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SEVERI'S RESULTS ON CORRESPONDENCES

Abstract. We analyze Severi's formula for the virtual number of fixed points of a correspondence T on a surface, and his notion of the rank of T. If the diagonal has valence zero, we verify Severi's formula with rank being the trace on the Neron-Severi group. Otherwise, we show that Severi's formula holds with a corrected notion of rank. We apply Severi's formula to complex surfaces with involution, both K3 surfaces and surfaces with $p_g = q = 0$.

1. Introduction

Severi developed a theory of correspondences in a series of papers which appeared in 1933, introducing the notions of *valences* and *indices*. One of the results achieved by Severi is a formula for the virtual number of fixed points of a correspondence on a smooth projective surface X. These papers are part of Severi's attempt to develop a theory of the series of equivalences on a surface. In fact Severi encountered (sometimes without being completely aware of what was going on!) the problem of not having a rigorous definition for the different equivalence relations among cycles, which are now known as rational, algebraic, homological and numerical equivalence. However as W. Fulton writes in [7, p.26]:

> *It would be unfortunate if Severi's pioneering works in this area were forgotten; and if incompleteness or the presence of errors are grounds for ignoring Severi's work, few of the subsequent papers on rational equivalence would survive.*

The above considerations indicate that Severi was often wrong and certainly too bold in making conjectures. However Severi was somehow able to perceive the *motivic* content of the matter, by considering correspondences and their action both on Chow groups and cohomology groups. In fact he was the first to relate the action of a correspondence $\Gamma \subset X \times X$ on the Chow group of 0-cycles on a smooth projective surface X to the cohomology class of Γ in $H^4(X \times X, \mathbb{C})$. In [15] (see also [4, 3.3]) he made a claim that in its original form is not correct but can be easily restated as what is now known as Bloch's conjecture.

CONJECTURE 1.1. *Let S be smooth projective surface over $\mathbb{C}$. If $p_g(S) = q(S) = 0$ the Chow group $CH_0(X)_0$ of 0-cycles of degree 0 vanishes.*

Bloch's conjecture is known to hold for all surfaces which are not of general type, see [2], and for many surfaces of general type, see [1] and [14].

In this note we will give a precise formulation of Severi's result on the virtual number of fixed points of a correspondence on a surface S (Theorem 4.7). We also provide a proof of what Severi claimed for the case when the diagonal Δ_S has valence

0 (Theorem 4.1). Then, in Sect. 5, we apply these results to a complex surface S with an involution, in the case $p_g(S) = q(S) = 0$ and in the case S is a K3 surface.

If X is a smooth projective variety, we will write $CH^i(X)$ for the Chow group of codimension i cycles on X, and write $A^i(X)$ for the Chow group with $\mathbb{Q}$-coefficients, $CH^i(X) \otimes \mathbb{Q}$. We write $NS(S)$ for the Nerón-Severi group of S, and $\rho(S)$ for the rank of $NS(S)$. If Γ is any correspondence on X, we write $\mathrm{trace}_{NS}(\Gamma)$ for the trace of Γ acting on the vector space $NS(X) \otimes \mathbb{Q}$. For example, $\mathrm{trace}_{NS}(\Delta_X)$ is the rank $\rho(X)$ of $NS(X)$.

2. Indices and valence of a correspondence

Let S be a smooth projective surface over C and let Γ be a correspondence in $CH^2(S \times S)$. The formula for the virtual number of fixed points of Γ is given in terms of the following numbers: the second Chern class $c_2(S)$, the trace of the action of Γ on the vector space $NS(S) \otimes \mathbb{Q}$, the *valence* v and the *indices* α, β of Γ (see Definition 2.2 below). The following formula appeared for the first time in 1933 in [17, p. 871]:

$$\deg(\Gamma \cdot \Delta_S) = \alpha(\Gamma) + \beta(\Gamma) + \delta(\Gamma) - v(\Gamma)(I + 1), \tag{2.1}$$

where $\delta(\Gamma)$ is the *rank* of Γ (see Definition 4.6) and I is the Zeuthen-Segre invariant, $I = c_2(S) - 4$. The same formula was reproduced in the first edition of Zariski's book on Algebraic Surfaces which appeared in 1935, and also in the second edition of it in 1971 (see [21, p. 246]).

