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POSTULATION OF DISJOINT UNIONS OF LINES AND A MULTIPLE POINT

Abstract. We study the postulation of a general union $X \subset \mathbb{P}^3$ of one m-point mP and t disjoint lines. Following a paper and a conjecture by E. Carlini, M. V. Catalisano and A. V. Geramita we solve the case $m = 3$ and discuss an approach to the general case.

1. Introduction

A scheme $X \subset \mathbb{P}^r$ is said to have *maximal rank* if for all integers $t > 0$ the restriction map $H^0(\mathcal{O}_{\mathbb{P}^r}(t)) \to H^0(X, \mathcal{O}_X(t))$ is either injective or surjective, i.e. if either $h^0(\mathcal{I}_X(t)) = 0$ or $h^1(\mathcal{I}_X(t)) = 0$, i.e. if X imposes the " expected " number of conditions to the vector space of all homogeneous degree t polynomials in $r+1$ variables. R. Hartshorne and A. Hirschowitz proved that for all $t > 0$ and $r \geq 3$ a general union $X \subset \mathbb{P}^r$ of t general lines has maximal rank. E. Carlini, M. V. Catalisano and A. V. Geramita considered several cases in which we allow unions of linear spaces with certain multiplicities (for zero-dimensional subspaces these are general unions of fat points) ([1], [2], [3]). We recall that for each $P \in \mathbb{P}^r$ the m-point mP of $\mathbb{P}^r$ is the closed subscheme of $\mathbb{P}^r$ with $(\mathcal{I}_P)^m$ as its ideal sheaf. In [3] they proved that for all $r \geq 4$, $m > 0$ and $t > 0$ a general union of an m-point and t disjoint lines has maximal rank. In the case $r = 3$ they proved that there are some exceptional cases; in [3] the failure of maximal rank for these cases is exactly described, i.e. all positive integers $h^0(\mathcal{I}_X(t))$ and $h^1(\mathcal{I}_X(t))$ are computed ([3, Theorem 4.2, part (ii)]). They conjectured in [3] that no other exceptional case arises and prove the conjecture in some cases. In particular they proved their conjecture for $m = 2$. In this paper we prove their conjecture in the case $m = 3$, i.e. we prove the following result.

THEOREM 1. *Let $X \subset \mathbb{P}^3$ be a general union of one 3-point and t disjoint lines. X has maximal rank if and only if either $t = 1$ or $t \geq 4$.*

For each fixed $m \geq 4$ we prove that there are only finitely many t such that we need to test if a general union in $\mathbb{P}^3$ of an m-point and t lines has maximal rank (Proposition 1).

For all integers $k > m \geq 0$ define the integers $e_{m,k}$ and $f_{m,k}$ by the following relations:

$$(1) \qquad (k+1)e_{m,k} + f_{m,k} + \binom{m+2}{3} = \binom{k+3}{3}, \ 0 \leq f_{m,k} \leq k.$$

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For all integers $m \geq 0$ and $t \geq 0$ let $Z(m,t)$ denote the set of all $X \subset \mathbb{P}^3$ which are the disjoint union of one m-point and t lines.

We prove the following result.

PROPOSITION 1. *Fix an integer $m \geq 4$ and assume that for each integer k such that $m+2 \leq k \leq m(m+1)(m+2)/2 - 1$ there are $A \in Z(m, e_{m,k})$ with $h^1(I_A(k)) = 0$ and $B \in Z(m, e_{m,k}+1)$ with $h^0(I_B(k)) = 0$. Then for any integer $t \geq m+1$ a general union of one m-point and t disjoint lines has maximal rank.*

If $f_{m,k} = 0$ we may avoid the check of the existence of B.

PROPOSITION 2. *Fix integers $t > 0$ and $k > m \geq 4$. Let $X \subset \mathbb{P}^3$ be a general union of one m-point and t disjoint lines. If $k \equiv m+1 \pmod 3$, then set $\alpha_{m,k} := \lfloor (m+4)(m+3)/6 \rfloor - m - 2$. If $k \equiv m+2 \pmod 3$, then set $\alpha_{m,k} := \lfloor (m+5)(m+4)/6 \rfloor - m - 3$. If $k \equiv m+3 \pmod 3$, then set $\alpha_{m,k} := \lfloor (m+6)(m+5)/6 \rfloor - 2m - 4$.*

(a) If $t \leq (k+3)(k+2)/6 - \alpha_{m,k}$, then $h^1(I_X(k)) = 0$.

(b) If $t > (k+4)(k+1)/6$, then $h^0(I_X(k)) = 0$.

Part (b) is a trivial consequence of [5], but is it put here to see that, for any fixed m,k with $k \gg m$, the set of all t not settled by either (a) or (b) is not very large and its cardinality depends only on m. This part is true if instead of mP we take an arbitrary zero-dimensional scheme and hence it works for several fat points.

For the proofs we use (as in [5] and [3]) a smooth quadric, but we also use a degree 3 ruled surface (see section 3 for its description).

We work over an algebraically closed field $\mathbb{K}$. As far as we understand none of our quotations of [3] require the characteristic zero assumption made in [3]. However, if someone uses computer algebra to check the conjecture made in [3] for some m (maybe using Proposition 1 to reduce the computational effort), then it is required to assume characteristic zero (working over $\mathbb{Z}$ seems to be too time-consuming for the usual softwares).

2. Preliminaries

Let $F \subset \mathbb{P}^3$ be any surface. Set $t := \deg(F)$. For each closed subscheme $Z \subset \mathbb{P}^3$ let $\mathrm{Res}_F(Z)$ denote the residual scheme of Z with respect to F, i.e. the closed subscheme of $\mathbb{P}^3$ with $I_Z : I_F$ as its ideal sheaf. If Z is reduced, then $\mathrm{Res}_F(Z)$ is the union of the irreducible components of Z not contained in F. Now assume $Z = mP$ for some $m > 0$ and some $P \in \mathbb{P}^3$. If $P \notin F$, then $\mathrm{Res}_F(mP) = mP$. If P is a smooth point of F, then $\mathrm{Res}_F(mP) = (m-1)P$ (with the convention $0P = \emptyset$). For any integer $x \geq t$ we have an exact sequence

$$0 \to I_{\mathrm{Res}_F(Z)}(x-t) \to I_Z(x) \to I_{Z\cap F,F}(x) \to 0.$$

Hence

- $h^0(\mathcal{I}_Z(x)) \le h^0(\mathcal{I}_{\mathrm{Res}_F(Z)}(x-t)) + h^0(F, \mathcal{I}_{Z\cap F}(x))$;
- $h^1(\mathcal{I}_Z(x)) \le h^1(\mathcal{I}_{\mathrm{Res}_F(Z)}(x-t)) + h^1(F, \mathcal{I}_{Z\cap F}(x))$.

As in [1], [2, Lemma 3.3] and [3] we will call " the Castelnuovo's inequality " any of these two inequalities.

A *sundial* $D \subset \mathbb{P}^3$ is a scheme $D = E \cup 2O$, where E is a reducible conic and O is the singular point of E ([1]). A sundial is a flat limit of a family of unions of two disjoint lines in $\mathbb{P}^3$ ([5, Example 2.1.1], [1, Lemma 2.5]).

We need the following obvious statement.

LEMMA 1. *Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. Fix $O \in Q$ and points $P_1, P_2 \in Q$ such that $P_1 \neq P_2$, $O \notin \{P_1, P_2\}$ and none of the 3 lines spanned by two of the points of $\{P_1, P_2, O\}$ is contained in Q. Set $Z := \{P_1, P_2\} \cup (2O, Q)$, where $(2O, Q)$ is the closed subscheme of Q with $(\mathcal{I}_{O,Q})^2$ as its ideal sheaf. Then there is a sundial $E \subset \mathbb{P}^3$ such that $Q \cap E = Z$.*

Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. A *fork* of Q is a degree 4 zero-dimensional scheme $Z \subset Q$, such that Z has 3 connected components and Z is contained in a smooth conic $D \subset Q$, i.e. in a smooth element $D \in |\mathcal{O}_Q(1,1)|$. Any fork of Q is obtained in the following way. Fix a smooth element $D \in |\mathcal{O}_Q(1,1)|$ and 3 distinct points P_1, P_2, P_3 of D. Let $v \subset D$ be the tangent vector of D with P_3 as its support; then $Z := \{P_1, P_2\} \cup v$ is a fork. A *smooth fork* of Q is a union $S \subset Q$ of 4 distinct points contained in a smooth conic $D \subset Q$. Fix any smooth conic $D \subset Q$ and any fork $Z \subset D$. Since D is a smooth curve, Z is the flat limit of a family of smooth forks contained in D. Take any two smooth conics $D, D' \subset Q$ such that $D \neq D'$. Since $\sharp(D \cap D') \le 2$, we get that a fork (resp. a smooth fork) Z is contained in a unique smooth conic $D \in |\mathcal{O}_Q(1,1)|$. The curve D is the complete intersection of Q and a plane H. The plane H is the plane spanned by Z. The existence of H shows that $h^1(\mathcal{I}_Z(1)) > 0$, i.e. $h^1(Q, \mathcal{I}_Z(1)) > 0$. This inequality gives us some technical problems, which we overcome using [5, Lemme 2.3] and Lemma 3 below.

LEMMA 2. *Let Q be a smooth quadric, let $Z \subset Q$ be a fork and let $S \subset Q$ be a smooth fork. Then there are reducible conics $B, B' \subset \mathbb{P}^3$ such that $S = Q \cap B$, the singular point of B' is contained in Q and $Z = Q \cap B'$ (scheme-theoretic intersection).*

Proof. Let $D \subset Q$ be the smooth conic containing Z and let $H \subset \mathbb{P}^3$ be the plane spanned by Z (it is the plane spanned by D). Write $Z = v \cup \{P_1, P_2\}$ with $\deg(v) = 2$. Let $P_3 \in D$ be the support of v. Since D is a smooth conic, the 3 points P_1, P_2, P_3 are not collinear. Let $B' \subset H$ be the union of the line containing $\{P_3, P_1\}$ and the line containing $\{P_2, P_3\}$. Since $Q \cap H = D$ and $B' \cap D = Z$ (as schemes), we have $Z = Q \cap B'$ (as schemes). Let $A \subset Q$ be the smooth conic containing S. Let N be the plane spanned by A. Write $S = \{O_1, O_2, O_3, O_4\}$ and take as B the union of the line containing $\{O_1, O_2\}$ and the line containing $\{O_3, O_4\}$. Since $Q \cap N = A$ and $B \cap A = S$, we have $S = Q \cap B$. $\square$

REMARK 1. Fix a smooth quadric Q and integers $s \geq 0$, $t \geq 0$, $(s,t) \neq (0,0)$. Let $Z \subset Q$ be a general union of t forks and s smooth forks. By Lemma 2 there is a disjoint union $B \subset \mathbb{P}^3$ of $s+t$ reducible conics such that exactly t of them have the singular points contained in Q and $Z = Q \cap B$. Conversely, let D be a general union of s reducible conics and t reducible conics with their singular points contained in Q. Then $Q \cap D$ is a general union of t forks and s smooth forks.

