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ON THE EXISTENCE OF SOLUTIONS FOR QUADRATIC INTEGRAL EQUATIONS ON UNBOUNDED INTERVALS FOR QUASIBOUNDED MAPS

Abstract. In this paper we investigate the existence of a solution on a semi-infinite interval for a quadratic integral equation of Volterra type in Fréchet spaces using a fixed point theorem due to Petryshyn and Fitzpatrick. The conditions we assume in this paper are relaxed with respect to the conditions assumed in a previous paper dealing with the same kind of integral equations.

Keywords: Fixed point property; Fréchet space; cohomology; acyclic valued; quadratic integral equations, ε-net.

1. Introduction and Notations.

In this paper, we try to approach the study of the solvability of a nonlinear quadratic integral equation of Volterra type

$$(I) \qquad x(t) = f(t) + (Ax)(t)\int_{t_0}^{t} u(s,t,x(s))ds, \quad t \in J = [0,+\infty).$$

We will look for solutions of these equations in the Fréchet space of real functions being defined and continuous on the semi-infinite interval $J = \mathbb{R}^+ = [0,+\infty)$.

The previous equation can be seen as a generalization of the Chandrasekhar H - equation in transport theory ([14]), in which $t \in [0,1]$. Moreover, quadratic integral equations have numerous other useful applications in describing events and problems in the real world. For example, quadratic integral equations are often applicable in the kinetic theory of gases, in the theory of neutron transport, and in traffic theory; (see [11], [13], [16]), [20], [21]). Of course integral equations of such a type are also often an object of just mathematical investigations ([1], [2], [3], [5], [6], [7], [8], [9], [12]) both in the case of t varying in a bounded interval or in an unbounded one. In the last 35 years or so, many authors have studied the existence of solutions for several classes of nonlinear quadratic integral equations with the study of these in the space of real functions which are defined, continuous and bounded also on an unbounded interval.

In general the proof of the existence of the solution of the integral equation depends on a suitable combination of the technique of measures of noncompactness and some fixed point principle.

Also more recently, a quadratic integral equation with linear modification of the argument is studied by many authors by means of a technique associated with measures of noncompactness, in order to prove the existence of solutions in a space like $C(J)$ (see, for istance, [12]). The conditions imposed in these papers are different than ours.

Motivated by some previous papers we try to extend the investigations to semi-infinite intervals by considering a kind of (quasi) linearity, instead of the sheer boundedness of the operator A, considered in a previous paper ([4]); some appropriate examples are also provided.

The main tool used in achieving our main result is a fixed point theorem for condensing acyclic (values) multivalued maps of an appropriate operator on these spaces.

2. Preliminaries

If $\mathcal{E}$ is a metric space we will denote by $\mathcal{BD}$ $(\mathcal{E})$ the family of all nonempty bounded subsets of $\mathcal{E}$.

In this paper we will denote by $F = (F, \{||\cdot||_n\}_{n\in\mathbb{N}})$ a Fréchet space, i.e. a locally convex space with the topology generated by a family of semi-norms $\{||\cdot||_n\}_{n\in\mathbb{N}}$, where $\mathbb{N}$ is the usual set of positive integer numbers. We assume that the family of semi-norms verifies: $||x||_1 \leq ||x||_2 \leq \cdots \leq ||x||_n \leq \cdots$, for every $x \in F$.

An example of Fréchet space is the space $F = C(J,\mathbb{R})$ of all continuous functions with the topology of the uniform convergence on compact subintervals of J. As a generating family of semi-norms for this topology one may consider $||x||_n = \max\{|x(t)| : t \in [0,n]\}$. We recall that the topology of F coincides with that of a complete metric space (F,d), where $d(x,y) = \sum_{n=1}^{+\infty} 2^{-n} \frac{||x-y||_n}{1+||x-y||_n}$, for $x,y \in F$ (see, for istance, [18]).

To the space F we associate a sequence of Banach spaces $\{F^n, ||\cdot||_n\}$ as follows: for every $n \in \mathbb{N}$, we consider the equivalence relations defined by $x \approx y$ if and only if $||x-y||_n = 0$ for $x,y,\in F$. We shall denote by $F^n = \{F_{\approx n}, ||\cdot||_n\}$, the quotient space., i.e. the completation of F^n with respect to $||\cdot||_n$.

A subset $M \subset F$ is said to be bounded if, for every $n \in \mathbb{N}$, there exists $k_n > 0$ such that $||x||_n \leq k_n$, for every $x \in M$.

