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SOME THEOREMS ON ϕ-CONTRACTIVE MAPPINGS IN MODULAR SPACE

Abstract. In the present paper we prove a fixed point theorem for ϕ-contractive mappings in modular space, then we examine the convergence and the satability for some iteration procedures for this fixed point.

1. Basic Preliminaries

Many authors have extended the well-known Banach contraction theorem. In addition to the authors specifically cited in this paper Albert et al.[1] on Hilbert by introducing weakly contraction, At the same time, Rhoades [8] has proved their results in arbitrary Banach spaces, then guaranteed the convergence and the stability for several fixed point iteration procedures. On the other hand, others authors have extended it to modular space. Khamsi [2], khamsi, Kozlowski and Riech [3] and recently Marzouki [5] has proved a fixed point theorem for contractive mapping in modular space without supposing that the modular verifies neither the Δ_2 condition nor the Fatou property.
In this work, we define ϕ-contractive mappings in modular space and we study the convergence and stability of some fixed point iteration procedures for ϕ-contractive mappings.
We will begin this section with a brief recollection of basic concepts and facts of the theory of modular spaces. For more details we refer to [7], then by defining ϕ-contraction in modular space.

DEFINITION 1. *Let X be an arbitrary vector space.*
(a) A functional $\rho : X \to [0,\infty]$ is call a modular if for arbitrary x, y in X,
(i) $\rho(x) = 0$ iff $x = 0$,
(ii) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$,
(iii) $\rho(\alpha x+\beta y) \leq \rho(x)+\rho(y)$ if $\alpha+\beta = 1$ and $\alpha \geq 0$, $\beta \geq 0$.
(b) If (iii) is replaced by

$$\rho(\alpha x+\beta y) \leq \alpha\rho(x)+\beta\rho(y) \text{ if } \alpha+\beta = 1 \text{ and } \alpha \geq 0,\ \beta \geq 0,$$

we say that ρ is a convex modular.

A modular ρ defines a corresponding modular space, i.e. the vector space X_ρ, given by

$$X_\rho = \Big\{x \in X : \ \rho(\mu x) \to 0 \text{ as } \mu \to 0\Big\}.$$

DEFINITION 2. *Let X_ρ be a modular space.*
(a) A sequence $\{x_n\}$ is called ρ-convergent to x if and only if

$$\rho(x_n - x) \to 0 \text{ as } n \to \infty.$$

(b) A sequence $\{x_n\}$ is called ρ-Cauchy whenever

$$\rho(x_n - x_m) \to 0 \text{ as } n, m \to \infty.$$

(c) The modular space X_ρ is called ρ-complete if any ρ-Cauchy sequence is ρ-convergent.
(d) ρ is said to satisfy the Δ_2 condition if

$$\rho(2x_n) \to 0 \quad \text{whenever } \rho(x_n) \to 0 \quad \text{as} \quad n \to \infty.$$

(e) The modular ρ is said to have the Fatou property if

$$\rho(x) \leq \liminf_{n\to\infty} \rho(x_n) \quad \text{whenever} \quad x_n \to x.$$

(f) A subset B of X_ρ is said to be ρ-closed if for any sequence $\{x_n\} \subset B$ ρ-convergent to $x \in X_\rho$, implies that x belongs to B.
(g) A subset B of X_ρ is called ρ-bounded if $\sup\{\rho(x-y) : x, y \in B\} < +\infty$.

EXAMPLE 1. We consider the vector space X defined by

$$X = \Big\{ f : [0,1] \to \Re \text{ measurable} \Big\},$$

we define in X the following modular: $f \in X$, we put

$$\rho(f) = \int_0^1 \varphi(t, |f(t)|)\, dt$$

where $\varphi : [0,1] \times \Re^+ \to \Re$ satisfies:
- φ is continuous, increasing and convex with respect to second variable.
- $\lim_{x\to+\infty} \varphi(t,x) = +\infty$, $\varphi(t,0) = 0$ and $\varphi(t,x) > 0\ \forall x > 0$.
- φ is measurable with respect to the first variable.