LEMMA 3. *Fix integers $a > 0$, $b > 0$, $v > 0$ and $z \geq 0$ such that $5v + 3z \leq (a+1)(b+1)$ and $h^1(Q, I_W(a,b)) = 0$ for a general union W of $v+z$ 2-points of Q. Let $U \subset Q$ be a general union of v forks and z 2-points. Then $h^1(Q, I_U(a,b)) = 0$.*

Proof. Let M be the union of U and the v 2-points of U whose support are the unreduced connected components of the v forks in U. Let $N \subset M$ be the union of the $v+z$ 2-points contained in M. Since $U \subset M$ and M is zero-dimensional, we have $h^1(Q, I_U(a,b)) \leq h^1(Q, I_M(a,b))$. Since $\deg(M) = 5v+3z \leq (a+1)(b+1)$, and $M \setminus N$ is general in Q, it is sufficient to prove that $h^1(I_N(a,b)) = 0$ ([1, lemma 2.2]). The last vanishing is true, because N is a general union of $v+z$ 2-points of Q. □

We usually apply Lemma 3 when $b > a > 0$ and $b \geq v+z-1$, because the very easy case $n = b$, $r = b-a$, $q'' = 0$, $d = 0$ and $t = v+z$ of [5, Lemme 2.3] shows that the hypothesis of Lemma 3 holds. Furthermore the case $n = b$, $r = b-a$, $q'' = 0$ and $t = 0$ of [5, Lemme 2.3] gives the following result.

LEMMA 4. *Fix integers $b > a \geq 0$, $v > 0$ such that $v \leq b+1$ and $4v \leq (a+1)(b+1)$. Let $Z \subset Q$ be a general union of v forks. Then $h^1(I_Z(a,b)) = 0$.*

3. The degree 3 ruled surface T

We describe the following degree 3 surface $T \subset \mathbb{P}^3$. We only need that T is an irreducible surface of degree 3 and that it contains a one-dimensional irreducible family $\mathcal{F}(T)$ of lines such that for each integer $e \geq 2$ a general $S \subset \mathcal{F}(T)$ with $\sharp(S) = e$ consists of e pairwise disjoint lines. We only use these properties of the surface T and the reader may forget how it is constructed. For any projective automorphism g of $\mathbb{P}^3$, $g(T)$ is a surface with the same properties and we call degree 3 ruled surface any of these projectively equivalent surfaces (and often denote with T any of these surfaces).

Let F_1 be the Hirzebruch surface isomorphic to the ruled surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1})$ over $\mathbb{P}^1$([4, §V.2]). We have $\mathrm{Pic}(F_1) \cong \mathbb{Z}^{\oplus 2}$ and we take as a basis of $\mathrm{Pic}(F_1)$ a fiber f of the ruling η of F_1 and the unique section h of η with negative self-intersection (in [4, §V.2] the basis is f and $H := h+f$). We have $h \cdot f = 1$, $f^2 = 0$ and $h^2 = -1$. We have $h^0(F_1, \mathcal{O}_{F_1}(h+2f)) = 5$ and $\mathcal{O}_{F_1}(h+2f)$ is very ample. We call $T' \subset \mathbb{P}^4$ the embedding of F_1 induced by the complete linear system $|\mathcal{O}_{F_1}(h+2f)|$. Since $(h+2f)\cdot(h+2f) = 3$, we have $\deg(T') = 3$. Let $O \in \mathbb{P}^4$ be a general point. Let $\ell_O : \mathbb{P}^4 \setminus \{O\} \to \mathbb{P}^3$ be the linear projection from O. Set $T := \ell_O(T')$. Since O is general, we have $O \notin T'$ and $\ell_O|T'$ is birational onto its image. Hence $T \subset \mathbb{P}^3$ is an irreducible degree 3 surface. Call $|f|$ the ruling of $T' \cong F_1$. Fix lines $L, L' \in |f|$ such that $L \neq L'$. Let $\mathcal{F}(T)$ be the set of all lines $\ell_O(L), L \in |f|$. We have $\ell_O(L) \cap \ell_O(L') \neq \emptyset$ if and only if O is contained in the hyperplane $\langle L \cup L' \rangle$ of $\mathbb{P}^4$ spanned by L and L'. Fix an integer $e \geq 2$ and a finite set $\{L_i\}$, $1 \leq i \leq e$, of distinct elements of $|f|$. We have $\ell_O(L_i) \cap \ell_O(L_j) \neq \emptyset$ for some $i \neq j$ if and only if $O \in \langle L_i \cup L_j \rangle$. Hence for each integer $e \geq 2$ a general $S \subset \mathcal{F}(T)$ with $\sharp(S) = e$ consists of e pairwise disjoint lines. Since $h \cdot (h + 2f) = 1$, the curve $h \subset T'$ is a line. We call D' its image in T. D' is a line of $\mathbb{P}^3$. The surface T has a singular line, D'', obtained in the following way. Take $E \in |\mathcal{O}_{T'}(h + f)|$. The curve $E \subset \mathbb{P}^4$ is a plane conic. If O is contained in the plane spanned by E, then $\ell_O(E)$ is a line, $\ell_O(E) \neq D'$ and $\ell_O(E)$ is contained in the singular locus of T. For a general O there is a unique $E \in |\mathcal{O}_{T'}(h + f)|$ such that O is in the plane spanned by E. The line D'' is the line $\ell_O(E)$ and $D'' \neq D'$.

LEMMA 5. *Fix integers t, e such that $0 < e \leq t+1$. Let $F \subset \mathbb{P}^3$ be any union of e pairwise disjoint lines. Then $h^1(\mathcal{I}_F(t)) = 0$, i.e. $h^0(\mathcal{I}_F(t)) = h^0(\mathcal{O}_{\mathbb{P}^3}(t)) - e(t+1)$.*

Proof. It is sufficient to do the case $e = t+1$. If $t = 1$, then the lemma is true, because two skew lines span $\mathbb{P}^3$. Hence we may assume $t \geq 2$ and use induction on t. Fix a line $L \subset F$ and take a plane $H \subset \mathbb{P}^3$ containing L. Since the elements of F are pairwise disjoint, H contains no other line of $F \setminus L$. Hence $F \cap H$ is the union of L and t distinct points. It is easy to check that $h^1(H, \mathcal{I}_S(t-1)) = 0$ for any $S \subset H$ such that $\sharp(S) = t$. Hence $h^1(H, \mathcal{I}_{F \cap H}(t)) = 0$. The inductive assumption gives $h^1(\mathcal{I}_{F \setminus L}(t-1)) = 0$. Hence the Castelnuovo's inequality gives $h^1(\mathcal{I}_F(t)) = 0$. □

The surface T contains two lines, D' and D'', such that each $L \in \mathcal{F}(T)$ intersects both D' and D''. Hence if Γ is the union of any $t+2$ pairwise disjoint elements of $\mathcal{F}(T)$, then $h^1(\mathcal{I}_\Gamma(t+1)) > 0$. Hence Lemma 5 is sharp and cannot be improved if we only look at disjoint lines contained in T.

LEMMA 6. *Fix a degree 3 ruled surface $T \subset \mathbb{P}^3$. Fix integers t, e such that $0 < e \leq t+1$ and a union $F \subset T$ of e general elements of $\mathcal{F}(T)$. Then we have $h^0(T, \mathcal{O}_T(t)) = \binom{t+3}{3} - \binom{t}{3}$ and $h^0(T, \mathcal{I}_F(t)) = h^0(T, \mathcal{O}_T(t)) - (t+1)e$, i.e. $h^1(T, \mathcal{I}_F(t)) = 0$.*

Proof. Since $h^1(\mathcal{O}_{\mathbb{P}^3}(t-3)) = 0$, the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(t-3) \to \mathcal{O}_{\mathbb{P}^3}(t) \to \mathcal{O}_T(t) \to 0$$

gives $h^0(T, \mathcal{O}_T(t)) = \binom{t+3}{3} - \binom{t}{3}$ (with the convention $\binom{t}{3} = 0$ if $t = 1, 2$). Hence it is sufficient to prove that $h^0(T, \mathcal{I}_F(t)) = h^0(T, \mathcal{O}_T(t)) - (t+1)e$. Since $h^0(F, \mathcal{O}_F(t)) = (t+1)e$, we have $h^0(T, \mathcal{I}_F(t)) \geq h^0(T, \mathcal{O}_T(t)) - (t+1)e$. So it is sufficient to prove that $h^0(T, \mathcal{I}_F(t)) \leq h^0(T, \mathcal{O}_T(t)) - (t+1)e$. For this, let $S \subset F$ be a set such that $\sharp(S \cap L) = t+1$ for each line $L \subseteq F$. Every degree t surface containing S contains F. Lemma 5 gives $h^0(\mathcal{I}_S(t)) = \binom{t+3}{3} - (t+1)e$, i.e. S gives $(t+1)e$ independent conditions to $H^0(\mathcal{O}_{\mathbb{P}^3}(t))$, then so to $H^0(T, \mathcal{O}_T(t))$. Hence we get $h^0(T, \mathcal{I}_S(t)) = h^0(T, \mathcal{O}_T(t)) - (t+1)e$. Hence we get $h^0(T, \mathcal{I}_F(t)) \leq h^0(T, \mathcal{O}_T(t)) - (t+1)e$. □

For any closed subscheme $Z \subset T$, each $t \in \mathbb{Z}$ and each linear subspace $V \subseteq H^0(T, \mathcal{O}_T(t))$ set $V(-Z) := V \cap H^0(T, \mathcal{I}_Z(t))$.

