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# ON SEMIGROUP IDEALS AND $(\sigma,\tau)$-$n$-DERIVATIONS IN NEAR-RINGS

**Abstract.** In the present paper, we investigate the commutativity of addition and ring behavior of prime near-rings satisfying certain conditions involving $(\sigma,\tau)$-$n$-derivations on semigroup ideals. We have also constructed some examples to justify that various restrictions imposed in the hypotheses of our theorems are not superfluous. Finally, few related results are also obtained.

## 1. Introduction

Throughout the paper, $N$ will denote a zerosymmetric left near-ring. $N$ is called zero symmetric if $0x = 0$ holds for all $x \in N$ ( Recall that in a left near-ring $x0 = 0$ holds for all $x \in N$ ). For any $x, y \in N$ the symbol $[x, y] = xy - yx$ stands for multiplicative commutator of $x$ and $y$ and the symbol $(x, y) = x + y - x - y$ stands for additive commutator of $x$ and $y$. For terminologies concerning near-rings, we refer to G.Pilz [14].

A derivation $d$ on $N$ is an additive mapping $d : N \longrightarrow N$ satisfying $d(xy) = d(x)y + xd(y)$ (or equivalently $d(xy) = xd(y) + d(x)y$) for all $x, y \in N$. This notion has been generalized by different authors in different directions ( see [1,2,3,4,7,8,9,12] for references ). An additive mapping $d : N \longrightarrow N$ is called a $(\sigma,\tau)$-derivation of $N$ if there exist functions $\sigma, \tau : N \longrightarrow N$ such that $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ for all $x, y \in N$ ( For reference see [8,9]). Very recently the authors [3,4] generalized notions of derivation and $(\sigma,\tau)$-derivation in two new directions by introducing the notions of $n$-derivation and $(\sigma,\tau)$-$n$-derivation, where $n$ is a positive integer. Following [3], a map $D : \underbrace{N \times N \times \cdots \times N}_{n-\text{times}} \longrightarrow N$ is said to be permuting (or symmetric) if the relation $D(x_1, x_2, \cdots, x_n) = D(x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(n)})$ holds for all $x_1, x_2, \cdots, x_n \in N$ and for every permutation $\pi \in S_n$, where $S_n$ is the permutation group on $\{1, 2, \cdots, n\}$. Let $n$ be a fixed positive integer. An $n$-additive (i.e.; additive in each argument) mapping $D : N \times N \times \cdots \times N \longrightarrow N$ is called an $n$-derivation if the relations

$D(x_1, x_2, \cdots, x_{i-1}, x_i x_i^{'}, x_{i+1}, \cdots, x_n)$
$= D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)x_i' + x_i D(x_1, x_2, \cdots, x_{i-1}, x_i^{'}, x_{i+1}, \cdots, x_n)$

hold for all $x_1, x_2, \cdots, x_{i-1}, x_i, x_i^{'}, x_{i+1}, \cdots, x_n \in N$, $i = 1, 2, 3, \cdots, n$. If in addition $D$ is a permuting map then all the above conditions are equivalent and in this case $D$ is called a permuting $n$-derivation of $N$ (see [3] for further reference). An $n$-additive mapping $D : N \times N \times \cdots \times N \longrightarrow N$ is called a $(\sigma,\tau)$-$n$-derivation of $N$ if there exist functions $\sigma, \tau : N \longrightarrow N$ such that the relations

$D(x_1, x_2, \cdots, x_{i-1}, x_i x_i^{'}, x_{i+1}, \cdots, x_n)$
$= D(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1} \cdots, x_n)\sigma(x_i^{'}) + \tau(x_i)D(x_1, x_2, \cdots, x_{i-1}, x_i^{'}, x_{i+1}, \cdots, x_n)$

hold for all $x_1,x_2,\cdots,x_{i-1},x_i,x_i^{'},x_{i+1},\cdots,x_n\in N$, $i=1,2,3,\cdots,n$. Further in addition if $D$ is a permuting map then all the above conditions are equivalent and in this case $D$ is called a permuting $(\sigma,\tau)$-$n$-derivation of $N$ (see [4] for further reference).

There are several results in the existing literature which assert that additive groups of prime near-rings with certain constrained derivation on semigroup ideals are abelian and some times under above constraints prime near-rings have ring like behavior. Recently several authors (see [5,10,11] for reference where further references can be found) have investigated ring behavior as well as commutativity of addition of prime near-rings satisfying certain conditions involving different types of derivations on semigroup ideals. Motivated by these results, now we shall consider $(\sigma,\tau)$-$n$-derivation on a near-ring $N$ and show that prime near-rings satisfying identities involving $(\sigma,\tau)$-$n$-derivations on semigroup ideals have their additive groups as abelian ones. We have established a result which shows that a prime near-ring with $(\sigma,\tau)$-$n$-derivation satisfying a condition on a semigroup ideal behaves like a ring. In fact, our theorems generalize, extend, improve and compliment several results obtained earlier on derivations, $(\sigma,\tau)$-derivations, symmetric bi-$(\sigma,\tau)$-derivations, permuting $n$-derivations and $(\sigma,\tau)$-$n$-derivations viz. Theorems $3.2-3.4$ of [3], Theorems $3.1-3.6$, $3.9-3.10$ of [4], Theorems 2.1&3.3 of [5], Theorem 6 of [8], Theorems 1 & 4 of [10] and Theorem 2 of [11] etc.- to mention a few only. Few related results have been also discussed.

## 2. Preliminary results

Throughout this paper, $\sigma$ and $\tau$ will represent automorphisms of $N$. We begin with the following lemmas which are essential for developing the proofs of our main results. Proofs of first two lemmas can be found in [5], while the next three ones have been essentially proved in [4].

