Rend. Sem. Mat. Univ. Politec. Torino
Vol. 71, 3–4 (2013), 505 – 534

A. Verra

SIX DIMENSIONAL PRYMS AND CONTE-MURRE THREEFOLDS

Abstract. The singular locus $\mathcal{S}$ of the universal Prym theta divisor over the Prym moduli space $\mathcal{R}_7$ of genus 7 curves is considered. An irreducible component of $\mathcal{S}$ is constructed which is unirational and dominates $\mathcal{R}_7$. This result relies on several classical constructions, related to the geometry of Enriques surfaces, offering a tour through them as a byproduct.

1. Introduction

The aim of this paper is to investigate some of the multiple relations between two, quite different, classes of complex algebraic varieties. From one side we consider Prym varieties of dimension 6 and, from the other side, the family of Enriques-Fano threefolds of genus 6.

An Enriques-Fano threefold X is a normal threefold containing an ample Cartier divisor H which is a minimal Enriques surface with at most Du Val singularities. Moreover it is assumed that X is not a generalized cone over H, that is, X is not a contraction of $\mathbf{P}(O_H \oplus O_H(H))$. The genus of X is $p := \frac{H^3}{2} + 1$.

The families of these threefolds were studied by Fano and by Godeaux in the 30's of the last century, when H is very ample. The adjunction map often provides a birational model of them which is a singular Fano threefold with special properties. In the 80's they were reconsidered in modern terms by Conte and Murre in the paper [6]. This was a starting point for a renewed long series of investigations, reaching several classification results and the sharp bound $p \leq 17$ for the genus. See for instance: [2], [3], [6], [20], [23], [17] among many other papers.

The family of threefolds of genus 6, more precisely their birational Fano models in $\mathbf{P}^5$, is the initial family studied in detail in [6]. Throughout all the paper, the members of this family will be called Conte-Murre threefolds.

From another point of view, Enriques-Fano threefolds appear also as quotients of Fano threefolds endowed with an involution having exactly 8 fixed points, [2]. Hence they are endowed with a quasi étale double covering branched at 8 points. If T is a Conte-Murre threefold, then T is a general linear section of the space $Q^{[2]} \subset |O_{\mathbf{P}^3}(2)|$, parametrizing pairs of planes of $\mathbf{P}^3$, and the covering of T is induced by the natural map $\mathbf{P}^{3^*} \times \mathbf{P}^{3^*} \to Q^{[2]}$. Note that T is embedded in $\mathbf{P}^6$ as a projectivitely normal threefold whose general hyperplane sections are smooth Enriques surfaces of degree 10. On the other hand $\mathbf{P}^6$ is the canonical space for curves of genus 7. Assume

$$C \subset T - Sing\ T \subset \mathbf{P}^6$$

is such a curve. Then the covering of T induces an étale double covering $\pi : \tilde{C} \to C$

which is defined by a non trivial 2-torsion η of Pic^0C. Hence the pair (C,η) is a Prym curve. This relates Prym curves of genus 7 and Conte-Murre threefolds.

Let us see why this relation is rich and interesting. More in general let (C,η) be a smooth Prym curve of genus g and let (P,Ξ) be its associated Prym variety. Then P is an abelian variety of dimension $g-1$, principally polarized by its theta divisor Ξ. We can comment on three related arguments from the theory of Prym varieties:

1) The study of the Prym Brill-Noether loci $P^r(C,\eta)$,

2) the Taylor expansion of Ξ at $o \in Sing\ \Xi$ and the Prym-Torelli problem,

3) the birational structure of the Prym moduli space $\mathcal{R}_g$.

1) $P^r(C,\eta)$ is the family of line bundles $\tilde{L} \in Pic^{2g-2}\tilde{C}$ such that $Nm\ \tilde{L} \cong \omega_C$, $h^0(\tilde{L}) \geq r+1$ and $h^0(\tilde{L}) = r+1 \mod 2$. Assume (C,η) is general. Then $P^r(C,\eta)$ is smooth of dimension $\rho = g-1-\binom{r+1}{2}$ if $\rho \geq 0$ and connected if $\rho \geq 1$, [25].

We can also consider the universal Prym Brill-Noether locus $\mathcal{P}^r_g$ over $\mathcal{R}_g$ that is the moduli space of triples $(C,\eta,\tilde{L})$. If $\rho \geq 0$ then $\mathcal{P}^r_g$ dominates $\mathcal{R}_g$ via the forgetful map. A unique irreducible component of $\mathcal{P}^r_g$ dominates $\mathcal{R}_g$ if $\rho \geq 1$. If $\rho = 0$, this latter result, though plausible, is not yet available in the literature.

2) Continuing with a general Prym curve, then $Sing\ \Xi$ is biregular to $P^3(C,\eta)$. A general point $o \in Sing\ \Xi$ is an ordinary double point. To give o is equivalent to give a line bundle $\tilde{L} \in Pic^{2g-2}\tilde{C}$ such that $h^0(\tilde{L}) = 4$ and $Nm\ \tilde{L} \cong \omega_C$, cfr. [1] App. C.

3) Finally various steps have been realized in the global study of $\mathcal{R}_g$. Farkas and Ludwig recently proved that $\mathcal{R}_g$ is of general type for $g \geq 14$, with the exception of $g = 15$ which is unknown, [13]. Nothing seems to be known on the Kodaira dimension of $\mathcal{R}_g$ for $9 \leq g \leq 13$. On the other hand the unirationality of $\mathcal{R}_g$ is well known for $g \leq 6$. See [15] and [21] for some general accounts on all this matter.

Coming to genus 7 the unirationality of $\mathcal{R}_7$ is known, see [14]. The case $g = 7$ is also the case where $Sing\ \Xi$ is finite for a general (C,η): Debarre proves in [7] that then $Sing\ \Xi$ consists of 16 ordinary double points. Since $Sing\ \Xi = P^3(C,\eta)$ these are the 16 elements $\tilde{L}$ of $P^3(C,\eta)$. We will see along this paper that the Petri map

$$\mu : H^0(\tilde{L}) \otimes H^0(\omega_{\tilde{C}} \otimes \tilde{L}^{-1}) \to H^0(\omega_{\tilde{C}})$$

is surjective and that $Ker\ \mu$ is the invariant eigenspace of the involution induced by $\pi : \tilde{C} \to C$. To summarize the situation, let $\mathbf{P}^3 := \mathbf{P}H^0(\tilde{L})^*$. Then $Ker\ \mu$ defines a 6-dimensional linear space $\mathbf{P}^6 \subset |\mathcal{O}_{\mathbf{P}^3}(2)|$ so that, with the previous notations,

$$C \subset T = \mathbf{P}^6 \cap Q^{[2]} \subset |\mathcal{O}_{\mathbf{P}^3}(2)|.$$

C is canonically embedded and T is a Conte-Murre threefold. Conversely, an embedding $C \subset T$ reconstructs a Prym curve (C,η) and two elements $\tilde{L}$ and $\omega_{\tilde{C}} \otimes \tilde{L}^{-1} \in P^3(C,\eta)$. Building on the geometry of this construction, and of Enriques surfaces as well, we explicitely describe an irreducible component $\mathcal{P}$ of $\mathcal{P}^3_7$ and show that

THEOREM $\mathcal{P}$ is unirational and dominates $\mathcal{R}_7$.

As an immediate consequence, we obtain a new proof that

COROLLARY $\mathcal{R}_7$ *is unirational.*

Since $\rho = 0$ the existence of other irreducible and dominant components of $\mathcal{P}^3_7$ is a priori possible. It seems plausible, with more substantial effort, to exclude them. Assuming this property, the ideal geometric picture for $g = 7$ appears to be as follows: let $C \subset \mathbf{P}^6$ be a general canonical curve. Consider the map

$$t : Sing\ \Xi / < i^* > \to T(C) := \{ \text{Conte-Murre threefolds through } C \}$$

defined by the condition $t(\tilde{L}, i^*\tilde{L}) = T$. One expects that t is bijective and that, over an open neighborhood of C in its Hilbert scheme, the monodromy of $T(C)$ is irreducible.

To conclude this introduction we want to outline a second natural step of the program started in this paper. To this purpose let us stress once more that a Conte-Murre threefold T is defined by a quadratic singularity o of the theta divisor Ξ of a general 6-dimensional Prym P. Our claim is that the Fano model of T, described in [6], is exactly the complete intersection, in the projectivized tangent space to P at o, of the quadratic and the cubic terms of the Taylor expansion of Ξ at o. We intend to inquire this claim and possibly prove it in a future paper.

In the 70's several young persons, among them the author of this paper, were introduced to beautiful algebraic varieties, like Enriques surfaces or Pryms or conic bundles, thanks to Alberto Conte and his passion for geometry. Time is passing, but the beautiful geometry learned at that time is not passing. This paper is dedicated to Alberto for his 70th birthday, with gratitude.

2. Preliminary results and basic notations

We will work over the complex field. A *Prym curve of genus g* is a pair (C, η) such that C is a smooth, integral projective curve of genus g and η is a non trivial 2-torsion element of $Pic\, C$. We fix from now on the following notations:

- $\pi : \tilde{C} \to C$ *is the étale double covering defined by* η.
- $i : \tilde{C} \to \tilde{C}$ *is the involution exchanging the two sheets of* π.

Then i is fixed point free and $\tilde{C}$ is a smooth integral curve of genus $2g-1$. As is well known the Prym variety of (C, η) is a principally polarized abelian variety of dimension $g-1$, naturally associated to (C, η). It is a pair (P, Ξ) such that P is an abelian variety of dimension $g-1$ and Ξ is a principal polarization on it. To construct P consider the Norm map $Nm : Pic\, \tilde{C} \to Pic\, C$ sending $\mathcal{O}_{\tilde{C}}(d)$ to $\mathcal{O}_C(\pi_* d)$. $\forall m \in \mathbb{Z}$ each fibre of

$$Nm : Pic^m\, \tilde{C} \to Pic^m\, C$$

is the disjoint union of two copies of the same abelian variety of dimension $g-1$. By definition this is P. To define Ξ we previously introduce the Prym Brill-Noether loci.

DEFINITION 1. *The r-th Prym Brill-Noether locus of* (C,η) *is*

$$P^r(C,\eta) := \{\tilde{L} \in Pic^{2g-2}\tilde{C} \ / \ Nm\,\tilde{L} \cong \omega_C,\ h^0(\tilde{L}) \geq r+1,\ h^0(\tilde{L}) \equiv r+1 \mod 2\}.$$

$P^r(C,\eta)$ has the scheme structure defined in [25]. Note that i^* acts on $P^r(C,\eta)$ and that $i^*\tilde{L} \cong \omega_{\tilde{C}} \otimes \tilde{L}^{-1}$. Let $\tilde{L} \in P^r(C,\eta)$, we fix the natural identification

$$H^0(\tilde{L}) := i^* H^0(\tilde{L}) = H^0(i^*\tilde{L}).$$

Then i^* acts on $H^0(\tilde{L}) \otimes H^0(\tilde{L})$ by exchanging the factors and the decomposition

$$H^0(\tilde{L}) \otimes H^0(\tilde{L}) = Sym^2 H^0(\tilde{L}) \oplus \wedge^2 H^0(\tilde{L}).$$

is the direct sum of the 1 and -1 eigenspaces of the involution i^*. Let

$$\mu_{\tilde{L}} : H^0(\tilde{L}) \otimes H^0(\tilde{L}) \to H^0(\omega_{\tilde{C}})$$

be the Petri map, that is $s \otimes t \to s i^* t$. Then $\mu_{\tilde{L}}$ is the direct sum of the maps

$$\mu^+_{\tilde{L}} : Sym^2 H^0(\tilde{L}) \to H^0(\omega_{\tilde{C}})^+ \ , \ \mu^-_{\tilde{L}} : \wedge^2 H^0(\tilde{L}) \to H^0(\omega_{\tilde{C}})^-$$

where $\pm$ denotes the $\pm$-eigenspace of i^*. We fix the identifications

$$H^0(\omega_{\tilde{C}})^+ = \pi^* H^0(\omega_C) = H^0(\omega_C) \text{ and } H^0(\omega_{\tilde{C}})^- = \pi^* H^0(\omega_C \otimes \eta) = H^0(\omega_C \otimes \eta).$$

Some frequently used conventions:
- $X \subset \mathbf{P}^n$, then $< X >$ is its linear span.
- X is not degenerate if $< X > = \mathbf{P}^n$.
- $X \subset Y$, then $I_{X/Y}$ is the ideal sheaf of X.
- V a vector space: $\mathbf{P}V$ is its projectivization, V^* the dual.
- $V^\perp := \{h \in V^* \ / \ h/V \text{ is zero}\}$.
- $\mathcal{L}$ a line bundle, $V \subset H^0(\mathcal{L})$: $|V|$ is the linear system defined by V.
- $|\mathcal{L}|$ is the linear system defined by $H^0(\mathcal{L})$.

3. Prym theory in genus 7 and Conte-Murre threefolds

From now on C will be a *general* curve of genus 7, which implies that (C,η) will be a *general* Prym curve for each η. In particular we can assume that $\omega_C \otimes \eta$ is very ample, see [5] 0.6. Moreover we can assume that ω_C is very ample and that the canonical model of C is generated by quadrics. $\tilde{C}$ has genus 13. Since C is general it follows

$$P^r(C,\eta) = \emptyset, r \geq 4$$

from Prym Brill-Noether theory, [25]. From Debarre, [7], we have

THEOREM 1. *Sing* $\Xi = P^3(C,\eta)$ *is a smooth* 0*-dimensional scheme of length* 16.

COROLLARY 1. *The forgetful map $f : \mathcal{P}^3_7 \to \mathcal{R}_7$ is generically étale of degree* 16.