LEMMA 7. *Fix an integer $t > 0$, the union $F \subset T$ of $t+1$ general elements of $\mathcal{F}(T)$ and a linear subspace $V \subseteq H^0(T, \mathcal{I}_F(t))$. Let $A \subset T$ be a general union of 3 collinear points. Then $\dim(V(-A)) = \max\{0, \dim(V) - 3\}$.*

Proof. For any $P \in \mathbb{P}^3$ let $\ell_P : \mathbb{P}^3 \setminus \{P\} \to \mathbb{P}^2$ denote the linear projection from P. For any $P_1, P_2 \in \mathbb{P}^3$ such that $P_1 \neq P_2$ let $\langle \{P_1, P_2\} \rangle$ denote the line spanned by P_1 and P_2. Since $\sharp(A) = 3$, we have $\dim(V(-A)) \geq \max\{0, \dim(V) - 3\}$. Since for general points $o, o' \in T$ there is a point $P \in T \setminus \{o, o'\}$ such that P, o and o' are collinear, we have $\dim(V(-A)) \leq \max\{0, \dim(V) - 2\}$. Assume $\dim(V) \geq 3$ and $\dim(V(-A)) = \dim(V) - 2$. It means that for general points $P_1, P_2 \in T$ the third point of $\langle \{P_1, P_2\} \rangle \cap T$ is in the base locus of $V(-\{P_1, P_2\})$. Fix P_1 and take a general P_2. We get that $V(-P_1)$ induces a rational map $T \dashrightarrow \mathbb{P}^r$, $r := \dim(V) - 2$, which factors through ℓ_{P_1} and makes $V(-P_1)$ a linear subspace of $\ell_{P_1}^*(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(t)))$. Since $V \subseteq H^0(T, \mathcal{I}_F(t))$, we have $V(-P_1) \subseteq \ell_{P_1}^*(H^0(\mathbb{P}^2, \mathcal{I}_{\ell_{P_1}(F)}(t)))$. Since $F \subset T$ and $P_1 \in T$ are general, $\ell_{P_1}(F)$ is a union of $t+1$ distinct lines. Hence $H^0(\mathbb{P}^2, \mathcal{I}_{\ell_{P_1}(F)}(t)) = 0$. Therefore, $V(-P_1) = 0$, a contradiction. □

4. The inductive set-up

For all integers $k > m \geq 0$ define the integers $u_{m,k}$ and $v_{m,k}$ by the following relations:

$$(2) \qquad (k+1)u_{m,k} - v_{m,k} + \binom{m+2}{3} = \binom{k+3}{3}, \; 0 \leq v_{m,k} \leq k.$$

We have $v_{m,k} = 0$ if and only if $f_{m,k} = 0$; in this case we have $e_{m,k} = u_{m,k}$. If $f_{m,k} > 0$, then $u_{m,k} = e_{m,k} + 1$ and $v_{m,k} = k + 1 - f_{m,k}$. The integers $e_{0,k}$ and $f_{0,k}$ are the integers listed in [5], top of page 173, i.e. $e_{0,k} = (k+2)(k+3)/6$ and $f_{0,k} = 0$ if $k \equiv 0, 1 \pmod 3$, while $e_{0,k} = (k+1)(k+4)/6$ and $f_{0,k} = (k+1)/3$ if $k \equiv 2 \pmod 3$. Now assume $m > 0$. Since $m > 0$ if $k \geq \binom{m+2}{3} - 1$ and $k \equiv 0, 1 \pmod 3$, then $e_{m,k} = e_{0,k} - 1 = (k+3)(k+2)/6 - 1$ and $f_{m,k} = k + 1 - \binom{m+2}{3}$. If $k \geq m(m+1)(m+2)/2 - 1$ and $k \equiv 2 \pmod 3$, then $e_{m,k} = e_{0,k}$ and $f_{m,k} = (k+1)/3 - \binom{m+2}{3}$.

Fix a surface $A \subset \mathbb{P}^3$ and let O be a smooth point of A. Let $(2O, A)$ denote the closed subscheme of A with $(\mathcal{I}_{O,A})^2$ as its ideal sheaf, i.e. the 2-point of A with O as its support. We have $(2O, A)_{\mathrm{red}} = \{O\}$. Since we assumed that O is a smooth point of A, $(2O, A)$ has degree 3.

For all integers $k \geq m \geq 0$ we consider the following assertions $A(m,k)$, $B(m,k)$ and $N(m,k)$:

$A(m,k)$: There is (E, S, L) such that $E = mP \sqcup A \in Z(m, e_{m,k})$, L is a line, $S \subset L$, $\sharp(S) = f_{m,k}$ and $h^0(\mathcal{I}_{E \cup S}(k)) = 0$.

$B(m,k)$: $u_{m,k} \geq 2v_{m,k}$ and there is (P,A,B,Q), where $P \in \mathbb{P}^3$, Q is a smooth quadric surface, A is a disjoint union of $u_{m,k} - 2v_{m,k}$ lines, B is a disjoint union of $v_{m,k}$ reducible conics with their singular points contained in Q, $P \notin A \cup B$ and $h^0(\mathcal{I}_{mP \cup A \cup B}(k)) = 0$.

$N(m,k)$: There is $E \in Z(m, e_{m,k})$ such that $h^1(\mathcal{I}_E(k)) = 0$, i.e. $h^0(\mathcal{I}_E(k)) = f_{m,k}$.

Obviously $A(m,k)$ implies $N(m,k)$.

LEMMA 8. *Fix integers $k > m > 0$ and $0 < t \leq e_{m,k} < s$ and take a general $E \in Z(m,t)$ and a general $F \in Z(m,s)$. If $A(m,k)$ is true, then $h^1(\mathcal{I}_E(k)) = 0$ and $h^0(\mathcal{I}_F(k)) = 0$. If $N(m,k)$ is true, then $h^1(\mathcal{I}_E(k)) = 0$.*

Proof. Take $(mP \sqcup A, S, L)$ satisfying $A(m,k)$. Let B be the union of any t lines of A. Since B is a union of some of the connected components of $A \cup S$ and $h^1(\mathcal{I}_{mP \cup A \cup S}(k)) = 0$, we have $h^1(\mathcal{I}_{mP \cup B}(k)) = 0$. By the semicontinuity theorem for cohomology ([4, III.12.8]) we have $h^1(\mathcal{I}_E(k)) = 0$. Since $h^0(\mathcal{I}_{mP \cup A \cup S}(k)) = 0$, we have $h^0(\mathcal{I}_{mP \cup A \cup L}(k)) = 0$. Since $s \geq \deg(A \cup L)$, we take a general union of $mP \cup A \cup L$ and $s - e_{m,k} - 1$ lines and then use the semicontinuity theorem for cohomology ([4, III.12.8]).

Now assume that $N(m,k)$ is true and take any solution $mP \sqcup A'$ of $N(m,k)$. Let $B \subseteq A'$ be any union of t lines. We have $h^1(\mathcal{I}_{mP \cup B}(k)) \leq h^1(\mathcal{I}_{mP \cup A'}(k)) = 0$. □

LEMMA 9. *Fix integers $k > m > 0$ and $t \geq u_{m,k}$. Take a general $E \in Z(m,t)$. If $B(m,k)$ is true, then $h^0(\mathcal{I}_E(k)) = 0$.*

Proof. Take (P,A,B,Q) satisfying $B(m,k)$. Let $G \subset \mathbb{P}^3$ be any disjoint union of $t - u_{m,k}$ lines such that $G \cap (A \cup B) = \emptyset$. Set $Z := \cup_{O \in \mathrm{Sing}(B)} 2O$. Since $h^0(\mathcal{I}_{mP \cup A \cup B}(k)) = 0$, we have $h^0(\mathcal{I}_{mP \cup A \cup B \cup Z \cup G}(k)) = 0$. Since $B \cup Z$ is a flat limit of a family of disjoint unions of $2v_{m,k}$ lines, the scheme $A \cup G \cup B \cup Z$ is a flat limit of a family of t disjoint lines. By the semicontinuity theorem for cohomology ([4, III.12.8]) we have $h^0(\mathcal{I}_E(k)) = 0$. □

The following key result was proved in [3].

LEMMA 10. *For all $m > 0$, $A(m,m+1)$, $N(m,m+1)$ and $B(m,m+1)$ are true.*

Proof. We have $e_{m,m+1} = m+2$ and $f_{m,m+1} = 0$. Hence $u_{m,m+1} = e_{m,m+1}$, $v_{m,m+1} = 0$, and $A(m,m+1)$, $N(m,m+1)$ and $B(m,m+1)$ are equivalent. $N(m,m+1)$ is true ([3, part (c) of Theorem 4.2]). □

Taking the difference of (1) (resp. (2)) and the same equation for the integer $k' := k+3$ we get

$$3e_{m,k} + (k+4)(e_{m,k+3} - e_{m,k}) + f_{m,k+3} - f_{m,k} = \binom{k+6}{3} - \binom{k+3}{3}; \tag{3}$$

$$3u_{m,k} + (k+4)(u_{m,k+3} - u_{m,k}) - v_{m,k+3} + v_{m,k} = \binom{k+6}{3} - \binom{k+3}{3}. \tag{4}$$

REMARK 2. Take $E \in Z(m, e_{m,k})$ and any $S \subset \mathbb{P}^3 \setminus E$ such that $\sharp(S) = f_{m,k}$. Since $h^0(E \cup S, \mathcal{O}_{E\cup S}(k)) = \binom{k+3}{3}$ (use (1)), we have $h^0(\mathcal{I}_{E\cup S}(k)) = h^1(\mathcal{I}_{E\cup S}(k))$. Hence (E,S,L) satisfies $A(m,k)$ if and only if $S \subset L$ and $h^1(\mathcal{I}_{E\cup S}(k)) = 0$. Take (P,A,B,Q) as in $B(m,k)$, but without assuming that $h^0(\mathcal{I}_{mP\cup A\cup B}(k)) = 0$. By (2) we have $h^0(mP \cup A \cup B, \mathcal{O}_{mP\cup A\cup B}(k)) = \binom{k+3}{3}$. Hence $h^0(\mathcal{I}_{mP\cup A\cup B}(k)) = h^1(\mathcal{I}_{mP\cup A\cup B}(k))$.

LEMMA 11. *Fix an integer $k \geq m(m+1)(m+2)/2 - 1$ such that $k \equiv 2 \pmod 3$. If $A(m,k)$ is true, then $A(m,k+3)$ is true.*

Proof. Notice that $e_{m,k} = e_{0,k}$, $f_{m,k} = (k+1)/3 - \binom{m+2}{3}$ and $k+4 = e_{m,k+3} - e_{m,k}$. Fix a ruled degree 3 surface T. Fix (F,S,L) satisfying $A(m,k)$ with $F = mP \sqcup E$. Without loss of generality we may assume that $P \notin T$, that $E \cup L$ is a general union of $e_{m,k}+1$ lines and that $S \cap T = \emptyset$. Since L is general, we have $\sharp(L \cap T) = 3$. Fix $O \in L \cap T$ and set $S' := S \cup \{O\}$. Let $G \subset T$ be a general union of $k+4$ lines. By Lemma 6 we have $h^1(T, \mathcal{I}_G(k+3)) = 0$. We have $h^0(T, \mathcal{I}_G(k+3)) = \binom{k+6}{3} - \binom{k+3}{3} - (k+4)^2 > 3 \cdot \deg(E)$ (use (3) and that $f_{m,k+3} - f_{m,k} = 1$, because $k \equiv k+3 \equiv 2 \pmod 3$). Since L is general, we may take as O a general point of T. Hence $h^1(T, \mathcal{I}_{G\cup\{O\}}(k+3)) = 0$. Applying $\deg(E)$ times Lemma 7 we get $h^1(T, \mathcal{I}_{G\cup\{O\}\cup(F\cap T)}(k+3)) = 0$, where $G \cup \{O\} \cup (F \cap T) = (F \cup G \cup S') \cap T$. Since $\mathrm{Res}_T(F \cup G \cup S') = F \cup S$ and $h^1(\mathcal{I}_{F\cup S}(k)) = 0$, the Castelnuovo's inequality gives $h^1(\mathcal{I}_{F\cup G\cup S'}(k+3)) = 0$, that is $(F \cup G, S', L)$ satisfies $A(m,k+3)$. □

LEMMA 12. *Fix an integer $k \geq \binom{m+2}{3} - 1$ such that $k \equiv 0,1 \pmod 3$. If $A(m,k)$ is true, then $A(m,k+3)$ is true.*

