-dependent for all $w\in\mathcal{V}$. If $T^{-1}qw=0$ then $\{w,T^{-1}qw\}$ is linearly $\mathcal{D}$-dependent. Suppose on contrary that w_0 and $T^{-1}qw_0$ are linearly $\mathcal{D}$-independent for some $w_0\in\mathcal{D}$.

If $T^{-1}w_0\notin Span_\mathcal{D}\{w_0,T^{-1}qw_0\}$ then $\{w_0,T^{-1}qw_0,T^{-1}w_0\}$ are linearly $\mathcal{D}$-independent. By the density of R there exist $x,y\in R$ such that

$$\begin{array}{lll} xw_0=0, & xT^{-1}qw_0=T^{-1}w_0, & xT^{-1}w_0=0 \\ yw_0=w_0, & yT^{-1}qw_0=0, & yT^{-1}w_0=T^{-1}w_0. \end{array}$$

With all these, we obtained from (3),

$$\begin{aligned} w_0=&\big((TxT^{-1}q-qx)y+TxT^{-1}(TyT^{-1}q-qy)+(TyT^{-1}q-qy)x\\ &+TyT^{-1}(TxT^{-1}q-qx)-(xy+yx)^n\big)w_0\neq 0, \text{ a contradiction.}\end{aligned}$$

If $T^{-1}w_0\in Span_\mathcal{D}\{w_0,T^{-1}qw_0\}$ then $T^{-1}w_0=w_0\beta+T^{-1}qw_0\gamma$ for some $\beta,\gamma\in\mathcal{D}$ and $\beta\neq 0$. Since w_0 and $T^{-1}qw_0$ are linearly $\mathcal{D}$-independent, by the density of R there exist $x,y\in R$ such that

$$\begin{array}{ll} xw_0=0, & xT^{-1}qw_0=w_0\beta+T^{-1}qw_0\gamma \\ yw_0=w_0, & yT^{-1}qw_0=0. \end{array}$$

The application of (3) implies that

$$\begin{aligned} 0=&\big((TxT^{-1}q-qx)y+TxT^{-1}(TyT^{-1}q-qy)+(TyT^{-1}q-qy)x\\ &+TyT^{-1}(TxT^{-1}q-qx)-(xy+yx)^n\big)w_0=Tw_0\beta=w_0\beta\neq 0,\end{aligned}$$

and we arrive at a contradiction. So we conclude that $\{w_0,T^{-1}qw_0\}$ are linearly $\mathcal{D}$-dependent, for all $w_0\in\mathcal{V}$ as claimed.

Step 2. By using the arguments presented above, we prove that $T^{-1}qw_0 = w_0\mu(w)$, for all $w \in \mathcal{V}$, where $\mu(w) \in \mathcal{D}$ depends on $w \in \mathcal{V}$. In fact, it is easy to check that $\mu(w)$ is independent of choice $w \in \mathcal{V}$. Indeed, for any $w, z \in \mathcal{V}$, in view of above situation, there exist $\mu(w), \mu(z), \mu(w+z) \in \mathcal{D}$ such that

$$T^{-1}qw = w\mu(w),\ T^{-1}qz = z\mu(z),\ T^{-1}q(w+z) = (w+z)\mu(w+z)$$

and therefore,

$$w\mu(w) + z\mu(z) = T^{-1}q(w+z) = (w+z)\mu(w+z).$$

Hence,

$$w(\mu(w) - \mu(w+z)) + z(\mu(z) - \mu(w+z)) = 0.$$

Since w and z are $\mathcal{D}$-independent, then $\mu(w) = \mu(z) = \mu(w+z)$. Otherwise, w and z are $\mathcal{D}$-dependent, say $w = \lambda z$ for some $\lambda \in \mathcal{D}$. Thus,

$$w\mu(w) = T^{-1}qw = T^{-1}q\lambda z = \lambda T^{-1}qz = \lambda z\mu(z) = w\mu(z)$$

i.e., $\mathcal{V}(\mu(w) - \mu(z)) = 0$. Since $\mathcal{V}$ is faithful, we get $\mu(w) = \mu(z)$. Hence, we conclude that there exists $\chi \in \mathcal{D}$ such that $T^{-1}qw = w\chi$ for all $w \in \mathcal{V}$.

At last, we want to show that $\chi \in Z(\mathcal{D})$ (the center of $\mathcal{D}$). Indeed, for any $\eta \in \mathcal{D}$, we have

$$T^{-1}q(w\eta) = (w\eta)\chi = w(\eta\chi),$$

and on the other hand,

$$T^{-1}q(w\eta) = (T^{-1}qw)\eta = (w\chi)\eta = w(\chi\eta).$$

Therefore, $\mathcal{V}(\eta\chi - \chi\eta) = 0$ and thus $\eta\chi = \chi\eta$, which implies that $\chi \in Z(\mathcal{D})$. Hence, $T^{-1}q \in C$, a contradiction. Hence $dim(\mathcal{V}_{\mathcal{D}}) = 1$, which implies that R is commutative.