Moreover each $o \in Sing\ \Xi$ is a stable singularity with respect to (C, η). This just means that o is a line bundle $\tilde{L} \in P^3(C, \eta)$ such that $Nm\ \tilde{L} \cong \omega_C$ and $h^0(\tilde{L}) = 4$. Let $V := H^0(\tilde{L})$, we want to study the Petri map

$$\mu_{\tilde{L}} : V \otimes i^* V \to H^0(\omega_{\tilde{C}}).$$

LEMMA 1. *Let (C, η) be any Prym curve of genus 7 and let $\tilde{L} \in P^3(C, \eta) - P^5(C, \eta)$. Assume that the Petri map of $\tilde{L}$ is surjective, then its Prym-Petri map is an isomorphism.*

Proof. Recall that $Coker\ \mu_{\tilde{L}}$ is the cotangent space at $\tilde{L}$ to the Brill-Noether scheme $W^3_{12}(\tilde{C}) := \{\tilde{M} \in Pic^{12}\tilde{C}\ /\ h^0(\tilde{M}) \geq 4\}$. Then $\mu_{\tilde{L}}$ surjective $\Rightarrow \tilde{L}$ is an isolated point of $W^3_{12}(\tilde{C})$, hence of $P^3(C, \eta)$. Hence $Coker\ \mu^- = 0$, that is, μ^- is an isomorphism. □

Since $\tilde{C}$ is not general, the Petri map $\mu_{\tilde{L}}$ needs not to be injective. Actually, the size of this paper is due to the steps for proving that, for a general (C, η), the subset

$$\{\tilde{L} \in P^3(C, \eta)\ /\ \mu_{\tilde{L}} \text{ is surjective}\}$$

is not empty. Next we remark that the linear system $|Im\ \mu_{\tilde{L}}|$ defines the product map

$$\phi := f_{\tilde{L}} \times (f_{\tilde{L}} \cdot i) : \tilde{C} \to \mathbf{P}^3 \times \mathbf{P}^3,$$

where $\mathbf{P}^3 := \mathbf{P}V^*$ and $f_{\tilde{L}} : \tilde{C} \to \mathbf{P}^3$ is the map defined by $\tilde{L}$. Let

$$\mathbf{P}^3 \times \mathbf{P}^3 \subset \mathbf{P}^{15} := \mathbf{P}(V^* \otimes V^*),$$

be the Segre embedding. We consider the projective involution $I : \mathbf{P}^{15} \to \mathbf{P}^{15}$ induced by the linear map $s \otimes t \to t \otimes s$. The projectivized eigenspaces of I are

$$\mathbf{P}^{5-} := \mathbf{P}(\wedge^2 V^*)\ ,\ \mathbf{P}^{9+} := \mathbf{P}(Sym^2 V^*).$$

Moreover it is clear that $i : \tilde{C} \to \tilde{C}$ is induced by I, in other words we have:

$$I \circ \phi = \phi \circ i.$$

Notice also that $\mathbf{P}^{5-}$ is the Plücker space for the Grassmannian

$$\mathbf{G} := G(1, \mathbf{P}V) \subset \mathbf{P}^{5-}$$

of pencils of planes of $\mathbf{P}^3$. On the other hand $\mathbf{P}(Sym^2 V^*)$ is the linear system

$$\mathbf{P}^9 + = |\mathcal{O}_{\mathbf{P}V}(2)|$$

of quadrics of $\mathbf{P}V = \mathbf{P}^{3^*}$. In particular this space admits the rank stratification

$$\mathbb{Q}^{[1]} \subset \mathbb{Q}^{[2]} \subset \mathbb{Q}^{[3]} \subset \mathbf{P}^{9+},$$

where $\mathbb{Q}^{[r]}$ denotes the locus of quadrics of rank $\leq r$. Let us consider the linear projections $\pi^+ : \mathbf{P}^{15} \to \mathbf{P}^{9+}$ and $\pi^- : \mathbf{P}^{15} \to \mathbf{P}^{9+}$ respectively of centers $\mathbf{P}^{5-}$ and $\mathbf{P}^{9+}$. We are interested to the commutative diagram

$$\begin{array}{ccccc}
\mathbf{P}^{5-} & \xleftarrow{\pi^-} & \mathbf{P}^{15} & \xrightarrow{\pi^+} & \mathbf{P}^{9+} \\
\uparrow & & \uparrow & & \uparrow \\
\mathbf{G} & \xleftarrow{\pi^-/\mathbf{P}^3\times\mathbf{P}^3} & \mathbf{P}^3\times\mathbf{P}^3 & \xrightarrow{\pi^+/\mathbf{P}^3\times\mathbf{P}^3} & \mathbb{Q}^{[2]} \\
\uparrow & & \uparrow \phi & & \uparrow \\
C^- & \xleftarrow{\pi^-\circ\phi} & \tilde{C} & \xrightarrow{\pi^+\circ\phi} & C^+
\end{array}$$

where the horizontal arrows are surjective and the top vertical ones injective, cfr. [22]. We observe that the quotient map of the involution $I/\mathbf{P}^3 \times \mathbf{P}^3$ is exactly

$$\pi^+/\mathbf{P}^3 \times \mathbf{P}^3 : \mathbf{P}^3 \times \mathbf{P}^3 \to \mathbb{Q}^{[2]}.$$

$\mathbb{Q}^{[2]}$ is a determinantal 6-fold of degree 10 parametrizing pairs of planes of $\mathbf{P}V$. As is well known the previous map is a quasi étale 2:1 cover of $\mathbb{Q}^{[2]}$, branched on $\mathbb{Q}^{[1]}$. Note that $\phi^* O_{\mathbf{P}^{15}}(1) \cong \omega_{\tilde{C}}$, because ϕ is defined by $Im\,\mu_{\tilde{L}} \subseteq H^0(\omega_{\tilde{C}})$. Notice also that

$$\tilde{L} \cong \phi^* O_{\mathbf{P}^3\times\mathbf{P}^3}(1,0) \text{ and } i^*\tilde{L} \cong O_{\mathbf{P}^3\times\mathbf{P}^3}(0,1).$$

Moreover $\pi^- \cdot \phi$ and $\pi^+ \cdot \phi$ are respectively defined by $|Im\,\mu^-_{\tilde{L}}|$ and $|Im\,\mu^+_{\tilde{L}}|$. In particular it follows that the linear spans $< C^+ >$ and $< C^- >$ are the orthogonal spaces:

$$< C^+ >= \mathbf{P}(Im\,\mu^+_{\tilde{L}})^\perp \subset \mathbf{P}H^0(\omega_C)^* \text{ and } < C^- >= \mathbf{P}(Im\,\mu^-_{\tilde{L}})^\perp \subset \mathbf{P}H^0(\omega_C \otimes \eta)^*.$$

Therefore we have:

LEMMA 2. *Assume that $\mu_{\tilde{L}}$ is surjective, then the projective models*

$$C^+ \subset < C^+ > \text{ and } C^- \subset < C^- >$$

are exactly the canonical and the Prym canonical embeddings of (C, η).

Proof. By lemma 3.3 the surjectivity of $\mu_{\tilde{L}}$ implies that the Prym-Petri map $\mu^-_{\tilde{L}}$ is an isomorphism. Then, since i^* acts on $Ker\,\mu_{\tilde{L}}$, it follows that $Ker\,\mu_{\tilde{L}} = Ker\,\mu^+_{\tilde{L}}$. This implies that $\mu^+_{\tilde{L}}$ is surjective. Then $|Im\,\mu^-_{\tilde{L}}| = |\omega_C \otimes \eta|$ and $|Im\,\mu^+_{\tilde{L}}| = |\omega_C|$, which implies the statement. □

Now it is the appropriate moment to introduce Conte-Murre threefolds:

DEFINITION 2. *A Conte-Murre threefold is a transversal 3-dimensional linear section of $Q^{[2]}$.*

More informations on the family of Conte-Murre threefolds are available in the introduction. In particular for such a threefold T we have $Sing\ T = T \cap Q^{[1]}$ and $Sing\ T$ consists of 8 points of multiplicity 4. The study of the family of triples $(C,\eta,\tilde{L})$ is one of the main goals of the next sections.

In what follows C is always a smooth, irreducible curve of genus 7 which is not hyperelliptic nor trigonal and (C,η) is a Prym curve. It is useful to fix the following

DEFINITION 3. *A triple $(C,\eta,\tilde{L})$ is a good triple if:*

1 $h^0(\tilde{L}) = 4$.

2 *$\tilde{L}$ is very ample.*

3 *The Petri map $\mu_{\tilde{L}}$ is surjective.*

4 *$< C^+ >$ is transversal to $Q^{[2]}$.*

Let $(C,\eta,\tilde{L})$ be a good triple. By (1) we have $h^0(\tilde{L}) = 4$. Since $\mu_{\tilde{L}}$ is surjective then $< C^+ >$ is a 6-dimensional space in $\mathbf{P}^{9+}$. Therefore C^+ is the canonical embedding of C and $< C^+ >$ is its canonical space. Moreover the map $\phi : \tilde{C} \to \mathbf{P}^3 \times \mathbf{P}^3$, considered in the previous diagram, is an embedding because $\tilde{L}$ is very ample. Let us also point out that $\tilde{L} \cong \phi^* O_{\mathbf{P}^3\times\mathbf{P}^3}(1,0)$.

Therefore a good triple defines an embedding of the canonical model of C in a Conte-Murre threefold. Conversely let us start from an embedding

$$C \subset T - Sing\ T,$$

where T is a Conte-Murre threefold and C is a smooth, integral curve of genus 7 which is canonically embedded in $< T >$. Then the map $\pi^+ : \mathbf{P}^3 \times \mathbf{P}^3 \to Q^{[2]}$ induces an étale double covering $\pi : \tilde{C} \to C$, where $\tilde{C} := \pi^{+^*}C$. The next lemma is standard.

LEMMA 3. *$\pi : \tilde{C} \to C$ is not the trivial étale double covering.*

Proof. For $m >> 0$ consider a general $S \in |O_T(m)|$ containing C. Then $\tilde{S} := \pi^*S$ is a complete intersection in $\mathbf{P}^3 \times \mathbf{P}^3$ of four very ample divisors $D, H_1, H_2 H_3$ such that D has bidegree (m,m) and H_i has bidegree $(1,1)$, $i = 1,2,3$. In particular $\tilde{S}$ is a smooth, connected surface of general type. Its canonical sheaf is $O_{\tilde{S}}(m-1,m-1)$. Now assume π is trivial. Then $\tilde{C} = C_1 + C_2$, where C_1, C_2 are disconnected copies of C. Hence it follows $C_1^2 = C_2^2 = 12(2-m)$ and $C_1 C_2 = 0$: against Hodge index theorem. □

By the lemma $\pi : \tilde{C} \to C$ defines a Prym curve (C,η). Let $\tilde{L} := O_{\tilde{C}}(1,0)$, then it is easy to see that $Nm\,\tilde{L} \cong \omega_C \cong Nm\, i^*\tilde{L}$. Starting from the embedding $C \subset T - Sing\ T$, we have constructed a triple $(C,\eta,\tilde{L})$ satisfying $h^0(\tilde{L}) \geq 4$ and conditions 3) and 4) of definition 3.6. $(C,\eta,\tilde{L})$ is expected to be a good triple.

DEFINITION 4. *An embedding $C \subset T - Sing\ T$ is good if $(C,\eta,\tilde{L})$ is a good triple.*

The next assumption is proved to be true in the last section of this paper, from now on we keep it:

ASSUMPTION 2. *Good triples* $(C,\eta,\tilde{L})$ *do exist.*

4. Genus 7 canonical curves in a Conte-Murre threefold

We will denote a good triple by $(C,\eta,\tilde{L})$. To simplify notations we will also set

$$\Lambda := < C^+ > \ , \ T := \mathbb{Q}^{[2]} \cdot \Lambda \text{ and } C^+ = C.$$

Since Λ is transversal to $Q^{[2]}$, then T is a Conte-Murre threefold and, moreover, the embedding $C \subset T - Sing\ T$ is good. Many properties of T have a classical flavor. We review some of them to be used, cfr. [9] 6.3. The orthogonal space of Λ is a plane

$$\Lambda^{\perp} \subset |\mathcal{O}_{\mathbf{P}^{3*}}(2)|$$

in the space of quadrics of $\mathbf{P}^{3*} = \mathbf{P}V$. Therefore, up to projective equivalence, to give $\Lambda^{\perp}$ is equivalent to give an even spin curve (Q,θ), where $Q \subset \Lambda^{\perp}$ is a smooth plane quartic and θ is an even theta characteristic on Q. In other words the threefold

$$\tilde{T} := \pi^{+^*} T$$

is the base scheme of a net $N := |I_{\tilde{T}/\mathbf{P}^3 \times \mathbf{P}^3}(1,1)|$ of symmetric bilinear forms of $\mathbf{P}^3 \times \mathbf{P}^3$. Giving a plane $\Lambda^{\perp}$ is equivalent to give a net N. Let $(x,y) = (x_1 : \ldots : x_4) \times (y_1 : \ldots : y_4)$ be coordinates on $\mathbf{P}^3 \times \mathbf{P}^3$, this means that N is generated by three bilinear forms

$$q^k(x,y) = \sum_{1 \leq i,j \leq 4} q^k_{ij} x_i y_j, \ (k = 1,2,3),$$

such that (q^k_{ij}) is a 4×4 symmetric matrix. We can also view $\tilde{T}$ as the graph of a very well known Cremona involution. We denote such a birational map as

$$\phi_{\tilde{T}} : \mathbf{P}^3 \to \mathbf{P}^3$$

and summarize without proofs its realization. Let $\pi_i : \tilde{T} \to \mathbf{P}^3$ be the projection onto the i-th factor, $i = 1,2$. Then its fibre at $\bar{x}$, is defined by the linear equations

$$q^1(\bar{x},y) = q^2(\bar{x},y) = q^3(\bar{x},y) = 0.$$

Since $\tilde{T}$ is integral, it follows that $\pi_1^*(\bar{x})$ is a point for a general $\bar{x}$. Hence the map

$$\pi_1^{-1} : \mathbf{P}^3 \to \tilde{T}$$

is birational. The fundamental locus of p_1^{-1} is the rank 2 locus of the 3×4 matrix of linear forms $M_x := (q^k_{1j} x_1 + \cdots + q^k_{4j} x_4)$. Such a fundamental locus is the embedding

$$Q_1 \subset \mathbf{P}^3$$

of the plane quartic Q by the map associated to $\omega_Q(\theta)$. The 3×3 minors of M_x generate the ideal of Q_1. The same properties hold for $\pi_2 : \mathbf{P}^3 \to \tilde{T}$. Again its fundamental locus is the same embedding of Q. In the coordinates $(y_1 : \ldots : y_4)$ this is the curve

$$Q_2 \subset \mathbf{P}^3$$

whose ideal is generated by the 3×3 minors of $M_y := (q^k_{i1}y_1 + \cdots + q^k_{i4}y_4)$. In particular it follows that $\pi_i^{-1} : \mathbf{P}^3 \to \tilde{T}$ is the blowing of Q_i, $i = 1,2$. By definition we will have

$$\phi_{\tilde{T}} := \pi_2 \cdot \pi_1^{-1} \text{ and } \phi_{\tilde{T}}^{-1} = \pi_1 \cdot \pi_2^{-1}.$$

THEOREM 3. *The birational maps $\phi_{\tilde{T}}$ and $\phi_{\tilde{T}}^{-1}$ are respectively defined by the linear systems of cubic surfaces $|I_{Q_1/\mathbf{P}^3}(3)|$ and $|I_{Q_2/\mathbf{P}^3}(3)|$.*

REMARK 1. It is well known that π_2, π_1 are respectively the contractions of the unions of the trisecant lines to Q_1 and to Q_2. Notice also that $\phi^2_{\tilde{T}} = id_{\mathbf{P}^3}$. By far $\phi_{\tilde{T}}$ is the most famous example of cubo-cubic birational involution of $\mathbf{P}^3$.

