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SOME REMARKS ON OPERATORS PRESERVING OSCILLATIONS

Abstract. We prove some properties of nonlinear operators preserving oscillations. We consider the particular cases of Preisach operators and Generalized Plays, of which we give a short introduction.

1. Introduction

The aim of this paper is to state and prove some simple results, concerning nonlinear operators $\mathscr{W} : D(\mathscr{W}) \subseteq C([0,T]) \to C([0,T])$, preserving oscillations, in the sense that they satisfy the following conditions (AA1)-(AA2), for some $\alpha \in (0,1]$. We show that, in case $1 < p \leqslant \infty$ and $\frac{1}{p} < \beta < 1$, they map $D(\mathscr{W}) \cap W^{\beta,p}((0,T))$ into $W^{\alpha\beta,\frac{p}{\alpha}}((0,T))$. This result seems to be new. In case $\alpha = 1$, $\mathscr{W}$ preserves also the belonging to $W^{1,p}((0,T))$ $(1 \leqslant p \leqslant \infty)$ and to $BV((0,T))$. We are not aware of a general statement in this sense, although this result is certainly well known in the case of our two main examples, namely Preisach operators and Generalized Plays. In order to make the paper more or less self contained, we have written in Sections 3 and 4 short introductions to this classes of nonlinear operators, which are quite important in applications.

2. Nonlinear operators and oscillations

We begin by introducing some notations and conventions. We shall indicate with C a positive constant, which may be different from time to time. If we want to stress the fact che C depends on $\alpha, \beta, \dots$, we shall write $C(\alpha, \beta, \dots)$. We shall consider only real valued functions and we shall not precise this in the following.

Let $-\infty < a < b < \infty$. We shall indicate with $C([a,b])$ the Banach space of continuous functions with domain $[a,b]$, equipped with its natural norm $\| \cdot \|_{C([a,b])}$. If $s,t \in [a,b]$, we indicate with $os(f;s,t)$ the oscillation of f in $[\min\{s,t\}, \max\{s,t\}]$, defined as

$$os(f;s,t) := \sup\{|f(\sigma) - f(\tau)| : \sigma, \tau \in [\min\{s,t\}, \max\{s,t\}]\}. \tag{1}$$

Clearly, if $s,t \in [a,b]$,

$$|f(t) - f(s)| \leqslant os(f;s,t) \leqslant V_{\min\{s,t\}}^{\max\{s,t\}}(f),$$

with

$$V_{\min\{s,t\}}^{\max\{s,t\}}(f) := \sup\{\sum_{j=1}^{n} |f(t_j) - f(t_{j-1})| : n \in \mathbb{N}, \min\{s,t\} = t_0 < \dots < t_n = \max\{s,t\}\}.$$

We shall indicate with $BV([a,b])$ the class of real valued functions with bounded variation in $[a,b]$. If I is an open interval in $\mathbb{R}$ and $1 \leqslant p \leqslant \infty$, we shall indicate with $W^{1,p}(I)$ the class of elements f in $L^p(I)$ such that the derivative in the sense of distributions f' is in $L^p(I)$. This space will be equipped with its natural norm

$$\|f\|_{W^{1,p}(I)} := \|f\|_{L^p(I)} + \|f'\|_{L^p(I)} \tag{2}$$

If $p = 1$ and I is bounded, $W^{1,1}(I)$ coincides with the class $AC(I)$ of absolutely continuous functions in I, which means that, $\forall \varepsilon \in \mathbb{R}^+$, there exists $\delta(\varepsilon)$ in $\mathbb{R}^+$, such that, if $[a_k, b_k]$ $(1 \leqslant k \leqslant n,\ n \in \mathbb{N})$ are pairwise disjoint subintervals of I such that $\sum_{k=1}^n (b_k - a_k) \leqslant \delta(\varepsilon)$, then $\sum_{k=1}^n |f(b_k) - f(a_k)| \leqslant \varepsilon$.

If $p = \infty$, $W^{1,\infty}(I)$ coincides with the class of bounded, Lipschitz continuous functions with domain I. We observe that

$$\|f'\|_{L^\infty(I)} = [f]_{Lip(I)} := \sup_{s,t \in I, s \neq t} |t-s|^{-1} |f(t) - f(s)|.$$

Let I be an open interval in $\mathbb{R}$, $p \in [1,\infty]$, $\beta \in (0,1)$. We indicate with $W^{\beta,p}(I)$ the class of elements in $L^p(I)$ such that

$$[f]_{W^{\beta,p}(I)} < \infty, \tag{3}$$

with

$$[f]_{W^{\beta,p}(I)} := \begin{cases} \int_{I \times I} \frac{|f(t)-f(s)|^p}{|t-s|^{1+\beta p}} ds dt & \text{if} \quad 1 \leqslant p < \infty, \\ \sup_{t,s \in I, s \neq t} |t-s|^{-\beta} |f(t) - f(s)| & \text{if} \quad p = \infty. \end{cases} \tag{4}$$

We set

$$\|f\|_{W^{\beta,p}(I)} := \|f\|_{L^p(I)} + [f]_{W^{\beta,p}(I)}. \tag{5}$$

We observe that $W^{\beta,\infty}(I)$ coincides with the class of bounded, Hölder continuous functions $C^\beta(I)$. In case $\beta > \frac{1}{p}$, $W^{\beta,p}(I)$ is continuously embedded into $BC(I)$, the class of real valued, continuous and bounded functions of domain I. For these definitions and embeddings, see [2], Chapters 3.4.2, 3.2.2, 2.7.1; in case $1 < p < \infty$ and $0 < \beta < 1$, $W^{\beta,p}(I)$ coincides with both the spaces $B^\beta_{p,p}(I)$ and $F^\beta_{p,p}(I)$, which are elements of the scales of (respectively) Besov and Triebel-Lizorkin spaces (see [2]). Obviously, we have also $W^{1,1}(I) \hookrightarrow BC(I)$.

The following fact will be important for us:

LEMMA 1. *Let $p \in (1,\infty)$, $\beta \in (\frac{1}{p}, 1)$. Then there exists $C \in \mathbb{R}^+$ such that, if $f \in W^{\beta,p}(\mathbb{R})$,*

$$\int_{\mathbb{R} \setminus \{0\}} \Big(\int_{\mathbb{R}} \sup_{|\rho| \leqslant h} |f(x+\rho) - f(x)|^p dx \Big) |h|^{-1-\beta p} dh \leqslant C(\beta, p) \|f\|^p_{W^{\beta,p}(I)}.$$

Proof. See [2], Chapter 2.5.10. □

As a simple consequence, we obtain the following

COROLLARY 1. *Let I be an open interval in $\mathbb{R}$ and $p \in (1,\infty]$. Let $\beta \in (\frac{1}{p},1)$. Then there exists $C = C(I,p,\beta)$ such that, $\forall f \in W^{\beta,p}(I)$,*

$$\int_{I\times I} \frac{os(f;s,t)^p}{|t-s|^{1+\beta p}} dsdt \leqslant C\|f\|^p_{W^{\beta,p}(I)}, \quad \text{if} \quad 1 < p < \infty,$$

$$\sup_{s,t\in I, s\neq t} os(f;s,t)(t-s)^{-\beta} \leqslant [f]_{W^{\beta,\infty}(I)}, \quad \text{if} \quad p = \infty.$$

Proof. The case $p = \infty$ is trivial. We assume $1 < p < \infty$. First, we observe that

$$os(f;s,t) \leqslant 2 \sup_{|\rho|\leqslant \frac{|t-s|}{2}} |f(\frac{s+t}{2}+\rho) - f(\frac{s+t}{2})|.$$

By Chapter 3.3.4 in [2], there exists a bounded linear extension operator E, mapping $W^{\beta,p}(I)$ into $W^{\beta,p}(\mathbb{R})$. So, if we set $\widetilde{f} := Ef$, we have

$$\int_{I\times I} \frac{os(f;s,t)^p}{|t-s|^{1+\beta p}} dsdt \leqslant \int_{\mathbb{R}\times\mathbb{R}} \frac{os(\widetilde{f};s,t)^p}{|t-s|^{1+\beta p}} dsdt \leqslant 2^p \int_{\mathbb{R}\times\mathbb{R}} \frac{\sup_{|\rho|\leqslant\frac{|t-s|}{2}} |\widetilde{f}(\frac{t+s}{2}+\rho) - \widetilde{f}(\frac{t+s}{2})|^p}{|t-s|^{1+\beta p}} dsdt$$

$$= 2^{-\beta p} \int_{\mathbb{R}\times\mathbb{R}} \frac{\sup_{|\rho|\leqslant|h|} |\widetilde{f}(x+\rho) - \widetilde{f}(x)|^p}{|h|^{1+\beta p}} dxdh \leqslant C(\beta,p)\|\widetilde{f}\|^p_{W^{\beta,p}(\mathbb{R})}$$

$$\leqslant C(\beta,p)\|E\|^p_{\mathscr{L}(W^{\beta,p}(I),W^{\beta,p}(\mathbb{R}))}\|f\|^p_{W^{\beta,p}(I)}.$$