Then it is known (see for example J. Musielak [7]) that ρ is a convex modular.
- $L^\varphi = \Big\{ f \in X : \lim_{\eta\to 0} \rho(\eta f) = 0 \Big\}$, is a modular space.
- L^φ is ρ- complete.
- L^φ is a Banach space equipped with the Luxembourg norm defined by

$$||x||_\rho = \inf\Big\{ \alpha : \rho(\frac{x}{\alpha}) \leq 1 \Big\}.$$

- ρ satisfies the Δ_2 condition and Fatou property.
-

$$\lim_{n\to+\infty} ||x_n - x||_\rho = 0 \Longleftrightarrow \lim_{n\to+\infty} \rho(x_n - x) = 0,$$

note that for an arbitrary modular in which ρ does not satisfy the Δ_2 condition, we have

$$\lim_{n\to+\infty}||x_n - x||_\rho = 0 \Longleftrightarrow \lim_{n\to+\infty}\rho(\alpha(x_n - x)) = 0 \ \ \forall\ \alpha > 0.$$

As, many examples on some subsets of modular functions spaces have been given in Khamsi et al. [3] to prove a fixed point property for mapping with a strict contraction with respect to the modular. So, in modular spaces and for more general class of maps with a ϕ-contraction, we will claim the same result. we introduce this class of maps by the definition belong:

DEFINITION 3. *Let X_ρ be a modular space, B be a nonempty and ρ-bounded subset of X_ρ and let $\phi : [0,\infty) \to [0,\infty)$ be an upper semi-continuous function such that $\phi(0) = 0$ and $\phi(s) < 2s$ for $s > 0$. In this context a mapping $T : X \to X$ is ϕ-contractive if*

$$\rho(2(Tx - Ty)) \le \phi(\rho(x - y)) \quad \forall\ x, y \in B.$$

2. Main Results

THEOREM 1. *Let B be a ρ-closed and ρ-bounded subset of a ρ-complete modular space X_ρ and $T : B \to B$ be a ϕ-contractive mapping, we suppose also that ρ is convex, then T has a unique fixed point. Moreover, the Picard iterative scheme $x_0 \in X_\rho$ $x_{n+1} = Tx_n$ converges to this fixed point*

Proof. Let x_0 be some point in B, define:

$$\begin{aligned} x_n &= T^n x_0, \qquad n = 0,1,2,... \\ \tau_n &= \rho(x_n - x_{n+1}), \qquad n = 0,1,2,... \end{aligned}$$

We have

$$\begin{aligned} \tau_n &= \rho(x_n - x_{n+1}) \\ &= \rho(Tx_{n-1} - Tx_n) \le \frac{1}{2}\rho(2(Tx_{n-1} - Tx_n)) \\ &\le \frac{1}{2}\phi(\rho(x_n - x_{n-1})) \\ &= \frac{1}{2}\phi(\tau_{n-1}) \quad (*) \\ &< \tau_{n-1}, \end{aligned}$$

so $\{\tau_n\}$ is a strictly decreasing sequence. Let $lim_n \tau_n = \tau$, assume that $\tau > 0$. Since ϕ is upper semi-continuous function as $n \to \infty$ in $(*)$ we obtain

$$\tau \le \frac{1}{2}\phi(\tau) < \tau$$

which is a contradiction. Consequently $\tau = 0$. Now, we wish to show that $\{x_n\}$ is a Cauchy sequence since $\tau = 0$ we proceed by contradiction. Then there exists $\varepsilon > 0$, and two sequences of natural numbers $\{m(i)\}, \{n(i)\}$ such that

$$n(i) > m(i) \geq i \qquad \text{and} \quad \rho(x_{n(i)} - x_{m(i)}) > \varepsilon \quad (**)$$

For each integer i, let $n(i)$ be the least integer exceeding $m(i)$ and satisfy $(**)$. Then

$$\rho(x_{n(i)} - x_{m(i)}) > \varepsilon \qquad \text{and} \quad \rho(x_{n(i)-1} - x_{m(i)}) \leq \varepsilon$$