Given our good triple $(C,\eta,\tilde{L})$ and putting $\tilde{\Lambda} :=< \tilde{T} >$ we have, in the Segre embedding of $\mathbf{P}^3 \times \mathbf{P}^3$, the following situation:

$$\tilde{C} \subset \tilde{T} = \tilde{\Lambda} \cdot \mathbf{P}^3 \times \mathbf{P}^3 \subset \mathbf{P}^{15}$$

Since the triple is good, $Im\,\mu_{\tilde{L}} = H^0(\omega_{\tilde{C}})$. Hence the next proposition is immediate.

PROPOSITION 1. *$\tilde{C}$ is canonically embedded in $\tilde{\Lambda}$.*

Now our program is to link $\tilde{C}$ to a second curve $\tilde{B}$ by a complete intersection of two smooth surfaces in $\tilde{T}$. Then we will use the family of curves $\tilde{B}$ to construct a rational variety, which in turn dominates a family of good triples $(C,\eta,\tilde{L})$ such that (C,η) has general moduli.

To realize this program we work with the contractions $\pi_1, \pi_2 : \tilde{T} \to \mathbf{P}^3$. Let $H_i \in |\pi_i^* \mathcal{O}_{\mathbf{P}^3}(1)|$ and let E_i be the exceptional divisor of π_i, $i = 1,2$. Then we have the following relations in $Pic\,\tilde{T}$:

$$3H_1 - E_1 \sim H_2 \text{ , } 3H_2 - E_2 \sim H_1.$$

We consider the linear systems $|H_i + H_1 + H_2|$, $i = 1,2$. Due to these relations we have

$$|H_i + H_1 + H_2| = |5H_i - E_i| = \pi_i^* |I_{Q_i/\mathbf{P}^3}(5)|.$$

Since the ideal of Q_i is generated by cubic forms, then $I_{Q_i}(5)$ is globally generated and hence $|5H_i - E_i|$ is base point free. It is easy to conclude that:

PROPOSITION 2. *A general $D \in |H_i + H_1 + H_2|$ is a smooth surface such that $\pi_i : D \to \mathbf{P}^3$ is an embedding of D as a smooth quintic surface passing through Q_i.*

REMARK 2. For completeness we mention that $|H_i + H_1 + H_2| = |7H_j - 2E_j|$ where $i \neq j$ and $i, j = 1, 2$. In particular let $\overline{D} = \pi_j(D)$, then $\overline{D}$ is a septic surface passing with multiplicity two through Q_j and $\pi_j : D \to \overline{D}$ is the normalization map of $\overline{D}$.

From now on let $\tilde{C}_i$ be the image of the embedding $\pi_i : \tilde{C} \to \mathbf{P}^3$, then we have

$$|I_{\tilde{C}/\tilde{T}}(H_i + H_1 + H_2)| \ = \ |I_{\tilde{C}/\tilde{T}}(5H_i - E_i)| \ = \ \pi_i^* |I_{Q_i \cup \tilde{C}_i}(5)|.$$

PROPOSITION 3. *The ideal sheaf* $I_{\tilde{C}/\tilde{T}}(H_i + H_1 + H_2)$ *is globally generated.*

Proof. We have $H^0(O_{\tilde{C}}(H_1)) = H^0(\tilde{L}$ and $H^0(O_{\tilde{C}}(H_2)) = H^0(i^*\tilde{L})$. Putting $W_k := H^0(O_{\tilde{C}}(H_k)), k = 1, 2$, we consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Ker\ m_1 & \longrightarrow & W_1 \otimes W_1 \otimes Sym^2 W_2 & \xrightarrow{m_1} & W_1 \otimes H^0(L \otimes \omega_{\tilde{C}}) & \longrightarrow & 0 \\ & & \downarrow n & & \downarrow l & & \downarrow m_3 & & \\ 0 & \longrightarrow & Ker\ m_2 & \longrightarrow & Sym^2 W_1 \otimes Sym^2 W_2 & \xrightarrow{m_2} & H^0(\omega_{\tilde{C}}^{\otimes 2}) & \longrightarrow & 0, \end{array}$$

where m_1, m_2, m_3 are the natural multiplication maps and $l(a \otimes b \otimes c) := (a \otimes b + b \otimes a) \otimes c$. Since the Petri map $\mu_{\tilde{L}} : W_1 \otimes W_2 \to H^0(\omega_{\tilde{C}})$ is surjective, the multiplication map

$$\mu_{\tilde{L}}^2 : W_1 \otimes W_1 \otimes W_2 \otimes W_2 \to H^0(\omega_{\tilde{C}}^{\otimes 2})$$

is surjective as well. This easily implies the surjectivity of m_1, m_2. Since l is clearly surjective, the surjectivity of m_3 also follows. Hence the diagram is exact and, by standard diagram chase, the map $n : Ker\ m_1 \to Ker\ m_2$ is surjective. Now let $I_{\tilde{C}}$ be the ideal sheaf of $\tilde{C}$ in $\mathbf{P}^3 \times \mathbf{P}^3$, then $Ker\ m_1 = W_1 \otimes H^0(I_{\tilde{C}}(H_1 + 2H_2))$. On the other hand we have $Ker\ m_2 = H^0(I_{\tilde{C}}(2H_1 + 2H_2))$ and n is surjective. Hence $I_{\tilde{C}}(H_1 + 2H_2)$ is globally generated if $I_{\tilde{C}}(2H_1 + 2H_2)$ is globally generated. Since the Segre embedding $\mathbf{P}^3 \times \mathbf{P}^3$ is generated by quadrics, $I_{\tilde{C}}(2H_1 + 2H_2)$ is globally generated if $\tilde{C}$ is not hyperelliptic, which is obvious, nor trigonal. Notice that $\tilde{C}$ is not trigonal because a non hyperelliptic trigonal curve of genus $g \geq 4$ has no fixed point free involution. Finally $\tilde{T}$ is a linear section of $\mathbf{P}^3 \times \mathbf{P}^3$, hence the sheaf $I_{\tilde{C}/\tilde{T}}(H_1 + 2H_2)$ is globally generated as well: we omit the easy details of this last step. The same proof works for $I_{\tilde{C}/\tilde{T}}(2H_1 + H_2)$. □

The next proposition is a straighforward, though useful, corollary.

PROPOSITION 4. *A general* $D \in |I_{\tilde{C}}(H_i + H_1 + H_2)|$ *is a smooth, connected surface. Moreover* $\pi_i : D \to \mathbf{P}^3$ *is an embedding of* D *as a smooth quintic surface containing* $Q_i \cup \tilde{C}_I$.

Proof. The argument is standard, cfr. [22]. By the lemma $\tilde{C}$ is locally complete intersection of two elements of $\mathbb{I} := |I_{\tilde{C}/\tilde{T}}(H_i + H_1 + H_2)|$ at every point $o \in \tilde{C}$. Since $\tilde{C}$ is smooth, then $\mathbb{I}_o := \{D \in \mathbb{I}\ /\ o \in Sing\ D\}$ has codimension ≥ 2 in $\mathbb{I}$. Therefore, counting dimensions, it follows that a general $D \in \mathbb{I}$ is smooth along $\tilde{C}$. Then D is smooth by Bertini theorem and obviously connected because it is very ample. Finally

$\tilde{C}$ is smooth and $\pi_i/\tilde{C}$ is an embedding. Hence π_i is an embedding along the scheme $\tilde{C}\cdot E_i$. Since $I_{\tilde{C}}(D)$ is globally generated and D is general, we can then assume that π is an embedding along $D\cdot E_i$, so that $\pi_i : D \to \mathbf{P}^3$ is an embedding. □

Definitely we fix now $i = 1$ for simplicity, then we consider a general smooth

$$\tilde{S} \in |I_{\tilde{C}/\tilde{T}}(2H_1 + H_2)|.$$

DEFINITION 5. *$NS(\tilde{S})$ is the Neron-Severi lattice of $\tilde{S}$. For any line bundle $O_{\tilde{S}}(D)$ we set:*

$$d := \text{ numerical class of } D\ .$$

Using the classical description of $\tilde{T}$ it is easy to compute that

$$h_1^2 = 5\ ,\ h_2^2 = 7\ ,\ h_1h_2 = 9\ ,\ \tilde{c}^2 = \tilde{c}h_1 = \tilde{c}h_2 = 12.$$

in the lattice $NS(\tilde{S})$. Since $\iota^*\tilde{C} = \tilde{C}$, we will also consider the smooth surface

$$\iota^*\tilde{S} \in |I_{\tilde{C}/\tilde{T}}(H_1 + 2H_2)|.$$

PROPOSITION 5. *A general element of $|O_{\tilde{S}}(H_1 + 2H_2 - \tilde{C})|$ is a smooth, connected curve of genus* 11 *and bidegree* $(11,11)$.

Proof. We know from proposition 4.6 that $I_{\tilde{C}/\tilde{T}}(\iota^*\tilde{S})$ is globally generated. Then the same remarks given to prove lemma 4.7 imply that $O_{\tilde{S}}(H_1+2H_2-\tilde{C})|$ is base point free. Since $(h_1 + 2h_2 - \tilde{c})^2 > 0$ the first part of the statement follows from Bertini theorem. Let $\tilde{B} \in |O_{\tilde{S}}(H_1 + 2H_2 - \tilde{C})|$. The canonical class of $\tilde{S}$ is h_1. Then, by adjunction formula, we have $2p_a(\tilde{B}) - 2 = (2h_1 + 2h_2 - \tilde{c})(h_1 + 2h_2 - \tilde{c}) = 20$. Hence $p_a(\tilde{B}) = 11$. Notice also that $h_1(h_1 + 2h_2 - \tilde{c}) = h_2(h_1 + 2h_2 - \tilde{c}) = 11$. □

The next result, though elementary, highlights a very interesting feature:

THEOREM 4. *For every $\tilde{B} \in |O_{\tilde{S}}(H_1 + 2H_2 - \tilde{C})|$ there exists exactly one hyperplane of $\tilde{\Lambda}$ containing it.*

Proof. A general $\tilde{B}$ is smooth of genus 11. Then $O_{\tilde{B}}(1)$ is a non special line bundle for degree reasons and $h^0(O_{\tilde{B}}(1)) = 12$. Since we have $h^0(O_{\tilde{T}}(1)) = 13$ and $\tilde{\Lambda} =< \tilde{T} >$, it follows that $\tilde{B}$ is contained in a hyperplane. By semicontinuity this holds for every $\tilde{B}$. To prove the uniqueness result it suffices to show that $h^0(O_{\tilde{S}}(H_1 + H_2 - \tilde{B})) = 1$. For this notice that $h_1(h_1 + h_2 - \tilde{b}) = 3$. Since h_1 is very ample and $\tilde{S}$ of general type, every curve D satsfying $h_1d = 3$ is isolated; hence $h^0(O_{\tilde{S}}(H_1 + H_2 - \tilde{B})) = 1$. □

5. Quintic surfaces through a genus 3 sextic and a skew cubic

This section is a useful digression to study the linear system $|\tilde{C}|$ on $\tilde{S}$. We will use the notation $\tilde{S}_i$ for $\pi_i(\tilde{S})$, $i = 1,2$. In particular $\tilde{S}_1$ is a smooth quintic surface containing a genus 3 sextic Q_1. As we will see in a moment, $\tilde{S}_1$ also contains a skew cubic which is 4-secant to Q_1. In what follows a skew cubic is a reduced, connected curve of degree 3 and arithmetic genus 0, possibly reducible. By the latter theorem we can consider on $\tilde{S}$ the hyperplane section

$$\tilde{F} + \tilde{B} := < \tilde{B} > \cdot \tilde{S}.$$

for every $\tilde{B} \in |O_{\tilde{S}}(H_1 + 2H_2 - \tilde{C})|$.