If $U_n = \{x \in F : ||x||_n < 1,\ n \in \mathbb{N},\}$ then εU_n, $\varepsilon > 0$, $n \in \mathbb{N}$, is a basis of convex and symmetric neighbourhoods of the origin in F.

Let M be a subset of the Fréchet space F and let $T : M \longrightarrow F$ be a map. Let $\{\varepsilon_n\}$ be a sequence of positive real numbers tending to zero. A sequence $\{T_n\}$ of maps $T_n : M \longrightarrow F$ is said to be an ε_n-approximation of T on M if $||T_n(x) - T(x)||_n \leq \varepsilon_n$ for every $n \in \mathbb{N}$ and $x \in M$.

Let E be a Banach space. The Hausdorff measure of noncompactness of a bounded set $X \subset E$ is defined by $\chi(X) = \inf\{\varepsilon > 0 : X$ has a finite ε-net in $E\}$. In the Banach space $C([a,b],\mathbb{R})$ it is possible to show that $\chi(X) = \frac{1}{2}\omega_0(X)$, where $\omega_0(X) = \lim_{\varepsilon\to 0} \sup\{\omega(x,\varepsilon),\ x \in X\}$ and

$\omega(x,\varepsilon) = \sup\{|x(t)-x(s)|;\ t,s \in [a,b],\ |t-s| < \varepsilon\}$(see [6]).

Let $F = C(J,\mathbb{R})$ be the Frechét space of all continuous functions defined in $J = [0,+\infty)$. A measure of noncompactness of a bounded set $X \subset F$ is defined by $\chi(X) = \{\chi_1(X),\ldots,\chi_n(X),\ldots\}$, where $\chi_n(X)$ is a measure of noncompactness of a bounded set X in the Banach space $\{F^n, ||\cdot||_n\}$, above defined.

Moreover, we say that a mapping $\varphi : F \longrightarrow F$ satisfies the *Darbo condition* with a constant $k \geq 0$, with respect to a measure of noncompactness χ, if for any bounded subset $X \in F$ we have $\varphi(X)$ is bounded and $\chi(\varphi(X)) \leq k\,\chi(X)$.

We shall say that the operator $\phi : F \longrightarrow F$ is *condensing* if $\chi(\phi(X)) < \chi(X)$ for every non precompact and bounded $X \in F$.

Let W and V be metric spaces; W is a compact absolute retract (AR) if W is compact and, for every homeomorphism $f : W \longrightarrow V$, then $f(W) \subset V$ is a retract of V. It follows from the Dugundji extension theorem ([10]) that every compact and convex subset of a Fréchet space is a compact AR. We say that $A \subset W$ is an R_δ –set in the space W if A is the intersection of countable decreasing sequence of absolute retracts contained in W. It is known that R_δ –set is an acyclic set, i.e. it is acyclic with respect to any continuous theory of cohomology (see, for istance, [19]).

Now, we will denote by $\mathcal{KA}\ (\mathcal{E})$ the family of all nonempty compact acyclic subsets of $\mathcal{E}$.

Let X be a subset of F; a multivalued map $S : X \longrightarrow \mathcal{P}\ (F)$, where $\mathcal{P}\ (F)$ is the family of all nonempty compact subsets of F, is said to be uppersemicontinuous (u.s.c.) if the graph of S is closed in $X \times F$.

The following results will be useful in the proof of our main theorem.

Proposition 1([15]): Let assume that the function $u : J \times J \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the following conditions:

i) $u(s,\cdot,\cdot) : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous for every $s \in J$;

ii) $u(\cdot,t,x) : J \longrightarrow \mathbb{R}$ is Lebesgue measurable for every $(t,x) \in J \times \mathbb{R}$;

iii) there exist locally integrable functions $l, m : J \longrightarrow J$ such that $|u(s,t,x)| \leq l(s) + m(s)|x|$ for all $(s,t,x) \in J \times J \times \mathbb{R}$;

iv) there exists a locally integrable function $\alpha : J \longrightarrow J$ such that *supp* $u \subset \Omega_\alpha$ where $\Omega_\alpha = \{(s,t,x) \in J \times J \times \mathbb{R} : |x| \leq \alpha(s)\}$.