Proof. Since $k \equiv 0,1 \pmod 3$, we have $e_{m,k} = e_{0,k} - 1$, $f_{m,k} = k+1-\binom{m+2}{3}$ and $k+4 = e_{m,k+3} - e_{m,k}$. Fix a ruled degree 3 surface T. Take (F,S,L) satisfying $A(m,k)$ with $F = mP \sqcup E$. Without loss of generality we may assume that $P \notin T$, that $E \cup L$ is a general union of $e_{m,k}+1$ lines and that $S \cap T = \emptyset$. Set $S' := S \cup (L \cap T)$. Let $G \subset T$ be a general union of $k+4$ lines. By Lemma 6 we have $h^1(T, \mathcal{I}_G(k+3)) = 0$. We have $h^0(T, \mathcal{I}_G(k+3)) = \binom{k+6}{3} - \binom{k+3}{3} - (k+4)^2 = 3 \cdot (\deg(E)+1)$ (use (3) and that $f_{m,k+3} - f_{m,k} = 3$, because $k \equiv k+3 \equiv 0,1 \pmod 3$). Since L is general, the set $T \cap L$ is a general union of 3 collinear points. Applying $(\deg(E)+1)$ times Lemma 7 we get $h^1(T, \mathcal{I}_{G\cup(L\cap T)\cup(F\cap T)}(k+3)) = 0$, where $G \cup (L \cap T) \cup (F \cap T) = (F \cup G \cup S') \cap T$. Since $\mathrm{Res}_T(F \cup G \cup S') = F \cup S$ and $h^1(\mathcal{I}_{F\cup S}(k)) = 0$, the Castelnuovo's inequality gives $h^1(\mathcal{I}_{F\cup G\cup S'}(k+3)) = 0$, that is $(F \cup G, S', L)$ satisfies $A(m,k+3)$. □

LEMMA 13. *Take $m > 0$ and $k > m+1$ such that $k \equiv 1 \pmod 3$ and $k \geq \binom{m+2}{3} - 1$. If $B(m,k)$ is true, then $B(m,k+2)$ is true.*

Proof. Notice that $v_{m,k} = v_{m,k+2} = \binom{m+2}{3}$. Take (P,A,B,Q) satisfying $B(m,k)$. We may assume that $P \notin Q$ and that $(A \cup B) \cap Q$ is a general union of $v_{m,k}$ forks and $2(u_{m,k} - 2v_{m,k})$ general points of Q. Let $G \subset Q$ be a general union of $u_{m,k+2} - u_{m,k} = (k+5)(k+4)/6 - (k+3)(k+2)/6 = (2k+7)/3$ lines of type $(1,0)$, i.e. lines of the same ruling of Q. We have $\mathrm{Res}_Q(mP \cup A \cup G \cup B) = mP \cup A \cup B$ and $h^1(\mathcal{I}_{mP\cup A\cup B}(k)) = 0$. So to prove that $(P, A\cup G, B, Q)$ satisfies $B(m,k+2)$ it is sufficient to show $h^1(Q, \mathcal{I}_{(mP\cup A\cup G\cup B)\cap Q}(k+2)) = 0$, i.e. $h^1(Q, \mathcal{I}_{B\cap Q}((k-1)/3, k+2)) = 0$, (where $B\cap Q$ is a general union of $v_{m,k}$ forks of Q), and then apply the Castelnuovo's inequality. We have $u_{m,k} \ge 2v_{m,k}$ (because $B(m,k)$ is true), that is $4v_{m,k} \le ((k+2)/3)(k+3)$. Apply Lemma 4. □

5. m=3

In this section we assume $m = 3$ and prove Theorem 1. Q always denotes a smooth quadric.

REMARK 3. We have $e_{3,4} = u_{3,4} = 5$, $f_{3,4} = v_{3,4} = 0$; $e_{3,5} = 7$, $u_{3,5} = 8$, $f_{3,5} = 4$, $v_{3,5} = 2$; $e_{3,6} = 10$, $u_{3,6} = 11$, $f_{3,6} = 4$, $v_{3,6} = 3$; $e_{3,7} = 13$, $u_{3,7} = 14$, $f_{3,7} = 6$, $v_{3,7} = 2$; $e_{3,8} = 17$, $u_{3,8} = 18$, $f_{3,8} = 2$, $v_{3,8} = 7$; $e_{3,9} = u_{3,9} = 21$, $f_{3,9} = v_{3,9} = 0$; $e_{3,10} = 25$, $u_{3,10} = 26$, $f_{3,10} = 1$, $v_{3,10} = 10$; $e_{3,11} = 29$, $u_{3,11} = 30$, $f_{3,11} = 6$, $v_{3,11} = 6$; $e_{3,12} = 34$, $f_{3,12} = 3$, $u_{3,12} = 35$, $v_{3,12} = 10$; $e_{3,14} = 44$, $u_{3,14} = 45$, $f_{3,14} = 10$, $v_{3,14} = 5$; $e_{3,17} = 62$, $u_{3,17} = 63$, $f_{3,17} = 14$, $v_{3,17} = 4$; $e_{3,20} = 83$, $u_{3,20} = 84$, $f_{3,20} = 18$, $v_{3,20} = 3$; $e_{3,23} = 107$, $u_{3,23} = 108$, $f_{3,23} = 22$, $v_{3,23} = 2$; $e_{3,26} = 134$, $u_{3,26} = 135$, $f_{3,26} = 26$, $v_{3,26} = 1$; $e_{3,29} = u_{3,29} = 165$, $f_{3,29} = v_{3,29} = 0$.

LEMMA 14. *Fix $P \in \mathbb{P}^3$ and a general union $A \subset \mathbb{P}^3$ of 4 lines. Then $h^0(\mathcal{I}_{3P\cup A}(4)) = 5$. For a general $F \subset \mathbb{P}^3$ with $\sharp(F) = 5$ we have $h^i(\mathcal{I}_{3P\cup A\cup F}(4)) = 0$, $i = 0,1$.*

Proof. Fix a general line $L \subset \mathbb{P}^3$. The case $m = 3$ of Lemma 10 gives $h^0(\mathcal{I}_{3P\cup A\cup L}(4)) = 0$, i.e. $h^1(\mathcal{I}_{3P\cup A\cup L}(4)) = 0$. Hence $h^1(\mathcal{I}_{3P\cup A}(4)) = 0$, i.e. $h^0(\mathcal{I}_{3P\cup A}(4)) = 5$. Therefore for a general $F \subset \mathbb{P}^3$ with $\sharp(F) = 5$ we have $h^i(\mathcal{I}_{3P\cup A\cup F}(4)) = 0$, $i = 0,1$. □

LEMMA 15. *Fix $P \in \mathbb{P}^3$ and a general disjoint union $B \subset \mathbb{P}^3$ of 7 lines. Then $h^0(\mathcal{I}_{3P\cup B}(5)) = 4$, that is $N(3,5)$ is true. For a general $G \subset \mathbb{P}^3$ with $\sharp(G) = 4$ we have $h^i(\mathcal{I}_{3P\cup B\cup G}(5)) = 0$, $i = 0,1$.*

Proof. Fix P, A, F as in Lemma 14. Write $F = S \sqcup S'$ with $\sharp(S) = 3$ and let H be a plane containing S. Since A and F are general, H is spanned by S, $P \notin H$, $S' \cap H = \emptyset$ and $H \cap A$ is formed by 4 general points of H. Set $Z := \cup_{O\in S} 2O$. Let $D \subset H$ be the union of the 3 lines of H spanned by each subset of S with two elements. By the generality of $A\cap H$ we have $A \cap D = \emptyset$. Take a general $G' \subset H$ with $\sharp(G') = 2$ and set $G := S' \cup G'$ and $B' := A \cup D \cup Z$. Notice that $D\cup Z$ is a flat limit of a family of disjoint unions of 3 lines. Hence B' is a flat limit of a family of disjoint unions of 7 lines. By the semicontinuity theorem for cohomology ([4, III.12.8]) it is sufficient to prove that $h^i(\mathcal{I}_{3P\cup B'\cup G}(5)) = 0$, $i = 0,1$. We have $\mathrm{Res}_H(3P \cup B' \cup G) = 3P \cup A \cup S \cup S'$. Hence $h^i(\mathcal{I}_{\mathrm{Res}_H(3P\cup B'\cup G)}(4)) = 0$, $i = 0,1$. By the Castelnuovo's inequality it is sufficient to prove that $h^i(H, \mathcal{I}_{(3P\cup B'\cup G)\cap H}(5)) = 0$, $i = 0,1$. The scheme $(3P \cup B' \cup G) \cap H$ is the union of D and of the 6 general points $(A\cap H) \cup G'$ of H. Hence $h^i(H, \mathcal{I}_{(3P\cup B'\cup G)\cap H}(5)) = 0$, $i = 0,1$. □

LEMMA 16. *Fix $P \in \mathbb{P}^3$, a general disjoint union $A \subset \mathbb{P}^3$ of 13 lines, a general line $L \subset \mathbb{P}^3$, a general $S' \subset L$ such that $\sharp(S') = 4$ and a general $S'' \subset \mathbb{P}^3$ such that*

$\sharp(S'') = 2$. *Then* $h^i(\mathcal{I}_{3P\cup A\cup S'\cup S''}(7)) = 0$, $i = 0,1$, *and hence* $N(3,7)$ *is true*.

Proof. Fix a smooth quadric Q. Take P, B, G as in Lemma 15. Write $G = G_1 \sqcup G_2$ with $\sharp(G_1) = 2$ and let L be the line spanned by G_1. For general P, B, G we may assume that $P \notin Q$, $G \cap Q = \emptyset$ and that $(B \cup L) \cap Q$ is formed by 16 general points of Q. Let $E \subset Q$ be a general union of 6 lines of type $(1,0)$. Set $A := B \cup E$, $S' := G_1 \cup (L \cap Q)$ and $S'' := G_2$. We have $\mathrm{Res}_Q(3P \cup A \cup S' \cup S'') = 3P \cup B \cup G$. Hence by the Castelnuovo's inequality it is sufficient to prove that $h^i(Q, \mathcal{I}_{(3P\cup A\cup S'\cup S'')\cap Q}(7)) = 0$, $i = 0,1$, i.e. that $h^i(Q, \mathcal{I}_{E\cup(L\cap Q)\cup(B\cap Q)}(7)) = 0$, i.e. that $h^i(Q, \mathcal{I}_{(L\cap Q)\cup(B\cap Q)}(1,7)) = 0$, $i = 0,1$. This is true, because the 16 points $(L \cap Q) \cup (B \cap Q)$ are general in Q. □

LEMMA 17. *$B(3,5)$ is true*.