PROPOSITION 6. *$\tilde{F}$ is a connected curve of arithmetic genus 0 and bidegree $(3,5)$ in $\mathbf{P}^3 \times \mathbf{P}^3$. It is isolated in $\tilde{S}$ and embedded by $\pi_1 : \tilde{F} \to \mathbf{P}^3$ as a skew cubic.*

Proof. We have $\tilde{f} = \tilde{c} - h_2$, hence $\tilde{f}^2 = -5$ and $\tilde{f}h_1 = 3$ so that $p_a(\tilde{F}) = 0$. Since $\tilde{f}h_2 = 5$, the bidegree is $(3,5)$. The map $p_1 : \tilde{S} \to \mathbf{P}^3$ embeds $\tilde{F}$ has a curve of degree 3. Then it is easy to deduce from $p_a(\tilde{F}) = 0$ that $\tilde{F}$ is reduced and connected. It is obviously isolated on $\tilde{S}$ and embedded in $\mathbf{P}^3$ as a skew cubic. □

DEFINITION 6. *$\tilde{F}_i$ is the image of the morphism $\pi_i : \tilde{F} \to \mathbf{P}^3$, $i = 1,2$.*

In particular $\tilde{F}_1$ is a skew cubic in the smooth quintic surface $\tilde{S}_1 := \pi_1(\tilde{S})$. Let us summarize the situation so far: from the previous remarks we have seen that

$$O_{\tilde{S}}(H_1 + H_2 - \tilde{F}) \cong O_{\tilde{S}}(H_1 + 2H_2 - \tilde{C}).$$

Therefore we have:

THEOREM 5. *Let $(C, \eta, \tilde{L})$ be a good triple. Then, with the previous notations, one has*

$$O_{\tilde{S}}(\tilde{C}) \cong O_{\tilde{S}}(\tilde{F} + H_2).$$

It is easy to see that a general element of $|O_{\tilde{S}}(H_2)|$ is a smooth curve of genus 9 and bidegree $(9,7)$ in $\mathbf{P}^3 \times \mathbf{P}^3$. Since the cubo-cubic Cremona transformation

$$p_2 \cdot p_1^{-1} : \mathbf{P}^3 \to \mathbf{P}^3$$

is defined by the linear system $|I_{Q_1/\mathbf{P}^3}(3)|$, we can see the elements of $|H_2|$ as follows.

LEMMA 4. *Let $\tilde{G} \in |H_2|$ be general and let $\tilde{G}_1 \subset \tilde{S}_1$ be its embedding via π_1. Then $\tilde{G}_1 \cup Q_1$ is a nodal complete intersection of $\tilde{S}_1$ and a general cubic surface $X \in |I_{Q_1/\mathbf{P}^3}(3)|$.*

REMARK 3. Taking a curve $\tilde{G}_1$ which is transversal to the skew cubic $\tilde{F}_1$ we obtain a nodal curve $\tilde{G}_1 + \tilde{F}_1$ which is linearly equivalent to $\tilde{C}$. We point out that

$$h_2\tilde{f} = 5 \text{ and } e_1\tilde{f} = 4,$$

where e_1 is the class of $\pi_1^* Q_1$ in $\tilde{S}$. The latter equality follows from $3h_1 - h_2 = e_1$. Then $\tilde{G}_1 \cap \tilde{F}_1$ is a set of 5 points. Counting multiplicities, $Q_1 \cap \tilde{F}_1$ is a set of 4 points.

So far we have seen that a good triple $(C,\eta,\tilde{L})$ defines a curve $\tilde{C}$ which moves in a linear system $|\mathcal{O}_{\tilde{S}}(\tilde{F}+H_2)|$ as above, where $\tilde{S}$ is biregular to a smooth quintic surface in $\mathbf{P}^3$. Now we partially reverse the construction and study smooth quintic surfaces endowed with a linear system of this type and its properties.

Thus we start from a non hyperelliptic even spin curve (Q,θ) of genus 3 and from its embedding $Q_1 \subset \mathbf{P}^3$ by $\omega_Q(\theta)$. We confirm, with the same meaning, all the previous notations. The spin curve (Q,θ) defines, via the cubo-cubic transformation defined by the linear system $|\mathcal{I}_{Q_1/\mathbf{P}^3}(3)|$, a smooth threefold

$$\tilde{T} \subset \mathbf{P}^3 \times \mathbf{P}^3$$

as above. Next we consider a general, smooth quintic surface $\tilde{S}_1 \in |\mathcal{I}_{Q_1/\mathbf{P}^3}(5)|$ and its pull-back $\tilde{S} := \pi_1^*\tilde{S}_1 \in |\mathcal{O}_{\tilde{T}}(2H_1+H_2)|$. We know that $\pi_1 : \tilde{S} \to \tilde{S}_1$ is biregular. Then we specialize $\tilde{S}$ paying attention to the latter remarks on skew cubics. We choose four independent points $o_1 \dots o_4 \in Q_1$ and a smooth skew cubic $\tilde{F}_1$ such that $Q_1 \cup \tilde{F}_1$ is nodal and *Sing* $Q_1 \cup \tilde{F}_1 = \{o_1 \dots o_4\}$. The proof of the next lemma is standard, see 4.7.

LEMMA 5. *A general element of* $|\mathcal{I}_{Q_1\cup\tilde{F}_1}(5)|$ *is a smooth quintic surface.*

Let $\tilde{F}$ be the strict transform of $\tilde{F}_1$ by $\pi_1 : \tilde{S} \to \tilde{S}_1$: it is now natural to consider

$$|\mathcal{O}_{\tilde{S}}(\tilde{F}+H_2)|.$$

LEMMA 6. *A general curve* $\tilde{C} \in |\mathcal{O}_{\tilde{S}}(\tilde{F}+H_2)|$ *is smooth, connected of genus* 13 *and bidegree* $(12,12)$.

Proof. It is standard to compute $p_a(\tilde{C}) = 13$ and that $\tilde{C}$ has bidegree $(12,12)$. Moreover $\tilde{c}^2 = 12$ is positive. To prove that $\tilde{C}$ is smooth and connected, we show that $|\mathcal{O}_{\tilde{S}}(\tilde{F}+H_2)|$ is base point free. Consider the standard exact sequence

$$0 \to \mathcal{O}_{\tilde{S}}(H_2) \to \mathcal{O}_{\tilde{S}}(H_2+\tilde{F}) \xrightarrow{r} \mathcal{O}_{\tilde{F}}(H_2+\tilde{F}) \to 0$$

and its associated long exact sequence. The proof that $h^1(\mathcal{O}_{\tilde{S}}(H_2) = 0$ is standard, so we omit it. Then $h^0(r)$ is surjective. Since the system is clearly base point free on $\tilde{S}-\tilde{F}$, it suffices to check that $\mathcal{O}_{\tilde{F}}(H_2+\tilde{F})$ is very ample. This is the very ample sheaf $\mathcal{O}_{\mathbf{P}^1}(8)$. □

LEMMA 7. *Let* $\tilde{C} \in |\mathcal{O}_{\tilde{S}}(\tilde{F}+H_2)|$ *be smooth, then* $\mathcal{O}_{\tilde{C}}(H_1+H_2) \cong \omega_{\tilde{C}}$ *and moreover* $\tilde{C}$ *is canonically embedded in* $< \tilde{T} >$.

Proof. Since *deg* $\mathcal{O}_{\tilde{C}}(H_1+H_2) =$ *deg* $\omega_{\tilde{C}}$, it suffices to show that $h^0(\mathcal{O}_{\tilde{C}}(H_1+H_2)) = h^0(\omega_{\tilde{C}}) = 13$ and that $\tilde{C}$ is not contained in a hyperplane of $< \tilde{T} >$. Since $\mathcal{O}_{\tilde{S}}(\tilde{C}) \cong \mathcal{O}_{\tilde{S}}(\tilde{F}+H_2)$, we have the exact sequence

$$0 \to \mathcal{O}_{\tilde{S}}(H_1-\tilde{F}) \to \mathcal{O}_{\tilde{S}}(H_1+H_2) \xrightarrow{r} \mathcal{O}_{\tilde{C}}(H_1+H_2) \to 0.$$

It suffices to show that $h^0(O_{\tilde{S}}(H_1-\tilde{F}))=h^1(O_{\tilde{F}}(H_1-\tilde{F}))=0$. Then $h^0(r)$ is an isomorphism. We have $h^0(O_{\tilde{S}}(H_1-\tilde{F}))=0$ because $\tilde{F}_1$ is a skew cubic. By Serre duality we then have $h^2(O_{\tilde{S}}(\tilde{F}))=0$ and $h^1(O_{\tilde{S}}(H_1-\tilde{F}))=h^1(O_{\tilde{S}}(\tilde{F}))$. Since $\tilde{F}=\mathbf{P}^1$, $\tilde{f}^2=-5$ and $\tilde{S}$ is a regular surface, the vanishing of $h^1(O_{\tilde{S}}(\tilde{F}))$ then follows from the standard exact sequence

$$0\to O_{\tilde{S}}\to O_{\tilde{S}}(\tilde{F})\to O_{\mathbf{P}^1}(-5)\to 0.$$

□

THEOREM 6. *For a smooth $\tilde{C}\in|O_{\tilde{S}}(\tilde{F}+H_2)|$ one has*

1 $h^0(\tilde{L})=4$.

2 *$\tilde{L}$ is very ample.*

3 *The Petri map $\mu_{\tilde{L}}$ is surjective.*

4 *$<\tilde{C}>$ is transversal to $\mathbf{P}^3\times\mathbf{P}^3$.*

Proof. Every non hyperelliptic, smooth genus 3 sextic $Q_1\subset\mathbf{P}^3$ defines a cubo-cubic Cremona transformation whose graph in $\mathbf{P}^3\times\mathbf{P}^3$ is smooth. Since $<\tilde{C}>$ cuts such a graph on $\mathbf{P}^3\times\mathbf{P}^3$, then 4) follows. Notice also that, by construction, $<\tilde{C}>=<\tilde{T}>$. To prove 3) we observe that $\omega_{\tilde{C}}\cong O_{\tilde{C}}(H_1+H_2)$, so that we have $\omega_{\tilde{C}}\otimes\tilde{L}^{-1}$ is $O_{\tilde{C}}(H_2)$. In particular it follows that $<\tilde{C}>$ is the linear space orthogonal to $Im\ \mu_{\tilde{L}}$. Then the equality $<\tilde{C}>=<\tilde{T}>$ implies 3). 2) follows because $\tilde{C}$ is smooth and $\pi_1:\tilde{S}\to\mathbf{P}^3$ is an embedding. 3) Finally, to prove 1), consider the standard exact sequence

$$0\to O_{\tilde{S}}(H_1-\tilde{F}-D)\to O_{\tilde{S}}(H_1)\to O_{\tilde{F}\cup D}(H_1)\to 0$$

where $D\in|H_2|$ is smooth and transversal to $\tilde{F}$. Since $\tilde{F}$ is a skew cubic it follows $h^0(O_{\tilde{S}}(H_1-\tilde{F}-D))=0$. Hence, the same exact sequence implies that $h^0(\tilde{L})\geq 4$ for every $\tilde{C}\in|O_{\tilde{S}}(\tilde{F}+H_2)|$. On the other hand we have $h^1(O_{\tilde{S}}(H_1))=0$. We show that $h^0(O_{\tilde{F}\cup D}(H_1))=4$. Then this implies $h^1(O_{\tilde{S}}(\tilde{F}+D))=0$ and hence $h^0(O_{\tilde{C}}(H_1))=4$ for every $\tilde{C}\in|O_{\tilde{S}}(\tilde{F}+D)|$. We remark that $\tilde{F}$ is an integral skew cubic and that $\tilde{F}\cdot D$ is not contained in a plane. Hence it suffices to show that $O_D(H_1))=4$. But D is linked to Q_1 by the complete intersection of $\tilde{S}_1$ and a general, smooth cubic $X\in|I_{Q_1/\mathbf{P}^3}(3)|$. Let E be a plane section of X: we then have $O_X(D)\cong O_X(5E-Q_1)$. On the other hand consider the standard exact sequence

$$0\to O_X(E-D)\to O_X(E)\to O_D(E)\to 0.$$

It is easy to check that $h^i(O_X(E-D))=0$ for $i=0,1$. Hence, considering the associated long exact sequence, it follows $h^0(O_X(E))==h^0(O_D(E))=4$. □

Now assume that a smooth $\tilde{C}\in|O_{\tilde{S}}(\tilde{F}+H_2)|$ is endowed with a fixed point free involution i, induced by the standard involution ι of $\mathbf{P}^3\times\mathbf{P}^3$. Let $C:=\tilde{C}/<i>$, $\eta\in Pic^0C$ the line bundle defining the quotient map, $\tilde{L}:=O_{\tilde{C}}(H_1)$. Then we have:

COROLLARY 2. *The triple* $(C,\eta,\tilde{L})$ *is a good triple.*

REMARK 4. The corollary explains why the construction realized in this section is not yet effective to produce good triples. We can now easily construct smooth curves $\tilde{C}$ of genus 13 endowed with a line bundle $\tilde{L}$, so that the previous conditions are satisfied. But we need to recognize in this family those curves $\tilde{C}$ such that $\iota/\tilde{C}$ defines a fixed point free involution. To realize this latter step, the very special feature observed in theorem 4.10 still has to be spoiled. We do this in the next section.

6. Symmetroids and Reye congruences in the play

To begin let us give the due motivations to the title of this section. A symmetroid is a very well known surface with a quartic birational model in $\mathbf{P}^3$, namely:

DEFINITION 7. *A symmetroid is a codimension four linear section* $\tilde{W}$ *of* $\mathbf{P}^3\times\mathbf{P}^3$ *which is complete intersection of four symmetric bilinear forms.*

By definition a bilinear form $q\in H^0(O_{\mathbf{P}^3\times\mathbf{P}^3}(1,1))$ is symmetric if $\iota^*q=q$. Hence we have $q=\sum m_{ij}x_iy_j$ so that (m_{ij}) is a 4×4 symmetric matrix. Fixing a basis $q^1\dots q^4$ of $H^0(I_{\tilde{W}/\mathbf{P}^3\times\mathbf{P}^3}(1,1))$, it follows that $\tilde{W}$ is the base locus of the linear system

$$z_1q^1+z_2q^2+z_3q^3+z_4q^4=0.$$

Restricting it to the diagonal $\{x=y\}$ we obtain a web of quadrics of $\mathbf{P}^3$ whose general member is smooth. To give $\tilde{W}$ is equivalent to give such a web. Let $\tilde{W}_x:=\pi_1(\tilde{W})$ and $\tilde{W}_y:=\pi_2(\tilde{W})$, then $\tilde{W}_x,\tilde{W}_y$ are quartic surfaces. It is easy to see that $\tilde{W}_x$ is defined by the equation $det\ M_x$, where M_x is a 4×4 symmetric matrix of linear forms. Replacing x by y in M_x, we obtain a matrix M_y such that $\tilde{W}_y=\{det\ M_y=0\}$. To give a symmetroid $\tilde{W}$ is therefore equivalent to give a quartic surface $\tilde{W}_x$ whose equation is the determinant of a symmetric 4×4 matrix of linear forms on $\mathbf{P}^3$. These quartic surfaces are known as quartic symmetroids, [9]. In a moment we will briefly recollect the main properties of symmetroids which are needed here.