Then for every $\varepsilon > 0$ and $b > 0$ there exists a map $\tilde{u} : J \times J \times \mathbb{R} \longrightarrow \mathbb{R}$ which satisfies i), ii), iii), iv) and the following conditions:

v) there exists an integrable function $\phi : [0,b] \longrightarrow J$, $b > 0$, such that $|\ \tilde{u}(s,t,x) - u(s,t,x)| \leq \phi(s)$ for all $(s,t,x) \in [0,b] \times [0,b] \times \mathbb{R}$ and $\int_0^b \phi(s)ds < \varepsilon$;

vi) there exists a locally integrable function $L : J \longrightarrow J$ such that $|\ \tilde{u}(s,t,x) - \tilde{u}(s,t,x')| \leq L(s)|x - x'|$ for all $(s,t) \in J \times J$ and $x, x' \in \mathbb{R}$.

Let M be a closed subset of a Fréchet space F; a map $T : M \longrightarrow F$ is said to be a compact map if $T(M)$ is a precompact subset of the space F.

Proposition 2([15]): Let F be a Fréchet space and let M a closed and nonempty subset of F and let $T : M \longrightarrow F$ be a compact map. Let $\{T_n\}$ be an ε_n-approximation of F on M, where $T_n : M \longrightarrow F$ are compact maps. If the equation $x - T_n(x) = y$ has at most one solution for every $n \in \mathbb{N}$ and $y \in \varepsilon_n U_n$, then the set of fixed points of T is a compact R_δ-set.

Proposition 3: (Gronwall inequality). Let $v, k, h : J \longrightarrow J$ be continuous (non-negative) functions such that $v(t) \leq k(t) + \int_0^t h(s)v(s)ds,\ t \in J$.

So we have $v(t) \leq k(t)\exp(\int_0^t h(s)ds),\ t \in J$.

The following proposition can be deduced from Theorem 1 of ([17]).

Proposition 4 (fixed point theorem): Let F be a Fréchet space with M a closed and convex subset of F. Assume that $T : M \longrightarrow \mathcal{KA}\ (M)$ be an uppersemicontinuous multivalued map; then, if $T(M) \subset M$ and T is condensing, T has a fixed point.

3. Main result

Now we are ready to deal with the existence of solutions of an integral equation of the type:

(I) $$x(t) = f(t) + (Ax)(t)\int_0^t k(t,s)F(s,x(s))ds, \quad t \in J.$$

THEOREM 1. *Suppose that the following conditions concerning equation (I) are satisfied:*

1) $(s,x) \longrightarrow F(s,x) : J \times J \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function such that:

- 1_i) *$F(s,\cdot) : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous for every $s \in J$ and $F(\cdot,x) : J \longrightarrow \mathbb{R}$ is Lebesgue measurable for every $x \in \mathbb{R}$;*
- 1_{ii}) *$|F(s,x)| \leq \gamma(s) + \beta(s)|x|$ for all $(s,x) \in J \times \mathbb{R}$, where $\gamma, \beta : J \longrightarrow J$ are locally integrable functions;*
- 1_{iii}) *$k : J \times J \longrightarrow \mathbb{R}$ is a continuous function such that there exists a bounded function $w : J \longrightarrow J$ with $\lim_{t \to 0^+} w(t) = 0$, $\int_0^{+\infty} |k(t_2,s) - k(t_1,s)|ds \leq w(|t_2 - t_1|)$, $\forall t_1, t_2 \in J$ and $|k(t,s)| \leq h(s)$ where $h : J \longrightarrow J$ is a locally integrable function, bounded on every $[0,b] \subset J$.*

2) The map $f : J \longrightarrow \mathbb{R}$ is a continuous function.

3) A is a continuous operator from the Fréchet space $F = C(J,\mathbb{R})$ into itself such that there exists a real number $a > 0$ for which $|(Ax)(t)| \leq a|x(t)|$ for every $t \in J$.

Moreover, if we denote by A_n the restrictions of the operator A to the space $\{F^n, ||\cdot||_n\}$, we suppose that A_n satisfy the Darbo property with a positive constant k_n.

4) the following inequalities hold

$4_i)$ $||f||_n < \dfrac{(a\bar{\gamma}-1)^2}{4a\bar{\beta}}$ *for every* $n \in \mathbb{N}$,

$4_{ii})$ $a\bar{\gamma} < 1$,

$4_{iii})$ $k_n \leq \dfrac{a}{1+a\bar{\gamma}}, \forall n \in \mathbb{N}$,

where $\bar{\beta} = \sup\{\int_0^t h(s)\beta(s)\, ds,\ t \in J\}$, $\bar{\gamma} = \sup\{\int_0^t h(s)\gamma(s)\, ds,\ t \in J\}$.
Then the integral equation (I) has at least one solution in $F = C(J,\mathbb{R})$.