Then we have

$$\begin{aligned}
\varepsilon < \rho_i &= \rho(x_{n(i)} - x_{m(i)}) \\
&\leq \frac{1}{2}\Big\{\rho(2(x_{n(i)} - x_{m(i)+1})) + \rho(2(x_{m(i)+1} - x_{m(i)})\Big\} \\
&\leq \frac{1}{2}\Big\{\phi(\rho(x_{n(i)-1} - x_{m(i)})) + \phi(\rho((x_{m(i)} - x_{m(i)-1}))\Big\} \quad (***) \\
&\leq \rho(x_{n(i)-1} - x_{m(i)}) + \frac{1}{2}\phi(\rho((x_{m(i)} - x_{m(i)-1})) \\
&\leq \varepsilon + \tau_{m(i)-1},
\end{aligned}$$

since $\{\tau_n\}$ converges to 0, $\rho_i \to \varepsilon$ and $\rho(x_{n(i)-1} - x_{m(i)}) \to \varepsilon$. It follows that, by $(***)$ we have

$$\varepsilon \leq \frac{1}{2}\phi(\rho(x_{n(i)-1} - x_{m(i)})) + \tau_{m(i)-1}$$

Letting $i \to \infty$, we obtain:

$$\varepsilon \leq \frac{1}{2}\phi(\varepsilon) < \varepsilon$$

This is a contradiction. Hence $\{x_n\}$ is a ρ-Cauchy sequence consequently there exists a point $x \in B$ such that

$$x_n \to x \qquad \text{as} \quad n \to \infty.$$

Now we shall show that $Tx = x$.

$$\begin{aligned}
\rho(\frac{x - Tx}{2}) &= \rho(\frac{x - Tx_n}{2} + \frac{Tx_n - Tx}{2}) \\
&\leq \rho(x - Tx_n) + \rho(Tx_n - Tx) \\
&\leq \rho(x - x_{n+1}) + \rho(2(Tx_n - Tx)) \\
&\leq \rho(x - x_{n+1}) + \phi(\rho((x_n - x))),
\end{aligned}$$

thus, letting $n \to \infty$, we have

$$\rho(\frac{x - Tx}{2}) = 0$$

hence $x = Tx$. We claim that x is the unique fixed point of T. For this suppose that y ($x \neq y$) is another fixed point of T. Then

$$\begin{aligned}
\rho(x - y) = \rho(Tx - Ty) &\leq \frac{1}{2}\rho(2(Tx - Ty)) \\
&\leq \frac{1}{2}\phi(\rho(x - y)),
\end{aligned}$$

since B is ρ-bounded, we obtain a contradiction. This completes the proof of the theorem. □

We will give the following example to illustrate the obtained result.

EXAMPLE 2. Let $X = [0,\infty)$ be a vector space, ρ be an application defined as follows:

$$\begin{aligned} \rho \quad : \quad & X \to X \\ & t \to t^2 \end{aligned}$$

So, we can see that ρ is not a norm but is a modular since the function $t \to t^2$ is convex. Then the associated modular space is $X_\rho = [0,\infty)$ by continuity of the modular ρ.
Now, take the function $\phi : [0,\infty) \to [0,\infty)$, such that $\phi(t) = \frac{t}{1+t}$, wich is clearly continuous, $\phi(0) = 0$ and $\phi(t) < t$ for every $t > 0$.
Consider $B = [0,1]$ the closed interval in $[0,\infty)$ which is ρ-closed, ρ-bounded and ρ-complete, since ρ is continuous. Then the following map

$$\begin{aligned} T \quad : \quad & B \to B \\ & t \to \frac{1}{2}\frac{t}{1+t} \end{aligned}$$

is ϕ-contractive. Consequently, by Theorem 1, it has a unique fixed point in B which is the point 0.

REMARKS 1. 1) If we suppose only $\rho(x_0 - Tx_0) < \infty$ for a certain $x_0 \in B$ instead B is ρ-bounded in the proof of Theorem 1, then the sequence $(\tau_n)_n$ will be well defined and the existence of fixed point for the map T will be obtained also but the uniqueness of the fixed point in this case can be fail.
In more particular case, when the modular ρ is with values in $[0,\infty)$. we get our main result without assuming that B is ρ-bounded.
2) Under the same hypothesis of Theorem 1 but in the case that $\phi(s) < s$ for $s > 0$. We claim our main result without The hypothesis that ρ should be convex.