Now let $(C,\eta,\tilde{L})$ be a good triple and $\tilde{C}\subset\tilde{T}$ the corresponding embedding. By proposition 4.7 a general $\tilde{S}\in|I_{\tilde{C}/\tilde{T}}(2H_1+H_2)|$ is smooth and such that

$$\tilde{S}\cdot\iota^*\tilde{S}=\tilde{C}\cup\tilde{B}$$

where $\tilde{B}$ is a curve of arithmetic genus 11 and bidegree $(11,11)$. We also know that $<\tilde{B}>$ is a hyperplane of $\tilde{\Lambda}:=<\tilde{T}>$. We fix from now on the following notation

$$\tilde{Y}:=<\tilde{B}>\cdot\tilde{T}.$$

LEMMA 8. $\tilde{Y}$ *is a symmetroid.*

Proof. By proposition 4.9 a general $D\in|\tilde{B}|$ is smooth, connected of genus 11 and degree 22. This implies $h^1(O_D(H_1+H_2))=0$ and $h^0(O_D(H_1+H_2))=12$. Moreover we have $h^i(\mathcal{O}_{\tilde{S}}(H_1+H_2))=0, i\geq 1$. Then the standard exact sequence

$$0\to \mathcal{O}_{\tilde{S}}(H_1+H_2-\tilde{B})\to \mathcal{O}_{\tilde{S}}(H_1+H_2)\to \mathcal{O}_{\tilde{B}}(H_1+H_2)\to 0$$

implies $h^0(\mathcal{O}_{\tilde{S}}(H_1+H_2-\tilde{B}))=1$ and $h^i(\mathcal{O}_{\tilde{S}}(H_1+H_2-\tilde{B}))=0, i\geq 1$. Then it follows $h^1(\mathcal{O}_{\tilde{B}}(H_1+H_2))=0$ and $h^0(\mathcal{O}_{\tilde{B}}(H_1+H_2))=12$. Note that ι acts on $\tilde{B}$ as a fixed point free involution. Let $\pi_B:\tilde{B}\to B=\tilde{B}/<\iota>$ be the quotient map. Then $p_a(B)=6$ and π_B is defined by a non trivial element $\eta_B\in Pic^0_2B$. Let $M\in Pic^{11}B$ such that $\pi^*_BM\cong \mathcal{O}_{\tilde{S}}(H_1+H_2)$, then we have

$$H^i(\mathcal{O}_{\tilde{B}}(H_1+H_2))=\pi^*_BH^i(M)\oplus\pi^*_BH^i(M\otimes\eta_B).$$

Here the summands are the eigenspaces of ι. Since $h^1(\mathcal{O}_{\tilde{B}}(H_1+H_2))=0$, it follows $h^1(M)=h^1(M\otimes\eta_B))=0$ and $h^0(M)=h^0(M\otimes\eta_B)=6$. Finally note that the action of ι on $H^0(\mathcal{O}_{\tilde{T}}(H_1+H_2))$ has a $+$ eigenspace E^+ of dimension 7. Hence the restriction $E^+\to\pi^*H^0(M)$ has 1-dimensional Kernel. Then $<\tilde{B}>\cdot\tilde{T}$ is a symmetroid. □

The next assumption will be not restrictive to the goals of this paper.

ASSUMPTION 7. *$\tilde{Y}$ is a smooth and general symmetroid.*

The assumption implies that $\iota:\tilde{Y}\to\tilde{Y}$ is a fixed point free involution. Hence $Y:=\tilde{Y}/<\iota>$ is a smooth Enriques surface. Note that $|\mathcal{O}_{\tilde{Y}}(\tilde{B})|$ is a ι-invariant linear system on $\tilde{Y}$ of curves of genus 11 and bidegree $(11,11)$. Moreover $\tilde{B}$ is a ι-invariant element of it. We have the diagram of projection maps

$$\mathbf{P}^{5-}\supset\mathbf{G}\supset Y_-\xleftarrow{\pi^-}\tilde{Y}\xrightarrow{\pi^+}Y_+\subset T\subset\mathbf{P}^{9+}.$$

In particular $Y_-:=\pi^-(\tilde{Y})$ and $Y_+:=\pi^+(\tilde{Y})$ are two copies of Y. Moreover we have

$$\mathbf{P}^{5-}\supset\mathbf{G}\supset B_-\xleftarrow{\pi^-}\tilde{B}\xrightarrow{\pi^+}B_+\subset T\subset\mathbf{P}^{9+},$$

where $B_-:=\pi^-(\tilde{B})$ and $B_+:=\pi^+(\tilde{B})$. B_+,B_- are two copies of the curve

$$B:=\tilde{B}/<\iota>.$$

B is connected, $B^2=10$ so that $p_a(B)=6$ and the quotient map $\pi_B:\tilde{B}\to B$ is an étale double covering. Both B_+ and B_- have degree 11. As in the proof of 6.2 we consider $M:=\mathcal{O}_{B_+}(1)$ and η_B the line bundle defining π_B. Then it follows that $M\otimes\eta_B\cong\mathcal{O}_{B_-}(1)$. The proof of the next lemma is already part the proof of 6.2.

LEMMA 9. *$h^0(M)=h^0(M\otimes\eta_B)=6$ and $h^1(M)=h^1(M\otimes\eta_B)=0$.*

We need to add more informations on Y, which is an Enriques surface of special type, and on its linear system $|\mathcal{O}_Y(B)|$. Y_-, Y_- are copies of Y embedded as surfaces of degree 10. We fix the notations $H_+\in|\mathcal{O}_{Y_+}(1)|$ and $H_-\in|\mathcal{O}_{Y-}(1)|$ for their general hyperplane sections. Both H_+,H_- are smooth curves of genus 6 Prym canonically embedded. Notice also that $H_+-H_-\sim K_Y$.

Every Enriques surface admits projective embeddings in $\mathbf{P}^5$ as a surface of degree 10. They are called *Fano models* of the surface. For every Fano model the surface contains exactly 20 pair curves which are embedded in the Fano model as plane cubics. They subdivide in 10 pairs: the difference of the two curves of each pair is a canonical divisor.

Actually Y is a general member of a special family of Enriques surfaces: those containing smooth, integral rational curves. Since each of these curves can be contracted to a rational double point, Y is also said to be a nodal Enriques surfaces. The models Y_- and Y_- are special Fano models of Y.

Y_- is characterized by the condition that there exists a quadric, namely the Klein quadric $\mathbf{G}$, containing it. This is equivalent to the non quadratic normality of Y^-. The Fano models of this type are congruences of lines in the Grassmannian $\mathbf{G}$: they are known as *Reye congruences* and form an irreducible family. As is well known:

PROPOSITION 7. *A general Y_- does not contain smooth rational curves of degree ≤ 3.*

Since $H_+ \sim H_- + K_Y$, the same is true for a general Y_+. We recall that Fano models Y_+ are characterized by the condition that the family of their trisecant lines is 3-dimensional, [8]. Equivalently these Fano models are the smooth hyperplane sections of Conte-Murre threefolds.

For a Fano model Y_+ we will denote the 10 pairs of its plane cubics as E_n, E'_n, with $n = 1 \dots 10$. As for every Enriques surface they intersect as follows

$$E_m E_n = E_m E'_n = 1 - \delta_{mn}, \ 1 \leq m, n \leq 10.$$

Every Enriques surface X admits morphisms $f : X \to \mathbf{P}^3$ whose schematic image is a sextic surface passing doubly through the edges of a tetrahedron Z. We say that $f(X)$ is an *Enriques model*, and $|f^* \mathcal{O}_{\mathbf{P}^3}(1)|$ is an *Enriques linear system*, if f is generically injective and Z is $\{z_1 z_2 z_3 z_4 = 0\}$, where $z_1 \dots z_4$ are independent linear forms. Now let

$$B \subset Y_+$$

be any connected curve of degree 11 and arithmetic genus 6. Then we have:

THEOREM 8. *$|2H_+ - B|$ is an Enriques linear system. Moreover such a linear system is either $|E_a + E_b + E_c|$ or $|E_a + E_b + E_c + K_Y|$, for some (a, b, c) such that $1 \leq a < b < c \leq 10$.*

Proof. We have $(2H_+ - B)^2 = 6$. We claim that $|2H_+ - B|$ is base point free. Then, by the classification in [5] IV 9 and its corollary 1 p.283, it follows that either $|2H_+ - B|$ is an Enriques linear system or it is superelliptic. In the latter case $|2H - B|$ defines a morphism $f : Y \to f(X) \subset \mathbf{P}^3$ which is a $2:1$ cover of a 4-nodal cubic surface, [5] IV 7. Applying the classification for this case, [5] corollary IV 7.1, one can check that there exists $R_1 + R_2 \in |2H_+ - B|$ such that $p_a(R_1) = 0$ and $R_1(2H_+ - B) \leq 3$. But then Y_+ contains a smooth, connected rational curve of degree ≤ 3, which is excluded for a general Y_+. Let us prove the previous claim: assume that $|2H-B|$ has fixed components, then one of them is an integral, smooth rational curve R of degree $HR \geq 4$. Consider the standard exact sequence

$$0 \to O_Y(H-B) \to O_Y(2H-B) \xrightarrow{\rho} O_H(2H-B) \to 0,$$

where $H \in |H_+|$ is smooth. Since $\tilde{Y}_+$ does not contain lines and $H(B-H)=1$, we have $|B-H| = |B-H+K_Y| = \emptyset$. This, by Riemann-Roch and Serre duality, implies $h^i(O_Y(H-B)) = 0, i \geq 0$. Hence the map $h^0(\rho)$ is an isomorphis. Then $O_H(H-R)|$ is linear series of dimension ≥ 3 and degree ≤ 6. Hence H is hyperelliptic by Clifford's theorem. This is excluded for a very ample divisor on an Enriques surface. In particular $2H-B$ is nef. Finally assume $|2H-B|$ has a base point. Then $|2H-B|$ is hyperelliptic and its classification is known, cfr. [5] IV 5.1. Applying it and the equality $H(2H-B)$ $= 9$ one can check that then Y_+ contains a smooth, connected rational curve of degree ≤ 3, which is excluded for a general Y_+. □

Keeping (a,b,c) as above, we fix the definition

$$\mathcal{E}(Y_+) := \{|D| \ / \ |D| = |E_a+E_b+E_c| \text{ or} |E_a+E_b+E_c+K_Y|\}.$$

We will say that $|D|$ is an Enriques linear system of Y_+. From $\mathcal{E}_{Y_+}$ we also define

$$\mathcal{F}(Y_+) := \{|2D-H_+| \ / \ |D| \in \mathcal{E}(Y_+)\}$$

and

$$\mathcal{B}(Y_+) := \{|2H_+-D| \ / \ \in \mathcal{E}(Y_+)\}.$$

REMARK 5. For any Fano model $X \subset \mathbf{P}^5$ of any Enriques surface one can define the previous sets. Let $H \in |O_X(1)|$ and $D \in \mathcal{E}(X)$, then $p_a(2D-H) = 0$. The non emptyness of $|2D-H|$ is discussed later.

7. Reye congruences and some unirationality results

Let $(C_k, \eta_k, \tilde{L}_k)$, $(k=1,2)$, be two good triples, they define the embeddings

$$\tilde{C}_k \subset \tilde{T}_k \subset \mathbf{P}^3 \times \mathbf{P}^3.$$

By definition they are isomorphic if there exists an isomorphism $\tilde{u}: C_1 \to C_2$ such that $i_2 \cdot \tilde{u} = \tilde{u} \cdot i_1$ and $\tilde{u}^* \tilde{L}_2 \cong \tilde{L}_1$, where i_k is the fixed point free involution defined on $\tilde{C}_k$. The isomorphism class of a good triple $(C, \eta, \tilde{L})$ will be denoted as $[C, \eta, \tilde{L}]$. We omit the very easy proof of the next lemma:

LEMMA 10. *Two good triples as above are isomorphic iff there exists* $\alpha \in$ *Aut* $\mathbf{P}^3 \times$ *Aut* $\mathbf{P}^3$ *such that* $\alpha(\tilde{C}_1) = \tilde{C}_2$.

We consider now good triples $(C, \eta, \tilde{L})$ satisfying the following conditions:

- $\tilde{C} \subset \tilde{S} \subset \tilde{T}$, *where* $\tilde{S} \in |I_{\tilde{B}/\tilde{T}}(2H_1 + H_2)|$.
- $\tilde{B} = \pi_+^* B \subset \tilde{Y}$, *where* $|B| \in \mathcal{B}(Y_+)$.
- $\tilde{Y}$ *is a general symmetroid*.
- $\tilde{S} \cdot \iota^* \tilde{S} = \tilde{B} \cup \tilde{C}$.
- $C = \tilde{C}/ < \iota >$, *the covering* $\pi : \tilde{C} \to C$ *is defined by* η, $\tilde{L} = O_{\tilde{C}}(H_1)$.

Clearly, the 4-tuple $x := (\tilde{T}, \tilde{Y}, \tilde{B}, \tilde{S})$ uniquely defines the good triple $(C, \eta, \tilde{L})$.