LEMMA 1. *Let $N$ be a prime near-ring. If $z\in Z\setminus\{0\}$ and $x$ is an element of $N$ such that $xz\in Z$ or $zx\in Z$, then $x\in Z$.*

LEMMA 2. *Let $N$ be a prime near-ring. If $Z$ contains a nonzero semigroup right ideal, then $N$ is a commutative ring.*

LEMMA 3. *Let $N$ be a near-ring. Then $D$ is a $(\sigma,\tau)$-$n$-derivation of $N$ if and only if*

$$D(x_1,x_2,\cdots,x_{i-1},x_ix_i^{'},x_{i+1},\cdots,x_n)$$
$$=\tau(x_i)D(x_1,x_2,\cdots,x_{i-1},x_i^{'},x_{i+1},\cdots,x_n)+D(x_1,x_2,\cdots,x_{i-1},x_i,x_{i+1},\cdots,x_n)\sigma(x_i^{'})$$

*for all $x_1,x_2,\cdots,x_{i-1},x_i,x_i^{'},x_{i+1},\cdots,x_n\in N$, $i=1,2,3,\cdots,n$.*

LEMMA 4. *Let $N$ be a near-ring and $D$ be a $(\sigma,\tau)$-$n$-derivation of $N$. Then*

$$\{D(x_1,x_2,\cdots,x_{i-1},x_i,x_{i+1},\cdots,x_n)\sigma(x_i^{'})+\tau(x_i)D(x_1,x_2,\cdots,x_{i-1},x_i^{'},x_{i+1},\cdots,x_n)\}y$$
$$=D(x_1,x_2,\cdots,x_{i-1},x_i,x_{i+1},\cdots,x_n)\sigma(x_i^{'})y+\tau(x_i)D(x_1,x_2,\cdots,x_{i-1},x_i^{'},x_{i+1},\cdots,x_n)y$$

*for all $x_1,x_2,\cdots,x_{i-1},x_i,x_i^{'},x_{i+1},\cdots,x_n,y\in N$, $i=1,2,3,\cdots,n$.*

LEMMA 5. *Let* $N$ *be a near-ring and* $D$ *be a* $(\sigma,\tau)$*-n-derivation of* $N$*. Then*
$\{\tau(x_i)D(x_1,x_2,\cdots,x_{i-1},x_i^{'},x_{i+1},\cdots,x_n)+D(x_1,x_2,\cdots,x_{i-1},x_i,x_{i+1},\cdots,x_n)\sigma(x_i^{'})\}y$
$=\tau(x_i)D(x_1,x_2,\cdots,x_{i-1},x_i^{'},x_{i+1},\cdots,x_n)y+D(x_1,x_2,\cdots,x_{i-1},x_i,x_{i+1},\cdots,x_n)\sigma(x_i^{'})y$
*for all* $x_1,x_2,\cdots,x_{i-1},x_i,x_i^{'},x_{i+1},\cdots,x_n,y\in N$*,* $i=1,2,3,\cdots,n$*.*

LEMMA 6. *Let* $N$ *be a prime near-ring and* $D$ *a nonzero* $(\sigma,\tau)$*-n-derivation of* $N$*. If* $U_1,U_2,...,U_n$ *are nonzero semigroup right ideals ( or nonzero semigroup left ideals) of* $N$*, then* $D(U_1,U_2,...,U_n)\neq\{0\}$*.*

*Proof.* Let $U_1,U_2,...,U_n$ be nonzero semigroup right ideals of $N$ and $D(U_1,U_2,...,U_n)=\{0\}$. This gives us that

$$D(u_1,u_2,...,u_n)=0 \tag{2.1}$$

for all $u_1\in U_1,u_2\in U_2,...,u_n\in U_n$. Putting $u_1r_1$, where $r_1\in N$, for $u_1$ in the relation (2.1) and using it again we have $\tau(u_1)D(r_1,u_2,...,u_n)=0$. Now replacing $u_1$ by $u_1r$ where $r\in N$ in the preceding relation we have $\tau(u_1)\tau(r)D(r_1,u_2,...,u_n)=0$ i.e.; $\tau(U_1)ND(r_1,u_2,...,u_n)=\{0\}$. But $\tau(U_1)\neq\{0\}$ and $N$ is a prime near-ring, we conclude that

$$D(r_1,u_2,...,u_n)=0. \tag{2.2}$$

Now putting $u_2r_2\in U_2$ in place of $u_2$, where $r_2\in N$, in relation (2.2) and proceeding as above we get $D(r_1,r_2,u_3,...,u_n)=0$. Proceeding inductively as before we conclude that $D(r_1,r_2,...,r_n)=0$ for all $r_1,r_2,...,r_n\in N$. This shows that $D(N,N,\cdots,N)=\{0\}$, leading to a contradiction as $D$ is a nonzero $(\sigma,\tau)$-$n$-derivation. Therefore, $D(U_1,U_2,...,U_n)\neq\{0\}$. We can also say that $D(U_1,U_2,...,U_n)=\{0\}$ implies that $D(N,N,...,N)=\{0\}$. Similar arguments can be given for semi group left ideals also. □

LEMMA 7. *Let* $N$ *be a prime near-ring,* $D$ *a nonzero* $(\sigma,\tau)$*-n-derivation of* $N$ *and* $U_1,U_2,\cdots,U_n$ *be nonzero semigroup ideals of* $N$*.*