Remark: Inequalities similar to those established in the above main result can be analogously found in a previous paper like [6].

Remark: If there is a constant λ_n such that $||A_n(x)-A_n(y)||_n \leq \lambda_n||x-y||_n$, then A satisfies the Darbo condition, see e.g. [9] for similar assumptions on the operator A.

Proof. We put $M = \{y \in F : ||y||_n \leq M_n\}$, where $M_n < \dfrac{1}{a\bar{\beta}}$. Then M is a suitable bounded, closed and convex set of F (which will be defined later on). If $q \in M$ we consider the equation:

$$(I_q) \qquad x(t) = f(t) + (Aq)(t)\int_0^t k(t,s)F(s,x(s))ds, \quad t \in J.$$

Let S be the (multivalued) map from M into F which associates to any $q \in M$ the solutions of (I_q). Clearly, the fixed points of the map S are the solutions of equation (I).

Let $q \in M$ be fixed; let x be a solution of the equation (I_q): in order to prove our theorem, we need the conditions of Proposition 4 be satisfied.

To that aim, the following steps in the proof have to be established:

- $S(M) \subset M$.
- the map S is uppersemicontinous map and condensing;
- the set $S(q)$ is an acyclic set, for every $q \in M$.

We have from 1) and 3)

$|x(t)| \leq |f(t)| + |(Aq)(t)| \int_0^t |k(t,s)F(s,x(s))|ds \leq |f(t)| + a|q(t)| \int_0^t |k(t,s)|(\gamma(s) + \beta(s)|x(s)|)ds \leq |f(t)| + a|q(t)| \int_0^t h(s)(\gamma(s)+\beta(s)|x(s)|)ds$.

This implies that $||x||_n \leq ||f||_n + a||q||_n(\bar{\gamma}+\bar{\beta}||x||_n)$.

Since $||q||_n \leq M_n < \frac{1}{a\bar{\beta}}$, we obtain $||x||_n \leq \dfrac{||f||_n + a\bar{\gamma}M_n}{1-a\bar{\beta}M_n}$.

The estimation we want to get is $||x||_n \le M_n$: to this aim is necessary that the (algebraic) inequality $a\overline{\beta}M_n^2 + (a\overline{\gamma} - 1)M_n + ||f||_n \le 0$ be satisfied when M_n are positive value. This inequality is satisfied when $\Delta > 0$ where $\Delta = (a\overline{\gamma} - 1)^2 - 4a\overline{\beta}||f||_n$ and this is true when the condition 4_i) holds.

So, if we recall $M = \{y \in F : ||y||_n \le M_n\}$, we have $S(M) \subset M$.

Now we prove that the map S is uppersemicontinuous.

Let $q_n, q_0 \in M$ and let $q_n \longrightarrow q_0$; let $x_n \in S(q_n)$ and $x_n \longrightarrow x_0 \in M$. We need to show that $x_0 \in S(q_0)$.

From the continuity of operator A, it follows that $\lim_{n\to+\infty} A(q_n) = A(q_0)$; from the continuity of the function $k(t,s)F(s,x)$ for every $s \in J$, from 1)- 3) (of Theorem 1) and the Dominated Lebesgue Convergence Theorem it follows:

$$\lim_{n\to+\infty} \int_0^t k(t,s)F(s,x_n(s))\, ds = \int_0^t \lim_{n\to+\infty} k(t,s)F(s,x_n(s))\, ds = \int_0^t k(t,s)F(s,x_0(s))\, ds.$$

Hence we have:

$$\lim_{n\to+\infty} x_n(t) = \lim_{n\to+\infty} (f(t) + (Aq_n)(t) \int_0^t k(t,s)F(s,x_n(s))\, ds) =$$

$$f(t) + (Aq_0)(t) \int_0^t k(t,s)F(s,x_0(s))\, ds.$$

The latter means that $\lim_{n\to+\infty} x_n(t) = x_0(t)$; hence $x_0 \in S(q_0)$, so that the map S is uppersemicontinuous.

Now we want to show that the map S is condensing.

