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REMARK ON A DOUBLE-INEQUALITY FOR THE Q-GAMMA FUNCTION

Abstract. We prove: Let Γ_q denote the $q-$gamma function and let a and q be real numbers with $0 < a \neq 1$ and $q > 1$. Then we have for all real numbers $x \in (0,1)$:

$$\min\Big(1, \frac{1}{\Gamma_q(1+a)}\Big) < \frac{\Gamma_q(1+x)^a}{\Gamma_q(1+ax)} < \max\Big(1, \frac{1}{\Gamma_q(1+a)}\Big),$$

both bounds being sharp. This complements a result of Sellami, Brahim, and Bettaibi, who established the double-inequality for $q \in (0,1)$.
Keywords: q-gamma function, sharp inequalities

In 2005, Alsina and Tomás [1] published an interesting double-inequality for the classical gamma function,

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt = \frac{1}{x} \prod_{n=1}^{\infty} \Big\{ \Big(1 + \frac{1}{n}\Big)^x \Big(1 + \frac{x}{n}\Big)^{-1} \Big\}.$$

They offered a geometrical proof for

$$(1) \qquad \frac{1}{\Gamma(1+a)} < \frac{\Gamma(1+x)^a}{\Gamma(1+ax)} < 1,$$

where $a \in \mathbf{N} \setminus \{1\}$ and $x \in (0,1)$. Sándor [10] and Mercer [7] extended this result and proved that (1) is valid for all real parameters $a > 1$. We remark that if $a \in (0,1)$, then (1) holds with "$>$" instead of "$<$". Similar inequalities were obtained by Shabani [13]; see also [6] and [9]. In the recent past, many research papers appeared providing remarkable inequalties for the Γ-function and its relatives. We refer to Sándor's detailed bibliography on this subject [11].

Jackson [3] introduced the q-analogue of the gamma function as

$$\Gamma_q(x) = (1-q)^{1-x} \frac{(q;q)_\infty}{(q^x;q)_\infty} \quad (0 < q < 1)$$

and

$$\Gamma_q(x) = (q-1)^{1-x} q^{\binom{x}{2}} \frac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty} \quad (q > 1),$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n).$$

The connection between the gamma and the q-gamma functions is given by the limit relations

$$\lim_{q \to 1^-} \Gamma_q(x) = \lim_{q \to 1^+} \Gamma_q(x) = \Gamma(x).$$

It is well-known that the Γ-function is strictly log-convex on $(0,\infty)$. Askey [2] and Moak [8] proved that Γ_q has the same property. Moreover, they discovered counterparts of the Bohr-Mollerup theorem for the q-gamma function. Since $\Gamma_q(1)=\Gamma_q(2)=1$, the strict convexity of Γ_q leads to

$$\Gamma_q(1+a)<1, \quad \text{if} \quad a\in(0,1) \tag{2}$$

and

$$\Gamma_q(1+a)>1, \quad \text{if} \quad a>1. \tag{3}$$

In 2007, Sellami et alii [12] presented a noteworthy q-analogue of (1).

Proposition. *Let a and q be real numbers with $0<a\neq 1$ and $q\in(0,1)$. Then we have for all real numbers $x\in(0,1)$:*

$$\min\Big(1,\frac{1}{\Gamma_q(1+a)}\Big)<\frac{\Gamma_q(1+x)^a}{\Gamma_q(1+ax)}<\max\Big(1,\frac{1}{\Gamma_q(1+a)}\Big). \tag{4}$$

Both bounds are sharp.

This extends a result of Kim and Adiga [4], who only considered the case $a>1$.

Does there exist a counterpart of (4) which holds for $q>1$? It is the aim of this note to give an affirmative answer to this question. A key role in the proof of our theorem plays an elegant identity which is due to Kocić [5]:

$$\Gamma_p(x)=q^{-\binom{x-1}{2}}\Gamma_q(x) \quad (p,q>0;pq=1;x>0). \tag{5}$$

The following complement of the Proposition is valid.

Theorem. *Let a and q be real numbers with $0<a\neq 1$ and $q>1$. Then, the double-inequality* (4) *holds for all real numbers $x\in(0,1)$. The given upper and lower bounds are the best possible.*

Proof. Let $0<a\neq 1$, $q>1$, and $x\in(0,1)$. From (5) we get

$$\frac{\Gamma_{1/q}(1+x)^a}{\Gamma_{1/q}(1+ax)}=q^{\frac{1}{2}a(a-1)x^2}\frac{\Gamma_q(1+x)^a}{\Gamma_q(1+ax)}. \tag{6}$$

Next, we apply (4) (with $1/q$ instead of q) and (6). Then,

$$q^{\frac{1}{2}a(1-a)x^2}\min\Big(1,\frac{1}{\Gamma_{1/q}(1+a)}\Big)<\frac{\Gamma_q(1+x)^a}{\Gamma_q(1+ax)}<q^{\frac{1}{2}a(1-a)x^2}\max\Big(1,\frac{1}{\Gamma_{1/q}(1+a)}\Big). \tag{7}$$

We denote the expressions on the left-hand side and on the right-hand side of (7) by $L_q(a,x)$ and $R_q(a,x)$, respectively.

Case 1. $0<a<1$.

Using (2) gives

$$L_q(a,x)=q^{\frac{1}{2}a(1-a)x^2}>1=\min\Big(1,\frac{1}{\Gamma_q(1+a)}\Big). \tag{8}$$

An application of (5) yields

$$\frac{1}{\Gamma_{1/q}(1+a)} = q^{\binom{a}{2}} \frac{1}{\Gamma_q(1+a)},$$

so that we obtain

$$R_q(a,x) = q^{\frac{1}{2}a(1-a)x^2} \frac{1}{\Gamma_{1/q}(1+a)} = q^{\frac{1}{2}a(1-a)(x^2-1)} \frac{1}{\Gamma_q(1+a)}$$

$$< \frac{1}{\Gamma_q(1+a)} = \max\left(1, \frac{1}{\Gamma_q(1+a)}\right). \tag{9}$$

From (7), (8), and (9) we conclude that (4) is valid.

Case 2. $a > 1$.

Applying (3) and (5) leads to

$$L_q(a,x) = q^{\frac{1}{2}a(1-a)x^2} \frac{1}{\Gamma_{1/q}(1+a)} = q^{\frac{1}{2}a(1-a)(x^2-1)} \frac{1}{\Gamma_q(1+a)}$$

$$> \frac{1}{\Gamma_q(1+a)} = \min\left(1, \frac{1}{\Gamma_q(1+a)}\right) \tag{10}$$

and

$$R_q(a,x) = q^{\frac{1}{2}a(1-a)x^2} < 1 = \max\left(1, \frac{1}{\Gamma_q(1+a)}\right). \tag{11}$$

Combining (7), (10), and (11) reveals that (4) holds.

Letting x tend to 0 and 1, respectively, we conclude from (4) that the given bounds are sharp. □

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