DEFINITION 8. *x is a good 4-tuple, $\mathfrak{G}$ is the family of all good 4-tuples.*

Along this section we will study the moduli map

$$m : \mathfrak{G} \to \mathcal{P}_7^3$$

defined as follows: $m(x) := [C, \eta, \tilde{L}]$. To begin we point out once more that:

LEMMA 11. *Let $(C, \eta, \tilde{L})$ be a good triple, then $\tilde{L}$ is an isolated point of $P^3(C, \eta)$.*

Proof. Since the Petri map $\mu_{\tilde{L}}$ is surjective, the Prym-Petri map $\mu_{\tilde{L}}^-$ is an isomorphism. Then the tangent space at $\tilde{L}$ to the Prym Brill-Noether locus $P^3(C, \eta)$ is 0-dimensional, since it is isomorphic to *Coker* $\mu_{\tilde{L}}^-$. □

Let $\mathcal{Z}$ be an integral variety whose points are isomorphism classes of good triples. Let $f : \mathcal{Z} \to \mathcal{R}_7$ be the natural forgetful map sending $[C, \eta, \tilde{L}]$ to $[C, \eta]$. Then:

PROPOSITION 8. *The morphism $f : \mathcal{Z} \to \mathcal{R}_7$ is finite over its image.*

Proof. The tangent space to the fibre of $f : \mathcal{Z} \to \mathcal{R}_7$ at $[C, \eta]$ is the tangent space to $P(C, \eta)$ at $\tilde{L}$. Hence, by the lemma, it is 0-dimensional. □

LEMMA 12. *Let $x = (\tilde{T}, \tilde{Y}, \tilde{B}, \tilde{S}) \in \mathfrak{G}$ be a good 4-tuple, then*

$$dim\, m^{-1}(m(x)) \; \leq \; dim\, |I_{\tilde{C}/\tilde{T}}(2H_1 + H_2)| + dim\, Aut\, \mathbf{P}^3 \times Aut\, \mathbf{P}^3.$$

Proof. Let $x' = (\tilde{T}', \tilde{Y}', \tilde{B}', \tilde{S}') \in m^{-1}(m(x))$ and $m(x') = [C', \eta', \tilde{L}']$. Then there exists $\alpha \in Aut\, \mathbf{P}^3 \times Aut\, \mathbf{P}^3$ such that $\alpha(\tilde{C}) = \tilde{C}'$. Since $\tilde{T}' =< \tilde{C}' > \cdot \mathbf{P}^3 \times \mathbf{P}^3$, it follows that $\alpha(\tilde{T}) = \tilde{T}'$ and that $\alpha^* |I_{\tilde{C}'/\tilde{T}'}(2,1)| = |I_{\tilde{C}/\tilde{T}}(2,1)|$. Moreover $\tilde{Y}$ and $\tilde{B}$ are uniquely defined from $\tilde{S}$, because $\tilde{C} \cup \tilde{B} = \tilde{S} \cdot \iota^* \tilde{S}$ and $\tilde{Y} =< \tilde{B} > \cdot \tilde{T}$. Hence the fibre $m^{-1}(m(x))$ is dominated by the family of pairs $(\tilde{S}, \alpha)$ and the statement follows. □

LEMMA 13. $dim\, |I_{\tilde{C}/\tilde{T}}(2H_1 + H_2)| = 3$.

Proof. By propositions 4.6 and 4.7 we can choose a smooth $\tilde{S} \in |I_{\tilde{C}/\tilde{T}}(2H_1 + H_2)|$ such that $|O_{\tilde{S}}(2H_1 + H_2 - \tilde{C})|$ is base point free. Let $D \in |O_S(2H_1 + H_2 - \tilde{C})|$, then $D^2 = 3$ and a general D is smooth, connected of genus 6. Consider the standard exact sequence

$$0 \to O_{\tilde{S}} \to O_{\tilde{S}}(D) \to O_D(D) \to 0.$$

Its associated long exact sequence yelds $dim\,|D| \leq 2$ and hence $dim\,|I_{\tilde{C}/\tilde{T}}(2H_1 + H_2)| \leq 3$. Actually the equality holds. □

LEMMA 14. *$dim\ I_{\tilde{B}}(2H_1 + H_2)| = 4$.*

Proof. The codimension in $|I_{\tilde{B}/\tilde{T}}(2H_1 + H_2)|$ of the linear system $\tilde{Y} + |O_{\tilde{T}}(H_1)|$ equals $h^0(O_{\tilde{Y}}(2H_1 + H_2 - \tilde{B}))$. So it suffices to show that it is 1. Let $\tilde{f} := c_1(O_{\tilde{Y}}(2H_1 + H_2 - \tilde{B}))$, we remark that $\tilde{f} + \iota^* \tilde{f}$ is $\pi_+^* c_1(O_{Y_+}(2D - H_+)$. It is shown in the next proposition 8.9 that $2D - H_+$ is the class of an isolated curve F of degree 8 and arithmetic genus 0. Hence $\pi_+^* F = \tilde{F} + \iota^* \tilde{F}$, where the two summands are isolated copies of F. This implies that $h^0(O_{\tilde{Y}}(2H_1 + H_2 - \tilde{B})) = h^0(O_{\tilde{Y}}(\tilde{F})) = 1$. □

Now we can count dimensions:

(1) The family of pairs $(\tilde{T}, \tilde{Y})$ is naturally identified to the flag variety of pairs $< \tilde{T} >, < \tilde{Y} >$ of linear spaces of $\mathbf{P}^{9+}$. It is a rational variety of dimension 27. We denote it as $\mathfrak{F}$.

(2) We denote by $\mathfrak{P}$ the family of triples $(\tilde{T}, \tilde{Y}, O_{\tilde{Y}}(\tilde{B}))$. This latter family is a finite covering of the family of pairs $(\tilde{T}, \tilde{Y})$.

(3) The variety of triples $(\tilde{T}, \tilde{Y}, \tilde{B})$ is open in a projective bundle over $\mathfrak{P}$, with fibre $\pi_+^* |O_{Y_+}(B)|^+$ at $(\tilde{T}, \tilde{Y}, O_{\tilde{Y}}(\tilde{B}))$. In other words we only take in $|O_{\tilde{Y}}(\tilde{B})|$ those curves $\tilde{B}$ defined by a ι-invariant global section. We denote such a family as $\mathfrak{B}$.

(4) Finally $\mathfrak{G}$ embeds as an open set in the $\mathbf{P}^4$-bundle over $\mathfrak{B}$ whose fibre at $(\tilde{T}, \tilde{Y}, \tilde{B})$ is $|I_{\tilde{B}}(2H_1 + H_2)|$.

Due to the previous discussion we can conclude that $dim\ \mathfrak{G} = 36$ and moreover that $\mathfrak{G}$ is unirational if $\mathfrak{P}$ is irreducible and unirational. We will prove this latter property after some preparation. The proof relies on the special geometry of Reye congruences and of their Enriques models. As a preparation we consider a good 4-tuple $x = (\tilde{T}, \tilde{Y}, \tilde{B}, \tilde{S})$ and

$$Y_+ \subset \mathbf{P}^5.$$

From now on $Z := \{z_1 z_2 z_3 z_4 = 0\}$ will denote the fundamental tetrahedron of $\mathbf{P}^3$ and $e(Z)$ the union of its edges. Since we are assuming that $\tilde{Y}$ is general, every Enriques linear system $|D| \in \mathcal{E}(Y_+)$ defines a generically injective morphism $f : Y \to \mathbf{P}^3$ whose image is a sextic Enriques surface $Y' \in |I^2_{e(Z)/\mathbf{P}^3}(6)|$. In this sense the equation of Y' appears to be 'general':

$$z_1 z_2 z_3 z_4 q + a(z_1 z_2 z_3)^2 + b(z_1 z_2 z_4)^2 + c(z_1 z_3 z_4)^2 + d(z_2 z_3 z_4)^2 = 0.$$

The special feature of Y_+ is however present. Consider indeed $|2D - H_+| \in \mathcal{F}(Y_+)$. Differently from the case of a general Enriques surface, this linear system is in our case not empty: the next result is a private communication of I. Dolgachev, [10].

PROPOSITION 9. *$2H_+ - D \sim F$, where F is a connected curve of degree* 8 *and genus* 0.

Proof. Fano models Y_+ are also characterized by the non emptyness of $|H_+ - 2E_n| = |H_+ - 2E'_n|$, cfr. [11, 8]. Now $(2D - H_+) - (H_+ - 2E_n)$ is divisible by 2 in *Pic Y*. Then the existence of F follows from the next lemma. □

The next result is due to E. Loijenga:

LEMMA 15. *Let X be an Enriques surface, R a (-2)-curve on it of class r. Assume that $f - r$ is divisible by* 2 *in Pic X and that $f^2 = -2$. Then f is the class of an effective curve F.*

Proof. Putting $f = r + 2y$ consider the K3 cover $\pi : \tilde{X} \to X$. We have $\pi^* R = R_1 + R_2$, where R_1, R_2 are disjoint copies of R. Let r_i be the class of R_i, then $\pi^* f = r_1 + r_2 + 2\pi^* y = (r_1 + \pi^* y) + (r_2 + \pi^* y)$. Each summand is a -2 class on the K3 surface $\tilde{X}$. Hence it is effective. Then $\pi^* f$ is an effective class and the same is true for f. □

REMARK 6. In our situation we started with $x = (\tilde{T}, \tilde{Y}, \tilde{B}, \tilde{S})$. Then we can also observe directly that $\tilde{S} \cdot \tilde{Y} + \iota^* \tilde{S} \cdot \tilde{Y} = 2\tilde{B} + \tilde{F} + \iota^* \tilde{F} \in \mathcal{O}_{\tilde{Y}}(3H_1 + 3H_2)|$, where $\tilde{F}$ is effective. Putting $F = \pi_+(\tilde{F})$, it follows $2B + F \sim 3H_+$, that is, $F \sim 2D - H_+$.

Starting from our good 4-tuple $x = (\tilde{T}, \tilde{Y}, \tilde{B}, \tilde{S})$ let us consider the morphism

$$f : Y \to \mathbf{P}^3,$$

defined by $|D| = |2H_+ - B_+|$. Y_+ contains a connected curve $F \sim 2D - H_+$ such that $deg\, F = 8$ and $p_a(F) = 0$. Since Y_+ is general F does not contain components of degree ≤ 3. On the other hand $FD = 3$. The next lemma is almost immediate:

LEMMA 16. *$f : F \to \mathbf{P}^3$ embeds F as a skew cubic curve intersecting in one point each edge of the fundamental tetrahedron Z.*

Proof. Under our assumptions, either F is integral or it is the union of two integral smooth rational quartic curves. Assume that f/F is not an embedding. The, since $DF = 3$, the curve $f(F)$ is a plane cubic and hence $D - F$ is an effective divisorial class. Since $H_+(D - F) = 1$, then Y_+ contains a line: a contradiction. □

The lemma highlights a further characterization of a general Reye congruence, we briefly summarize. The family $\mathcal{Y}_-$ of Reye congruences Y_- is irreducible and a general member of it is constructed as follows. Let $X' \in |\mathcal{I}^2_{Z/\mathbf{P}^3}(6)|$ be a general sextic among those containing a general skew cubic F' which intersects each edge of the tetrahedron Z in one point. Let $f : X \to X'$ be the normalization map and let

$$|D| := |O_{X'}(1)| \, , \, F := f^*F'.$$

Then X is a smooth Enriques surface and $O_X(2D - F)$ defines an embedding of X

$$X_- \subset \mathbf{G} \subset \mathbf{P}^{5-}$$

as a Reye congruence. Actually the construction defines a stable rank two vector bundle $\mathcal{V}$ on X such that $c_1(\mathcal{V}) = O_X(2D - F)$ and $deg\, c_2(\mathcal{V}) = 3$. It turns out that $h^0(\mathcal{V}) = 4$ and that X_- is the image of the classifying map of $\mathcal{V}$.

$\mathcal{V}$ is known as a *Reye bundle*, cfr. [11] and [8]. Now we construct a rational family which is specially useful to dominate the family $\mathfrak{P}$:

At first we consider the edges $Z_1 \dots Z_6$ of the fundamental tetrahedron Z of $\mathbf{P}^3$ and the rational family $\mathcal{S}$ of general 6-tuples $s := (o_1, \dots, o_6) \in Z_1 \times \cdots \times Z_6$. As is well known s uniquely defines an integral skew cubic curve F_s. It is very easy to show that the restriction map $\rho_s : H^0(I^2_{e(Z)/\mathbf{P}^3}(6)) \to H^0(O_{F_s}(6))$ is surjective. Let $\mathbf{P}^{13} := |I_{e(Z)/\mathbf{P}^3}(6)|$; then, over the variety $\mathcal{S}$, we have the projective bundle

$$\mathcal{R} := \{(s, X') \in \mathcal{F} \times \mathbf{P}^{13} \; / \; F'_s \subset X'\},$$

whose fibre at s is $\mathbf{P}Ker\; \rho_s$. By the previous remarks the family $\mathcal{R}$ dominates the moduli space of Reye congruences. Indeed let (o, X') be general in $\mathcal{R}$ and let $f : X \to X'$ be the normalization map. Then, keeping the previous notations, X is an Enriques surface and $O_X(2D - F + K_X)$ embeds X as a Reye congruence.

Secondly we construct, over a suitable open set U of $\mathcal{R}$, the natural universal family $p : \mathcal{X} \to U$. The fibre of p at (s, X') is the normalization X of X' and a smooth Enriques surface. Let $f : \mathcal{X} \to \mathcal{X}'$ be the normalization map, over $\mathcal{X}$ there exist two line bundles $\mathcal{F}$ and $\mathcal{D}$ such that

$$\mathcal{D} \otimes O_X := f^* O_{X'}(1) \text{ and } \mathcal{F} := O_X(f^* F'_o).$$

As is well known the 1-dimensional space $H^1(O_X(F)) \cong Ext^1(O_X(D), \omega_X(D - F))$ uniquely reconstructs the Reye bundle $\mathcal{V}$ as an extension. Therefore, globalizing this construction, there exists a rank two vector bundle $\mathcal{V}_{\mathcal{X}}$ over $\mathcal{X}$, whose restriction to X is $\mathcal{V}$. We consider the rank 4 vector bundle $p_* \mathcal{V}^*_{\mathcal{X}}$, with fibre $H^0(\mathcal{V})^*$ at (s, X') and its Grassmann bundle $\mathcal{G} \to U$, with fibre the Grassmannian $G(2, H^0(\mathcal{V})^*)$ at (s, X'). Let

$$\phi : \mathcal{X} \to \mathcal{G} \subset \mathbf{P} p_* \mathcal{V}^*_{\mathcal{X}},$$

be the classifying map. Clearly ϕ embeds each fibre X as a Reye congruence. Up to shrinking U, we can assume that $\mathcal{G}$ is trivial, that is we can assume that

$$\mathcal{G} = U \times \mathbf{G} \subset U \times \mathbf{P}^{5-}.$$

Let $\mathcal{Y}_-$ be the family of Reye congruences of $\mathbf{G}$, now we consider the rational map

$$\psi : U \times Aut\; \mathbf{G} \to \mathcal{Y}_-$$

which is so defined: $\psi(o, X', a) := X_-$ is the surface $X_- = a \cdot \phi(X)$ of the family.