(*i*) *If* $x\in N$ *and* $D(U_1,U_2,\cdots,U_n)x=\{0\}$*, then* $x=0$*.*

(*ii*) *If* $x\in N$ *and* $xD(U_1,U_2,\cdots,U_n)=\{0\}$*, then* $x=0$*.*

*Proof.* (*i*) By our hypothesis $D(U_1,U_2,...,U_n)x=\{0\}$ i.e

$$D(u_1,u_2,...,u_n)x=0,\ \text{ for all } u_i\in U_i;1\leq i\leq n. \tag{2.3}$$

Putting $r_1u_1$ in place of $u_1$, where $r_1\in N$, in relation (2.3) we get $D(r_1u_1,u_2,...,u_n)x=0$. Using Lemma 4 previous relation takes the form $D(r_1,u_2,...,u_n)\sigma(u_1)x+\tau(r_1)D(u_1,u_2,...,u_n)x=0$. Using the hypothesis again we get $D(r_1,u_2,...,u_n)\sigma(u_1)x=0$. Replacing $u_1$ by $u_1s$ where $s\in N$ in preceding relation we obtain $D(r_1,u_2,...,u_n)\sigma(u_1)\sigma(s)x=0$ i.e.; $D(r_1,u_2,...,u_n)\sigma(u_1)Nx=\{0\}$. Since $N$ is a prime near-ring, either $D(r_1,u_2,...,u_n)\sigma(u_1)=0$ or $x=0$. Our claim is that $D(r_1,u_2,...,u_n)\sigma(u_1)\neq 0$, for some $r_1\in N,u_1\in U_1,u_2\in U_2,...,u_n\in U_n$. For other wise, if $D(r_1,u_2,...,u_n)\sigma(u_1)=0$ for all $r_1\in N,u_1\in U_1,u_2\in U_2,...,u_n\in U_n$, then replacing $u_1$ by $tu_1$, where $t\in N$ in the latter relation, we arrive at $D(r_1,u_2,...,u_n)\sigma(t)\sigma(u_1)=0$ i.e.; $D(r_1,u_2,...,u_n)N\sigma(U_1)=$

$\{0\}$. As $\sigma(U_1) \neq \{0\}$, primeness of $N$ yields $D(r_1,u_2,...,u_n) = 0$ for all $r_1 \in N, u_2 \in U_2,...,u_n \in U_n$. Now by Lemma 6, we obtain that $D(N,N,...,N) = \{0\}$, leading to a contradiction. Therefore, we conclude that $x = 0$.

$(ii)$ It can be proved in a similar way. □

## 3. Commutativity of near-rings

In 2007, Gölbasi et al. [10, Theorem 1] proved that if a prime near-ring $N$ admits a nontrivial $(\sigma,\tau)$-derivation $d$ for which $d(U) \subseteq Z$, where $U$ is a nonzero semigroup right ideal, then $N$ is a commutative ring. Later this result was generalized for symmetric bi-$(\sigma,\tau)$-derivation in 2010 by Öztürk et al. [11 ,Theorem 2] who proved that if $N$ is a 2-torsion free prime near-ring which admits a nonzero symmetric bi-$(\sigma,\tau)$-derivation $D$ such that $D(U,U) \subseteq Z$, then $N$ is a commutative ring, where $U$ is a nonzero semigroup ideal of $N$. We have obtained its analogue in the setting of $(\sigma,\tau)$-$n$-derivation. We have also shown that symmetric and 2-torsion freeness properties used by Öztürk are redundant. In fact, we have obtained the following.

THEOREM 1. *Let $N$ be a prime near-ring, $U_1,U_2,...,U_n$ be nonzero semigroup right ideals of $N$ and let $D$ be a nonzero $(\sigma,\tau)$-$n$-derivation of $N$. If $D(U_1,U_2,...,U_n) \subseteq Z$, then $N$ is a commutative ring.*

*Proof.* For all $u_1,u_1' \in U_1, u_2 \in U_2,...,u_n \in U_n$, we get

$$D(u_1u_1',u_2,...,u_n) = D(u_1,u_2,...,u_n)\sigma(u_1') + \tau(u_1)D(u_1',u_2,...,u_n) \in Z. \tag{3.1}$$

Now commuting the equation (3.1) with the element $\sigma(u_1')$ we have

$$\{D(u_1,u_2,...,u_n)\sigma(u_1') + \tau(u_1)D(u_1',u_2,...,u_n)\}\sigma(u_1') = \sigma(u_1')\{D(u_1,u_2,...,u_n)\sigma(u_1') + \tau(u_1)D(u_1',u_2,...,u_n)\}.$$

Using our hypothesis and Lemma 4, we get
$D(u_1',u_2,...,u_n)(\tau(u_1)\sigma(u_1') - \sigma(u_1')\tau(u_1)) = 0$ for all $u_1,u_1' \in U_1, u_2 \in U_2,...,u_n \in U_n$. Replacing $u_1$ by $u_1v_1$, where $v_1 \in U_1$ in the latter relation and using the same again we arrive at

$$D(u_1',u_2,...,u_n)\tau(u_1)[\tau(v_1),\sigma(u_1')] = 0. \tag{3.2}$$

for all $u_1,u_1',v_1 \in U_1, u_2 \in U_2,...,u_n \in U_n$. Putting $u_1r$, where $r \in N$ for $u_1$ in the relation (3.2), we obtain that $D(u_1',u_2,...,u_n)\tau(u_1)N[\tau(v_1),\sigma(u_1')] = \{0\}$. For given $u_1' \in U_1$, primeness of $N$ yields either $D(u_1',u_2,\cdots,u_n)\tau(u_1) = 0$ or $[\tau(v_1),\sigma(u_1')] = 0$. If the first case holds i.e.; $D(u_1',u_2,\cdots,u_n)\tau(u_1) = 0$, then hypothesis gives us $D(u_1',u_2,\cdots,u_n)N\tau(u_1) = \{0\}$. Now using the facts that $\tau(U_1) \neq \{0\}$ and $N$ is prime, we conclude that $D(u_1',u_2,\cdots,u_n) = 0$ for all $u_2 \in U_2,\cdots,u_n \in U_n$. In this case, the relation (3.1) takes the form $D(u_1u_1',u_2,\cdots,u_n) = D(u_1,u_2,\cdots,u_n)u_1' \in Z$ for all $u_1 \in U_1, u_2 \in U_2,\cdots,u_n \in U_n$; and by Lemmas 1 and 6, we get $u_1' \in Z$. On the other hand if second case holds i.e.; $[\tau(v_1),\sigma(u_1')] = 0$ for all $v_1 \in U_1$, then replacing $v_1$ by

$v_1s$, where $s \in N$ in the latter relation, we find that $\tau(v_1)[\tau(s),\sigma(u_1^{'})] = 0$. This shows that $\tau(U_1)N[\tau(s),\sigma(u_1^{'})] = \{0\}$. Now primeness of $N$ yields $[\tau(s),\sigma(u_1^{'})] = 0$ for $s \in N$. This implies that $\sigma(u_1^{'}) \in Z$ i.e.; $u_1^{'} \in Z$. Including both the cases, we conclude that $U_1 \subseteq Z$ and $N$ is therefore a commutative ring by Lemma 2. □

REMARK 1. The above theorem includes Theorems 3.1 & 2.1 of [4] & [5] respectively.