The notion of the indices and valence of a correspondence Γ on a smooth projective variety X are about 100 years old and were well known to Severi and Lefschetz; see [11]. We give their precise definition below, following [7, §16], using the notion of degenerate correspondences in the Chow group of algebraic cycles modulo rational equivalence.

However, Severi's notion of the rank of Γ, as given in *op. cit.*, is rather obscure. Also, Severi's formula is based on the assumption that the correspondence Δ_S has rank 1, i.e., if Δ_S does not belong to the ideal $\mathcal{J}(S)$ of degenerate correspondences (see Definition 2.3). We will give the correct definition of the rank of a correspondence in Definition 4.6.

DEFINITION 2.2. *Let X be a smooth projective variety over a field k. The indices of a correspondence $\Gamma \subset X \times X$ are the numbers $\alpha(\Gamma) = \deg(\Gamma \cdot [P \times X])$ and $\beta(\Gamma) = \deg(\Gamma \cdot [X \times P])$, where P is any rational point on X; see [7, 16.1.4].*

The indices are additive in Γ, and $\beta(\Gamma) = \alpha({}^t\Gamma)$.

DEFINITION 2.3. *A correspondence is said of valence zero if it belongs to the ideal $\mathcal{J}(X)$ in $A^n(X \times X)$ of degenerate correspondences, i.e., the ideal generated by correspondences of the form $[V \times W]$, with V or W proper subvarieties of X. We say that a correspondence Γ has valence v if $\Gamma + v\Delta_X$ has valence 0.*

For example, Δ_X always has valence -1, but it may also have valence 0, as is the case for $X = \mathbb{P}^1$. If Γ_1, Γ_2 in $A^d(X \times X)$ have valences v_1, v_2 then $\Gamma = \Gamma_1 + \Gamma_2$ has valence $v_1 + v_2$, and $\Gamma_1 \circ \Gamma_2$ has valence $-v_1 v_2$ by [7, 16.1.5(a)].

If Δ_X does not have valence zero then the valence of a correspondence Γ is either unique or undefined. On the other hand, if Δ_X has valence zero and the valence of a correspondence Γ is defined, then $\Gamma \in \mathcal{J}(X)$, hence it has valence v for every $v \in \mathbb{Q}$. This is, for example, the case if X is a rational surface.

EXAMPLE 2.4 (Chasles-Cayley-Brill-Hurwitz). Let C be a curve of genus g. If $T \in A^1(C \times C)$ is a correspondence with valence v, then the Cayley-Brill formula is: $\deg(T \cdot \Delta_C) = \alpha(T) + \beta(T) + 2vg$. This is proven in [Fu 16.1.5(e)].

We thank the Referee for pointing out the following Lemma; cf. [19, 2.2.1].

LEMMA 2.5. *If Δ_X has valence zero, then rational, algebraic, homological and numerical equivalence coincide in $A^*(X)$.*

Proof. If $v(\Delta_S) = 0$ then the diagonal decomposes as $\sum_j c_j [V_j \times W_j]$. Hence for any cycle $Z \in A^*(X)$ we have

$$Z = \Delta_X \cdot Z = \sum_j c_j (Z \cdot V_j)[W_j]$$

It follows that numerically equivalent cycles are rationally equivalent. □

3. The Chow motive of a surface

Let $\mathcal{M}_{rat}(k)$ be the (covariant) category of *Chow motives* with $\mathbb{Q}$-coefficients over an algebraically closed field k of characteristic 0 and let $h(X)$ be the motive associated to a smooth projective variety X. If S is a smooth projective surface then the motive $h(S)$ has a *reduced Chow-Künneth decomposition* as in [10, 7.2.2] of the form