□

Now we introduce a (nonlinear) operator $\mathscr{W} : D(\mathscr{W}) \to C([0,T])$, for some $T \in \mathbb{R}^+$. We shall assume that $\mathscr{W}$ satisfies the following conditions:

(AA1) $D(\mathscr{W}) \subseteq C([0,T])$;

(AA2) there exists $\alpha \in (0,1]$ such that, $\forall R \in \mathbb{R}^+$ there exists $C(R)$ in $\mathbb{R}^+$ so that, if $f \in D(\mathscr{W})$ and $\|f\|_{C([0,T])} \leqslant R$, $\forall s,t \in [0,T]$,

$$os(\mathscr{W}(f);s,t) \leqslant C(R) os(f;s,t)^\alpha.$$

THEOREM 1. *Let $\mathscr{W}$ satisfy (AA1)-(AA2). Let $p \in (1,\infty]$, $\beta \in (\frac{1}{p},1)$ and let $f \in D(\mathscr{W}) \cap W^{\beta,p}((0,T))$. Then $\mathscr{W}(f) \in W^{\alpha\beta,\frac{p}{\alpha}}((0,T))$. Moreover, $\forall R \in \mathbb{R}^+$, there exists $C(R) \in \mathbb{R}^+$ such that, if $f \in D(\mathscr{W}) \cap W^{\beta,p}((0,T))$ and $\|f\|_{C([0,T])} \leqslant R$,*

$$[\mathscr{W}(f)]_{W^{\alpha\beta,\frac{p}{\alpha}}((0,T))} \leqslant C(R)\|f\|^\alpha_{W^{\beta,p}((0,T))}.$$

Proof. In fact, by Corollary 1, we have, setting $I := (0,T)$,

$$[\mathscr{W}(f)]^{\frac{p}{\alpha}}_{W^{\alpha\beta,\frac{p}{\alpha}}((0,T))} \leqslant \int_{I\times I} \frac{os(\mathscr{W}(f);s,t)^{p/\alpha}}{|t-s|^{1+\beta p}} dtds$$

$$\leqslant C_0(R)\int_{I\times I} \frac{os(f;s,t)^p}{|t-s|^{1+\beta p}} dtds \leqslant C_1(R)\|f\|^p_{W^{\beta,p}((0,T))}.$$

□

In case $\alpha = 1$, we have also the following

THEOREM 2. *Assume that (AA1)-(AA2) hold, with $\alpha = 1$. Then:*
(I) $\forall p \in [1,\infty]$, $\mathscr{W}$ maps $D(\mathscr{W}) \cap W^{1,p}((0,T))$ into $W^{1,p}((0,T))$;
(II) $|\mathscr{W}(f)'(t)| \leqslant C(|f\|_{C([0,T])})|f'(t)|$ a. e. in $(0,T)$.
(III) $\mathscr{W}$ maps $D(\mathscr{W}) \cap BV((0,T))$ into $BV((0,T))$. Moreover, if $f \in D(\mathscr{W}) \cap BV((0,T))$,

$$V_0^T(\mathscr{W}(f)) \leqslant C(\|f\|_{C([0,T])})V_0^T(f).$$

Proof. Assume that $f \in W^{1,1}([0,T]) \cap D(\mathscr{W})$. Let $n \in \mathbb{N}$ and $0 \leqslant a_1 \leqslant b_1 \leqslant a_2 \leqslant b_2 \leqslant \cdots \leqslant a_n \leqslant b_n \leqslant T$. Then

$$\sum_{j=1}^n |\mathscr{W}(f)(b_j) - \mathscr{W}(f)(a_j)| \leqslant \sum_{j=1}^n os(\mathscr{W}(f);a_j,b_j)$$

$$\leqslant C(\|f\|_{C([0,T])})\sum_{j=1}^n os(f;a_j,b_j)$$

$$\leqslant C(\|f\|_{C([0,T])})\int_{\cup_{j=1}^n(a_j,b_j)} |f'(t)|dt \leqslant \varepsilon$$

if $\sum_{j=1}^n(b_j - a_j) \leqslant \delta(\varepsilon)$, by the absolute continuity of the integral. We deduce that $\mathscr{W}(f) \in AC((0,T)) = W^{1,1}([0,T])$ and so it is differentiable almost everywhere. Observe that, if f is differentiable in t and $t+h \in [0,T]$,

$$os(f;t,t+h) \leqslant |f'(t)||h| + o(|h|) \quad (h \to 0).$$

So, if f and $\mathscr{W}(f)$ are differentiable in t

$$|\tfrac{\mathscr{W}(f)(t+h)-\mathscr{W}(f)(t)}{h}| \leqslant |h|^{-1}os(\mathscr{W}(f);t,t+h)$$

$$\leqslant C(\|f\|_{C([0,T])})|h|^{-1}os(f;t,t+h) = C(\|f\|_{C([0,T])})(|f'(t)| + o(1)) \quad (h \to 0).$$

So (II) is proved and (I) follows easily.

Concerning (III), let $0 = t_0 < \cdots < t_n = T$. Then

$$\sum_{j=1}^n |\mathscr{W}(f)(t_j) - \mathscr{W}(f)(t_{j-1})| \leqslant \sum_{j=1}^n os(\mathscr{W}(f);t_{j-1},t_j)$$

$$\leqslant C(\|f\|_{C([0,T])})\sum_{j=1}^n os(f;t_{j-1},t_j)$$

$$\leqslant C(\|f\|_{C([0,T])})\sum_{j=1}^n V_{t_{j-1}}^{t_j}(f) = C(\|f\|_{C([0,T])})V_0^T(f)$$

□

3. Preisach operators

The first class of operators to which we apply the results of Section 2 is the class of Preisach operators, which were introduced by F. Preisach in 1935, in connection to problem in ferromagnetism (see [3], Historical Notes). Here we present an introduction to it, which seems simpler than the one in [3], Chapter IV.

Let $T \in \mathbb{R}^+$, $\rho := (\rho_1, \rho_2) \in \mathbb{R}^2$, with $\rho_1 < \rho_2$, $\xi \in \{-1, 1\}$. Given $u \in C([0,T])$, we set

$$A_t := \{s \in [0,t] : u(s) \in \{\rho_1, \rho_2\}\}. \tag{1}$$

and

$$h_\rho(u,\xi)(t) = \begin{cases} -1 & \text{if } A_t \neq \emptyset, u(\max A_t) = \rho_1, \text{ or } u(s) < \rho_1 \forall s \in [0,t] \\ & \text{ or } \rho_1 < u(s) < \rho_2 \quad \forall s \in [0,t] \text{ and } \xi = -1, \\ & \\ 1 & \text{if } A_t \neq \emptyset, u(\max A_t) = \rho_2, \text{ or } u(s) > \rho_2 \forall s \in [0,t] \\ & \text{ or } \rho_1 < u(s) < \rho_2 \quad \forall s \in [0,t] \text{ and } \xi = 1. \end{cases} \tag{2}$$

h_ρ is called *relay operator*. Roughly speaking we start from 1 if $u(0) \geqslant \rho_2$ or $u(0) > \rho_1$ and $\xi = 1$, from -1 if $u(0) \leqslant \rho_1$ or $u(0) < \rho_2$ and $\xi = -1$. Then, we pass from 1 to -1 whenever we meet t such that $u(t) = \rho_1$, we pass from -1 to 1 whenever we meet t such that $u(t) = \rho_2$. It is easily seen that, for fixed values of ρ and ξ, if $\delta \in \mathbb{R}^+$ is such that, whenever $|t-s| \leqslant \delta$, $|u(t) - u(s)| < \rho_2 - \rho_1$, $h_\rho(u,\xi)$ changes sign $[\frac{T}{\delta}] + 1$ times, at most.