Proof. Since any 2 points of $\mathbb{P}^3$ are contained in a smooth quadric, it is sufficient to find $P \in \mathbb{P}^3$, a disjoint union A of 4 lines and the union D of 2 disjoint reducible conics with $h^0(\mathcal{I}_{3P\cup A\cup D}(5)) = 0$. By [3, part (e) of Theorem 4.2] we have $h^0(\mathcal{I}_{2P\cup E}(4)) = 1$, where E is a general union of 6 lines. Hence $h^0(\mathcal{I}_{2P\cup E\cup\{O\}}(4)) = 0$ for a general $O \in \mathbb{P}^3$. Let $H \subset \mathbb{P}^3$ be a general plane containing P and O. Moving if necessary E we may assume that $E \cap H$ is formed by 6 general points of H. Fix $P_1, P_2 \in E \cap H$ with $P_1 \neq P_2$. Let D_i, $i = 1,2$, be the line of H spanned by O and P_i. Call L_i the line of E containing P_i. Hence $L_i \cup D_i$ is a reducible conic whose singular point is different from O. As in [5, Example 2.1.1] or in [1, lemma 2.5] it is easy to check that $2O \cup (L_1 \cup D_1) \cup (L_2 \cup D_2)$ is a flat limit of a family of unions of two disjoint reducible conics. Hence by the semicontinuity theorem for cohomology it is sufficient to prove that $h^0(\mathcal{I}_{3P\cup 2O\cup D_1\cup D_2\cup E}(5)) = 0$. We have $\mathrm{Res}_H(3P \cup 2O \cup D_1 \cup D_2 \cup E) = 2P \cup E \cup \{O\}$. Hence by the Castelnuovo's inequality it is sufficient to prove $h^0(H, \mathcal{I}_{(3P,H)\cup D_1\cup D_2\cup(E\setminus(L_1\cup L_2)\cap H)}(5)) = 0$, i.e. $h^0(H, \mathcal{I}_{(3P,H)\cup(E\setminus(L_1\cup L_2)\cap H)}(3)) = 0$. Since $(E \setminus (L_1 \cup L_2)) \cap H$ is a general union of 4 points of H, we are done. □

LEMMA 18. *$B(3,7)$ is true*.

Proof. Fix (P, A, B, Q) satisfying $B(3,5)$, i.e. A is a disjoint union of 4 lines, B is a disjoint union of 2 reducible conics with their singular points contained in Q and $h^0(\mathcal{I}_{3P\cup A\cup B}(5)) = 0$. Take a general smooth quadric Q'. We may assume $P \notin Q'$. Let $E \subset Q'$ be a general union of 6 lines of type $(1,0)$ and set $A' := A \cup E$. Since $\mathrm{Res}_{Q'}(3P \cup A' \cup B) = (3P \cup A \cup B)$, to prove that (P, A', B, Q) satisfies $B(3,7)$, by the Castelnuovo's inequality it is sufficient to prove $h^0(Q', \mathcal{I}_{E\cup((A\cup B)\cap Q')}(7)) = 0$ ([1, lemma 2.2]), i.e. $h^0(Q', \mathcal{I}_{(A\cup B)\cap Q'}(1,7)) = 0$. Since (for any fixed B), $A \cap Q'$ is a general union of 8 points of Q', it is sufficient to prove $h^0(Q', \mathcal{I}_{B\cap Q'}(1,7)) = 8$, i.e. $h^0(Q', \mathcal{I}_{B\cap Q'}(1,7)) = h^0(Q', \mathcal{O}_{Q'}(1,7)) - \sharp(B \cap Q')$, i.e. $h^1(Q', \mathcal{I}_{B\cap Q'}(1,7)) = 0$, where $B \cap Q'$ is a general union of 2 smooth forks of Q'. Since a fork is a degeneration of a family of smooth forks, by the semicontinuity theorem for cohomology it is sufficient to prove that $h^1(Q', \mathcal{I}_W(1,7)) = 0$, where W is a general union of 2 forks of Q'. Apply the case $(a,b,v) = (1,7,2)$ of Lemma 4. □

LEMMA 19. *$A(3,9)$ and $B(3,9)$ are true.*

Proof. Since $e_{3,9}=21$, $f_{3,9}=0$, $u_{3,9}=21$, and $v_{3,9}=0$, $A(3,9)$ and $B(3,9)$ are equivalent. Fix a smooth quadric Q, a general $P\in\mathbb{P}^3\setminus Q$, a general union A of 10 disjoint lines and a general union B of two reducible conics with their singular points contained in Q. We have $h^i(I_{3P\cup A\cup B}(7))=0$, $i=0,1$ (Lemma 18). Let $E\subset Q$ be a general union of 7 lines of type $(1,0)$. Set $Z:=\cup_{O\in\mathrm{Sing}(B)}2O$. The scheme $B\cup Z$ is a disjoint union of 2 sundials and so it is a flat limit of a family of disjoint unions of 4 lines. Therefore $A\cup B\cup Z$ is a flat limit of a family of disjoint unions of 14 lines. Hence by the semicontinuity theorem and the Castelnuovo's inequality it is sufficient to prove $h^i(Q,I_{(Z\cap Q)\cup(B\cap Q)\cup(A\cap Q)\cup E}(9))=0$, $i=0,1$, i.e. $h^i(Q,I_{(Z\cap Q)\cup(B\cap Q)\cup(A\cap Q)}(2,9))=0$, $i=0,1$. Since $(Z\cap Q)\cup(B\cap Q)$ is a general union of two 2-points of Q and 4 points of Q, $(Z\cap Q)\cup(B\cap Q)\cup(A\cap Q)$ is a general union of two 2-points of Q and 24 points of Q. Therefore it is sufficient to prove $h^1(Q,I_W(2,9))=0$, where W is a general union of two 2-points of Q. Use either [5, Lemme 2.3] or [6], [7]. □

LEMMA 20. *$A(3,11)$ and $B(3,11)$ are true.*

Proof. Take (P,A,B,Q) satisfying $B(3,9)$ (hence $B=\emptyset$). We may assume that $P\notin Q$ and that $A\cap Q$ is formed by 42 general points of Q. We first prove $B(3,11)$. Let $E\subset Q$ be a union of 9 lines of type $(1,0)$ such that six of them each meets $A\cap Q$ at one point. Since $A\cup E$ is a disjoint union of 18 lines and 6 reducible conics, to get $B(3,11)$, by the Castelnuovo's inequality, it is sufficient to prove that $h^0(Q,I_{E\cup(A\cap Q)}(11))=0$, i.e. $h^0(Q,I_{A\cap Q\setminus A\cap E}(2,11))=0$. This is true, because $A\cap Q\setminus A\cap E$ is a general union of 36 points of Q and $h^0(Q,\mathcal{O}_Q(2,11))=36$.

Now we prove $A(3,11)$. Let $F\subset Q$ be a union of 8 lines of type $(1,0)$ with $F\cap A=\emptyset$. Take another line $L\subset Q$ of type $(1,0)$ with $A\cap L=\emptyset$ and fix $S\subset L$ with $\sharp(S)=6$. Since $A\cup F$ is a disjoint union of 29 lines and $A\cap Q$ is a general union of 42 points of Q, to prove that $(3P\cup A\cup F,S,L)$ satisfies $A(3,11)$ it is sufficient to use the Castelnuovo's inequality and observe that $h^0(Q,I_S(3,11))=42$. □

LEMMA 21. *$B(3,6)$ and $N(3,6)$ are true.*

Proof. We first prove $B(3,6)$. $B(3,4)$ is true (Lemma 10). Take (P,A,B,Q) satisfying $B(3,4)$. We have $B=\emptyset$, $3P\sqcup A\in Z(3,5)$ and $h^0(I_{3P\cup A}(4))=0$. We have $u_{3,6}=11$ and $v_{3,6}=3$. We may assume that $P\notin Q$ and that $A\cap Q$ is a general union of 10 points of Q. Let $F\subset Q$ be a disjoint union of 6 lines of type $(1,0)$ on Q such that 3 of them each meets $A\cap Q$ at one point. The scheme $G:=3P\cup A\cup F$ is the disjoint union of $3P$, 6 lines and 3 reducible conics with their singular points contained in Q. Hence to prove $B(3,6)$ it is sufficient to prove that $h^0(I_G(6))=0$. The scheme $G\cap Q$ is the disjoint union of F and 7 general points of Q. Hence $h^0(Q,I_{G\cap Q}(6))=0$. The Castelnuovo's inequality gives $h^0(I_G(6))=0$.

Now we prove $N(3,6)$. We have $e_{3,4}=5$, $f_{3,4}=0$, $e_{3,6}=10$ and $f_{3,6}=4$. Fix a smooth quadric $Q\subset\mathbb{P}^3$. Take (E,S,L) satisfying $A(3,4)$ with $S=\emptyset$ and $E=3P\sqcup A$. We have $h^i(I_E(4))=0$, $i=0,1$. For a general (E,S,L) we may assume $P\notin Q$ and that $A \cap Q$ is a general union of 10 points of Q. Let $F' \subset Q$ be a general union of 5 lines of type $(1,0)$. To prove that $h^1(\mathcal{I}_{E\cup F'}(6)) = 0$ and hence to prove $N(3,6)$, by the Castelnuovo's inequality it is sufficient to notice that $h^1(Q, \mathcal{I}_{A\cap Q}(1,6)) = 0$ and hence $h^1(Q, \mathcal{I}_{(E\cup F')\cap Q}(6)) = 0$. □

LEMMA 22. *$A(3,8)$ is true.*

Proof. Since $f_{3,8} = 2$ and any two distinct points of $\mathbb{P}^3$ are collinear, it is sufficient to prove that $h^0(\mathcal{I}_E(8)) = 2$ for a general $E \in Z(3,17)$. Take (P,A,B,Q) satisfying $B(3,6)$ (Lemma 21) with $P \notin Q$, A a disjoint union of 5 lines and B a disjoint union of 3 reducible conics with their singular points contained in Q. By the semicontinuity theorem for cohomology ([4, III.12.8]), we may deform A and B so that $(A\cup B)\cap Q$ is zero-dimensional and general, with the only restriction that each singular point of B is contained in Q. Set $Z := \cup_{O\in \mathrm{Sing}(B)} 2O$. Let $G \subset Q$ be a general union of 6 lines of type $(1,0)$. The scheme $B\cup Z$ is a disjoint union of 3 sundials and so it is a flat limit of a family of unions of 6 disjoint lines. Hence by the semicontinuity theorem it is sufficient to prove $h^0(\mathcal{I}_{3P\cup A\cup G\cup B\cup Z}(8)) = 2$. The scheme $(3P\cup G\cup A\cup B\cup Z)\cap Q$ is the disjoint union of G, the scheme $Z\cap Q = \cup_{O\in \mathrm{Sing}(B)}(2O,Q)$ (i.e. a general union of 3 2-points of Q) and 16 general points of Q (Lemma 1). Since $\mathrm{Res}_Q(3P\cup A\cup G\cup B\cup Z) = 3P\cup A\cup B$ and $h^0(\mathcal{I}_{3P\cup A\cup B}(6)) = 0$, by the Castelnuovo's inequality it is sufficient to prove that $h^0(Q, \mathcal{I}_{Q\cap Z}(2,8)) = 2$. We have $h^0(Q, \mathcal{I}_{Q\cap Z}(2,8)) = h^0(Q, \mathcal{O}_Q(2,8)) - 9$ ([5, Lemma 2.3], [6], [7]) □