By construction U dominates the moduli of Reye congruences. Hence $U \times$ *Aut* **G** dominates $\mathcal{Y}_-$, which is therefore unirational. We can finally prove our theorem:

THEOREM 9. *$\mathfrak{P}$ is unirational so that $\mathfrak{G}$ is a unirational variety of dimension* 36.

Proof. To prove the unirationality result we use the rational family $U \times$ *Aut* **G** and the preceeding construction. We keep everywhere the same notations. Over $U \times$ *Aut* **G** we have the projective bundle $\mathbb{T} \to U \times$ *Aut* **G** defined as follows. Let $(s, X', a) \in U \times$ *Aut* **G**, consider the embedded Reye congruence $X_- = a \cdot \phi(X) \subset \mathbf{G}$. Via our usual diagram of linear projections we construct from X the symmetroid $\tilde{X} := \pi_-^{-1}(X_-) \subset \mathbf{P}^3 \times \mathbf{P}^3$. Then, by definition, the fibre of $\mathbb{T}$ at (s, X', a) is $\mathbb{T}_{(s,X',a)} := |I_{\tilde{X}/\mathbf{P}^3 \times \mathbf{P}^3}(1,1)|^*$.

We remark that a general element of this fibre is a smooth threefold $\tilde{T}$, complete intersection of three general symmetric bilinear forms vanishing on $\tilde{X}$. Moreover $\tilde{X}$ is endowed with the polarization $\mathcal{O}_{\tilde{X}}(\tilde{B}) := \pi_+^* \mathcal{O}_{X_+}(2H_+ - D)$ where $X_+ := \pi_+(\tilde{X})$, $|D| := |f^* \mathcal{O}_{X'}(1)|$ and $H_+ \in |\mathcal{O}_{X_+}(1)|$. Hence a general point of $\mathbb{T}$ defines a triple $(\tilde{T}, \tilde{X}, \mathcal{O}_{\tilde{X}}(\tilde{B})) \in \mathfrak{P}$. This defines a rational map $\tau : \mathbb{T} \to \mathfrak{P}$ sending (s, X', a) to $(\tilde{T}, \tilde{X}, \mathcal{O}_{\tilde{X}}(\tilde{B}))$.

The surjectivity of τ is clear and follows from the the previous results: a general triple $t := (\tilde{T}, \tilde{X}, \mathcal{O}_{\tilde{X}}(\tilde{B})$ is in the image of τ as follows. Let $X = \tilde{X}/ < \iota >$, choose $X' \in |I^2_{e(Z)/\mathbf{P}^3}(6)|$ so that $X' = f(X)$, where $f : X \to \mathbf{P}^3$ is defined by $\mathcal{O}_X(2H_+ - B_+)$. Furthermore consider $F \in |2D - H_+|$ and choose the 6-tuple s so that s defines the skew cubic curve $f(F)$. Then, we have $t = \tau(s, X', a)$ for some $a \in$ *Aut* **G**. □

We can finally conclude our discussion proving some new unirationality results, which are a goal of this paper. Let $\mathcal{P}$ be the image of the map $m : \mathfrak{G} \to \mathcal{P}_7^3$, we have:

THEOREM 10. *$\mathcal{P}$ is a unirational component of $\mathcal{P}_7^3$ which dominates $\mathcal{R}_7$.*

Proof. We have *dim* $\mathfrak{G} = 36$. Moreover the general fibre of m has dimension ≤ 18. Hence $\mathcal{P}$ has dimension ≥ 18. But $\mathcal{P}$ is a family of isomorphism classes of good triples. Hence, by proposition 8.4, the forgetful map $f : \mathcal{P} \to \mathcal{R}_7$ is finite onto its image. Since *dim* $\mathcal{R}_7 = 18$, it follows that $\mathcal{P}$ is 18-dimensional and dominates $\mathcal{R}_7$ via f. The unirationality of $\mathcal{P}$, and of $\mathcal{R}_7$, then follows from the previous theorem. □

We believe that $\mathcal{P}$ is the unique irreducible component of $\mathcal{P}_7^3$. The theorem implies the unirationality of $\mathcal{R}_7$. As remarked, this latter result has been recently proved in [14] by a different, much simpler method.

8. Appendix: existence of good triples and good 4-tuples

Finally, to show that the work performed so far is not empty, we prove that good triples and 4-tuples do exist. We prove that there exists a flat family

$$\tilde{\mathcal{U}} := \{\tilde{C}_u,\ u \in U\}$$

of connected curves $\tilde{C}_u \subset \mathbf{P}^3 \times \mathbf{P}^3$ of arithmetic genus 13 such that:

(1) U is smooth, connected with a distinguished point $o \in U$.
(2) $\tilde{C}_u$ is canonically embedded in $< \tilde{C}_u >$, it is smooth for $u \neq o$.
(3) ι acts on each $\tilde{C}_u$ as a fixed point free involution $i_u := \iota/\tilde{C}_u$.
(4) $(C_u, \eta_u, \tilde{L}_u)$ is a good triple for $u \neq o$.

Here we set $\tilde{L}_u := O_{\tilde{C}_u}(1,0)$, $C_u := \tilde{C}_u / < \iota >$. $\eta_u \in Pic^0 C$ defines the quotient cover $\pi_u : \tilde{C}_u \to C_u$. Moreover let $\tilde{T}_u :=< \tilde{C}_u > \cdot \mathbf{P}^3 \times \mathbf{P}^3$, we prove the existence of families $\tilde{\mathcal{U}}$ such that:

(5) $\forall u \in U,\ \tilde{T}_u = \tilde{T}$ where $\tilde{T}$ is general in its family.
(6) $\forall u \in U$, the linear system $|I_{\tilde{C}_u/\tilde{T}}(2H_1 + H_2)|$ is 3-dimensional.

Let $\{\tilde{S}_u, u \in U\}$ be any family such that $\tilde{S}_u \in |I_{\tilde{C}_u/\tilde{T}}(2H_1 + H_2)|$, then we have:

$$\tilde{S}_u \cdot \iota^* \tilde{S}_u := \tilde{B}_u \cup \tilde{C}_u \text{ and } \tilde{Y}_u :=< \tilde{B}_u > \cdot \tilde{T}.$$

In particular the latter family yelds the family of 4-tuples

$$x_u := (\tilde{T}, \tilde{Y}_u, \tilde{B}_u, \tilde{S}_u),\ u \in U.$$

For a family $\mathcal{U}$ as above, we will also see that:

(7) Y_u is a smooth and general symmetroid, provided $u \in U$ is general.

Since the previous conditions are satisfied, it follows:

THEOREM 11. *A general x_u is a good 4-tuple, defining the good triple $(C_u, \eta_u, \tilde{L}_u)$.*

Performing such a program is not difficult if a convenient curve $\tilde{C}_o$ is chosen to begin, we construct $\tilde{C}_o$ as follows.

We fix a smooth and general symmetroid $\tilde{Y}_o$ and a smooth 3-dimensional linear section $\tilde{T}$ of $\mathbf{P}^3 \times \mathbf{P}^3$ containing it. Let $P \subset \mathbf{P}^3$ be a general plane. By transversality we can assume that $P \times P$ is transversal to $\tilde{Y}_o$ and to $\tilde{T}$. This implies that

$$\tilde{E} := P \times P \cap \tilde{T}$$

is a smooth, elliptic sextic curve and that $\iota/\tilde{E}$ is a fixed point free involution on it. Furthermore $\tilde{Y}_o$ defines a transversal hyperplane section of $\tilde{E}$, we denote as

$$n := n_1 + n_2 + n_3 + \iota(n_1 + n_2 + +n_3),$$

We have $n = \tilde{Y}_o \cdot \tilde{E}$, let us point out that $\tilde{E}$ is uniquely constructed from the choice of n_1, n_2, n_3 in $\tilde{Y}_o$. Indeed the plane P is determined by $\pi_1(n_1)$, $\pi_1(n_2)$, $\pi_1(n_3)$. In particular we have the ι-invariant 4-dimensional linear space

$$N := <\tilde{Y}_o> \cap <\tilde{E}> = <n> .$$

The construction also yelds a line and a plane. Respectively they are:

$$N_+ := \pi_+(N) \text{ and } N_- := \pi_-(N).$$

Now we fix an Enriques linear system $|D_o| \in \mathcal{E}(\tilde{Y}_{o+})$ and then $\pi_+^*|D_o|$. A general element of $\pi_+^*|D_o|$ is a smooth connected curve of bidegree $(9,9)$ and genus 7 endowed with the fixed point free involution induced by ι. Since $dim\, \pi_+^*|D_o| = 3$ there exists $\tilde{D} \in \pi_+^*|D_o|$ passing through n_1, n_2, n_3 and hence through $\iota(n_1)$, $\iota(n_2)$, $\iota(n_3)$ as well. Up to moving n_1, n_2, n_3 in $\tilde{Y}_o$, we can assume that $\tilde{D}$ is smooth. This implies that:

PROPOSITION 10. *$\tilde{C}_o := \tilde{D} \cup \tilde{E}$ is a nodal connected curve of bidegree $(12,12)$. We have $p_a(\tilde{C}_o) = 13$ and $Sing\, \tilde{C}_o = \tilde{E} \cap \tilde{D}$; $i_o := \iota / \tilde{C}_o$ is a fixed point free involution on $\tilde{C}_o$.*

We fix the notations $C_o := \tilde{C}_o / < i_o >$ and η_o for the line bundle defining the quotient cover $\pi_o : \tilde{C}_o \to C_o$. The curves $C_{o-} := \pi_-(\tilde{C}_o)$ and $C_{o+} := \pi_+(\tilde{C}_o)$ are copies of C_o. We have also to consider the curves $E := \tilde{E} / < \iota >$, $D := \tilde{D} / < \iota >$, then

$$E_- := \pi_-(\tilde{E}) \text{ , } E_+ := \pi_+(\tilde{E}) \text{ and } D_- := \pi_-(\tilde{D}) \text{ , } D_+ := \pi_+(\tilde{D})$$

are respectively copies of E, D. Let $T = \pi_+(T)$, we have the usual diagram

$$\mathbf{P}^{5-} \supset \mathbf{G} \supset D_- \cup E_- \xleftarrow{\pi_-} \tilde{C}_o \xrightarrow{\pi_+} E_+ \cup D_+ \subset T - Sing\, T \subset <T>.$$

PROPOSITION 11.
(1) $\tilde{C}_o$ is canonically embedded in $<\tilde{T}>$.
(2) C_{o+} is canonically embedded in $<T>$ and $Sing\, C_{o+}$ spans N_+.
(3) C_{o-} is Prym canonically embedded in $\mathbf{P}^{5-}$ and $Sing\, C_{o-}$ spans N_-.

Proof. (1) Consider $\tilde{C}_o$: it follows from its construction that $h^0(\mathcal{O}_{\tilde{C}_o}(1)) = 13$. Moreover its two components $\tilde{D}$ and $\tilde{E}$ are glued along a hyperplane section of $\tilde{E}$, namely n. It is a standard property that then $\mathcal{O}_{\tilde{C}_o}(1) \cong \omega_{\tilde{C}_o}$. Since $<\tilde{E}>$ is not contained in $<\tilde{Y}_o>$, we have $<\tilde{C}_o> = <\tilde{T}>$ and $dim\ <\tilde{C}_o> = 12$. Hence (1) follows. (2) We have $Sing\, C_{o+} = E_+ \cap D_+$. Now $E_+ \cap D_+$ consists of the points $\pi_+(n_1), \pi_+(n_2), \pi_+(n_3)$. They span the line $N_+ = \pi_+(N)$ considered above, which is therefore trisecant to D_+. Then the same argument used in (1) implies that $C_{o+} = E_+ \cup D_+$ is canonically embedded in $<T>$. (3) The proof is very similar to the previous ones: we omit it. □

Consider the line bundle $\tilde{L}_o := \mathcal{O}_{\tilde{C}_o}(H_1)$, we have $\tilde{L} \otimes i_o^* \tilde{L} \cong \omega_{\tilde{C}_o}(1)$. $\tilde{C}_o$ is not smooth, however all the conditions in the definition of good triple are satisified:

PROPOSITION 12.

1 $h^0(\tilde{L}_o) = 4$.