In the following, we shall indicate with $\mathscr{P}$ the Preisach plane, defined as

$$\mathscr{P} := \{\rho = (\rho_1, \rho_2) \in \mathbb{R}^2 : \rho_1 < \rho_2\}. \tag{3}$$

The following properties are easy to prove and are left to the reader:

PROPOSITION 1. *Let $T \in \mathbb{R}^+$, $\xi \in \{-1,1\}$, $u, u_1, u_2 \in C([0,T])$, $\rho = (\rho_1, \rho_2)$, $\rho' = (\rho'_1, \rho'_2) \in \mathscr{P}$, $c \in \mathbb{R}$. Then:*

(I) if $t \in [0,T]$ and $u(t) \leqslant \rho_1$, then $h_\rho(u,\xi)(t) = -1$, if $u(t) \geqslant \rho_2$, $h_\rho(u,\xi)(t) = 1$;

(II) if $t \in [0,T]$ and $u_1(s) \leqslant u_2(s)$ $\forall s \in [0,t]$, then $h_\rho(u_1,\xi)(t) \leqslant h_\rho(u_2,\xi)(t)$;

(III) $\forall t \in [0,T]$, $h_\rho(u+c,\xi)(t) = h_{\rho_1 - c, \rho_2 - c}(u,\xi)(t)$;

(IV) if $\rho_1 \leqslant \rho'_1$ and $\rho_2 \leqslant \rho'_2$, $h_{\rho'}(u,\xi)(t) \leqslant h_\rho(u,\xi)(t)$ $\forall t \in [0,T]$;

(V) $\lim_{r \to \rho_1^+} h_{r,\rho_2}(u,\xi)(t) = h_\rho(u,\xi)(t) = lim_{r \to \rho_2^-} h_{\rho_1, r}(u,\xi)(t)$;

(VI) $h_{-\rho_2, -\rho_1}(-u, -\xi)(t) = -h_\rho(u,\xi)(t)$.

(VII) If $\phi : [0,T] \to [0,T]$ is an increasing homeomorphism of $[0,T]$ into itself, then $h_\rho(u \circ \phi, \xi) = h_\rho(u,\xi) \circ \phi$.

(VIII) If $u, v \in C([0,T])$ and, for some $t \in [0,T]$, $u_{|[0,t]} = v_{|[0,t]}$, then $h_\rho(u,\xi)_{|[0,t]} = h_\rho(v,\xi)_{|[0,t]}$.

REMARK 1. Property (VII) in Proposition 1 is known as the *rate independence property*. Property (VIII) in Proposition 1 is known as the *memory property*. Properties (VII) and (VIII) together characterise *hysteresis operators*.

Now we study the behaviour of $\rho \to h_\rho(u,\xi)(t)$, for fixed values of u, ξ, t.

LEMMA 2. *Let* $u \in C([0,T])$*,* $\xi \in \{-1,1\}$*,* $t \in [0,T]$*. Then:*
(I) if $\rho_1 \geqslant u(t)$*,* $h_{\rho_1,\rho_2}(u,\xi)(t) = -1\ \forall \rho_2 \in (\rho_1,\infty)$*;*
(II) if $\rho_1 < \min_{[0,t]}(u)$*,* $h_{\rho_1,\rho_2}(u,1)(t) = 1\ \forall \rho_2 \in (\rho_1,\infty)$*;*
(III) if $\rho_1 < u(t)$*, there exists* $\psi(\rho_1) \in (\rho_1,\infty)$*, such that* $h_{\rho_1,\rho_2}(u,-1)(t) = 1$ *if* $\rho_2 \in (\rho_1,\psi(\rho_1)]$*,* $h_{\rho_1,\rho_2}(u,-1)(t) = -1$ *if* $\rho_2 > \psi(\rho_1)$*;*
(IV) if $\min_{[0,t]}(u) \leqslant \rho_1 < u(t)$*, there exists* $\psi(\rho_1) \in (\rho_1,\infty)$*, such that* $h_{\rho_1,\rho_2}(u,1)(t) = 1$ *if* $\rho_2 \in (\rho_1,\psi(\rho_1)]$*,* $h_{\rho_1,\rho_2}(u,1)(t) = -1$ *if* $\rho_2 > \psi(\rho_1)$*.*
(V) The function ψ *defined in (III) or (IV), the domain of which we indicate with* $D(\psi)$*, is non increasing; moreover,* $\forall \rho_1 \in D(\psi)$ *there exists* $\delta \in \mathbb{R}^+$ *such that* $[\rho_1,\rho_1+\delta) \subseteq D(\psi)$ *and* ψ *is constant in this interval.*
(VI) As a consequence, ψ *has an at most countable range and is right continuous.*

Proof. The proof is elementary, employing Proposition 1 (IV)-(V).

□

Analogously, we have:

LEMMA 3. *Let* $u \in C([0,T])$*,* $\xi \in \{-1,1\}$*,* $t \in [0,T]$*. Then:*
(I) if $\rho_2 \leqslant u(t)$*,* $h_{\rho_1,\rho_2}(u,\xi)(t) = 1\ \forall \rho_1 \in (\infty,\rho_2)$*;*
(II) If $\max_{[0,t]}(u) < \rho_2$*,* $h_{\rho_1,\rho_2}(u,-1)(t) = -1\ \forall \rho_1 \in (\infty,\rho_2)$*;*
(III) if $u(t) < \rho_2 \leqslant max_{[0,t]}(u)$*, there exists* $\chi(\rho_2) \in (-\infty,\rho_2)$*, such that* $h_{\rho_1,\rho_2}(u,-1)(t) = 1$ *if* $\rho_1 \in (\infty,\chi(\rho_2))$*,* $h_{\rho_1,\rho_2}(u,-1)(t) = -1$ *if* $\chi(\rho_2) \leqslant \rho_1 < \rho_2$*;*
(IV) if $u(t) < \rho_2$*, there exists* $\chi(\rho_2) \in (-\infty,\rho_2)$*, such that* $h_{\rho_1,\rho_2}(u,1)(t) = 1$ *if* $\rho_1 \in (\infty,\chi(\rho_2))$*,* $h_{\rho_1,\rho_2}(u,1)(t) = -1$ *if* $\chi(\rho_2) \leqslant \rho_1 < \rho_2$*;*
(V) The function χ *defined in (III) or (IV), the domain of which we indicate with* $D(\chi)$*, is non increasing; moreover,* $\forall \rho_2 \in D(\chi)$ *there exists* $\delta \in \mathbb{R}^+$ *such that* $(\rho_2-\delta,\rho_2] \subseteq D(\chi)$ *and* χ *is constant in this interval.*
(VI) As a consequence, χ *has an at most countable range and is left continuous.*

REMARK 2. From Lemma 2 it is clear that, if $\rho_1 < \rho_2 < \rho_2'$ and $h_{\rho_1,\rho_2}(u,\xi)(t) \neq h_{\rho_1,\rho_2'}(u,\xi)(t)$, $\rho_1 \in D(\psi)$ and $\rho_2 \leqslant \psi(\rho_1) \leqslant \rho_2'$. Analogously, From Lemma 3 it follows that, if $\rho_1 < \rho_1' < \rho_2$ and $h_{\rho_1,\rho_2}(u,\xi)(t) \neq h_{\rho_1',\rho_2}(u,\xi)(t)$, $\rho_2 \in D(\chi)$ and $\rho_1 \leqslant \chi(\rho_2) \leqslant \rho_1'$.

Given $u \in C([0,T])$, $\xi \in \{-1,1\}$, $t \in [0,T]$, we get the functions $\psi = \psi(u,\xi,t)$ and $\chi = \chi(u,\xi,t)$ (which may have empty domain). Therefore we can consider the sets Ψ and X of the functions $\psi(u,\xi,t)$ and $\chi(u,\xi,t)$:

$$\Psi := \{\psi(u,\xi,t) : u \in C([0,t]), \xi \in \{-1,1\}, t \in [0,T]\}, \tag{4}$$

$$X := \{\chi(u,\xi,t) : u \in C([0,t]), \xi \in \{-1,1\}, t \in [0,T]\}. \tag{5}$$

If $\psi : D(\psi) \to \mathbb{R}$ is in Ψ and $\delta \in \mathbb{R}^+$, we set

$$\psi_\delta := \{(\rho_1, \rho_2) \in \mathscr{P} : \rho_1 \in D(\psi), \psi(\rho_1) - \delta \leqslant \rho_2 \leqslant \psi(\rho_1) + \delta\}. \tag{6}$$

Analogously, if $\chi : D(\chi) \to \mathbb{R}$ is in X and $\delta \in \mathbb{R}^+$, we set

$$\chi_\delta := \{(\rho_1, \rho_2) \in \mathscr{P} : \rho_2 \in D(\chi), \chi(\rho_2) - \delta \leqslant \rho_1 \leqslant \chi(\rho_2) + \delta\}. \tag{7}$$

REMARK 3. As example, we take $u : [0,T] \to \mathbb{R}$, $u(t) = c$, with $c \in \mathbb{R}$. We have $D(\psi(u,1,t)) = \emptyset$, $D(\psi(u,-1,t)) = (-\infty, c)$, $\psi(u,-1,t)(\rho_1) = c$, $D(\chi(u,1,t)) = (c,\infty)$, $\chi(u,1,t)(\rho_2) = c$, $D(\chi(u,-1,t)) = \emptyset$.