In the sequel, B is a ρ-closed subspace of X_ρ and ρ is with values in $[0,\infty)$.

3. Iterative Schemes for ϕ-contractive maps

We shall now investigate the convergence of other iterative procedures applied to T. The Mann iterative scheme is defined by

$$x_0 \in X_\rho, \;\; x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \;\; n \geq 0, \;\; (a)$$

where $0 \leq \alpha_n \leq 1$ for each n.

THEOREM 2. *Suppose the hypotheses of Theorem 1 are satisfied and also that ϕ is continuous. Then the Mann iterative scheme (a), with (i) $0 \leq \alpha_n \leq 1$ and (ii) $\sum \alpha_n = \infty$, converges to the unique fixed point of T*

Proof. From Theorem 1, T has a unique fixed point. Call it x. Using (a),

$$\begin{aligned}\rho(x_{n+1}-x) &= \rho((1-\alpha_n)x_n+\alpha_n Tx_n-x)\\ &\leq (1-\alpha_n)\rho(x_n-x)+\alpha_n\rho(Tx_n-x)\\ &\leq (1-\alpha_n)\rho(x_n-x)+\frac{\alpha_n}{2}\rho(2(Tx_n-Tx))\\ &\leq (1-\alpha_n)\rho(x_n-x)+\frac{\alpha_n}{2}\phi(\rho(x_n-x)) \quad (1a)\\ &\leq \rho(x_n-x).\end{aligned}$$

Therefore $\{\rho(x_n-x)\}$ is a non-increasing sequence, which converge to a limit τ. Suppose that $\tau>0$.

Put $\tau_n=\rho(x_n-x)$ and $\mu=\frac{1}{2}(\tau-\frac{1}{2}\phi(\tau))>0$. From $(1a)$ it follows that, there exists a fixed integer N,

$$\sum_{n=N}^{\infty}\alpha_n\mu\leq\sum_{n=N}^{\infty}\alpha_n(\tau_n-\frac{1}{2}\phi(\tau_n))\leq\sum_{n=N}^{\infty}(\tau_n-\tau_{n+1})\leq\tau_N,$$

contradicting (ii). Therefore $\tau=0$. □

The Ishikawa iteration scheme is defined by

$$\begin{aligned}x_0\in X_\rho,\ x_{n+1} &= (1-\alpha_n)x_n+\alpha_n Ty_n \quad (2a)\\ y_n &= (1-\beta_n)x_n+\beta_n Tx_n,\ \ n\geq 0,\end{aligned}$$

where $0\leq\alpha_n,\ \beta_n\leq 1$ for all n.

THEOREM 3. *Suppose the hypotheses of Theorem 2 are satisfied. Then the Ishikawa iterative scheme, with* (i) $0\leq\alpha_n,\ \beta_n\leq 1$ *and* (ii) $\sum\alpha_n\beta_n=\infty$, *converges to the unique fixed point of* T

The proof of Theorem 3 is similar to that of Theorem 2, and will therefore be omitted.

Let X_ρ be a modular space, T a self-map of X_ρ. Then, for a fixed integer N the map S is defined by

$$S=\sum_{i=1}^{N}\alpha_i T^i,\ \text{ where each } \alpha_i\geq 0,\ \sum_{i=1}^{N}\alpha_i=1,\ \text{and } \alpha_1\neq 0. \quad (3a)$$

Then the function iteration scheme of Kirk [4] is defined as follows.

$$x_0\in X_\rho\ \ x_{n+1}=S^n x_0.$$

THEOREM 4. *Suppose the hypotheses of Theorem 1 are satisfied. Then the iterative scheme* $\{S^n x_0\}$, *converges to the unique fixed point of* T

Proof. Let x be the unique fixed point of T. Then x is the unique fixed point of S. We shall show that there exists ψ mapping such that S is ψ-contractive mapping.