For arbitrary $t_1, t_2 \in [0,n]$, we have, for $x \in X$ (bounded subset of M)

$$|x(t_2) - x(t_1)| \le |f(t_2) - f(t_1)| + |Aq(t_2) \int_0^{t_2} k(t_2,s)F(s,x(s))ds -$$

$$(Aq)(t_1) \int_0^{t_1} k(t_1,s)F(s,x(s))\, ds| \le$$

$$|f(t_2) - f(t_1)| + |(Aq)(t_2) \int_0^{t_2} k(t_2,s)F(s,x(s))\, ds - (Aq)(t_2) \int_0^{t_2} k(t_1,s)\, F(s,x(s))\, ds| +$$

$$|(Aq)(t_2) \int_0^{t_2} k(t_1,s)F(s,x(s))\, ds - (Aq)(t_2) \int_0^{t_1} k(t_1,s)F(s,x(s))\, ds| +$$

$$|(Aq)(t_2)\int_0^{t_1} k(t_1,s)F(s,x(s))\,ds - (Aq)(t_1)\int_0^{t_1} k(t_1,s)F(s,x(s))\,ds| \leq$$

$$|f(t_2)-f(t_1)| + |(Aq)(t_2)|\int_0^{t_2} |k(t_2,s)-k(t_1,s)||F(s,x(s))\,|ds+$$

$$|(Aq)(t_2)||\int_{t_1}^{t_2} h(s)(\gamma(s)+\beta(s)|x(s)|)\,ds|+$$

$$|(Aq)(t_2)-(Aq)(t_1)|\int_0^{t_1} h(s)(\gamma(s)+\beta(s)|x(s)|)\,ds \leq$$

$$\omega(f,|t_2-t_1|) + a||q||_n w(|t_2-t_1|)\overline{\gamma} + a||q||_n\overline{\beta}||x||_n w(|t_2-t_1|)+$$

$$a||q||_n|\int_{t_1}^{t_2} h(s)(\gamma(s)+\beta(s))||x||_n ds| + \omega(Aq,|t_2-t_1|)(\overline{\gamma}+\overline{\beta}||x||_n).$$

Hence, by using assumptions 1), 2), 3) and the absolute continuity of the integral, we have

$$\omega_0(S(X)) \leq (\overline{\gamma}+\overline{\beta}M_n)\omega_0(A(X));$$

thus

$$\chi(S(X)) \leq (\overline{\gamma}+\overline{\beta}M_n)k_n\chi(X)).$$

Now we observe that, since $k_n \leq \dfrac{a}{1+a\overline{\gamma}}$ then $1-\overline{\gamma}k_n > 0$.

Moreover, from $M_n < \dfrac{1}{a\overline{\beta}}$. it follows that $M_n < \dfrac{1-\overline{\gamma}k_n}{k_n\overline{\beta}}$.

So that S is condensing.

Now we have to show that the set $S(q)$ is an acyclic set for every fixed $q \in M$.

We have $|x(t)| \leq |f(t)| + a|q(t)|\int_0^t h(s)\gamma(s)ds + a|q(t)|\int_0^t h(s)\beta(s)|x(s)|ds$.

By Gronwall lemma, we have:

$|x(t)| \leq (|f(t)| + a|q(t)|\int_0^t h(s)\gamma(s)ds)\exp(a|q(t)|\int_0^t h(s)\beta(s)ds) = \alpha(t)$.

We put now $u(s,t,x) = k(t,s)F(s,x)$.

Let $\psi : \mathbb{R} \longrightarrow [0,1]$ be a continuous function such that $\psi(x) = 1$, for $|x| \leq 1$ and $\psi(x) = 0$ for $|x| \geq 2$.

We put $\overline{u}(s,t,x) = \psi(\dfrac{x}{1+\alpha(s)})u(s,t,x)$; that map satisfies conditions i), ii), iii), iv) of Proposition 1, where $l(s) = h(s)\gamma(s)$ and $m(s) = h(s)\beta(s)$.

Since $\overline{u}(s,t,x) = u(s,t,x)$ when $|x| \leq \alpha(s)$, then the set of solutions of our equation coincides with the set of solutions of the equation

$x(t) = f(t) + (Aq)(t)\int_0^t \overline{u}(s,t,x(s))\,ds$.

Moreover $supp\,\overline{u}(s,t,x) \subset \Omega_{\overline{\alpha}}$ where $\overline{\alpha}(s) = 2\alpha(s) + 2$.