$$h(S) = h_0(S) \oplus h_1(S) \oplus h_2(S) \oplus h_3(S) \oplus h_4(S),$$

where $h_i(S) = (S, \pi_i, 0)$; each π_i is a projector whose cohomology class is the $(i, 4-i)$ component in the Künneth decomposition of Δ_S in $H^4(S \times S)$. Here H^* means any classical Weil cohomology theory, such as Betti cohomology with $\mathbb{Q}$ coefficients when $k = \mathbb{C}$.

The motive $h_2(S)$ further decomposes as $h_2^{\mathrm{alg}}(S) \oplus t_2(S)$ where $h_2^{\mathrm{alg}}(S)$ is the algebraic part of $h_2(S)$ and $t_2(S)$ is the *transcendental motive*. The motive $h_2^{\mathrm{alg}}(S)$ may be constructed by choosing a basis $\{E_1, \cdots, E_\rho\}$ for the $\mathbb{Q}$-vector space $NS(S)_{\mathbb{Q}}$ which is orthogonal in the sense that $E_i \cdot E_j = 0$ for $i \neq j$ and the self-intersections E_i^2 are nonzero. The correspondences $\varepsilon_i = \frac{[E_i \times E_i]}{(E_i)^2}$ are orthogonal and idempotent, so their sum

$$\pi_2^{\mathrm{alg}} = \varepsilon_1 + \cdots + \varepsilon_\rho = \sum_{1 \leq i \leq \rho} \frac{[E_i \times E_i]}{(E_i)^2} \tag{3.1}$$

is also an idempotent correspondence, and $h_2^{\mathrm{alg}}(S) = (S, \pi_2^{\mathrm{alg}}, 0)$. Since $\{E_i/(E_i)^2\}$ is a dual basis to the $\{E_i\}$, π_2^{alg} and $h_2^{\mathrm{alg}}(S)$ are independent of the choice of basis.

Setting $M_i = (S, \varepsilon_i, 0)$, $h_2^{\mathrm{alg}}(S)$ is the direct sum of the M_i, and each M_i is isomorphic to the Lefschetz motive $\mathbb{L}$ (see [10, 7.2.3]), so $h_2^{\mathrm{alg}}(S) \cong \mathbb{L}^{\oplus\rho}$. Setting $H^2_{\mathrm{alg}}(S) = \pi_2^{\mathrm{alg}} H^2(S)$, we also have isomorphisms $H^2_{\mathrm{alg}}(S) \cong NS(S)_{\mathbb{Q}}$.

The transcendental motive $t_2(S)$ is defined as $t_2(S) = (S, \pi_2^{\mathrm{tr}}, 0)$, where

$$\pi_2^{\mathrm{tr}} = \pi_2 - \pi_2^{\mathrm{alg}}.$$

Setting $H^2_{\mathrm{tr}}(S) = \pi_2^{\mathrm{tr}} H^2(S)$, we have $H^2(S) = H^2_{\mathrm{alg}}(S) \oplus H^2_{\mathrm{tr}}(S)$, so $H^2(t_2(S))$ is the "transcendental part" of $H^2(S)$. In addition, $A_0(t_2(S))$ is the Albanese Kernel $T(S)$; see [10, 7.2.3]. The Chow motive $t_2(S)$ does not depend on the choices made to define the refined Chow-Künneth decomposition, it is functorial on S for the action of correspondences, and it is a birational invariant of S (see [KMP]).

If $p_g(S) = 0$ then $H^2_{tr}(S) = 0$ and $t_2(S) = 0$ iff $T(S) = 0$, i.e., S satisfies Bloch's conjecture 1.1. The condition $t_2(S) = 0$ is also equivalent to the finite dimensionality of $h(X)$, see [8]. If $p_g(S) > 0$ then $T(S) \neq 0$, hence $t_2(S) \neq 0$.