Now we fix in $\mathscr{P}$ a bounded positive Borel measure μ and a Borel measurable function ξ, with values in $\{-1,1\}$. Given $u \in C([0,T])$ and $t \in [0,T]$, the function $\rho \to h_\rho(u, \xi_\rho)(t)$ is Borel and bounded in $\mathscr{P}$. It follows that we can define the Preisach operator

$$\mathscr{H}(u,\xi)(t) := \int_{\mathscr{P}} h_\rho(u,\xi_\rho)(t)\,d\mu(\rho). \tag{8}$$

We set

$$\mathscr{P}_\pm := \{\rho \in \mathscr{P} : h_\rho = \pm 1\}. \tag{9}$$

Then

$$\mathscr{H}(u,\xi)(t) = \int_{\mathscr{P}^+} h_\rho(u,1)(t)\,d\mu(\rho) + \int_{\mathscr{P}^-} h_\rho(u,-1)(t)\,d\mu(\rho).$$

Clearly $\mathscr{H}(u,\xi)$ is bounded in $[0,T]$, as

$$|\mathscr{H}(u,\xi)(t)| \leqslant \mu(\mathscr{P}) \quad \forall t \in [0,T].$$

From Proposition 1 (VII)-(VIII), we immediately obtain:

PROPOSITION 2. *Let μ and ξ as above. Then:*
(I) If $\phi : [0,T] \to [0,T]$ is an increasing homeomorphism of $[0,T]$ into itself, then $\mathscr{H}(u \circ \phi, \xi) = \mathscr{H}_\rho(u,\xi) \circ \phi$.
(II) If $u, v \in C([0,T])$ and, for some $t \in [0,T]$, $u_{|[0,t]} = v_{|[0,t]}$, then $\mathscr{H}(u,\xi)_{|[0,t]} = \mathscr{H}(v,\xi)_{|[0,t]}$.

We show that, under suitable conditions, $\mathscr{H}(u,\xi) \in C([0,T])$ $\forall u \in C([0,T])$:

THEOREM 3. *Suppose that μ is such that, $\forall r \in \mathbb{R}$,*

$$\mu((\{r\} \times (r,\infty)) \cup ((-\infty, r) \times \{r\})) = 0. \tag{10}$$

Then $\mathscr{H}(\cdot,\xi)$ maps $C([0,T])$ into itself.

Proof. Let $u \in C([0,T])$. Clearly, if $t, t+h \in [0,T]$,

$$|\mathscr{H}(u,\xi)(t+h) - \mathscr{H}(u,\xi)(t)|$$

$$\leqslant \int_{\mathscr{P}} (|h_\rho(u,1)(t+h) - h_\rho(u,1)(t)| + |h_\rho(u,-1)(t+h) - h_\rho(u,-1)(t)|) d\mu(\rho)$$

If, for some $\rho \in \mathscr{P}$, $h_\rho(u,1)(t+h) \neq h_\rho(u,1)(t)$ or $h_\rho(u,-1)(t+h) \neq h_\rho(u,-1)(t)$, necessarily $u([t,t+h]) \cap \{\rho_1,\rho_2\} \neq \emptyset$. Let $\delta \in \mathbb{R}^+$, be such that $u([t,t+h]) \subseteq [u(t) - \delta, u(t)+\delta]$. Then

$$|\mathscr{H}(u,\xi)(t+h) - \mathscr{H}(u,\xi)(t)| \leqslant 4\mu\{(\rho_1,\rho_2) \in \mathscr{P} : [u(t)-\delta, u(t)+\delta] \cap \{\rho_1,\rho_2\} \neq \emptyset\}.$$

As

$$\bigcap_{\delta>0} \{(\rho_1,\rho_2) \in \mathscr{P} : [u(t)-\delta, u(t)+\delta] \cap \{\rho_1,\rho_2\} \neq \emptyset\}$$

$$= \{u(t)\} \times (u(t),\infty)) \cup ((-\infty, u(t)) \times \{u(t)\}),$$

from standard properties of measures, we deduce that

$$\lim_{\delta \to 0} \mu(\{(\rho_1,\rho_2) \in \mathscr{P} : [u(t)-\delta, u(t)+\delta] \cap \{\rho_1,\rho_2\} \neq \emptyset\})$$

$$= \mu(\{u(t)\} \times (u(t),\infty)) \cup ((-\infty, u(t)) \times \{u(t)\}) = 0,$$

which implies the conclusion. □

Now we give conditions ensuring that oscillations are "preserved".

THEOREM 4. *Suppose that μ is such that for certain $L \in \mathbb{R}^+$, $\alpha \in (0,1]$, $\forall r \in \mathbb{R}$, $\forall \delta \in \mathbb{R}^+$,*

$$\mu(\{(\rho_1,\rho_2) \in \mathscr{P} : \{\rho_1,\rho_2\} \cap [r-\delta, r+\delta] \neq \emptyset\}) \leqslant L\delta^\alpha. \tag{11}$$

Then, $\forall u \in C([0,T])$, $\forall a,b \in [0,T]$,

$$os(\mathscr{H}(u,\xi); a,b) \leqslant 4L \cdot os(u;a,b)^\alpha.$$

Proof. Let $a \leqslant s \leqslant t \leqslant b$. Then, as in the proof of Theorem 3,

$$|\mathscr{H}(u,\xi)(t) - \mathscr{H}(u,\xi)(s)|$$

$$\leqslant \int_{\mathscr{P}} (|h_\rho(u,1)(t) - h_\rho(u,1)(s)| + |h_\rho(u,-1)(t) - h_\rho(u,-1)(s)|) d\mu(\rho)$$

If, for some ρ, $|h_\rho(u,1)(t) - h_\rho(u,1)(s)| + |h_\rho(u,-1)(t) - h_\rho(u,-1)(s)| \neq 0$, necessarily $u([s,t]) \cap \{\rho_1,\rho_2\} \neq \emptyset$, so that, a fortiori, as $u([s,t]) \subseteq [u(a)-\delta, u(a)+\delta]$, with $\delta = os(u;a,b)$, we have

$$|\mathscr{H}(u,\xi)(t) - \mathscr{H}(u,\xi)(s)|$$

$$\leqslant 4\mu(\{(\rho_1,\rho_2) \in \mathscr{P} : \{\rho_1,\rho_2\} \cap [u(a)-\delta, u(a)+\delta] \neq \emptyset\}) \leqslant 4L\delta^\alpha.$$

The conclusion follows.

□

So from Theorems 1 and 2, we deduce the following

COROLLARY 2. *Assume that the assumptions of Theorem 6 are satisfied, for some* $\alpha \in (0,1]$. *Then*

(I) $\forall p \in (1,\infty]$, $\forall \beta \in (\frac{1}{p},1)$, $\mathscr{H}(\cdot,\xi)$ *maps* $W^{\beta,p}((0,T))$ *into* $W^{\alpha\beta,\frac{p}{\alpha}}((0,T))$*; moreover, there exists* $C \in \mathbb{R}^+$ *(depending on* μ, ξ, β, p, T*) such that*

$$[\mathscr{H}(u,\xi)]_{W^{\alpha\beta,\frac{p}{\alpha}}((0,T))} \leqslant C\|u\|^{\alpha}_{W^{\beta,p}((0,T))}.$$

In case $\alpha = 1$,

(II) $\forall p \in [1,\infty]$, $\mathscr{H}(\cdot,\xi)$ *maps* $W^{1,p}((0,T))$ *into itself;*

(III) $|\mathscr{H}(u,\xi)'(t)| \leqslant C|u'(t)|$ *a. e. in* $(0,T)$.

(IV) $\mathscr{H}(\cdot,\xi)$ *maps* $C([0,T]) \cap BV((0,T))$ *into itself. Moreover,*

$$V_0^T(\mathscr{H}(u,\xi)) \leqslant CV_0^T(u).$$

In applications, it is often important to know that a certain nonlinear operator enjoys certain regularity properties, one of which is the fact that it is Lipschitz continuous. So we exhibit a sufficient condition, in order that $\mathscr{H}(\cdot,\xi)$ is Lipschitz continuous in $C([0,T])$.