We shall First show the following claim For each i, $1 \leq i \leq N$, T^i is ϕ-contractive The proof will be by finite induction. It is trivially true for $i = 1$. Assume it is true for $i = j$. Then, for $x, y \in X_\rho$, since T^j is ϕ-contractive,

$$\begin{aligned}\rho(2(T^{j+1}x - T^{j+1}y)) &\leq \phi(\rho(T^j x - T^j y)) \\ &\leq 2\rho(T^j x - T^j y) \\ &\leq \rho(2(T^j x - T^j y)) \\ &\leq \phi(\rho((x-y)))\end{aligned}$$

Then

$$\begin{aligned}\rho(2(Sx - Sy)) &= \rho(\sum_{i=1}^{N} 2\alpha_i (T^i x - T^i y)) \\ &\leq +\sum_{i=1}^{N} \alpha_i \rho(2(T^i x - T^i y)) \\ &\leq \sum_{i=1}^{N} \alpha_i \phi(\rho(x-y))\end{aligned}$$

and S is ψ-contractive with $\psi(t) = \sum_{i=1}^{N} \alpha_i \phi(t)$. The iteration $S^n x_0$ can be defined by $x_{n+1} = Sx_n$. Therefore, convergence follows from Theorem 1. □

The function iteration process of Massa [6] is defined by:

$$S = \sum_{i=1}^{\infty} \alpha_i T^i, \text{ where each } \alpha_i \geq 1, \ \sum_{i=1}^{\infty} \alpha_i = 1, \text{ and } \alpha_j \alpha_{j+1} \neq 0. \quad (4a)$$

for at least one integer j. The iteration is then $S^n x_0$ for any x_0 in X_ρ.
If ρ has the Fatou property, then using the same proof as that of Theorem 4. yields the following.

THEOREM 5. *Suppose the hypotheses of Theorem 1 are satisfied and* ρ *has the Fatou property. Then for any* $x_0 \in X_\rho$ *the iterative scheme* $\{S^n x_0\}$*, where S is defined by* $(4a)$*, converges to the unique fixed point of T*

4. Stability

Let $f(T, x_n)$ be an iteration procedure involving T which yields a sequence $\{x_n\}$ of point from X_ρ. For example $f(T, x_n) = Tx_n$ is an iteration procedure. Suppose that $\{x_n\}$ converges to fixed point x of T. Let $\{y_n\} \subset B(x, r) \subset X_\rho$, and $\varepsilon_n = \rho(y_{n+1} - f(T, y_n))$. If $\lim \varepsilon_n = 0$ implies that $\lim y_n = x$, then the iteration procedure $x_{n+1} = f(T, x_n)$ is said to be T-stable.
Where $B(x, r)$ is a ball of center x and radius r.

THEOREM 6. *Suppose the hypotheses of Theorem 1 are satisfied and* ρ *satisfies the* Δ_2 *condition. Then iterative scheme, defined by* $f(T,x_n) = Tx_n$ *is* T*-stable*

Proof. Let $\{y_n\} \subset X_\rho$ and define $\varepsilon_n = \rho(y_{n+1} - f(T,y_n))$. Let x be the unique fixed point of T. We shall show that $\lim y_n = x$ iff $\lim \varepsilon_n = 0$.
Suppose that $\{y_n\}$ converge to x. Then

$$\begin{aligned} \varepsilon_n &= \rho(y_{n+1} - Ty_n) \\ &\leq \frac{1}{2}\rho(2(y_{n+1} - x)) + \frac{1}{2}\rho(2(x - Ty_n)) \\ &\leq \rho(2(y_{n+1} - x)) + \rho(x - y_n) \end{aligned}$$

Since ρ satisfies the Δ_2 condition, $\lim \varepsilon_n = 0$.