Following example demonstrates that the primeness in the hypothesis of Theorem 1 can not be omitted.

EXAMPLE 1. Let $\mathbb{C}$ be the usual ring of complex numbers and $(\mathbb{Z},+,*)$ be a left near-ring of integers, where '$*$' is defined by $a*b = |a|b$. Assume $N = \mathbb{C}\times\mathbb{Z}$ and $U_1 = \mathbb{C}\times m_1\mathbb{Z}, U_2 = \mathbb{C}\times m_2\mathbb{Z},\cdots,U_n = \mathbb{C}\times m_n\mathbb{Z}$, where $m_1,m_2,\cdots,m_n$ are different positive integers. Then it can be easily verified that $N$ is a zerosymmetric left near-ring with regard to componentwise addition and multiplication, having $U_1,U_2,\cdots,U_n$ its nonzero semigroup right ideals. Define $D : N\times N\times\cdots\times N \longrightarrow N, \sigma : N \longrightarrow N$ and $\tau : N \longrightarrow N$ such that

$$D((z_1,a_1),(z_2,a_2),\cdots,(z_n,a_n)) = (\lambda(z_1-\bar{z_1})(z_2-\bar{z_2})\cdots(z_n-\bar{z_n}),0),$$

where $\lambda$ is any complex number, $\sigma(z,a) = (\bar{z},a)$ and $\tau(z,a) = (z,a)$ respectively. It is easy to show that $N$ is a semiprime near-ring but not a prime near-ring and $D$ is a nonzero $(\sigma,\tau)$-$n$-derivation of $N$, where $\sigma$ and $\tau$ are automorphisms of $N$ such that $D(U_1,U_2,\cdots,U_n) \subseteq Z$. However, $N$ is not a commutative ring.

## 4. Additive commutativity of near-rings

In this section, we prove some results which show that the additive groups of prime near-rings satisfying certain conditions involving $(\sigma,\tau)$-$n$-derivations on semigroup ideals turn out to be commutative. Let $X$ and $Y$ be nonempty subsets of $N$, and $[X,Y] = \{[x,y] \mid x \in X, y \in Y\}$.

THEOREM 2. *Let $N$ be a prime near-ring. Let $D_1$ and $D_2$ be any two nonzero $(\sigma,\tau)$-$n$-derivations of $N$. If $[D_1(U_1,U_2,...,U_n),D_2(U_1,U_2,...,U_n)] = \{0\}$, where $U_1,U_2,...,U_n$ are nonzero semigroup ideals of $N$, then $(N,+)$ is abelian.*

*Proof.* It is straight forward to show that if $z \in N$ is such that $[z,D_2(U_1,U_2,...,U_n)] = [z+z,D_2(U_1,U_2,...,U_n)] = \{0\}$ and $u_1,u_1^{'} \in U_1$ are such that $u_1+u_1^{'} \in U_1$, then $zD_2(c,u_2,...,u_n) = 0$, where $c$ is the additive commutator $(u_1+u_1^{'}-u_1-u_1^{'})$, $u_2 \in U_2,\cdots,u_n \in U_n$. If $r,s \in U_1$ we have $rs \in U_1$ and $rs+rs = r(s+s) \in U_1$. Since $[D_1(U_1,U_2,...,U_n),D_2(U_1,U_2,...,U_n)] = \{0\}$, taking $z = D_1(rs,u_2^{'},...,u_n^{'})$ where $r$, $s \in U_1, u_2^{'} \in U_2,\cdots,u_n^{'} \in U_n$ gives $D_1(U_1^2,U_2,...,U_n)D_2(c,u_2,...,u_n) = \{0\}$ because for all $r,s \in U_1$ implies that $rs \in U_1^2$. But $U_1^2 = \{pq \mid p,q \in U_1\}$ is a nonzero semigroup ideal, so by Lemma 7$(i)$ we get

$$D_2(u_1+u_1^{'}-u_1-u_1^{'},u_2,u_3,...,u_n)=0 \tag{4.1}$$

for all $u_1,u_1^{'}\in U_1$ such that $u_1+u_1^{'}\in U_1, u_2\in U_2,\cdots,u_n\in U_n$. Now take $u_1=rx$ and $u_1^{'}=ry$ where $r\in U_1$ and $x,y\in N$, so that $u_1,u_1^{'}$ and $u_1+u_1^{'}$ are all in $U_1$. It follows from relation (4.1) that $D_2(rx+ry-rx-ry,u_2,u_3,...,u_n)=0$ for all $r\in U_1$ and for all $x$, $y\in N, u_2\in U_2,\cdots,u_n\in U_n$. Replacing $r$ by $wr, w\in U_1$, we get $D_2(U_1,U_2,...,U_n)$ $\sigma(rx+ry-rx-ry)=\{0\}$ for all $r\in U_1$ and $x,y\in N$. Now putting $zr$ for $r$, where $z\in N$ and using the fact that $\sigma$ is an automorphism we conclude that $D_2(U_1,U_2,\cdots,U_n)N\sigma(r)$ $\sigma(x+y-x-y)=\{0\}$ for all $x,y\in N, r\in U_1$. Primeness of $N$ and Lemma 6 insure that $\sigma(r)\sigma(x+y-x-y)=0$ for all $x,y\in N,r\in U_1$. Now using $rt$ for $r$, where $t\in N$ in the latter relation we obtain that $\sigma(r)N\sigma(x+y-x-y)=\{0\}$ for all $x,y\in N,r\in U_1$ i.e.; $\sigma(U_1)N\sigma(x+y-x-y)=\{0\}$ for all $x,y\in N$. Using the facts that $\sigma$ is an automorphism and $U_1\neq\{0\}$, primeness of $N$ insures that $(N,+)$ is abelian. □

REMARK 2. It can be easily seen that Theorem 3.2 of [4] and Theorem 3.3 of [5] become corollaries of the above result.