THEOREM 5. *Suppose that the measure* μ *is such that there exists* $L \in \mathbb{R}^+$*, so that, if* $\psi \in \Psi$, $\chi \in X$, $\delta \in \mathbb{R}^+$,

$$\mu(\psi_\delta \cup \chi_\delta) \leqslant L\delta. \tag{12}$$

Then $\mathscr{H}(\cdot,\xi)$ *is Lipschitz continuous from* $C([0,T])$ *into itself.*

Proof. By Remark 3, $\forall r \in \mathbb{R}$, $\forall k \in \mathbb{N}$,

$$\mu(\{(\rho_1,\rho_2) \in \mathscr{P} : r - 1/k \leqslant \rho_1 \leqslant r + 1/k, \rho_2 > r\}$$

$$\cup \{(\rho_1,\rho_2) \in \mathscr{P} : r - 1/k \leqslant \rho_2 \leqslant r + 1/k, \rho_1 < r\}) \leqslant Lk^{-1}.$$

Letting k go to ∞, we deduce that μ satisfies condition (10). So we know from Theorem 3 that $\mathscr{H}(\cdot,\xi)$ maps $C([0,T])$ into itself.

Next, let $u \in C([0,T])$ and let $\delta \in \mathbb{R}^+$. We estimate, for $t \in [0,T]$, $|\mathscr{H}(u+\delta,\xi)(t) - \mathscr{H}(u,\xi)(t)|$. By Proposition 1(III), we have

$$|\mathscr{H}(u+\delta,\xi)(t) - \mathscr{H}(u,\xi)(t)|$$

$$\leqslant \int_{\mathscr{P}} |h_{\rho_1-\delta,\rho_2-\delta}(u,1)(t) - h_{\rho_1-\delta,\rho_2}(u,1)(t)|d\mu(\rho)$$

$$+ \int_{\mathscr{P}} |h_{\rho_1-\delta,\rho_2}(u,1)(t) - h_{\rho_1,\rho_2}(u,1)(t)|d\mu(\rho)$$

$$+ \int_{\mathscr{P}} |h_{\rho_1-\delta,\rho_2-\delta}(u,-1)(t) - h_{\rho_1-\delta,\rho_2}(u,-1)(t)|d\mu(\rho)$$

$$+ \int_{\mathscr{P}} |h_{\rho_1-\delta,\rho_2}(u,-1)(t) - h_{\rho_1,\rho_2}(u,-1)(t)|d\mu(\rho).$$

Assume that

$$|h_{\rho_1-\delta,\rho_2-\delta}(u,1)(t) - h_{\rho_1-\delta,\rho_2}(u,1)(t)|$$

$$= |h_{\rho_1,\rho_2}(u+\delta,1)(t) - h_{\rho_1,\rho_2+\delta}(u+\delta,1)(t)| \neq 0.$$

Therefore,

$$\rho_2 \leqslant \psi(u+\delta,1,t)(\rho_1) \leqslant \rho_2 + \delta.$$

We deduce

$$\int_{\mathscr{P}} |h_{\rho_1-\delta,\rho_2-\delta}(u,1)(t) - h_{\rho_1-\delta,\rho_2}(u,1)(t)| d\mu(\rho) \leqslant 2L\delta.$$

Treating analogously the other summands, we obtain

$$|\mathscr{H}(u+\delta,\xi)(t) - \mathscr{H}(u,\xi)(t)| \leqslant 8L\delta, \quad \forall t \in [0,T].$$

Replacing u with $u-\delta$, we obtain also

$$|\mathscr{H}(u,\xi)(t) - \mathscr{H}(u-\delta,\xi)(t)| \leqslant 8L\delta, \quad \forall t \in [0,T].$$

Finally, let $v \in C([0,T])$. We set $\delta := \|u-v\|_{C([0,T])}$. Then

$$u(t) - \delta \leqslant v(t) \leqslant u(t) + \delta \quad \forall t \in [0,T].$$

So, by Proposition 1(II), we have

$$|\mathscr{H}(u,\xi)(t) - \mathscr{H}(v,\xi)(t)|$$

$$\leqslant \mathscr{H}(u+\delta,\xi)(t) - \mathscr{H}(u-\delta,\xi)(t)$$

$$\leqslant |\mathscr{H}(u,\xi)(t) - \mathscr{H}(u-\delta,\xi)(t)| + |\mathscr{H}(u+\delta,\xi)(t) - \mathscr{H}(u,\xi)(t)| \leqslant 8L\delta$$

$$= 8L\|u-v\|_{C([0,T])}.$$

The proof is complete.

□

REMARK 4. A sufficient condition implying (10) is $\mu = kd\rho$, with $k \in L^1(\mathscr{P})$. Assume, moreover, that, for some $p \in (1,\infty)$,

$$\int_{\mathbb{R}} \Big(\int_{(\rho_1,\infty)} k(\rho_1,\rho_2)^p d\rho_2\Big)^{1/p} d\rho_1 + \int_{\mathbb{R}} \Big(\int_{(-\infty,\rho_2)} k(\rho_1,\rho_2)^p d\rho_1\Big)^{1/p} d\rho_2 < \infty. \tag{13}$$

Let $r \in \mathbb{R}$, $\delta \in \mathbb{R}^+$. Then, identifying k with its trivial extension to $\mathbb{R}^2$, we have

$$\mu(\{(\rho_1,\rho_2) \in \mathscr{P} : \{\rho_1,\rho_2\} \cap [r-\delta, r+\delta] \neq \emptyset\})$$

$$\leqslant \textstyle\int_{\mathbb{R}} (\int_{r-\delta}^{r+\delta} k(\rho_1,\rho_2) d\rho_2) d\rho_1 + \int_{\mathbb{R}} (\int_{r-\delta}^{r+\delta} k(\rho_1,\rho_2) d\rho_1) d\rho_2$$

$$\leqslant \delta^{\alpha} 2^{1-1/p} [\textstyle\int_{\mathbb{R}} (\int_{r-\delta}^{r+\delta} k(\rho_1,\rho_2)^p d\rho_2)^{1-1/p} d\rho_1 + \int_{\mathbb{R}} (\int_{r-\delta}^{r+\delta} k(\rho_1,\rho_2)^p d\rho_1)^{1-1/p} d\rho_2],$$

with

$$\alpha = 1 - \frac{1}{p}. \tag{14}$$

So (11) holds.

Moreover, assume that there exist $K_1, K_2 \in L^1(\mathbb{R})$ such that

$$k(\rho_1,\rho_2) \leqslant \min\{K_1(\rho_1), K_2(\rho_2)\} \quad \text{a.e. in } \mathscr{P}. \tag{15}$$

Let $\psi \in \Psi$, $\delta \in \mathbb{R}^+$. Then

$$\psi_\delta \subseteq \cup_{j\in\mathbb{N}}(I_j \times [\beta_j - \delta, \beta_j + \delta]),$$

with $\{I_j : j \in \mathbb{N}\}$ pairwise disjoint intervals in $\mathbb{R}$, $\beta_j \in \mathbb{R}$. We deduce that

$$\mu(\psi_\delta) = \int_{\psi_\delta} k(\rho_1,\rho_2)d\rho_1 d\rho_2 \leqslant 2\delta \sum_{j\in\mathbb{N}} \int_{I_j} K_1(\rho_1)d\rho_1 \leqslant 2\delta \int_{\mathbb{R}} K_1(\rho_1)d\rho_1,$$

so that the assumptions of Theorem 5 are satisfied. We observe that (15) implies also the validity of (11) with $\alpha = 1$.

REMARK 5. We have assumed that μ is a real positive measure. We can replace this condition with the less restrictive assumption that μ is a real measure with bounded variation. Then Theorem 3, Theorem 6, Corollary 2, Theorem 5 can be extended to this more general situation, replacing the conditions (10), (11), (12) with corresponding conditions involving the variation measure $|\mu|$. The proofs can be easily adapted to this more general situation, employing Hahn's decomposition $\mu = \mu_+ - \mu_-$, with $\mu_\pm$ positive measures.

4. Generalized Plays

Another class of operators allowing applications of the results in Section 2 are Generalized Plays. The simplest case, with $\lambda_r(u) \equiv u$ and $\lambda_l(u) \equiv u + h$, with $h \in \mathbb{R}^+$, is known as "Ordinary Play" and models the interaction between the positions of a piston and of a cylinder (see [1], Chapter 1.2). Broader treatments of this class are given in [1], Chapter 2, and in [3], Chapter VI. We introduce the situation described in [3]. In [1] a slightly more general case is considered. So we introduce the assumption:

(AB) $\lambda_l, \lambda_r \in C(\mathbb{R})$, *they are nondecreasing and* $\lambda_r(u) \leqslant \lambda_l(u)$, $\forall u \in \mathbb{R}$.