Recall that $A^2(S \times S)$ is the endomorphism ring of $h(S)$ in $\mathcal{M}_{rat}$, with the diagonal correspondence Δ_S acting as the identity. Consider the ring projection

$$\Psi_S : A^2(S \times S) = \mathrm{End}_{\mathcal{M}_{rat}}(h(S)) \to \mathrm{End}_{\mathcal{M}_{rat}}(t_2(S)) \tag{3.2}$$

sending Γ to $\pi_2^{\mathrm{tr}} \circ \Gamma \circ \pi_2^{\mathrm{tr}}$. By construction, Ψ_S sends the class $[\Delta_S]$ of the diagonal to π_2^{tr}, which is the identity map of the motive $t_2(S)$. In fact, Ψ_S induces a ring isomorphism

$$\Psi_S : A^2(S \times S)/J_{\mathrm{nd}}(S) \xrightarrow{\cong} \mathrm{End}_{\mathcal{M}_{\mathrm{rat}}}(t_2(S)),$$

where $J_{\mathrm{nd}}(S)$ is the ideal of $A^2(S \times S)$ generated by the classes of correspondences which are not dominant over S by at least one of the two projections $S \times S \to S$, see [10, 7.4.3].

LEMMA 3.3. *Let S be a smooth projective surface with $q(S) = 0$. Then $\mathcal{J}(S) = J_{nd}(S)$ in $A^2(S \times S)$.*

Proof. From the definition of the ideals $\mathcal{J}(S)$ and $J_{\mathrm{nd}}(S)$ we get $\mathcal{J}(S) \subseteq J_{\mathrm{nd}}(S)$. Let $\Gamma \in J_{\mathrm{nd}}(S)$ such that Γ is not dominant over S under the first projection. We claim that Γ belongs to the ideal of degenerate correspondences. Γ vanishes on some $V \times S$, with V open in S, hence it has support on $W \times S$, with $\dim W \leq 1$. If $\dim W = 0$ then $\Gamma = \sum_i n_i [S \times P_i]$ in $A^2(S \times S)$, where P_i are closed points in S. Hence $\Gamma \in \mathcal{J}(S)$. If $\dim W = 1$ then $\Gamma \in A^1(W \times S)$, where $A^1(W \times S) = p_1^*(A^1(W)) \times p_2^*(A^1(S))$, with p_i the projections, because $H^1(S, O_S) = 0$ (see [9, p. 292]). Therefore $\Gamma \in \mathcal{J}(S)$. □

4. Severi's formula

In [16, p. 761], Severi claims that if on a surface S there exists a correspondence $\Gamma \in A^2(S \times S)$ with two distinct valences, i.e., $v(\Delta_S) = 0$, then S is "regular of genus 0', i.e $q(S) = p_g(S) = 0$. The following theorem verifies Severi's claim.

THEOREM 4.1. *Let S be a smooth projective surface. Then the following conditions are equivalent:*

(1) There exists a correspondence $\Gamma \in A^2(S \times S)$ with two distinct valences v and v';

(2) $v(\Delta_S) = 0$;

(3) $p_g(S) = q(S) = 0$ and S satisfies Bloch's conjecture;

(4) $A_0(S) \simeq \mathbb{Q}$.

Proof. Since both $\Gamma + v\Delta_S$ and $\Gamma + v'\Delta_S$ are in $\mathcal{J}(S)$, so is $(v - v')\Delta_S$. Therefore $\Delta_S \in \mathcal{J}(S)$, i.e., Δ_S has valence 0, which shows that (1) $\Longrightarrow$ (2). Clearly (2) $\Longrightarrow$ (1), by taking $\Gamma = \Delta_S$.