Let $n \in \mathbb{N}$ be arbitrarily chosen and let $b = n$, $\varepsilon = \frac{1}{n}$; Hence there exists a map $u_n : J \times J \times \mathbb{R} \longrightarrow \mathbb{R}$ with appropriate functions L_n, ϕ_n, l_n, m_n and α_n satisfying the same conditions i) – vi) of Proposition 1.

Now, for every fixed $q \in M$, let us define the operators $T, T_n : M \longrightarrow F$ as follows

$$T(x)(t) = f(t) + (Aq)(t)\int_0^t \overline{u}(s,t,x(s))\,ds, \quad t \in J$$

$$T_n(x)(t) = f(t) + (Aq)(t)\int_0^t u_n(s,t,x(s))\,ds, \quad t \in J$$

With a proof similar to [3], we can show that T and T_n are compact operators.

It is easy to see that $\{T_n\}$ is an ε_n-approximation of T on M, where $\varepsilon_n = \frac{h_n}{n}$, by putting $h_n = \sup\{a|q(t)|, t \in [0,n]\}$. In fact, for $t \in [0,n]$, we have:

$|T_n(x)(t) - T(x)(t)| = |(Aq)(t)\int_0^t u_n(s,t,x(s))\,ds - (Aq)(t)\int_0^t \overline{u}(s,t,x(s))\,ds| \leq$

$a|q(t)|\int_0^t |u_n(s,t,x(s)) - \overline{u}(s,t,x(s))|\,ds < \frac{a|q(t)|}{n}$.

Hence we obtain $||T_n(x)(t) - T(x)(t)||_n \leq \frac{h_n}{n} = \varepsilon_n$.

Now we have only to show that, for every $n \in \mathbb{N}$ and $y \in F$, the equation $x - T_n(x) = y$ has at most one solution.

Suppose that x_1 and x_2 are solutions of the previous equation; we have:

$|x_1(t) - x_2(t)| \leq |(Aq)(t)\int_0^t u_n(s,t,x_1(s))\,ds - (Aq)(t)\int_0^t u_n(s,t,x_2(s))\,ds|$

$\leq a|q(t)|\ \int_0^t |u_n(s,t,x_1(s)) - u_n(s,t,x_2(s))|\,ds$

$\leq a|q(t)|\ \int_0^t L_n(s)|x_1(s) - x_2(s)|\,ds$.

By the Gronwall inequality we obtain $|x_1(t) - x_2(t)| = 0$ for every $t \in J$.

□

4. Examples

Let us consider the following quadratic integral equation of the Volterra type:

$$(Q) \qquad x(t) = t\,\exp(-4t^2) + x(t)\int_0^t \frac{x(s)}{1 + t^2 + 4s^2}\,ds, \quad t \in [0, +\infty).$$

Clearly we have $\overline{\beta} = \frac{\pi}{4} = \sup\{\int_0^t \frac{1}{1+4s^2}\, ds,\ t \in J\}$,
$a = 1,\ \ \overline{\gamma} = 0,\ k_n \leq 1$. By direct computation, we have $||f||_n \leq \frac{1}{\sqrt{8\,e}}$; so, the requirement $||f||_n < \frac{(a\overline{\gamma}-1)^2}{4a\overline{\beta}}$ of the theorem is satisfied since $\frac{1}{\sqrt{8\,e}} < \frac{1}{\pi}$.

The hypotheses of Theorem 1 are verified and we can say that the equation (Q) has at least one solution.

Now let us consider the following quadratic integral equation of the Volterra type:

$$(Q_1)\quad x(t) = \frac{1}{2}\,\exp(-(t+2)^2) + \ln(1+x^2(t)) \int_0^t \frac{s\,x(s)}{16+t^2+s^4}\, ds, \quad t \in J = [0, +\infty).$$

Clearly we have $\overline{\beta} = \sup\{\int_0^t \frac{s}{16+s^4}\, ds,\ t \in J\} = \frac{\pi}{16}$
$a = 1,\ \ \overline{\gamma} = 0,\ k_n \leq 1$.

By direct computation, we have $||f||_n = \frac{\exp(-4)}{2}$; so, the last requirement of the theorem is satisfied since we have $||f||_n < \frac{(a\overline{\gamma}-1)^2}{4a\overline{\beta}}$,

The hypotheses of Theorem 1 are verified and we can say that the equation (Q_1) has at least one solution.

Remark: we note that, by assuming the conditions required in [4], we are not in position to say that the equations there approached admit at least one solution.

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