We set

$$\Omega := \{(u,x) \in \mathbb{R}^2 : \lambda_r(u) \leqslant x \leqslant \lambda_l(u)\}. \tag{1}$$

Let $u \in C([a,b])$, monotonic, with $a, b \in \mathbb{R}$, $a \leqslant b$ and let $x_0 \in \mathbb{R}$ be such that $(u(a), x_0) \in \Omega$. Then, we set

$$W(a,x_0,u)(t) := \begin{cases} \max\{x_0, \lambda_r(u(t))\} & \text{if } u \text{ is nondecreasing,} \\ \min\{x_0, \lambda_l(u(t))\} & \text{if } u \text{ is nonincreasing.} \end{cases} \tag{2}$$

REMARK 6. Observe that $(u(t), W(a,x_0,u)(t)) \in \Omega$ $\forall t \in [a,b]$. In fact, this is obviously true if $W(a,x_0,u)(t) \in \{\lambda_r(u(t)), \lambda_l(u(t))\}$. Moreover, if, for example, u is nondecreasing, in case $W(a,x_0,u)(t)) = x_0$,

$$\lambda_r(u(t)) \leqslant x_0 \leqslant \lambda_l(u(a)) \leqslant \lambda_l(u(t)).$$

Observe also that, in case u is constant, from (2) we obtain

$$W(a,x_0,u)(t) = x_0, \quad \forall t \in [a,b].$$

In general, $W(a,x_0,u)$ is monotonic of the same type of u.

Next, we extend the previous definition to the case that $u \in C([a,b])$ and is piecewise monotonic. Assume that

$$a = t_0 < t_1 < \cdots < t_N = b,$$

and, for each $j \in \{1,\ldots,N\}$, $u_{|[t_{j-1},t_j]}$ is monotonic. Then, if $x_0 \in \mathbb{R}$, and $(u(a),x_0) \in \Omega$, we can define $W(a,x_0,u)$ recursively in each interval $[t_{j-1},t_j]$, setting

$$W(a,x_0,u)(t) := \begin{cases} W(a,x_0,u_{|[t_0,t_1]})(t) & \text{if } t \in [t_0,t_1], \\ W(t_j, W(a,x_0,u)(t_j), u_{|[t_j,t_{j+1}]})(t) & \text{if } t \in [t_j,t_{j+1}], \\ & j \in \{1,\ldots,N-1\}, \end{cases} \tag{3}$$

where, of course, we employ (2). Observe that $W(t_j, W(a,x_0,u)(t_j), u_{|[t_j,t_{j+1}]})(t)$ is well defined by Remark 6. We leave to the reader the following

PROPOSITION 3. *Let $-\infty < a \leqslant b < \infty$, $x_0 \in \mathbb{R}$, $u,v \in C([a,b])$, piecewise monotonic and such that $(u(a),x_0) \in \Omega$. Then:*
(I) $W(a,x_0,u)$, defined in (3), is independent of the choice of the decomposition $\{a = t_0 < t_1 < \cdots < t_N \leqslant b\}$;
(II) $W(a,x_0,u)([a,b]) \subseteq \{x_0\} \cup (\lambda_r \circ u)([a,b]) \cup (\lambda_l \circ u)([a,b])$.
(III)$W(a,x_0,u) \in C([a,b])$;
(IV) if $a \leqslant \tau \leqslant t \leqslant b$,

$$W(a,x_0,u)(t) = W(\tau, W(a,x_0,u)(\tau), u_{|[\tau,b]})(t).$$

(V) If $\phi : [a,b] \to [a,b]$ is an increasing homeomorphism of $[a,b]$ into itself, then $W(a,x_0,u\circ\phi) = W(a,x_0,u)\circ\phi$.
(VI) If $u,v \in C([a,b])$ and, for some $t \in [a,b]$, $u_{|[a,t]} = v_{|[a,t]}$, then $W(a,x_0,u)_{|[a,t]} = W(a,x_0,v)_{|[a,t]}$.

REMARK 7. (V) and (VI) together state that W is a hysteresis operator (see Remark 1).

PROPOSITION 4. *Let* $-\infty < a < b < \infty$, $u,v \in C([a,b])$ *piecewise monotonic, such that* $u(t) \leqslant v(t)$ $\forall t \in [a,b]$, $x_0, x_1 \in \mathbb{R}$, $x_0 \leqslant x_1$, *so that* $(u(a), x_0)$, $(v(a), x_1)$ *are in* Ω. *Then*

$$W(a,x_0,u)(t) \leqslant W(a,x_1,v)(t), \quad \forall t \in [a,b]. \tag{4}$$

Proof. The core of the proof consists in showing that (4) holds in some right neighborhood of a in case u and v are monotonic. This is almost obvious, except in the case that $x_0 = x_1$, u is nondecreasing and v is nonincreasing. In this case, $W(a,x_0,u)(t) = W(a,x_1,v)(t) = x_0$ in some right neighborhood of a. We limit ourselves to show that $W(a,x_0,u)(t) = x_0$ in some right neighborhood of a. This is clear in case $\lambda_r(u(a)) < x_0$. So we assume that $\lambda_r(u(a)) = x_0$. If $u(a) = v(a)$, then $u(t) = v(t) = u(a)$ $\forall t \in [a,b]$, so that $W(a,x_0,u)(t) = x_0$ $\forall t \in [a,b]$. Finally, assume that $\lambda_r(u(a)) = x_0$ and $u(a) < v(a)$. Then, $u(t) \leqslant v(t) \leqslant v(a)$ $\forall t \in [a,b]$. So

$$x_0 \leqslant \lambda_r(u(t)) \leqslant \lambda_r(v(a)) \leqslant x_0$$

because $(v(a), x_0) \in \Omega$. We conclude that $W(a,x_0,u)(t) = x_0$.

Now we prove the general statement. We set

$$A := \{t \in [a,b] : W(a,x_1,v)(t) < W(a,x_0,u)(t)\}.$$

We assume, by contradiction, that $A \neq \emptyset$ and we put $\tau := \inf(A)$. Then, clearly, by continuity, $a \leqslant \tau < b$, and, from $x_0 \leqslant x_1$, $W(a,x_0,u)(t) \leqslant W(a,x_1,v)(t)$ $\forall t \in [a,\tau]$, $W(a,x_0,u)(\tau) = W(a,x_1,v)(\tau)$. By Proposition 3 (IV), we have, if $t \in [\tau,b]$,

$$W(a,x_0,u)(t) = W(\tau, W(a,x_0,u)(\tau),$$

$$u_{|[\tau,t]})(t), W(a,x_0,v)(t) = W(\tau, W(a,x_0,u)(\tau), v_{|[\tau,t]})(t).$$

So, from the first part of the proof, we deduce that $W(a,x_0,u)(t) \leqslant W(a,x_0,v)(t)$ for t in some right neighbourhood of τ, in contradiction with the definition of τ. □

Now we introduce the following notation: let I be an interval in $\mathbb{R}$, $f \in C(I)$, $\alpha, \beta \in I$, with $\alpha \leqslant \beta$, $\rho \in \mathbb{R}^+$. We set

$$\omega(f,\alpha,\beta,\rho) := \sup_{t,s\in[\alpha,\beta], |t-s|\leqslant\rho} |f(t) - f(s)|. \tag{5}$$

We have:

LEMMA 4. *Let* $u \in C([a,b])$ *be piecewise monotonic and let* $\rho \geqslant 0$, $x_0, x_1 \in \mathbb{R}$, *with* $x_0 \leqslant x_1$, $(u(a), x_0)$ *and* $(u(a)+\rho, x_1)$ *in* Ω. *Then*, $\forall t \in [a,b]$,

$$|W(a,x_1,u+\rho)(t) - W(a,x_0,u)(t)|$$

$$\leqslant \max\{x_1 - x_0, \omega(\lambda_r, \min_{[a,b]} u, \max_{[a,b]} u + \rho, \rho), \omega(\lambda_l, \min_{[a,b]} u, \max_{[a,b]} u + \rho, \rho)\}. \tag{6}$$