LEMMA 23. *$B(3,8)$ is true.*

Proof. Take (P,A,B,Q) satisfying $B(3,6)$ (Lemma 21) with $P \notin Q$, A a disjoint union of 5 lines and B a disjoint union of 3 reducible conics with their singular points contained in Q. By the semicontinuity theorem for cohomology ([4, III.12.8]), we may deform A and B so that $(A\cup B)\cap Q$ is zero-dimensional and general, with the only restriction that each singular point of B is contained in Q. Let $F \subset Q$ be a union of 7 different lines of type $(1,0)$ such that $F\cap(B\cap Q) = \emptyset$ and 4 of them each meets one point of $A\cap Q$. Set $G := 3P\cup A\cup B\cup F$. To prove $B(3,8)$ it is sufficient to prove that $h^1(\mathcal{I}_G(8)) = 0$. The scheme $G\cap Q$ is the disjoint union of F, 3 general forks and 6 general points. Apply the Castelnuovo's inequality and the case $(a,b,v) = (1,8,3)$ of Lemma 4. □

LEMMA 24. *$A(3,10)$ is true.*

Proof. Since $f_{3,10} = 1$, it is sufficient to find $E \in Z(3,25)$ such that $h^0(\mathcal{I}_E(10)) = 1$, i.e. $h^1(\mathcal{I}_E(10)) = 0$. Take (P,A,B,Q) satisfying $B(3,8)$ (Lemma 23) with $P \notin Q$, A a disjoint union of 4 lines and B a disjoint union of 7 reducible conics with their singular points contained in Q. Set $Z := \cup_{O\in \mathrm{Sing}(B)} 2O$. By the semicontinuity theorem and Lemma 1 we may assume that $(A\cup B\cup Z)\cap Q$ is a general union of 7 2-points of Q and 22 general points of Q. Let $F \subset Q$ be the general union of 7 lines of type $(1,0)$. Since $3P\cup A\cup B\cup Z\cup F$ is a flat limit of a family of elements of $Z(3,25)$ and $h^1(\mathcal{I}_{3P\cup A\cup B}(8)) = 0$, by the semicontinuity theorem and the Castelnuovo's inequality it is sufficient to prove that $h^1(Q, \mathcal{I}_{(Q\cap(A\cup B\cup Z))\cup F}(10)) = 0$, i.e.

$h^1(Q, I_{Q\cap(A\cup B\cup Z)}(3,10)) = 0$, i.e. $h^1(Q, I_{Z\cap Q}(3,10)) = 0$, where $Z\cap Q$ is a general union of 7 2-points. Use [5, Lemme 2.3] or [6], [7]. $\square$

LEMMA 25. *$B(3,10)$ is true.*

Proof. Take (P,A,B,Q) satisfying $B(3,8)$ with $P \notin Q$, A a disjoint union of 4 lines and B a disjoint union of 7 reducible conics with their singular points contained in Q. By the semicontinuity theorem for cohomology ([4, III.12.8]), we may deform A and B so that $(A\cup B)\cap Q$ is zero-dimensional and general, with the only restriction that each singular point of B is contained in Q. Let $F \subset Q$ be a union of 8 different lines of type $(1,0)$ with the restriction that $F\cap(B\cap Q) = \emptyset$ and 3 of them each meets $A\cap Q$ at one point. Set $G := 3P\cup A\cup B\cup F$. Since $A\cap Q$ is general, G is the disjoint union of $3P$, 6 lines and 10 reducible conics with their singular points contained in Q. We have $\mathrm{Res}_Q(G) = 3P\cup A\cup B$ and $h^0(I_{3P\cup A\cup B}(8)) = 0$ (Lemma 23). Hence by the Castelnuovo's inequality it is sufficient to prove that $h^0(Q, I_{G\cap Q}(10)) = 0$, i.e. $h^0(Q, I_Z(2,10)) = 0$, where $Z = Z_1 \sqcup Z_2$, Z_1 is a general union of 7 forks and Z_2 is a general union of 5 points. Hence it is sufficient to prove that $h^0(Q, I_{Z_1}(2,10)) = 5$. Apply Lemma 4 with $(a,b,v) = (2,10,7)$. $\square$

LEMMA 26. *$A(3,12)$ and $B(3,12)$ are true.*

Proof. We have $e_{3,12} = 34$, $f_{3,12} = 3$, $u_{3,12} = 35$ and $v_{3,12} = 10$. We first prove $B(3,12)$. Take (P,A,B,Q) satisfying $B(3,10)$ with $P \notin Q$, A a disjoint union of 6 lines and B a disjoint union of 10 reducible conics with their singular points contained in Q. By the semicontinuity theorem we may assume that $(A\cup B)\cap Q$ is a general union of 10 forks and 12 points. Let $F \subset Q$ be the union of 9 lines of type $(1,0)$ such that $F\cap(A\cup B) = \emptyset$. Set $G := 3P\cup A\cup B\cup F$. G is the disjoint union of $3P$, B (i.e. 10 reducible conics with their singular points contained in Q) and 15 disjoint lines. Hence by the Castelnuovo's inequality it is sufficient to prove $h^0(Q, I_{G\cap Q}(12)) = 0$, i.e. $h^0(Q, I_{(A\cup B)\cap Q}(3,12)) = 0$. Since $A\cap Q$ is general union of 12 points, it is sufficient to prove that $h^0(Q, I_{Q\cap B}(3,12)) = 12$, where $Q\cap B$ is a general union of 10 forks of Q. Apply Lemma 4 with $(a,b,v) = (3,12,10)$.

Now we prove $A(3,12)$. Set $W := \cup_{O\in \mathrm{Sing}(B)} 2O$. Hence $B\cup W$ is a general union of 10 sundials ([5, Example 2.2.1], [1, lemma 2.5], [2], [3]). Let $F' \subset Q$ be the union of 8 general lines of type $(1,0)$. Fix a general line $L\subset Q$ of type $(1,0)$ and a general $S\subset L$ such that $\sharp(S) = 3$. Set $G' := 3P\cup A\cup B\cup W\cup F'\cup S$. Since $B\cup W$ is a flat limit of a family of unions of 20 disjoint lines ([5, Example 2.1.1],[1, lemma 2.5], [3, lemma 2.6]), the semicontinuity theorem for cohomology says that it is sufficient to prove $h^1(I_{G'}(12)) = 0$. We have $\mathrm{Res}_Q(G') = 3P\cup A\cup B$ and $h^1(I_{3P\cup A\cup B}(10)) = 0$ (Lemma 25). Hence by the Castelnuovo's inequality it is sufficient to prove $h^1(Q, I_{G'\cap Q}(12)) = 0$, i.e. $h^1(Q, I_{(B\cap Q)\cup(W\cap Q)\cup(A\cap Q)\cup S}(4,12)) = 0$, i.e. $h^1(Q, I_{(W\cap Q)\cup S}(4,12)) = 0$, where $(W\cap Q)\cup S$ is a general union of 10 2-points of Q and S. Apply [5, Lemme 2.3]. $\square$

LEMMA 27. *$B(3,13)$ and $B(3,15)$ are true.*

Proof. We have $u_{3,11} = 30$, $v_{3,11} = 6$, $u_{3,13} = 40$ and $v_{3,13} = 10$. First we prove $B(3,13)$. Take (P,A,B,Q) satisfying $B(3,11)$ (Lemma 20) with $P \notin Q$. Let $F \subset Q$ be a union of 10 lines of type $(1,0)$ such that 4 of them each meets $A\cap Q$ at one point, and $F \cap (B \cap Q) = \emptyset$. Set $G := 3P \cup A \cup B \cup F$. By the Castelnuovo's inequality it is sufficient to prove that $h^1(Q, I_{G\cap Q}(13)) = 0$, i.e. $h^1(Q, I_{B\cap Q}(3,13)) = 0$, where $B\cap Q$ is a general union of 6 forks. Apply Lemma 4 with $(a,b,v) = (3,13,6)$.

$B(3,13)$ implies $B(3,15)$ by the case $m = 3$ and $k = 13$ of Lemma 13. □

LEMMA 28. *$B(3,14)$ and $N(3,14)$ are true.*

Proof. Recall that $u_{3,12} = 35$, $v_{3,12} = 10$, $u_{3,14} = 45$, $v_{3,14} = 5$, $e_{3,14} = 44$ and $f_{3,14} = 10$. Take (P,A,B,Q) satisfying $B(3,12)$ with $P \notin Q$. First we prove $N(3,14)$. Set $Z := \cup_{O\in \mathrm{Sing}(B)} 2O$. By Lemma 1 and the semicontinuity theorem we may assume that $(A\cup B\cup Z)\cap Q$ is a general union of 10 2-points of Q and 50 points of Q. Let $F \subset Q$ be a general union of 9 lines of type $(1,0)$ on Q. Since $3P\cup A\cup F\cup B\cup Z$ is a flat limit of a family of elements of $Z(3,44)$ and $\mathrm{Res}_Q(3P\cup A\cup F\cup B\cup Z) = 3P\cup A\cup B$, to prove $N(3,14)$ by the semicontinuity theorem and the Castelnuovo's inequality it is sufficient to prove $h^1(Q, I_{(Q\cap(A\cup B\cup Z))\cup F}(14)) = 0$, i.e. $h^1(Q, I_{Q\cap(A\cup B\cup Z)}(5,14)) = 0$, i.e. $h^1(Q, I_{Z\cap Q}(5,14)) = 0$, which is true by [5, Lemme 2.3].

Now we prove $B(3,14)$. Let $G \subset Q$ be a general union of 10 lines of type $(1,0)$. Fix $S \subset \mathrm{Sing}(B)$ such that $\sharp(S) = 5$ and set $Z' := \cup_{O\in S} 2O$. Since $3P\cup A\cup G\cup B\cup Z'$ is a flat limit of a family of disjoint unions of elements of $Z(3,35)$ and 5 reducible conics with their singular points contained in Q, by the semicontinuity theorem and the Castelnuovo's inequality, it is sufficient to prove that $h^1(Q, I_{(Q\cap(A\cup B\cup Z'))\cup G}(14)) = 0$, i.e. $h^1(Q, I_{Q\cap(A\cup B\cup Z')}(4,14)) = 0$. Since $A\cap Q$ is general and $\deg(Q\cap(A\cup B\cup Z')) = 75$, it is sufficient to prove $h^1(Q, I_{Q\cap(B\cup Z')}(4,14)) = 0$. This is true, because $h^1(Q, I_{Q\cap(B\cup Z')}(4,14)) = 0$ by [5, Lemme 2.3]. □

LEMMA 29. *$B(3,17)$ and $N(3,17)$ are true.*

Proof. Recall that $e_{3,17} = 62$, $u_{3,17} = 63$, and $v_{3,17} = 4$. Take (P,A,B,Q) satisfying $B(3,15)$ (Lemma 27) with $P \notin Q$. First we prove $N(3,17)$. Set $Z := \cup_{O\in \mathrm{Sing}(B)} 2O$. By Lemma 1 and the semicontinuity theorem we may assume that $(A\cup B\cup Z)\cap Q$ is a general union of 10 2-points of Q and 82 points of Q. Let $F \subset Q$ be a general union of 11 lines of type $(1,0)$ on Q. Since $3P\cup A\cup F\cup B\cup Z$ is a flat limit of a family of elements of $Z(3,62)$, to prove $N(3,17)$, by the semicontinuity theorem and the Castelnuovo's inequality, it is sufficient to prove $h^1(Q, I_{(Q\cap(A\cup B\cup Z))\cup F}(17)) = 0$, i.e. $h^1(Q, I_{Q\cap(A\cup B\cup Z)}(6,17)) = 0$, i.e. $h^1(Q, I_{Z\cap Q}(6,17)) = 0$. This is true by [5, Lemme 2.3].