Conversely, suppose that $\lim \varepsilon_n = 0$. For notational convenience define $\tau_n = \rho(y_n - x)$. Then we have

$$\begin{aligned} \tau_{n+1} &= \rho(y_{n+1} - x) \\ &\leq \frac{1}{2}\rho(2(y_{n+1} - Ty_n)) + \frac{1}{2}\rho(2(x - Ty_n)) \\ &\leq \rho(2(y_{n+1} - Ty_n)) + \frac{1}{2}\phi(\tau_n) \end{aligned}$$

Suppose that $\tau = \limsup \tau_n > 0$. So, by the fact that ρ satisfies the Δ_2 condition, it follows that

$$\tau \leq \frac{1}{2}\phi(\tau) < \tau,$$

this is a contradiction, it then follows that $\tau = 0$. Therefore $\lim y_n = x$. □

The fact the Kirk and Massa iteration schemes are modified function iteration makes the proofs of the next two results corollaries of Theorem 6.

THEOREM 7. *Suppose the hypotheses of Theorem 1 are satisfied and* ϕ *non-deceasing and* ρ *satisfies the* Δ_2 *condition. Then iterative scheme, defined by* $f(T,x_n) = Sx_n$*, where* S *is defined as in* (3*a*)*, is* T*-stable*

THEOREM 8. *Suppose the hypotheses of Theorem 1 are satisfied and* ϕ *non-deceasing and* ρ *has the Fatou property and satisfies the* Δ_2 *condition. Then iterative scheme, defined by* $f(T,x_n) = Sx_n$*, where* S *is defined as in* (4*a*)*, is* T*-stable*

THEOREM 9. *Suppose the hypotheses of Theorem 2 are satisfied and* $t \longmapsto t - \frac{1}{2}\phi(t)$ *is non-deceasing function. The Mann iteration, with* $0 < \alpha \leq \alpha_n \leq 1$*, converges to the unique fixed point* x *of* T*, with error estimate*

$$\rho(x_{n+1} - x) \leq \psi^{-1}(\psi(\rho(x_0 - x)) - \sum_{k=0}^{n} \alpha_k),$$

where ψ *is defined*

$$\psi(t) = \int \frac{dt}{t - \frac{1}{2}\phi(t)},$$

and ψ^{-1} *is the inverse function of* ψ

Proof. From Theorem 2, the Mann iteration converges to unique fixed point of T. Call it x. Using $(1a)$,

$$\alpha_n(\tau_n - \frac{1}{2}\phi(\tau_n)) \leq \tau_n - \tau_{n+1},$$

with $\tau_n = \rho(x_n - x)$. Thus

$$\psi(\tau_n) - \psi(\tau_{n+1}) = \int_{\tau_{n+1}}^{\tau_n} \frac{dt}{t - \frac{1}{2}\phi(t)}.$$

Since $t \longmapsto t - \frac{1}{2}\phi(t)$ is nondecreasing, we have for all t in $[\tau_{n+1}, \tau_n]$,

$$\frac{1}{t - \frac{1}{2}\phi(t)} \geq \frac{1}{\tau_n - \frac{1}{2}\phi(\tau_n)},$$

then

$$\psi(\tau_n) - \psi(\tau_{n+1}) = \int_{\tau_{n+1}}^{\tau_n} \frac{dt}{t - \frac{1}{2}\phi(t)} \geq \frac{\tau_n - \tau_{n+1}}{\tau_n - \frac{1}{2}\phi(\tau_n)} \geq \alpha_n.$$

Thus

$$\psi(\tau_{n+1}) \leq \psi(\tau_n) - \alpha_n \leq ... \leq \psi(\tau_0) - \sum_0^n \alpha_k.$$

□

In a similar manner one can prove the following result.

THEOREM 10. *Suppose the hypotheses of Theorem 2 are satisfied and* $t \longmapsto t - \frac{1}{2}\phi(t)$ *is non-deceasing function. The Ishikawa iteration, with* $0 \leq \beta_n,\ \alpha_n \leq 1,\ \sum \beta_n \alpha_n = \infty$ *converges to the unique fixed point* x *of* T*, with error estimate*

$$\rho(x_{n+1} - x) \leq \psi^{-1}(\psi(\rho(x_0 - x)) - \sum_{k=0}^n \alpha_k \beta_k),$$

where ψ *is defined*

$$\psi(t) = \int \frac{dt}{t - \frac{1}{2}\phi(t)},$$

and ψ^{-1} *is the inverse function of* ψ

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