Proof. Let M be the second term in (6). We assume that $a = t_0 < t_1 < \cdots < t_N$ and u is monotonic in each interval $[t_0, t_1], \ldots, [t_{N-1}, t_N]$. We observe firstly that, by Proposition 4, $W(a, x_1, u+\rho)(t) - W(a, x_0, u)(t) \geqslant 0 \ \forall t \in [a, b]$. We shall show that $W(a, x_1, u+\rho)(t) - W(a, x_0, u)(t) \leqslant M \ \forall t \in [a, b]$. We start by proving this inequality if $t \in [t_0, t_1]$, assuming (for example) that in this interval u is nondecreasing. Then, if $t \in [t_0, t_1]$,

$$W(a, x_1, u+\rho)(t) - W(a, x_0, u)(t) = \max\{x_1, \lambda_r(u(t)+\rho)\} - \max\{x_0, \lambda_r(u(t))\}$$

$$\leqslant \max\{x_1 - x_0, \lambda_r(u(t)+\rho) - \lambda_r(u(t))\} \leqslant M.$$

We assume that (6) holds if $t \leqslant t_j$, with $1 \leqslant j < N$. Then, it $t \in [t_j, t_{j+1}]$ and u is (say) non increasing in this interval, then

$$W(a, x_1, u+\rho)(t) - W(a, x_0, u)(t)$$

$$= \min\{W(a, x_1, u+\rho)(t_j), \lambda_l(u(t)+\rho)\} - \min\{W(a, x_0, u)(t_j), \lambda_l(u(t))\}$$

$$\leqslant \max\{W(a, x_1, u+\rho)(t_j) - W(a, x_0, u)(t_j), \lambda_l(u(t)+\rho) - \lambda_l(u(t))\} \leqslant M.$$

□

REMARK 8. Employing Lemma 4 in case $\rho = 0$, we get

$$|W(a, x_1, u)(t) - W(a, x_0, u)(t)| \leqslant x_1 - x_0 \quad \forall t \in [a, b]. \tag{7}$$

We deduce the following

PROPOSITION 5. *Let $u, v \in C([a, b])$ be piecewise monotonic, $x_0, x_1 \in \mathbb{R}$, with $(u(a), x_0), (v(a), x_1) \in \Omega$. We set*

$$\rho := \|u - v\|_{C([a,b])}.$$

Then

$$\|W(a, x_1, v) - W(a, x_0, u)\|_{C([a,b])}$$

$$\leqslant 2\max\{|x_1 - x_0|, \omega(\lambda_r, \min_{[a,b]} u - \rho, \max_{[a,b]} u + \rho, \rho),$$

$$\omega(\lambda_l, \min_{[a,b]} u - \rho, \max_{[a,b]} u + \rho, \rho)\}.$$

Proof. We set

$$x_2 := \max\{x_0, x_1, \lambda_r(u(a)+\rho)\},$$

$$x_3 := \min\{x_0, x_1, \lambda_l(u(a)-\rho)\}.$$

Then, $x_3 \leqslant x_2$, $(u(a)+\rho, x_2), (u(a)-\rho, x_3) \in \Omega$, as,

$$\lambda_r(u(a)+\rho) \leqslant x_2 \leqslant \max\{\lambda_l(u(a)), \lambda_l(v(a)), \lambda_r(u(a)+\rho)\} \leqslant \lambda_l(u(a)+\rho),$$

$$\lambda_r(u(a)-\rho) \leqslant \min\{\lambda_r(u(a)), \lambda_r(v(a)), \lambda_l(u(a)-\rho)\} \leqslant x_3 \leqslant \lambda_l(u(a)-\rho).$$

So, by Proposition 4 and Lemma 4, we have, $\forall t \in [a,b]$,

$$|W(a,x_1,v)(t) - W(a,x_0,u)(t)| \leqslant W(a,x_2,u+\rho)(t) - W(a,x_3,u-\rho)(t)$$

$$= [W(a,x_2,u+\rho)(t) - W(a,x_0,u)(t)] + [W(a,x_0,u)(t)] - W(a,x_3,u-\rho)(t)]$$

$$\leqslant \max\{x_2 - x_0, \omega(\lambda_r, \min_{[a,b]} u, \max_{[a,b]} u+\rho,\rho), \omega(\lambda_l, \min_{[a,b]} u, \max_{[a,b]} u+\rho,\rho)\}$$

$$+ \max\{x_0 - x_3, \omega(\lambda_r, \min_{[a,b]} u-\rho, \max_{[a,b]} u,\rho), \omega(\lambda_l, \min_{[a,b]} u-\rho, \max_{[a,b]} u,\rho)\}.$$

Moreover, in case $x_2 = \lambda_r(u(a)+\rho)$, we have

$$x_2 - x_0 \leqslant \lambda_r(u(a)+\rho) - \lambda_r(u(a)) \leqslant \omega(\lambda_r, \min_{[a,b]} u - \rho, \max_{[a,b]} u+\rho, \rho)$$

and, in case $x_3 = \lambda_l(u(a)-\rho)$,

$$x_0 - x_3 \leqslant \lambda_l(u(a)) - \lambda_l(u(a)-\rho) \leqslant \omega(\lambda_l, \min_{[a,b]} u - \rho, \max_{[a,b]} u+\rho, \rho).$$

The conclusion follows. □

COROLLARY 3. *Let $u \in C([a,b])$ and $x_0 \in \mathbb{R}$ be such that $(u(a),x_0) \in \Omega$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of continuous and piecewise monotonic functions, uniformly converging to u and $(x_n)_{n\in\mathbb{N}}$ a sequence in $\mathbb{R}$, converging to x_0, and such that $(u_n(a),x_n) \in \Omega$ $\forall n \in \mathbb{N}$. Then the sequence $(W(a,x_n,u_n))_{n\in\mathbb{N}}$ converges uniformly in $[a,b]$. The limit depends only on u and x_0.*

Proof. Let α and β be real numbers, such that $\alpha \leqslant u_n(t) \leqslant \beta$ if (say) $n \geqslant n_0$. Let $\varepsilon \in \mathbb{R}^+$ and let $\rho \in (0,1]$, be such that

$$\max\{\omega(\lambda_r, \alpha-1, \beta+1, \rho), \omega(\lambda_l, \alpha-1, \beta+1, \rho)\} \leqslant \frac{\varepsilon}{2}.$$

The existence of ρ is a consequence of the local uniform continuity of λ_r and λ_l. Then, if $\min\{m,n\} \geqslant n_0$, by Proposition 5, in case $|x_n - x_m| \leqslant \frac{\varepsilon}{2}$., we have

$$\|W(a,x_n,u_n) - W(a,x_m,u_m)\|_{C([a,b])} \leqslant \varepsilon.$$

The conclusion follows easily. □

So we are able to state and prove the following

THEOREM 6. *Let $-\infty < a \leqslant b < \infty$, $x_0 \in \mathbb{R}$, $u \in C([a,b])$ and such that $(u(a),x_0) \in \Omega$. Then:*

(I) there exists a sequence (x_n,u_n) in $\mathbb{R} \times C([a,b])$ such that $(u_n(a),x_n) \in \Omega$ $\forall n \in \mathbb{N}$, u_n is piecewise monotonic for every $n \in \mathbb{N}$ and $(x_n,u_n)_{n\in\mathbb{N}}$ converges to (x_0,u) in $\mathbb{R} \times C([a,b])$.

(II) The corresponding sequence $(W(a,x_n,u_n))_{n\in\mathbb{N}}$ converges uniformly to an element $W(a,x_0,u)$ of $C([a,b])$. Such element does not depend on the sequence $(x_n,u_n)_{n\in\mathbb{N}}$ with the declared properties.
(III) $\forall t \in [a,b]$, $W(a,x_0,u)(t) \in [\lambda_r(u(t)),\lambda_l(u(t))]$;
(IV) $W(a,x_0,u)([a,b]) \subseteq \{x_0\} \cup (\lambda_r \circ u)([a,b]) \cup (\lambda_l \circ u)([a,b])$;
(V) if $a \leqslant \tau \leqslant t \leqslant b$,

$$W(a,x_0,u)(t) = W(\tau, W(a,x_0,u)(\tau), u_{|[\tau,b]})(t);$$

(VI) let $v \in C([a,b])$ be such that $u(t) \leqslant v(t)$ $\forall t \in [a,b]$, $x_1 \in \mathbb{R}$, with $x_0 \leqslant x_1$ and $(v(a),x_1)$ in Ω. Then