If $v(\Delta_S) = 0$ then $\Delta_S \in \mathcal{J}(S)$. Since $\mathcal{J}(S) \subseteq J_{\mathrm{nd}}(S)$ we get $\Psi_S(\Delta_S) = 0$, i.e., the identity map on $t_2(S)$ is 0 in $\mathcal{M}_{rat}$. This is equivalent to $t_2(S) = 0$ and hence $T(S) = A_0(t_2(S))$ equals 0. The condition $T(S) = 0$ forces $p_g(S) = 0$ and is a form of Bloch's conjecture. By Lemma 2.5, $A^1(S)$ injects into $H^2(S, \mathbb{Q})$, so $q(S) = 0$. This shows that (2) implies (3).The equivalence (3) $\Longleftrightarrow$ (4) is well known.

If $p_g(S) = q(S) = 0$ and Bloch's conjecture holds for S, then, by [3, Prop. 1], there exist a closed $V \subset S$ of dimension 0 and a divisor D on S such that $\Delta_S = \Gamma_1 + \Gamma_2$ in $A^2(S \times S)$, with Γ_1 supported on $V \times S$ and Γ_2 supported on $S \times D$. Hence $\Delta_S \in J_{\mathrm{nd}}(S)$. By Lemma 3.3 we get $\Delta_S \in \mathcal{J}(S)$, because $q(S) = 0$, so that $v(\Delta_S) = 0$, i.e., (3) $\Longrightarrow$ (2).

The equivalence (3) $\Longleftrightarrow$ (4) is well known. □

REMARK 4.2. For a surface S, if $v(\Delta_S) = 0$, then $p_g(S) = q(S) = 0$ and also $t_2(S) = 0$. Therefore $h(S) = \mathbf{1} \oplus \mathbb{L}^{\oplus \rho(S)} \oplus \mathbb{L}^2$, so that $h(S)$ coincides with the motive of S in the category of numerical motives $\mathcal{M}_{num}$.

EXAMPLE 4.3. Let S be a hyperelliptic surface over $\mathbb{C}$, i.e., a smooth projective surface with $p_g(S) = 0$ and $q(S) = 1$, which is isomorphic to a quotient $E \times F/G$, with E, F elliptic curves and G a finite group. By [2] the Albanese kernel of S vanishes, hence, by [3, Prop.1], $\Delta_S = \Gamma_1 + \Gamma_2$ with $\Gamma_1 \subset V \times S$, $\Gamma_2 \subset S \times D$ and $V \neq S, D \neq S$. Therefore $\Delta_S \in J_{\mathrm{nd}}(S)$. Because $q(S) \neq 0$, Theorem 4.1 implies that $v(\Delta_S) \neq 0$, so that $\Delta_S \notin \mathcal{J}(S)$.

PROPOSITION 4.4. *Let S be a smooth projective surface. Then for every correspondence $T \in A^2(S \times S)$ of valence zero:*

$$\deg(T \cdot \Delta_S) = \alpha(T) + \beta(T) + \mathrm{trace}_{NS(S)}(T).$$

Proof. Since the correspondence T has valence 0, it belongs to the ideal $\mathcal{J}(S)$ of degenerate correspondences. Therefore we may write $T = T_0 + T_1$, where

$$T_0 = \sum p_j [P_j \times S] + \sum q_k [S \times Q_k], \qquad T_1 = \sum m_i [D_i \times D'_i];$$

here $D_i, D'_i \in A^1(S)$ and P_j, Q_j are points. We may move $[D_i \times D'_i]$ within its class in