2 $\tilde{L}_o$ *is very ample.*

3 *The Petri map of* $\tilde{L}_o$ *is surjective.*

4 $\tilde{T}$ *is smooth.*

Proof. Statements 2), 3), 4) follow easily from the definitions or from the proposition. To show 1) we recall that $\tilde{L}_o$ is $O_{\tilde{C}_o}(H_1)$. Let $\rho : H^0(\tilde{L}_o) \to H^0(O_{\tilde{D}}(H_1))$ be the natural restriction map. If $\rho(s) = 0$ then s is zero on $\tilde{D}$. Since $DE = 6$ and $\tilde{L}$ has degree 3 on $\tilde{E}$, it follows $s = 0$. Hence ρ is injective and it suffices to show that $h^0(O_{\tilde{D}}(H_1)) = 4$. Since $\tilde{D}$ lies in $\tilde{Y}_o$ we can use the standard exact sequence

$$0 \to O_{\tilde{Y}_o}(H_1 - \tilde{D}) \to O_{\tilde{Y}_o}(H_1) \to O_{\tilde{D}}(H_1) \to 0.$$

Passing to the long exact sequence, it suffices to show that $h^0(O_{\tilde{Y}_o}(H_1 - \tilde{D})) = 0$ and $h^1(O_{\tilde{Y}_o}(H_1 - \tilde{D})) = 0$. The former vanishing follows from $H_1(H_1 - \tilde{D}) = -1$. Moreover note that $(D - H_1)^2 = -2$. Hence $D - H_1 \sim R$ where R is effective and $p_a(R) = 0$. Since $H_2R = 3$ it follows that R is connected and isolated. Hence $h^1(O_{\tilde{Y}_o}(R)) = 0$. □

As it happens for a good triple $(C, \eta, \tilde{L})$ we have for the curve $\tilde{C}_o$:

LEMMA 17. $|I_{\tilde{C}_o/\tilde{T}}(2H_1 + H_2)|$ *is* 3*-dimensional.*

Proof. Let $\rho : H^0(I_{\tilde{C}_o/\tilde{T}}(2H - 1 + H_2)) \to H^0(O_{\tilde{Y}_o}(2H_1 + H_2 - \tilde{D}))$ be the natural restriction map. It is clear that there exists exists a unique reducible surface of class $2H_1 + H_2$ containing $\tilde{Y}_o \cup \tilde{C}_o$. This is $\tilde{Y}_o \cup X$, where $X = P \times \mathbf{P}^3 \cap \tilde{T}$ and P is the plane considered above. Hence *dim Ker* $\rho = 1$. Moreover $|O_{\tilde{Y}_o}(2H_1 - H_2 - \tilde{D})|$ is base point free of self intersection 2: we omit the standard proof of this property. But then *dim Im* $\rho \leq h^0(O_{\tilde{Y}_o}(2H_1 + H_2 - \tilde{D})) = 3$. Hence $h^0(I_{\tilde{C}_o/\tilde{T}}(2H_1 + H_2)) \leq 4$. On the other hand the opposite inequality is immediately checked. This implies the statement. □

Assume $\tilde{\mathcal{U}} := \{\tilde{C}_u,\ u \in U\}$ is the family we require. By semicontinuity and the lemma, it is not restrictive to assume that *dim* $|I_{\tilde{C}_u/\tilde{T}}(2H_1 + H_2)| = 3$, $\forall u \in U$. Therefore we can choose a family $\tilde{\mathcal{S}} := \{\tilde{S}_u, u \in U\}$ of surfaces $\tilde{S}_u \in |I_{\tilde{C}_u/\tilde{T}}(2H_1 + H_2)|$ so that $\tilde{S}_o = \tilde{Y}_o \cup X$ as in the latter proof. Let $u \neq o$, then we have $\tilde{S}_u \cdot \iota^*\tilde{S}_u = \tilde{C}_u \cup \tilde{B}_u$ as above. Moreover $\tilde{Y}_u := \tilde{T} \cdot < \tilde{B}_u >$ is a symmetroid, defined by $(\tilde{C}_u, \tilde{S}_u)$.

PROPOSITION 13. *Let* $\tilde{U}$ *and* $\tilde{Y}_u$ *be as above. Then* $\tilde{Y}_u$ *is smooth and general for a general* u.

Proof. Let $\tilde{B}' := \{(u,x) \in U \times \tilde{T} \ / \ x \in \tilde{S}_u \cdot \iota^*\tilde{S}_u - \tilde{C}_u\}$. Putting $\tilde{B}'_u := \{u\} \times \tilde{T} \cdot \tilde{B}'$, we have $\tilde{B}'_u = \tilde{S}_u \cdot \iota^*\tilde{S}_u - \tilde{C}_u$. Let $\tilde{B}_u$ be its closure, we know that $\tilde{S}_u \cdot \iota^*\tilde{S}_u = \tilde{C}_u \cup \tilde{B}_u$ for $u \neq o$. Moreover $\tilde{Y}_u :=< \tilde{B}_u > \cdot \tilde{T}$ is the symmetroid defined by $(\tilde{S}_u, \tilde{C}_u)$. For $u = o$ the construction yelds $\tilde{B}_o = \tilde{Y}_o$. We have constructed a family $\{\tilde{Y}_u,\ u \in U\}$ of symmetroids such that $\tilde{Y}_o$ is smooth and general. Then the same is true for a general u. □

Finally we can conclude our program: assume that the canonical curve

$$C_{o+} = E_+ \cup D_+ \subset T - Sing\ T$$

is smoothable in T. Then there exists a smooth, connected variety U as above and a flat family $\mathcal{U} := \{C_u \ /\ u \in U\}$ such that $C_o = E_+ \cup D_+$ and, for $u \neq o, C_u$ is a smooth, connected curve in $T - Sing\ T$ which is canonically embedded in $< T >$. Lifting this family by π_+ we obtain a flat family of curves

$$\tilde{\mathcal{U}} := \{\tilde{C}_u := \pi_+^* C_u,\ u \in U\}$$

such that $\tilde{C}_o = \tilde{E} \cup \tilde{D}$. By semicontinuity, a general $\tilde{C}_u$ satisfies the same properties proved for $\tilde{E} \cup \tilde{D}$ in the previous propositions 8.4 and 8.3. Hence the latter family is the family we aimed to have. Therefore, to prove theorem 8.1 and so to complete this paper, we are left to show that:

THEOREM 12. *$D_+ \cup E_+$ is smoothable in the Conte-Murre threefold T.*

Proof. For simplicity we put $C = C_o, D = D_+, E = E_+$. We denote the normal bundle of l$X \subset T - Sing\ T$ by N_X. Following [16] snd [24] it suffices to show that $h^1(N_C) = 0$ and that the natural map $H^0(N_C) \to T^1$ is surjective, where $T^1 := Coker\ (T_T \otimes O_C \to N_C)$. Applying theorem 4.5 of [16] and the identical proof used there for curves in $\mathbf{P}^3$, the following conditions are sufficient to deduce these properties: (i) $h^1(N_E) = h^1(N_D) = 0$, (ii) there exists a surjective map $H^0(M) \otimes O_D \to O_{Sing\ C}$ which factors through N_D, cfr. [16] proof of 4.5. To show (i) recall that D is contained in the hyperplane section $Y_+ := \pi_+(\tilde{Y}_o)$ of T and consider the standard exact sequence

$$0 \to O_D(D) \to N_D \to O_D(H_+) \to 0.$$

Since $O_D(D)$ is a Prym canonical line bundle, $h^1(O_D(D)) = 0$. The same is true for $O_D(H_+)$ because its degree is 9. Passing to the associated long exact sequence we obtain $h^1(N_D) = 0$. The same argument works for proving $h^1(N_E) = 0$. To prove (ii) we choose, among different possible ones, a proof highlighting the nice geometry of Reye congruences.

Finally $t := Sing\ C$ consists of three points. They are exactly the points on the trisecant line N_+ to the curve $D = \pi_+(\tilde{D})$ embedded in $\mathbf{P}^5 :=< Y_+ >$. Due to the previous exact sequence we have a morphism $\phi : H^0(O_D(D)) \otimes O_D \to O_S$, factoring through N_D. Note that $h^0(O_D(D)) = 3$. Hence ϕ is surjective if $h^0(O_D(D - t)) = 0$. This is indeed true: among different possible proofs we choose one highlighting again the nice geometry of Reye congruences.

Observe that $D \subset \mathbf{P}^5$ is a curve of genus 4 and degree 9 endowed with the trisecant line $N_+ =< t >$. This implies that $O_{D_+}(H_+ - t) \cong \omega_D$. On the other hand $\eta_D := O_D(K_{Y_+})$ is not trivial and $O_D(D) \cong \omega_D \otimes \eta_D$. Moreover the condition $h^0(O_D(D - t)) \geq 1$ is easily seen to be equivalent to $h^0(\eta_D(t)) \geq 1$.

Assume $h^0(\eta_D(t)) \geq 1$ and consider $t' \in |\eta_D(t)|$. Then we have $O_D(H_-) = O_D(H_+ + K_{Y_+}) \cong \omega_D(t')$. This means that, in the adjoint embedding $D_- \subset Y_- \subset \mathbf{P}^{5-}$, the image t'_- of t' spans a trisecant line to D_-. But Y_- is a Reye congruence model and it is well known that its only trisecant lines are trisecant to one of the 20 plane cubics of Y_-, [8, 11]. Such a curve is also embedded in Y_+ as a plane cubic A' and contains t'. It is also well known that, since A' is a plane cubic and D defines an Enriques linear system, then $DA \leq 3$. This follows because the classes of H_+, D, A' are well known in $Pic\ Y_+$, cfr. [5] IV 9. Hence we have $A' \cdot D = t'$. and $O_D(t) \cong O_D(A)$, where A is the unique element of $|A' + K_{Y_+}|$. It is standard to check that $h^0(O_D(A)) = 1$ and deduce that $< t >$ is the unique trisecant line to D.

We have shown that the non surjetivity of $\phi : H^0(O_D(D)) \otimes O_D \to O_t$ implies that $t = A \cdot D$, where A is a plane cubic. Now it is well known that the family $Trisec(Y_+)$ of trisecant lines to Y_+ is integral of dimension 3 and that the family of 0-dimensional schemes $\{l \cdot Y_+,\ l \in Trisec(Y_+)\}$ covers the surface Y_+, [8, 5]. Since $dim\ |D| = 3$ and D is chosen to be general in $|D|$, it is not restrictive to assume that no plane cubic of Y_+ contains t. Hence $h^0(O_D(D-t)) = 0$ and ϕ is surjective. □

References

[1] E. ARBARELLO, M. CORNALBA, P. GRIFFITHS, J.HARRIS, *Geometry of Algebraic Curves I* GMW series **267** 1-386 Springer (1985).

[2] L. BAYLE, *Classification des variétés complexes projectives de dimension trois dont une section hyperplane générale est une surface d' Enriques* J. Reine Angew. Math. **449** 9–63 (1994).

[3] I. A. CHELTSOV, *Rationality of an Enriques-Fano threefold of genus five* Izvestiya Mathematics **68**, 607–618 (2004).

[4] F. COSSEC, *Reye congruences* Trans. Am. Math. Soc. **280**, 737-751 (1983).

[5] F. COSSEC AND I. DOLGACHEV, *Enriques surfaces. I*, Progress in Mathematics series **76** 1-396 Birkhäuser Inc. Boston (1989).

[6] A. CONTE AND J. P. MURRE, *Algebraic varieties of dimension three whose hyperplane sections are Enriques surfaces*. Ann. Sc. Norm. Sup. Pisa **12**, 43-80 (1985).

[7] O. DEBARRE, *Sur le problème de Torelli pour les variétés de Prym* . American J. Math. **111** 111-134 (1989).

[8] A. CONTE AND A. VERRA, *Reye constructions for nodal Enriques surfaces* Transactions AMS **336**, 73-100 (1993).

[9] I. DOLGACHEV, *Classical Algebraic Geometry. A Modern View*. Cambridge UP 1-639 (2012).

[10] I. DOLGACHEV, *Private communication*.

[11] I. DOLGACHEV AND I. REIDER, *On rank* 2 *vector bundles with* $c_1^2 = 10$ $c_2 = 3$ *on Enriques surfaces*, Proc. USA-USSR Conf. on Algebraic Geometry, LNM Chicago (1989).

[12] R. DONAGI, *The fibers of the Prym map* Contemp. Math. **136**, 55-125 (1992).

[13] G. FARKAS AND K. LUDWIG, *The Kodaira dimension of the moduli space of Prym varieties* Journal of EMS **12**, 755-795 (2010).

[14] G. FARKAS AND A. VERRA, *Special K3 surfaces and moduli of Prym varieties* Preprint, see Oberwolfach reports **9**, 1857-1858 (2012).

[15] G. FARKAS, *Prym varieties and their moduli* in 'Contributions to Algebraic Geometry' EMS series of Congress Reports EMS Pub. House 215-257 (2012).

[16] R. HARTSHORNE AND A. HIRSCHOWITZ, *Smoothing algebraic space curves* in Algebraic Geometry, Sitges 1983, Lecture Notes Math. **1124**, 98–131 Springer Berlin (1985).

[17] A. KNUTSEN, A.F. LOPEZ, R. MUNOZ, *On the Extendability of Projective Surfaces and a Genus Bound for Enriques-Fano Threefolds*. J. Diff. Geometry **88**, 483-518 (2011).

[18] D. MUMFORD, *Prym varieties I* in Contributions to analysis, 325-350 Academic Press New York (1974).

[19] D.MUMFORD, *Varieties defined by quadratic equations* in 'Questions on algebraic varieties', Proceedings CIME 1970, Ed. Cremonese, Roma, 31-100 (1971).

[20] Y. PROKHOROV, *On Fano-Enriques varieties* . Mat. Sb. (english translation) **198**, 559-574 (2007).

[21] A. VERRA, *Rational prametrizations of moduli spaces of curves* in 'Handbook of Moduli' III ALM series 26, International Press, Boston, 431-501 (2013).

[22] A. VERRA, *On the universal principally polarized abelian variety of dimension 4* Contemp. Math. **465**, 253-274 (2008).

[23] T. SANO, *On classifications of non-Gorenstein Q-Fano 3-folds of Fano index 1* J. Math. Soc. Japan **47**, 369–380 (1995).

[24] E. SERNESI, *On the existence of certain families of curves* Invent. Math. **75**, 25-27 (1984)

[25] G. WELTERS, *A theorem of Gieseker-Petri type for Prym varieties*. Ann. Sci. École Norm. Sup. **18**, 671–683.(1985).

AMS Subject Classification: Primary 14K10, Secondary 14H10, 14H40

Alessandro VERRA
Universitá Roma Tre, Dipartimento di Matematica
Largo San Leonardo Murialdo
1-00146 Roma, ITALY
email:verra@mat.unirom3.it

Lavoro pervenuto in redazione il 02.07.2013.