$$W(a,x_0,u)(t) \leqslant W(a,x_1,v)(t), \quad \forall t \in [a,b].$$

(VII) If $\phi : [a,b] \to [a,b]$ is an increasing homeomorphism of $[a,b]$ into itself, then $W(a,x_0,u\circ\phi) = W(a,x_0,u)\circ\phi$.
(VIII) If $u,v \in C([a,b])$ and, for some $t \in [a,b]$, $u_{|[a,t]} = v_{|[a,t]}$, then $W(a,x_0,u)_{|[a,t]} = W(a,x_0,v)_{|[a,t]}$.
(IX) Let $u,v \in C([a,b])$, $x_0,x_1 \in \mathbb{R}$, with $(u(a),x_0),(v(a),x_1) \in \Omega$. We set

$$\rho := \|u-v\|_{C([a,b])}.$$

Then

$$\|W(a,x_1,v) - W(a,x_0,u)\|_{C([a,b])}$$

$$\leqslant 2\max\{|x_1-x_0|, \omega(\lambda_r, \min_{[a,b]} u - \rho, \max_{[a,b]} u + \rho, \rho),$$

$$\omega(\lambda_l, \min_{[a,b]} u - \rho, \max_{[a,b]} u + \rho, \rho)\}.$$

Proof. We prove only (I). The other statements can be proved passing to the limit in an appropriate sequence (x_n,u_n), with u_n piecewise monotonic. We take $z_n \in C([a,b])$, piecewise monotonic, such that $\|z_n - u\|_{C([a,b])} \leqslant \frac{1}{2n}$ and set $u_n := z_n - \frac{1}{n}$, in such a way that, $\forall t \in [a,b]$,

$$u_n(t) = z_n(t) - \frac{1}{n} \leqslant u(t) + \frac{1}{2n} - \frac{1}{n} < u(t),$$

and $x_n := \min\{x_0, \lambda_l(u_n(a))\}$, so that

$$\lambda_l(u_n(a)) \geqslant x_n \geqslant \min\{\lambda_r(u(a)), \lambda_l(u_n(a))\} \geqslant \lambda_r(u_n(a))$$

and $(u_n(a),x_n) \in \Omega$. □

Now, we introduce the following assumption

(AC) Let $\alpha \in (0,1]$. Then, $\forall M \in \mathbb{R}^+$ there exists $\Gamma(M) \in \mathbb{R}^+$, such that, if $u,v \in \mathbb{R}$ and $\max\{|u|,|v|\} \leqslant M$,

$$\max\{|\lambda_r(u) - \lambda_r(v)|, |\lambda_l(u) - \lambda_l(v)|\} \leqslant \Gamma(M)|u-v|^\alpha.$$

From Theorem 6 and (AC), we immediately deduce the following

COROLLARY 4. *Assume that (AC) holds. Then, if* $u, v \in C([a,b])$*,* $x_0, x_1 \in \mathbb{R}$*,* $(u(a), x_0), (v(a), x_1) \in \Omega$ *and* $\|u\|_{C([a,b])} \leqslant M$*,*

$$\|W(a,x_1,v) - W(a,x_0,u)\|_{C([a,b])}$$

$$\leqslant 2max\{|x_1 - x_0|, \Gamma(M + \|u-v\|_{C([a,b])})\|u-v\|^{\alpha}_{C([a,b])}\}.$$

Finally, we study how Generalized Plays modify oscillations. The following fact holds:

PROPOSITION 6. *(I) Assume that (AB) holds. Let* $u \in C([a,b])$ *and* $x_0 \in \mathbb{R}$ *be such that* $(u(a), x_0) \in \Omega$*. Then:*

$$os(W(a,x_0,u);a,b) \leqslant os(\lambda_r \circ u;a,b) + os(\lambda_l \circ u;a,b).$$

(III) Assume that (AC) holds. Then if $a \leqslant s \leqslant t \leqslant b$*,*

$$os(W(a,x_0,u);a,b) \leqslant C(\|u\|_{C([a,b])})os(u;s,t)^{\alpha}.$$

Proof. (I) If u is piecewise monotonic, it follows easily that, either $W(a,x_0,u)(t) = x_0$ $\forall t \in [a,b]$, or the range of $W(a,x_0,u)$ is contained in $(\lambda_r \circ u)([a,b]) \cup (\lambda_l \circ u)([a,b])$. Let $t \in [a,b)$, $h \in \mathbb{R}^+$ be such that $t < t+h \leqslant b$. Assume that $W(a,x_0,u)$ is not constant in $[t, t+h]$. We set

$$t_0 := \inf\{s \in [t,t+h] : W(a,x_0,u)(s) \neq W(a,x_0,u)(t)\},$$

$$t_1 := \sup\{s \in [t,t+h] : W(a,x_0,u)(s) \neq W(a,x_0,u)(t+h)\}.$$

Then, $t \leqslant t_0 < t_1 \leqslant t+h$, $W(a,x_0,u)(t) = W(a,x_0,u)(t_0)$,

$$W(a,x_0,u)(t+h) = W(a,x_0,u)(t_1)$$

and

$$W(a,x_0,u)(t_0) \in \{\lambda_r(u(t_0)), \lambda_l(u(t_0))\}, \quad W(a,x_0,u)(t_1) \in \{\lambda_r(u(t_1)), \lambda_l(u(t_1))\}.$$

Suppose that (for example)

$$W(a,x_0,u)(t_0) = \lambda_r(u(t_0)) \neq \lambda_l(u(t_0)), W(a,x_0,u)(t_1) = \lambda_l(u(t_1)) \neq \lambda_r(u(t_1)).$$

We set

$$t_2 := \sup\{s \in [t_0,t_1] : W(a,x_0,u)(s) = \lambda_r(u(s))\},$$

$$t_3 := \inf\{s \in [t_2,t_1] : W(a,x_0,u)(s) = \lambda_l(u(s))\}.$$

Then $W(a,x_0,u)(t_2) = \lambda_r(u(t_2)), W(a,x_0,u)(t_3) = \lambda_l(u(t_3)), t_0 \leqslant t_2 \leqslant t_3 \leqslant t_1$. It is clear that $W(a,x_0,u)$ is constant in $[t_2,t_3]$ and so $\lambda_r(u(t_2)) = \lambda_l(u(t_3))$. Therefore,

$$|W(a,x_0,u)(t+h) - W(a,x_0,u)(t)| = |W(a,x_0,u)(t_1) - W(a,x_0,u)(t_0)|$$

$$= |\lambda_l(u(t_1)) - \lambda_r(u(t_0)| \leqslant |\lambda_l(u(t_1)) - \lambda_l(u(t_3))| + |\lambda_r(u(t_2)) - \lambda_l(u(t_0))|$$

$$\leqslant os(\lambda_r \circ u, a; b) + os(\lambda_l \circ u; a, b).$$

So the case of u piecewise monotonic is done. The general case follows by approximation.

Concerning (II), from (I) and Theorem 6(V) we have

$$os(W(a,x_0,u);s,t) \leqslant os(\lambda_r \circ u;s,t) + os(\lambda_l \circ u;s,t) \leqslant C(\|u\|_{C([a,b])})(t-s)^{\alpha}.$$

$\square$

So from Theorems 1 and 2, we deduce the following

COROLLARY 5. *Assume that the assumptions (AB) and (AC) hold, for some* $\alpha \in (0,1]$. *Let* $x_0 \in \mathbb{R}$. *Then*

(I) $\forall p \in (1,\infty]$, $\forall \beta \in (\frac{1}{p},1)$, $W(a,x_0,\cdot)$ *maps* $\{u \in W^{\beta,p}((0,T)) : (u(a),x_0) \in \Omega\}$ *into* $W^{\alpha\beta,\frac{p}{\alpha}}((a,b))$*; moreover,*

$$[W(a,x_0,u)]_{W^{\alpha\beta,\frac{p}{\alpha}}((0,T))} \leqslant C(\|u\|_{C([a,b])})\|u\|^{\alpha}_{W^{\beta,p}((0,T))}.$$

In case $\alpha = 1$,

(II) $\forall p \in [1,\infty]$, $W(a,x_0,\cdot)$ *maps* $\{u \in W^{1,p}((a,b)) : (u(a),x_0) \in \Omega\}$ *into* $W^{1,p}((a,b))$*;*

(III) moreover, $|W(a,x_0,u)'(t)| \leqslant C(\|u\|_{C([a,b])})|u'(t)|$ *a. e. in* (a,b).

(IV) $W(a,x_0,\cdot)$ *maps* $\{u \in C([a,b]) \cap BV((a,b)) : (u(a),x_0) \in \Omega\}$ *into* $C([a,b]) \cap BV((a,b))$. *Moreover,*

$$V_a^b(W(a,x_0,\cdot)) \leqslant C(\|u\|_{C([a,b])})V_a^b(u).$$

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