$A^2(S\times S)$ in such a way that it does not meet any of the $P_j\times S$ or $S\times Q_k$, so that

$$[D_i\times D_i']\cdot[P_j\times S]=[D_i\times D_i']\cdot[S\times Q_k]=0$$

for all i,j,k. With this reduction, we have $\alpha(T_1)=\beta(T_1)=0$, $\alpha(T_0)=\deg(T_0\cdot[P\times S])=\sum q_k$, and $\beta(T_0)=\deg(T_0\cdot[S\times P])=\sum p_j$. We also have

$$\deg(T_0\cdot\Delta_S)=\sum p_j+\sum q_k=\alpha(T)+\beta(T).$$

Now for any divisor C on S, we have $[P_j\times S]_*(C)=[S\times Q_k]_*(C)=0$. Thus T_0 acts as zero on $NS(S)$, so $\text{trace}_{NS}(T_0)=0$. Therefore we may assume that $T=T_1$, and need to evaluate

$$\deg(T_1\cdot\Delta_S)=\deg(\sum m_i[D_i\times D_i']\cdot\Delta_S)=\sum m_i\,(D_i\cdot D_i').$$

Choose an orthogonal basis $\{E_\ell, 1\le\ell\le\rho(S)\}$ for the $\mathbb{Q}$-vector space $NS(S)\otimes\mathbb{Q}$. In terms of this basis,

$$D_i=\sum\nolimits_k a_{ik}\,E_k,\qquad D_i'=\sum\nolimits_\ell b_{i\ell}\,E_\ell.$$

Since $E_k\cdot E_\ell=0$ when $k\neq\ell$, we may expand $D_i\cdot D_i'$ to get

$$\deg(T\cdot\Delta_S)=\sum_i m_i\,(D_i\cdot D_i')=\sum_{i,k} m_i\,a_{ik}\,b_{ik}\,(E_k)^2.$$

Because $(D_i\times D_i')_*(E_k)=(D_i\cdot E_k)D_i'=a_{ik}(E_k)^2D_i'$,

$$T_*(E_k)=(\sum m_i[D_i\times D_i'])_*(E_k)=\sum m_i a_{ik} b_{i\ell}(E_k)^2E_\ell.$$

Thus $\text{trace}_{NS(S)}(T_*)=\sum m_i a_{ik} b_{ik}(E_k)^2$, and the result follows. □

REMARK 4.5. If S is a smooth projective surface over $\mathbb{C}$ and Γ is a correspondence in $A^2(S\times S)$ then, by the Lefschetz fixed point formula (see [7, 16.1.15]):

$$\deg(\Gamma\cdot\Delta_S)=\sum_{0\le i\le 4}(-1)^i\,\text{trace}_{H^i(S)}(\Gamma).$$

Note that $\alpha(\Gamma)$ and $\beta(\Gamma)$ are the traces of Γ acting on $H^4(S)$ and $H^0(S)$, respectively. This is immediate from Definition 2.2, since $\pi_0=[S\times P]$ and $\pi_4=[P\times S]$ in our covariant setting. If $v(\Gamma)=0$ then $\Psi_S(\Gamma)=0$, so Γ acts as 0 on $t_2(S)$ and on $H^2_{tr}(S,\mathbb{Q})$. Therefore Proposition 4.4 says that a correspondence of valence 0 has trace 0 on the odd cohomology of S.

In [17, p. 871], Severi gave a definition of the *rank* $\delta(T)$ of a correspondence T of valence 0 and gave an argument asserting that if T is a correspondence of valence 0 on a surface S, then

$$\deg(T\cdot\Delta_S)=\alpha(T)+\beta(T)+\delta(T).$$

Severi pointed out that $\delta(T)$ may be computed by taking any decomposition of T into the sum of degenerate correspondences. Proposition 4.4 shows that $\delta(T)=\text{trace}_{NS}(T)$.

Severi then defines the *rank* of a correspondence Γ of valence v to be the number $\delta(\Gamma)$ such that $\delta(\Gamma)+v$ is the rank of the correspondence $T=\Gamma+v\Delta_S$ of valence 0. He also sets the rank of the the diagonal Δ_S to be 1, which is consistent when Δ_S has valence -1. Thus we may reinterpret Severi's definition as follows.