THEOREM 3. *Let $U_1,U_2,...,U_n$ be nonzero semigroup ideals of a prime near-ring $N$. If it admits nonzero $(\sigma,\tau)$-n-derivations $D_1$ and $D_2$ such that $D_1(x_1,x_2,\cdots,x_n)$ $D_2(y_1,y_2,\cdots,y_n)=-D_2(x_1,x_2,\cdots,x_n)D_1(y_1,y_2,\cdots,y_n)$ for all $x_1,y_1\in U_1;x_2,$ $y_2\in U_2;\cdots;x_n,y_n\in U_n$, then $(N,+)$ is abelian.*

*Proof.* If $u_1,u_1^{'}\in U_1$ are such that $u_1+u_1^{'}\in U_1$, then replacing $y_1$ in the hypothesis by $u_1,u_1^{'}$ and $u_1+u_1^{'}$ respectively and using the same again we obtain,
$D_1(x_1,x_2,\cdots,x_n)D_2(u_1,y_2,\cdots,y_n) + D_1(x_1,x_2,\cdots,x_n)D_2(u_1^{'},y_2,\cdots,y_n)$ $+D_1(x_1,x_2,\cdots,x_n)D_2(-u_1,y_2,\cdots,y_n)+D_1(x_1,x_2,\cdots,x_n)D_2(-u_1^{'},y_2,\cdots,y_n)=0$ i.e.; $D_1(x_1,x_2,\cdots,x_n)D_2((u_1,u_1^{'}),y_2,\cdots,y_n)=0$. This implies that $D_1(U_1,U_2,\cdots,U_n)$ $D_2((u_1,u_1^{'}),y_2,\cdots,y_n)=\{0\}$. Now using Lemma 7$(i)$, we conclude that

$$D_2((u_1,u_1^{'}),y_2,\cdots,y_n)=0$$

for all $u_1,u_1^{'}\in U_1$ such that $u_1+u_1^{'}\in U_1,y_2\in U_2,\cdots,y_n\in U_n$. This is the same as the relation (4.1). Now using the similar arguments as used in the proof Theorem 2, we conclude that $(N,+)$ is abelian. □

REMARK 3. Theorem 3.3 of [4] is an immediate corollary of the above theorem.

THEOREM 4. *Let $N$ be a prime near-ring with nonzero $(\sigma,\tau)$-n-derivations $D_1$ and $D_2$ such that*

$$D_1(x_1,x_2,\cdots,x_n)\sigma D_2(y_1,y_2,\cdots,y_n)+\tau D_2(x_1,x_2,\cdots,x_n)D_1(y_1,y_2,\cdots,y_n)=0$$

*for all $x_1,y_1\in U_1;x_2,y_2\in U_2;\cdots;x_n,y_n\in U_n$, where $U_1,U_2,...,U_n$ are nonzero semi-group ideals of $N$. Then $(N,+)$ is abelian.*

*Proof.* If $u_1, u_1^{'} \in U_1$ are such that $u_1+u_1^{'} \in U_1$, then substituting $u_1, u_1^{'}$ and $u_1+u_1^{'}$ for $y_1$ respectively in the hypothesis and using the same again we obtain,
$D_1(x_1,x_2,\cdots,x_n)\sigma D_2(u_1,y_2,\cdots,y_n) + D_1(x_1,x_2,\cdots,x_n)\sigma D_2(u_1^{'},y_2,\cdots,y_n) + D_1(x_1,x_2,\cdots,x_n)\sigma D_2(-u_1,y_2,\cdots,y_n) + D_1(x_1,x_2,\cdots,x_n)\sigma D_2(-u_1{'},y_2,\cdots,y_n) = 0$. i.e.; $D_1(x_1,x_2,\cdots,x_n)\sigma D_2((u_1,u_1{'}),y_2,\cdots,y_n) = 0$. This gives us $D_1(U_1,U_2,\cdots,U_n)$ $D_2((u_1,u_1^{'}),y_2,\cdots,y_n) = \{0\}$. Now using Lemma 7$(i)$, we conclude that $\sigma D_2((u_1,u_1{'}),y_2,\cdots,y_n) = 0$. Since $\sigma$ is an automorphism of $N$, we have $D_2((u_1,u_1{'}),y_2,\cdots,y_n)) = 0$ for all $u_1,u_1^{'} \in U_1$ such that $u_1+u_1^{'} \in U_1, y_2 \in U_2,\cdots,$ $y_n \in U_n$. This is same as the relation (4.1). Now proceeding in the similar way as in the proof of Theorem 2, we conclude that $(N,+)$ is abelian. □

REMARK 4. The above theorem includes Theorem 3.4 of [4].