Case 2. Let δ is Q-outer, then I satisfies

$$sy + \varphi(x)t + tx + \varphi(y)s = (xy + yx)^n \text{ for all } x, y, s, t \in I. \tag{4}$$

Firstly, we assume that φ is not Q-inner, then for all $x, y, s, t, u, v \in I$, we have

$$sy + ut + tx + vs = (xy + yx)^n \text{ for all } x, y, s, t, u, v \in I.$$

In particular $x = t = 0$ and $v = y$, then I satisfied the following blended component $sy + ys = 0$, for all $s, y \in I$. Replacing y by yw and using the fact that $sy = -ys$, we find that $y[s, w] = 0$ for all $y, s, w \in I$ and hence $IR[s, w] = 0$ for all $s, w \in I$. Since I is a nonzero ideal of R and R is prime, we get $[s, w] = 0$ for all $s, w \in I$, hence R is commutative.

Secondly, if φ is Q-inner, then there exist an invertible element $T \in Q$, $\varphi(x) = TxT^{-1}$ for all $x \in R$. Thus from (4), we have

$$(sy + TxT^{-1}t) + (tx + TyT^{-1}s) = (xy + yx)^n \text{ for all } x, y, s, t \in I.$$

In particular $s = t = 0$, and using the same argument presented in the beginning of the proof, we get the required result. This completes the proof of the theorem.

We immediately get the following corollaries from the above theorem:

COROLLARY 1. *Let R be a prime ring, I a nonzero ideal of R and δ is a skew derivation associated with an automorphism φ of R. If $\delta(x)x+\varphi(x)\delta(x) = x^2$ for all $x \in I$, then R is commutative.*

Proof. By the given hypothesis, I satisfies

$$\delta(x)x+\varphi(x)\delta(x) = x^2. \tag{5}$$

Linearizing (5), one can obtain $\delta(x)y+\varphi(x)\delta(y)+\delta(y)x+\varphi(y)\delta(x) = (xy+yx)$ for all $x,y \in I$. Which can be rewritten as $\delta(x\circ y) = (x\circ y)$ for all $x,y \in I$. Now apply Theorem 6, for $n = 1$ we get the required conclusion. □

When $\delta = \varphi - 1_R$, we obtain the following results.

COROLLARY 2. *Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer. If φ is a non-identity automorphism of R such that $\varphi(x\circ y) = (x\circ y)^n$ for all $x,y \in I$, then R is commutative.*

Let R be a unital ring. For a unit $u \in R$, the map $\varphi_u : x \to uxu^{-1}$ defines an automorphism of R. If d is a derivation of R, then it is easy to see that the map $ud : x \to ud(x)$ defines a φ_u-derivation of R. So we have the following results.

COROLLARY 3. *Let R be a prime unital ring, u a unit in R, I a nonzero ideal of R and n a fixed positive integer. Suppose that φ_u is a derivation of R such that $\varphi_u(x\circ y) = (x\circ y)^n$ for all $x,y \in I$, then R is commutative.*

3. Proof of Theorem 7.

Assume that R is non-commutative, otherwise we have nothing to prove. If $\delta = 0$, then $x\circ_n y = 0$ for all $x,y \in I$. By Chuang [9, Theorem 1], this polynomial identity is also satisfied by Q and hence R as well. Note that this is a polynomial identity and thus there exists a field $\mathbb{F}$, such that $R \subseteq M_k(\mathbb{F})$, the ring of $k\times k$ matrices over a field $\mathbb{F}$, where $k \geq 1$. Moreover, R and $M_k(\mathbb{F})$ satisfy the same polynomial identity [16, Lemma 1], that is $x\circ_n y = 0$ for all $x,y \in M_k(\mathbb{F})$. But by choosing $x = e_{12}$, $y = e_{11}$ we get $0 = x\circ_n y = e_{12} \neq 0$, a contradiction.

Now we assume that $\delta \neq 0$ and I satisfies

$$\delta(x)\circ_m\delta(y) = x\circ_n y \text{ for all } x,y \in I. \tag{6}$$

Let δ is Q-inner, then $\delta(x) = \varphi(x)q - qx$ for all $x \in R$, $q \in Q$. From (6) and also for all $x,y \in Q$ (see [9, Theorem 2]), we have

$$(\varphi(x)q-qx)\circ_m(\varphi(y)q-qy) = x\circ_n y \text{ for all } x,y \in Q. \tag{7}$$

If φ is not Q-inner, then from (7), we have

$$(uq-qx)\circ_m(vq-qy)=(x\circ_n y) \text{ for all } x,y,u,v\in Q.$$

In particular $u=v=0$, then Q satisfies the following polynomial identity

$$(qx\circ_m qy)=(x\circ_n y), \text{ for all } x,y\in Q.$$

In this situation as already mention in Theorem 6, that is R and $M_k(\mathbb{F})$ satisfies the same polynomial identity[6], hence $M_k(\mathbb{F})$ satisfy $(qx\circ_m qy)=(x\circ_n y)$. By choosing $x=e_{12}, y=e_{22}, q=e_{21}$, we see that

$$\begin{aligned}0=(qx\circ_m qy)-(x\circ_n y)&=(e_{21}e_{12}\circ_m e_{21}e_{22})-(e_{12}\circ_n e_{22})\\&=-e_{12}\neq 0, \text{ a contradiction.}\end{aligned}$$