DEFINITION 4.6. *If $T\in A^2(X\times X)$ is a correspondence on a smooth projective variety X of valence 0, we define its rank $\delta(T)$ to be the trace* $\mathrm{trace}_{NS}(T)$ *of T acting on $NS(X)$. If Δ_X does not have valence 0, and Γ is a correspondence of valence v, we define the rank of Γ to be*

$$\delta(\Gamma)=\mathrm{trace}_{NS}(\Gamma)+v\,(\rho(X)-1).$$

With this definition of $\delta(\Gamma)$ we recover Severi's formula (2.1) for a surface S. If $v=v(\Gamma)$ then $T=\Gamma+v\Delta_S$ has valence 0 and by Proposition 4.4 we have

$$\begin{aligned}\delta(T)=\delta(\Gamma+v\Delta_S)&=\mathrm{trace}_{NS}(\Gamma+v\Delta_S)=\mathrm{trace}_{NS}(\Gamma)+v\,\mathrm{trace}_{NS}(\Delta_S)\\&=\mathrm{trace}_{NS}(\Gamma)+v\cdot\rho(S)=\delta(\Gamma)+v.\end{aligned}$$

Recall that the second Chern class of S satisfies $c_2(S)=\deg(\Delta_S\cdot\Delta_S)$ (see [7, 8.1.12]).

THEOREM 4.7. *Let S be a smooth projective surface. If Δ_S does not have valence 0 and $\Gamma\in A^2(S\times S)$ is a correspondence with valence v, then*

$$\deg(\Gamma\cdot\Delta_S)=\alpha(\Gamma)+\beta(\Gamma)+\mathrm{trace}_{NS}(\Gamma)+v\cdot(2+\rho(S)-c_2(S))\,.$$

Therefore

$$\deg(\Gamma\cdot\Delta_S)=\alpha(\Gamma)+\beta(\Gamma)+\delta(\Gamma)-v\cdot(c_2(S)-3)$$

as in (2.1).

Proof. By definition, $T=\Gamma+v\Delta_S$ has valence 0. From Proposition 4.4 we have

$$\deg(T\cdot\Delta_S)=\alpha(T)+\beta(T)+\mathrm{trace}_{NS}(T),$$

with $\alpha(T)=\alpha(\Gamma)+v$, $\beta(T)=\beta(\Gamma)+v$ and $\mathrm{trace}_{NS}(T)=\mathrm{trace}_{NS}(\Gamma)+v\,\rho(S)$ by additivity of the trace. We also have

$$\deg(T\cdot\Delta_S)=\deg(\Gamma\cdot\Delta_S)+v\cdot\deg(\Delta_S\cdot\Delta_S)=\deg(\Gamma\cdot\Delta_S)+v\cdot c_2(S).$$

Equating the formulas yields the desired formula for $\deg(\Gamma\cdot\Delta_S)$. □

5. Surfaces with an involution

We now consider the case of the correspondence on a smooth projective surface S over $\mathbb{C}$ which is the graph of an involution σ, i.e. $\Gamma_\sigma=\{(x,\sigma(x))\in S\times S\}$ and show that, if $p_g(S)=q(S)=0$, then $\deg(\Gamma_\sigma\cdot\Delta_S)$ is given by the same formula as in Proposition 4.4. We also apply Theorem 4.7 to the case of a K3 surface with an involution, see Example 5.5.

The fixed locus of σ consists of a 1-dimensional part D (possibly empty) and $k \geq 0$ isolated fixed points $\{P_1, \dots, P_k\}$. The images Q_i in S/σ of the P_i are nodes, and S/σ is smooth elsewhere. The blow-up X of S at the k set of isolated fixed points resolves these singularities, σ lifts to an involution on X (which we will still call σ), and the quotient $Y = X/\sigma$ is a desingularization of S/σ. The images $C_1, \dots, C_k$ in Y of the exceptional divisors of X are disjoint nodal curves, i.e., smooth rational curves with self-intersection -2. In summary, we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & S \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & S/\sigma. \end{array} \tag{5.1}$$

If $p_g(S) = q(S) = 0$ then $k = K_S \cdot D + 4$, see [5, 3.2].
For brevity, we write t for $\mathrm{trace}_{NS}(\Gamma_\sigma)$, the trace of the action of σ on $NS(S)_{\mathbb{Q}}$.