THEOREM 5. *Let N be a prime near-ring admitting a nonzero* $(\sigma,\tau)$*-n-derivation* $D_1$ *and a nonzero n-derivation* $D_2$ *such that* $D_1(x_1,x_2,\cdots,x_n)\sigma D_2(y_1,y_2,\cdots,y_n) + \tau D_2(x_1,x_2,\cdots,x_n)D_1(y_1,y_2,\cdots,y_n) = 0$ *for all* $x_1,y_1 \in U_1; x_2,y_2 \in U_2;\cdots;x_n,y_n \in U_n$*, where* $U_1,U_2,...,U_n$ *are nonzero semigroup ideals of N. Then* $(N,+)$ *is abelian.*

*Proof.* If $u_1, u_1^{'} \in U_1$ are such that $u_1+u_1^{'} \in U_1$, then substituting $u_1, u_1^{'}$ and $u_1+u_1^{'}$ for $y_1$ respectively in the hypothesis and using the same again we arrive at,
$D_1(x_1,x_2,\cdots,x_n)\sigma D_2(u_1,y_2,\cdots,y_n) + D_1(x_1,x_2,\cdots,x_n)\sigma D_2(u_1{'},y_2,\cdots,y_n) + D_1(x_1,x_2,\cdots,x_n)\sigma D_2(-u_1,y_2,\cdots,y_n) + D_1(x_1,x_2,\cdots,x_n)\sigma D_2(-u_1^{'},y_2,\cdots,y_n) = 0$. i.e.; $D_1(x_1,x_2,\cdots,x_n)\sigma D_2((u_1,u_1^{'}),y_2,\cdots,y_n) = 0$. This implies that $D_1(U_1,U_2,\cdots,U_n)$ $D_2((u_1,u_1^{'}),y_2,\cdots,y_n) = \{0\}$. Now using Lemma 7$(i)$ we conclude that $\sigma D_2((u_1,u_1^{'}),y_2,\cdots,y_n) = 0$. But $\sigma$ is an automorphism of $N$, we conclude that $D_2((u_1,u_1^{'}),y_2,\cdots,y_n) = 0$ for all $u_1,u_1^{'} \in U_1$ such that $u_1+u_1^{'} \in U_1, y_2 \in U_2,\cdots,$ $y_n \in U_n$. This is identical with the relation (4.1), if we treat $D_2$ as $(I,I)$-$n$-derivation of $N$, where $I$ is the identity automorphism of $N$. Now arguing on similar lines as in case of Theorem 2, we conclude that $(N,+)$ is abelian. □

REMARK 5. The above theorem includes Theorem 3.5 of [4].

THEOREM 6. *Let N be a prime near-ring admitting a* $(\sigma,\tau)$*-n-derivation D and a* $(\sigma,\tau)$*-derivation d such that* $dD(x_1,x_2,\cdots,x_n) = 0$ *for all* $x_1 \in U_1, x_2 \in U_2,\cdots,$ $x_n \in U_n$*, where* $U_1,U_2,...,U_n$ *are nonzero semigroup ideals of N. Then one of the following holds:*

$(i)$ $D = 0$,

$(ii)$ $d = 0$,

$(iii)$ $(N,+)$ *is abelian.*

*Proof.* By our hypothesis we have, $dD(x_1,x_2,\cdots,x_n) = 0$ for all $x_1 \in U_1, x_2 \in U_2,\cdots,$ $x_n \in U_n$. Replacing $x_1$ by $x_1x_1^{'}$, where $x_1^{'} \in U_1$ we have $d\{D(x_1,x_2,\cdots,x_n)\sigma(x_1^{'}) + \tau(x_1)D(x_1^{'},x_2,\cdots,x_n)\} = 0$ i.e; $dD(x_1,x_2,\cdots,x_n)\sigma^2(x_1^{'}) + \tau D(x_1,x_2,\cdots,x_n)d\sigma(x_1^{'}) +$

$d\tau(x_1)\sigma D(x_1',x_2,\cdots,x_n)+\tau^2(x_1)dD(x_1',x_2,\cdots,x_n)=0$. Using the hypothesis again we get

$$\tau D(x_1,x_2,\cdots,x_n)d\sigma(x_1')+d\tau(x_1)\sigma D(x_1',x_2,\cdots,x_n)=0 \tag{4.2}$$

for all $x_1',x_1\in U_1,x_2\in U_2,\cdots,x_n\in U_n$. If $u_1,u_1'\in U_1$ are such that $u_1+u_1'\in U_1$, then replacing $x_1'$ by $u_1+u_1'$ in the relation (4.2) and using it again we arrive at, $\tau D(x_1,x_2,\cdots,x_n)d\sigma(u_1)+\tau D(x_1,x_2,\cdots,x_n)d\sigma(u_1')+\tau D(x_1,x_2,\cdots,x_n)d\sigma(-u_1)+\tau D(x_1,x_2,\cdots,x_n)d\sigma(-u_1')=0$ i.e.; $\tau D(x_1,x_2,\cdots,x_n)d\sigma(u_1+u_1'-u_1-u_1')=0$. This implies that $D(x_1,x_2,\cdots,x_n)\tau^{-1}d\sigma(u_1+u_1'-u_1-u_1')=0$ i.e.; $D(U_1,U_2,\cdots,U_n)$ $\tau^{-1}d\sigma(u_1+u_1'-u_1-u_1')=\{0\}$. If $D=0$, then nothing to do. Suppose that $D\neq 0$, hence by Lemma 7$(i)$ we have $\tau^{-1}d\sigma(u_1+u_1'-u_1-u_1')=0$ i.e.;

$$d\sigma(u_1,u_1')=0 \tag{4.3}$$

for all $u_1,u_1'\in U_1$ such that $u_1+u_1'\in U_1$. If we take $u_1=rx$ and $u_1'=ry$ where $r\in U_1$ and $x,y\in N$, then it implies that $u_1,u_1'$ and $u_1+u_1'$ are elements of $U_1$. Thus substituting $u_1=rx$ and $u_1'=ry$ where $r\in U_1$ and $x,y\in N$ in the relation (4.3), we get