Now consider, if φ is Q-inner, then there exist an invertible element $T\in Q$, $\varphi(x)=TxT^{-1}$ for all $x\in R$. From (7) we can write,

$$(TxT^{-1}q-qx)\circ_m(TyT^{-1}q-qy)=(x\circ_n y) \text{ for all } x,y\in Q.$$

As in the proof Theorem 6, we see that

$$(TxT^{-1}q-qx)\circ_m(TyT^{-1}q-qy)=(x\circ_n y) \text{ for all } x,y\in R,$$

where R is a primitive ring with $\mathcal{D}$ as the associated division ring. If $\mathcal{V}$ is finite dimensional over $\mathcal{D}$ then the density of R implies that $R\cong\mathbb{M}_k(\mathcal{D})$, where $k=dim_{\mathcal{D}}\mathcal{V}$.

Assume first that $dim(V_{\mathcal{D}})\geq 2$.

We want to show that v and $T^{-1}qv$ are linearly $\mathcal{D}$-dependent for all $v\in\mathcal{V}$. If $v=0$ then $\{v,T^{-1}qv\}$ is linearly $\mathcal{D}$-dependent. Suppose on contrary that v_0 and $T^{-1}qv_0$ are linearly $\mathcal{D}$-independent for some $v_0\in\mathcal{D}$. By the density of R, there exist $x,y\in R$ such that

$$\begin{aligned}xv_0=0,\qquad xT^{-1}qv_0=-T^{-1}v\\ yv_0=0,\qquad yT^{-1}qv_0=T^{-1}v.\end{aligned}$$

These imply that

$$\begin{aligned}0&=\big((TxT^{-1}q-qx)\circ_m(TyT^{-1}q-qy)-(x\circ_n y)\big)v_0\\&=(-1)^m2^m v_0, \text{ a contradiction.}\end{aligned}$$

So we conclude that $\{v_0,T^{-1}qv_0\}$ are linearly $\mathcal{D}$-dependent for all $v_0\in\mathcal{V}$. So for each $v\in\mathcal{V}$, $T^{-1}qv=v\alpha_v$ for some $\alpha_v\in\mathcal{D}$. By a standard argument, it is easy to see that α_v is independent of the choice of $v\in\mathcal{V}$. Thus we can write $T^{-1}qv=v\beta$ for all $v\in\mathcal{V}$ and a fixed $\beta\in\mathcal{D}$. Reasoning as in the proof of Theorem 6, we conclude that $\delta=0$, again a contradiction.

Let δ is Q-outer, then from (6), I satisfies $(s\circ_m y)=(x\circ_n y)$ for all $x,y,s,t\in I$. In particular $x=0$, and using the same technique as presented above, we are done. □

In view of above theorem we can write the following corollaries.

COROLLARY 4. *Let R be a prime ring, I a nonzero ideal of R and δ is a skew derivation of R. If $\delta(x)^2 = x^2$ for all $x \in I$, then R is commutative.*

Proof. By the given hypothesis, I satisfies

$$\delta(x)^2 = x^2. \tag{8}$$

Linearizing (8), one can obtain $\delta(x)\delta(y)+\delta(y)\delta(x) = (xy+yx)$ for all $x,y \in I$. Which can be rewritten as $\delta(x)\circ\delta(y) = (x \circ y)$ for all $x,y \in I$. Now apply Theorem 7, for $m = n = 1$ we get the required conclusion. □

COROLLARY 5. *Let R be a prime ring with $char(R) \neq 2$, I a nonzero ideal of R and m,n are fixed positive integers. If φ is a non-identity automorphism of R such that $\varphi(x)\circ_m \varphi(y) = x \circ_n y$ for all $x,y \in I$, then R is commutative.*

COROLLARY 6. *Let R be a prime unital ring with $char(R) \neq 2$, u a unit in R, I a nonzero ideal of R and m,n are fixed positive integers. Suppose that φ_u is a derivation of R such that $\varphi_u(x)\circ_m \varphi_u(y) = x \circ_n y$ for all $x,y \in I$, then R is commutative.*

The following example demonstrates that the hypothesis of primeness of R is essential in Theorem 6 and Theorem 7.

EXAMPLE 1. Let S be the set of all integers. Consider $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a,b \in S \right\}$ and $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in S \right\}$. Define maps $\varphi : R \to R$ by $\varphi \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & 0 \end{pmatrix}$ and $\delta : R \to R$ by $\delta \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & -2b \\ 0 & 0 \end{pmatrix}$. The fact that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$ implies that R is not prime. It is easy to check that I is a nonzero ideal of R, (δ,φ) is a skew derivation of R and satisfying (1) $\delta(x\circ y) = (x\circ y)^n$ (2) $\delta(x)\circ_m \delta(y) = x \circ_n y$ for all $x,y \in I$. However, R is not commutative.

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