Now we prove $B(3,17)$. Let $G \subset Q$ be a general union of 12 lines of type $(1,0)$. Fix $S \subset \mathrm{Sing}(B)$ such that $\sharp(S) = 6$ and set $Z' := \cup_{O\in S} 2O$. Since $3P\cup A\cup G\cup B\cup Z'$ is a flat limit of a family of disjoint unions of elements of $Z(3,55)$ and 4 reducible conics whose singular points are contained in Q, by the semicontinuity theorem and the Castelnuovo's inequality, it is sufficient to prove that $h^1(Q, I_{(Q\cap(A\cup B\cup Z'))\cup G}(17)) =$

0, i.e. $h^1(Q, \mathcal{I}_{Q\cap(A\cup B\cup Z')}(5,17)) = 0$. Since $A\cap Q$ is general and $\deg(Q\cap(A\cup B\cup Z')) = 108$, it is sufficient to prove $h^1(Q, \mathcal{I}_{Q\cap(B\cup Z')}(5,17)) = 0$. This is true, because $h^1(Q, \mathcal{I}_{Q\cap(B\cup Z)}(5,17)) = 0$ by [5, Lemme 2.3]. □

LEMMA 30. *$B(3,16)$ and $B(3,18)$ are true.*

Proof. We have $u_{3,16} = 57$, $v_{3,16} = 10$, $u_{3,18} = 70$ and $v_{3,18} = 10$.

First we prove $B(3,16)$. Take (P,A,B,Q) satisfying $B(3,14)$ (Lemma 28) with $P \notin Q$. Let $F \subset Q$ be a general union of 12 lines of type $(1,0)$ such that 5 of them each meets $A\cap Q$ at one point, and $F\cap(B\cap Q) = \emptyset$. Set $G := 3P\cup A\cup B\cup F$. By the Castelnuovo's inequality it is sufficient to prove that $h^1(Q, \mathcal{I}_{G\cap Q}(16)) = 0$, i.e. $h^1(Q, \mathcal{I}_{B\cap Q}(4,16)) = 0$, where $B\cap Q$ is a general union of 5 forks of Q. Apply Lemma 4 with $(a,b,v) = (4,16,5)$.

$B(3,18)$ follows from $B(3,16)$ by the case $m = 3$ and $k = 16$ of Lemma 13. □

LEMMA 31. *$B(3,20)$ and $N(3,20)$ are true.*

Proof. Recall that $u_{3,18} = 70$, $v_{3,18} = 10$, $e_{3,20} = 83$, $u_{3,20} = 84$, $f_{3,20} = 18$ and $v_{3,20} = 3$. Take (P,A,B,Q) satisfying $B(3,18)$ (Lemma 30) with $P \notin Q$. Set $Z := \cup_{O\in \mathrm{Sing}(B)} 2O$. By the semicontinuity theorem and Lemma 1 we may assume that $(A\cup B\cup Z)\cap Q$ is a general union of 10 2-points of Q and 120 points of Q. Let $F \subset Q$ be a general union of 13 lines of type $(1,0)$ (hence $F\cap((A\cup B)\cap Q) = \emptyset$). Let $G \subset Q$ be a union of 14 lines of type $(1,0)$, exactly 3 of them each contains one point of $A\cap Q$, and $G\cap(B\cap Q) = \emptyset$. To prove that the scheme $3P\cup A\cup B\cup Z\cup F$ (resp. $3P\cup A\cup B\cup Z\cup G$) is a flat limit of a family of solutions of $N(3,20)$ (resp. $B(3,20)$), by the semicontinuity theorem and the Castelnuovo's inequality, it is sufficient to prove that $h^1(Q, \mathcal{I}_{Z\cap Q}(7,20)) = 0$ (resp. $h^1(Q, \mathcal{I}_{Z\cap Q}(6,20)) = 0$). Apply [5, Lemme 2.3] □

LEMMA 32. *$B(3,19)$ and $B(3,21)$ are true.*

Proof. Recall that $u_{3,19} = 77$, $v_{3,19} = 10$, $u_{3,21} = 92$, and $v_{3,21} = 10$. Take (P,A,B,Q) satisfying $B(3,17)$ with $P \notin Q$ and add a disjoint union $F \subset Q$ of 14 lines of type $(1,0)$, exactly 6 of them each contains one point of $A\cap Q$, and $F\cap(B\cap Q) = \emptyset$. To prove $B(3,19)$ it is sufficient to prove $h^1(Q, \mathcal{I}_{3P\cup A\cup F\cup B}(19)) = 0$. Since $\mathrm{Res}_Q(3P\cup A\cup F\cup B) = 3P\cup A\cup B$, by the Castelnuovo's inequality it is sufficient to prove that $h^1(Q, \mathcal{I}_{(A\cup F\cup B)\cap Q}(19)) = 0$, i.e. $h^1(Q, \mathcal{I}_{B\cap Q}(5,19)) = 0$, where $B\cap Q$ is a general union of 4 forks of Q. Apply Lemma 4 with $(a,b,v) = (5,19,4)$.

$B(3,19)$ implies $B(3,21)$ by the case $m = 3$, $k = 19$ of Lemma 13. □

LEMMA 33. *$B(3,23)$ and $N(3,23)$ are true.*

Proof. Recall that $e_{3,23} = 107$, $u_{3,23} = 108$, $f_{3,23} = 22$ and $v_{3,23} = 2$. Take (P,A,B,Q) satisfying $B(3,21)$ (Lemma 32) with $P \notin Q$. Set $Z := \cup_{O\in \mathrm{Sing}(B)} 2O$. Fix $S \subset \mathrm{Sing}(B)$ with $\sharp(S) = 8$ and set $Z' := \cup_{O\in S} 2O$. By the semicontinuity theorem and Lemma 1 we may assume that $(A\cup B\cup Z)\cap Q$ is a general union of 10 2-points of Q and 164 points of Q. Let $F \subset Q$ be a general union of 15 lines of type $(1,0)$. Let $G \subset Q$ be a general union of 16 lines of type $(1,0)$. To prove that the scheme $3P \cup A \cup B \cup Z \cup F$ (resp. $3P \cup A \cup B \cup Z' \cup G$) is a flat limit of a family of solutions of $N(3,23)$ (resp. $B(3,23)$), by the semicontinuity theorem and the Castelnuovo's inequality, it is sufficient to prove that $h^1(Q, \mathcal{I}_{Z\cap Q}(8,23)) = 0$, where $Z \cap Q$ is a general union of 10 2-points of Q (resp. $h^1(Q, \mathcal{I}_{(B\cap Q)\cup(Z'\cap Q)}(7,23)) = 0$, where $(B \cap Q) \cup (Z' \cap Q)$ is a general union of 2 forks of Q, 8 2-points of Q and 160 general points of Q). Apply [5, Lemme 2.3] (resp. Lemma 3 with $(a,b,v,z) = (7,23,2,8)$ and [5, Lemme 2.3]). □

LEMMA 34. *$B(3,22)$ and $B(3,24)$ are true.*

Proof. We have $u_{3,22} = 100$ and $v_{3,22} = 10$. First we prove $B(3,22)$. Take (P,A,B,Q) satisfying $B(3,20)$ (Lemma 31) with $P \notin Q$ and $(A\cup B) \cap Q$ a general union of 3 forks and 156 points of Q. Let $G \subset Q$ be a union of 16 lines of type $(1,0)$, exactly 7 of them each contains one point of $A \cap Q$ and $G \cap (B \cap Q) = \emptyset$. By the Castelnuovo's inequality it is sufficient to prove that $h^1(Q, \mathcal{I}_{(A\cup B\cup G)\cap Q}(22)) = 0$, i.e. $h^1(Q, \mathcal{I}_{B\cap Q}(6,22)) = 0$. Use Lemma 4 with $(a,b,v) = (6,22,3)$.

$B(3,24)$ follows from $B(3,22)$ by the case $m = 3$, $k = 22$ of Lemma 13. □

LEMMA 35. *$B(3,26)$ and $N(3,26)$ are true.*

Proof. Recall that $e_{3,26} = 134$, $u_{3,26} = 135$, $f_{3,26} = 26$ and $v_{3,26} = 1$. Take (P,A,B,Q) satisfying $B(3,24)$ (Lemma 34) with $P \notin Q$ and $(A\cup B) \cap Q$ a general union of 10 forks and 194 points of Q. Set $Z := \cup_{O\in \mathrm{Sing}(B)} 2O$. Let $F \subset Q$ be a union of 17 lines of type $(1,0)$ and $G \subset Q$ a union of 18 lines of type $(1,0)$, exactly one of them containing one point of $A \cap Q$. By the semicontinuity theorem and the Castelnuovo's inequality to prove $N(3,26)$ (resp. $B(3,26)$) using $3P \cup A \cup B \cup Z \cup F$ (resp. $3P \cup A \cup B \cup Z \cup G$) it is sufficient to observe that $h^1(Q, \mathcal{I}_{Z\cap Q}(9,26)) = h^1(Q, \mathcal{I}_{Z\cap Q}(8,26)) = 0$ ([5, Lemme 2.3]). □

LEMMA 36. *$B(3,25)$ and $B(3,27)$ are true.*

Proof. We have $u_{3,25} = 126$ and $v_{3,25} = 10$. First we prove $B(3,25)$. Take (P,A,B,Q) satisfying $B(3,23)$ (Lemma 33) with $P \notin Q$ and $(A\cup B) \cap Q$ a general union of 2 forks and 208 points of Q. Take a union $G \subset Q$ of 18 lines of type $(1,0)$, exactly 8 of them each contains one point of $A \cap Q$, and $G \cap (B \cap Q) = \emptyset$. To prove that $3P \cup A \cup B \cup G$ solves $B(3,25)$, by the Castelnuovo's inequality it is sufficient to prove that $h^1(Q, \mathcal{I}_{(A\cup B\cup G)\cap Q}(25)) = 0$, i.e. $h^1(Q, \mathcal{I}_{B\cap Q}(7,25)) = 0$. Apply Lemma 4 with $(a,b,v) = (7,25,2)$.