$$d\sigma(rx,ry)=0 \tag{4.4}$$

for all $r\in U_1$, $x,y\in N$. Now replacing $r$ by $wr$ where $w\in U_1$ in the relation (4.4) and using it again we obtain that $d(\sigma(w))\sigma^2(rx,ry)=0$ i.e.; $d(\sigma(w))\sigma^2(r)\sigma^2(x,y)=0$ for all $w,r\in U_1$, $x,y\in N$. Now putting $zr$, where $z\in N$ for $r$ in the latter relation, we arrive at $d(\sigma(w))N\sigma^2(r)\sigma^2(x,y)=\{0\}$ for all $w,r\in U_1$, $x,y\in N$. Now primeness of $N$ assures that either $d(\sigma(w))=0$ for all $w\in U_1$ or $\sigma^2(r)\sigma^2(x,y)=0$ for all $x,y\in N$ and for all $r\in U_1$. If first case holds i.e.; $d(\sigma(w))=0$ for all $w\in U_1$, then replacing $w$ by $wt$, where $t\in N$ and using this relation again we arrive at $\tau(\sigma(w))d(\sigma(t))=0$ i.e.; $\tau(\sigma(U_1))d(N)=\{0\}$. Since $\tau(\sigma(U_1))\neq\{0\}$, using the Lemma 7$(ii)$ we conclude that $d=0$. On the other hand if second case holds i.e; $\sigma^2(r)\sigma^2(x,y)=0$ for all $x,y\in N$ and for all $r\in U_1$, then putting $rp$, where $p\in N$ for $r$ we obtain that $\sigma^2(r)N\sigma^2(x,y)=\{0\}$ for all $x,y\in N$ and for all $r\in U_1$ i.e.; $\sigma^2(U_1)N\sigma^2(x,y)=\{0\}$ for all $x,y\in N$. Since $\sigma^2(U_1)\neq\{0\}$, primeness of $N$ provides us $\sigma^2(x,y)=0$ for all $x,y\in N$. This shows that $(N,+)$ is abelian. □

REMARK 6. Theorem 3.6 of [4] is an immediate consequence of the above theorem.

The following example shows that the restriction of primeness imposed on the hypothesis of Theorem 6 is not superfluous.

EXAMPLE 2. Let $\mathbb{C}$ be the usual ring of complex numbers and $(S_3,+)$ be the symmetric group of degree 3. If we define an operation '$*$' in $S_3$ by $a*b=b$, for all $0\neq a,b\in S_3$ and $0*b=0$ for all $b\in S_3$, where 0 stands for identity of group $(S_3,+)$. Let $N=\mathbb{C}\times S_3$ and $U_1=U_2=\cdots=U_n=\{0\}\times S_3$. Then it can be easily seen that $N$ is a zerosymmetric left near-ring with regard to componentwise addition and multiplication, having $U_1,U_2,\cdots,U_n$ its nonzero semigroup ideals. Define $D : N\times N\times\cdots\times N \longrightarrow N, d : N \longrightarrow N, \sigma : N \longrightarrow N$ and $\tau : N \longrightarrow N$ such that $D((z_1,a_1),(z_2,a_2),\cdots,(z_n,a_n)) = (\lambda(z_1-\bar{z_1})(z_2-\bar{z_2})\cdots(z_n-\bar{z_n}),0)$, where $\lambda$ is any complex number; $d(z,a) = (\eta(z-\bar{z}),0)$,where $\eta$ is any complex number; $\sigma(z,a) = (\bar{z},a)$ and $\tau(z,a) = (z,a)$ respectively. It can be easily shown that $(i)$ $N$ is a semiprime near-ring but not a prime near-ring.,$(ii)$ $D$ is a nonzero $(\sigma,\tau)$-$n$-derivation of $N$ and $(iii)$ $d$ is a nonzero $(\sigma,\tau)$-derivation of $N$, where $\sigma$ and $\tau$ are automorphisms of $N$. Here it can be also seen that $dD(U_1,U_2,\cdots,U_n) = \{0\}$. However, $(N,+)$ is not abelian.

It was shown in [10, Theorem 4] that if $d$ is non-trivial $(\sigma,\tau)$-derivation of a prime near-ring $N$ and $U$ is a nonzero semigroup ideal such that $\sigma d = d\sigma$, $\tau d = d\tau$. Let $a\in N$ and $[d(U),a]_{\sigma,\tau} = \{0\}$. Then $d(a) = 0$ or $a\in Z$. We have obtained its analogue in the setting of $(\sigma,\tau)$-$n$-derivation. In addition we have proved that the conditions $\sigma d = d\sigma$ and $\tau d = d\tau$ used by Gölbasi are superfluous. In fact we obtained the following.

THEOREM 7. *Let $U_1,U_2,...,U_n$ be nonzero semigroup ideals of a prime near-ring $N$, which admits a nonzero $(\sigma,\tau)$-$n$-derivation $D$. If $a\in N$ and $[D(x_1,x_2,\cdots,x_n),a]_{\sigma,\tau} = 0$ for all $x_1\in U_1, x_2\in U_2,\cdots,x_n\in U_n$, then $D(a,x_2,\cdots,x_n) = 0$ for all $x_2\in U_2,\cdots,x_n\in U_n$ or $a\in Z$.*

*Proof.* For all $x_1\in U_1$, we have $ax_1\in U_1$. Now replacing $x_1$ by $ax_1$ in the hypothesis, we get $D(ax_1,x_2,\cdots,x_n)\sigma(a) = \tau(a)D(ax_1,x_2,\cdots,x_n)$ for all $x_1\in U_1, x_2\in U_2,\cdots, x_n\in U_n$ and so by Lemma 4,

$$\begin{aligned}&D(a,x_2,\cdots,x_n)\sigma(x_1)\sigma(a)+\tau(a)D(x_1,x_2,\cdots,x_n)\sigma(a)\\&\quad=\tau(a)D(a,x_2,\cdots,x_n)\sigma(x_1)+\tau(a)\tau(a)D(x_1,x_2,\cdots,x_n)\end{aligned}$$

Using hypothesis again we have

$$D(a,x_2,\cdots,x_n)\sigma(x_1)\sigma(a) = \tau(a)D(a,x_2,\cdots,x_n)\sigma(x_1) \tag{4.5}$$

for all $x_1\in U_1, x_2\in U_2,\cdots,x_n\in U_n$. Now replacing $x_1$ by $x_1x_1'$, where $x_1'\in U_1$ in the relation (4.5) and using it again we arrive at $D(a,x_2,\cdots,x_n)\sigma(x_1)[\sigma(x_1'),\sigma(a)] = 0$. Substituting $x_1t$, where $t\in N$ for $x_1$ in the latter relation we get $D(a,x_2,\cdots,x_n)\sigma(x_1)N = \{0\}$. For given $a\in N$, primeness of $N$ yields either $D(a,x_2,\cdots,x_n)\sigma(x_1) = 0$ or $[\sigma(x_1'),\sigma(a)] = 0$. If first condition holds i.e.; $D(a,x_2,\cdots,x_n)\sigma(x_1) = 0$ for all $x_1\in U_1, x_2\in U_2,\cdots,x_n\in U_n$, then replacing here $x_1$ by $sx_1$, where $s\in N$, we arrive at $D(a,x_2,\cdots,x_n)\sigma(s)\sigma(x_1) = 0$ i.e.; $D(a,x_2,\cdots,x_n)N\sigma(x_1) = \{0\}$. This implies that $D(a,x_2,\cdots,x_n)N\sigma(U_1) = \{0\}$. Now using the facts that $N$ is prime and $\sigma(U_1)\neq\{0\}$, we conclude that $D(a,x_2,\cdots,x_n) = 0$ for all $x_2\in U_2,\cdots,x_n\in U_n$. On the other hand if second condition holds i.e.; $[\sigma(x_1'),\sigma(a)] = 0$ for all $x_1'\in U_1$, then $x_1'a = ax_1'$ for all $x_1'\in U_1$. Replacing $x_1'$ by $x_1'r$, where $r\in N$ in the latter relation and using the same again we arrive at $x_1'[r,a] = 0$ i.e.; $x_1'N[r,a] = \{0\}$. Now $U_1\neq\{0\}$ and primeness of $N$ insure that $a\in Z$. This completes the proof. □

REMARK 7. The above theorem has three very interesting corollaries, namely; Theorems 3.9 & 3.10 of [4] and Theorem 6 of [8].

The following example demonstrates that the primeness in the hypothesis of the above theorem is necessary.

EXAMPLE 3. Let $n$ be a fixed positive integer and $\mathbb{C}_1 = (\mathbb{C}, +, .)$, the ring of complex numbers with regard to usual addition $+$ and multiplication '.'. Next suppose that $\mathbb{C}_2 = (\mathbb{C}, +, *)$, where $+$ is the usual addition of complex numbers, and '$*$' is defined as $x * y = |x|y$, for all $x, y \in \mathbb{C}$. Then $\mathbb{C}_2$ is a zero symmetric left near-ring. Further, it can be easily verified that the set $S = \mathbb{C}_1 \times \mathbb{C}_2$ is a zerosymmetric left near-ring with regard to componentwise addition and multiplication. Now suppose that $N = \left\{ \begin{pmatrix} (x,x') & (y,y') \\ (0,0) & (0,0) \end{pmatrix} \mid (x,x'),(y,y'),(0,0) \in S \right\}$. It can be easily checked that $N$ is a zerosymmetric left near-ring with respect to matrix addition and matrix multiplication, which is not a prime near-ring. Define $D : N \times N \times \cdots \times N \longrightarrow N$ and $\sigma, \tau : N \longrightarrow N$ such that
$D\left( \begin{pmatrix} (x_1,x_1') & (y_1,y_1') \\ (0,0) & (0,0) \end{pmatrix}, \begin{pmatrix} (x_2,x_2') & (y_2,y_2') \\ (0,0) & (0,0) \end{pmatrix}, \cdots, \begin{pmatrix} (x_n,x_n') & (y_n,y_n') \\ (0,0) & (0,0) \end{pmatrix} \right)$
$= \begin{pmatrix} (0,0) & (\bar{x}_1\bar{x}_2\cdots\bar{x}_n,0) \\ (0,0) & (0,0) \end{pmatrix}$, $\sigma\begin{pmatrix} (x,x') & (y,y') \\ (0,0) & (0,0) \end{pmatrix} = \begin{pmatrix} (x,x') & (-y,-y') \\ (0,0) & (0,0) \end{pmatrix}$ and
$\tau\begin{pmatrix} (x,x') & (y,y') \\ (0,0) & (0,0) \end{pmatrix} = \begin{pmatrix} (\bar{x},\bar{x'}) & (\bar{y},\bar{y'}) \\ (0,0) & (0,0) \end{pmatrix}$, where $\bar{x}_1, \bar{x}_2, \cdots \bar{x}_n, \bar{x}, \bar{x'}, \bar{y}$ and $\bar{y'}$ denote the conjugates of the complex numbers $x_1, x_2, \cdots x_n, x, x', y$ and $y'$ respectively. It can be easily seen that $D$ is a nonzero $(\sigma,\tau)$-$n$-derivation of $N$, where $\sigma$ and $\tau$ are automorphisms of $N$. If we take $U_1 = \left\{ \begin{pmatrix} (0,0) & (x,x') \\ (0,0) & (0,0) \end{pmatrix} \mid (x,x'),(0,0) \in S \right\}$, $U_2 = U_3 = \cdots = U_n = N$, then it can be easily verified that $U_1, U_2, \cdots, U_n$ are nonzero semigroup ideals of $N$ such that $[D(u_1, u_2, \cdots, u_n), a]_{\sigma,\tau} = 0$ for all $u_1 \in U_1, u_2 \in U_2, \cdots, u_n \in U_n$; where $a = \begin{pmatrix} (x,0) & (0,0) \\ (0,0) & (0,0) \end{pmatrix}$ with $0 \neq x$.

However neither $D(a, U_2, U_3, \cdots, U_n) = \{0\}$ nor $a \in Z$.

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