$B(3,25)$ implies $B(3,27)$ by the case $m = 3$ and $k = 25$ of Lemma 13. □

LEMMA 37. *$A(3,29)$ and $B(3,29)$ are true.*

Proof. Since $f_{3,29} = 0$, we have $v_{3,29} = 0$ and so $A(3,29)$ and $B(3,29)$ are equivalent. We have $e_{3,29} = 165$. Take (P,A,B,Q) satisfying $B(3,27)$ (Lemma 36) with $P \notin Q$ and $(A\cup B) \cap Q$ a general union of 10 forks and 250 points of Q. Set $Z := \cup_{O\in \mathrm{Sing}(B)} 2O$.

Let $F \subset Q$ be a general union of 20 lines of type $(1,0)$. To prove that the scheme $3P \cup A \cup B \cup Z \cup F$ is a flat limit of solutions of $A(3,29)$, by the semicontinuity theorem and the Castelnuovo's inequality, it is sufficient to prove that $h^1(Q, \mathcal{I}_{Z\cap Q}(9,29)) = 0$. Apply [5, Lemme 2.3]. $\square$

Proof of Theorem 1. The exceptional cases $t = 2,3$ are known ([3, part (ii) of Theorem 4.2]). The cases $t = 1$ and $t = 5$ are covered by [3, Theorem 4.2 (i)]. Hence we may assume $t \geq 6$.

Let k be the only integer such that $e_{3,k-1} < t \leq e_{3,k}$. Since $Z(3,t)$ is irreducible, it is sufficient to find $A, B \in Z(3,t)$ such that $h^1(\mathcal{I}_A(k)) = 0$ and $h^0(\mathcal{I}_B(k-1)) = 0$. To prove the existence of A it is sufficient to prove $N(3,k)$ (or the stronger statement $A(3,k)$) (Lemma 8). Hence A exists if $k = 5$ (Lemma 15), $k = 6$ (Lemma 21), $k = 7$ (Lemma 16), $k = 8$ (Lemma 22), $k = 9$ (Lemma 19), $k = 10$ (Lemma 24), $k = 11$ (Lemma 20), $k = 14$ (Lemma 28), $k = 17$ (Lemma 29), $k = 20$ (Lemma 31), $k = 23$ (Lemma 33), $k = 26$ (Lemma 35), $k = 29$ (Lemma 37). Since $A(3,9)$ and $A(3,10)$ are true (Lemmas 19 and 24), Lemma 12 gives that $A(3,k)$ is true for all $k \geq 9$ such that $k \equiv 0,1 \pmod 3$. Since $A(3,29)$ is true (Lemma 37), $A(3,k)$ is true for all $k \geq 32$ such that $k \equiv 2 \pmod 3$ (Lemma 11). Hence A exists for all $t \geq 6$.

The scheme B exists if $B(3,k-1)$ is true (Lemma 9). Hence B exists if $k-1 = 5$ (Lemma 17), $k-1 = 6$ (Lemma 21), $k-1 = 7$ (Lemma 18), $k-1 = 8$ (Lemma 23), $k-1 = 9$ (Lemma 19), $k-1 = 11$ (Lemma 20), $k-1 = 14$ (Lemma 28), $k-1 = 17$ (Lemma 29), $k-1 = 20$ (Lemma 31), $k-1 = 23$ (Lemma 33), $k-1 = 26$ (Lemma 35) and $k-1 = 29$ (Lemma 37). The scheme B exists if $A(3,k-1)$ is true (Lemma 8). We proved that $A(3,k-1)$ is true for all $k \geq 30$ such that $k \equiv 0 \pmod 3$ (Lemma 11) and for all $k \geq 10$ such that $k \equiv 1,2 \pmod 3$ (Lemma 12). If $k = 5$ the scheme B exists, because $A(3,4)$ is true (Lemmas 8 and 10). Hence B exists for all $t \geq 6$. $\square$

6. Proofs of Propositions 1 and 2

LEMMA 38. *Fix integers $k > m > 0$. Assume $h^1(\mathcal{I}_A(k)) = 0$ for at least one $A \in Z(m, e_{m,k})$ and $h^0(\mathcal{I}_B(k)) = 0$ for at least one $B \in Z(m, e_{m,k}+1)$. Then $A(m,k)$ is true.*

Proof. By the semicontinuity theorem we have $h^1(\mathcal{I}_X(k)) = 0$ for a general $X \in Z(m, e_{m,k})$ and $h^0(\mathcal{I}_Y(k)) = 0$ for a general $Y \in Z(m, e_{m,k}+1)$. First assume $f_{m,k} = 0$. In this case for any line L the triple $(A, \emptyset, L)$ satisfies $A(m,k)$. Now assume $f_{m,k} > 0$. Fix a general $Y \in Z(m, e_{m,k}+1)$ and write $Y = W \sqcup L$ with L a line. Since Y is general, W is a general element of $Z(m, e_{m,k})$. Hence $h^1(\mathcal{I}_W(k)) = 0$, i.e. $h^0(\mathcal{I}_W(k)) = f_{m,k}$. Since $h^0(\mathcal{I}_{W\cup L}(k)) = 0$, we have $h^0(\mathcal{I}_{W\cup S}(k)) = 0$ for a general $S \subset L$ such that $\sharp(S) = f_{m,k}$. The triple (W, S, L) satisfies $A(m,k)$. $\square$

Proof of Proposition 1. $A(m, m+1)$ is true (Lemma 10). Lemma 38 gives that $A(m,k)$ is true for all k such that $m+2 \leq k \leq m(m+1)(m+2)/2 - 1$. Lemmas 11 and 12 give that $A(m,k)$ is true for all $k \geq m(m+1)(m+2)/2$. Apply Lemma 8. $\square$

Fix an integer $m \geq 4$. Set $e'_{m,m+1} := m+2$, $e'_{m,m+2} := m+3$ and $e'_{m,m+3} := 2m+4$. For each integer $k \geq m+4$ set $e'_{m,k} := e'_{m,c} + e_{0,k} - e_{0,c}$, where $c \in \{m+1, m+2, m+3\}$ is the integer with $k \equiv c \pmod 3$. Since $m \geq 4$, we have $e'_{m,c} < e_{0,c}$ for all $c \in \{m+1, m+2, m+3\}$. Hence $e'_{m,k} < e_{0,k}$ for all $k > m$. Notice that $e'_{m,k} = e_{0,k} - \alpha_{m,k}$. For all integers $k \geq m+1$ we define the following assertion $E(m,k)$:

$E(m,k), k \geq m+1$: $h^1(\mathcal{I}_E(k)) = 0$ for a general $E \in Z(m, e'_{m,k})$.

REMARK 4. $E(m,m+1)$ is true, because it is equivalent to $A(m,m+1)$ and we may quote Lemma 10, i.e. [3, Theorem 4.2 (i)(c)].

LEMMA 39. *$E(m,m+2)$ is true.*

Proof. Fix a plane H, a line $D \subset H$ and (A,S,L) satisfying $A(m,m+1)$ (Lemma 10). Hence $S = \emptyset$ and $A = mP \sqcup A'$ with $\deg(A') = m+2$. By the semicontinuity theorem for cohomology ([4, III.12.8]), we may assume $P \notin H$ and that $A \cap H$ is a general union of $m+2$ points of H. Hence $h^1(H, \mathcal{I}_{D \cup (A \cap H)}(m+2)) = h^1(H, \mathcal{I}_{A \cap H}(m+1)) = 0$. The Castelnuovo's inequality gives that $A \cup D$ is a solution of $E(m,m+2)$. □

LEMMA 40. *$E(m,m+3)$ is true.*

Proof. Fix a smooth quadric Q and a general $A = mP \cup A' \in Z(m,m+2)$. We have $h^1(\mathcal{I}_A(m+1)) = 0$ (Lemma 10), $P \notin Q$ and $A \cap Q$ is a general union of $2m+4$ points of Q. Let $F \subset Q$ be a union of $m+2$ lines of type $(1,0)$ with $A \cap F = \emptyset$. Since $h^0(Q, \mathcal{O}_Q(1,m+3)) = 2m+8 \geq 2m+4$ and $A \cap Q$ is general, we have $h^1(Q, \mathcal{I}_{F \cup (A \cap Q)}(m+3, m+3)) = h^1(Q, \mathcal{I}_{A \cap Q}(1, m+3)) = 0$. The Castelnuovo's inequality gives that $A \cup F$ is a solution of $E(m,m+3)$. □

LEMMA 41. *Fix integers $k > m \geq 4$. If $E(m,k)$ is true, then $E(m,k+3)$ is true.*

Proof. Fix a degree 3 ruled surface T. Fix a general $X = mP \sqcup Y \in Z(m, e'_{m,k})$ (hence $P \notin T$). Since $E(m,k)$ is true, the semicontinuity theorem implies that $h^1(\mathcal{I}_X(k)) = 0$. Let $F \subset T$ be a general union of $k+4$ elements of $\mathcal{F}(T)$. We have $h^0(T, \mathcal{I}_F(k+3)) = h^0(T, \mathcal{O}_T(k+3)) - (k+4)^2$ (Lemma 6). We have $e'_{m,k+3} - e'_{m,k} = e_{0,k+3} - e_{0,k} = k+4$. The case $m = 0$ of (3) gives

$$3e_{0,k} + (k+4)(e_{0,k+3} - e_{0,k}) + f_{0,k+3} - f_{0,k} = \binom{k+6}{3} - \binom{k+3}{3}. \tag{5}$$

Recall that $h^0(T, \mathcal{O}_T(k+3)) = \binom{k+6}{3} - \binom{k+3}{3}$. If $k \equiv 0,1 \pmod 3$, then $f_{0,k+3} = f_{0,k} = 0$. If $k \equiv 2 \pmod 3$, then $f_{0,k+3} - f_{0,k} = 1$. Since $e'_{m,k} < e_{0,k}$, (5) gives that $3 \cdot \deg(Y) \leq h^0(T, \mathcal{I}_F(k+3))$. Applying $\deg(Y)$ times Lemma 7 (for a general Y) and then applying Castelnuovo's inequality we get that $mP \cup Y \cup F$ satisfies $E(m,k+3)$. □

Proof of Proposition 2. Remark 4 and Lemmas 39, 40, 41 give that $E(m,x)$ is true for all $x > m$. Fix a general element $X = mP \sqcup Y \in Z(m,t)$. Since $t \leq (k+2)(k+$

$3)/6-\alpha_{m,k}$, we have $t \le \lfloor (k+2)(k+3)/6 \rfloor - \alpha_{m,k} = e'_{m,k}$. Hence $E(m,k)$ gives that $h^1(I_X(k)) = 0$.

Part (b) is true for the following reason. Set $a := (k+3)(k+2)/6$ if $k \equiv 0,1 \pmod 3$ and $a := (k^2+5k+10)/6$ if $k \equiv 2 \pmod 3$, i.e. let a be the minimal integer x such that $x > (k+4)(k+1)/6$. Let Y be a general element of $Z(0,x)$. By [5] we have $h^0(I_Y(k)) = 0$. Hence $h^0(I_{mP \cup Y \cup G}(k)) = 0$ for any $P \in \mathbb{P}^3$ and any union G of $t-a$ lines. $\square$

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