The following result is immediate from the Lefschetz fixed point formula and Remark 4.5, but we prefer to give an elementary proof.

THEOREM 5.2. *Let S be smooth projective surface over $\mathbb{C}$, with $p_g(S) = q(S) = 0$. Let σ be an involution on S and let $\Gamma_\sigma = (1 \times \sigma)\Delta_S$. Then*

$$\deg(\Gamma_\sigma \cdot \Delta_S) = \alpha(\Gamma_\sigma) + \beta(\Gamma_\sigma) + t = 2 + t = 4 - D^2.$$

Moreover σ acts as the identity on $H^2(S, \mathbb{Q})$ (i.e., $t = \rho(S)$) iff

$$K_S^2 = D^2 + 8.$$

Proof. Since S has no odd cohomology, the motive $h(S)$ has a Chow-Künneth decomposition

$$h(S) = h_0(S) \oplus h_2^{\mathrm{alg}}(S) \oplus t_2(S) \oplus h_4(S),$$

where $\pi_0 = [S \times P], \pi_4 = [P \times S]$, with P a rational point on S, so that

$$\Delta_S = [S \times P] + \pi_2^{\mathrm{alg}} + \pi_2^{\mathrm{tr}} + [P \times S].$$

Since $h_2^{\mathrm{alg}} = (S, \pi_2^{\mathrm{alg}}, 0)$, where $\pi_2^{\mathrm{alg}} = \sum \varepsilon_i$ is as in (3.1), the action of σ on h_2^{alg} is determined by $\Gamma_\sigma(\varepsilon_i) = \frac{[E_i \times \sigma(E_i)]}{(E_i)^2}$. Let a_{ij} be such that $\sigma(E_i) = \sum_j a_{ij} E_j$. Then

$$(1 \times \sigma)\pi_2^{\mathrm{alg}} \cdot \Delta_S = \sum_{1 \leq i \leq \rho} \frac{[E_i \times \sigma(E_i)]}{(E_i)^2} \cdot \Delta_S = \sum_{1 \leq i \leq \rho} a_{ii}.$$

Therefore $\deg(\Gamma_\sigma \cdot \pi_2^{\mathrm{alg}}) = \mathrm{trace}_{NS}(\sigma)$. We have $\pi_2^{\mathrm{tr}} H^2(S) = H^2_{\mathrm{tr}}(S) = 0$, because $p_g(S) = 0$, hence $((1 \times \sigma)\pi_2^{\mathrm{tr}}))H^2(S) = 0$. By [7, 19.2]

$$cl((1 \times \sigma)\pi_2^{\mathrm{tr}} \cdot \Delta_S) = cl((1 \times \sigma)\pi_2^{\mathrm{tr}}) \cdot cl(\Delta_S) = 0$$

in $H_0(S\times S)$, hence the 0-cycle $(\Gamma_\sigma\cdot\pi_2^{\mathrm{tr}})$ has degree 0 in $A_0(S\times S)$. We also have

$$\beta(\Gamma_\sigma)=\deg(\Gamma_\sigma\cdot\pi_0)=1\ ;\ \alpha(\Gamma_\sigma)=\deg(\Gamma_\sigma\cdot\pi_4)=1.$$

Summing up we get

$$\deg(\Gamma_\sigma\cdot\Delta_S)=\deg(\Gamma_\sigma\cdot\pi_0)+\deg(\Gamma_\sigma\cdot\pi_2^{\mathrm{alg}})+\deg(\Gamma_\sigma\cdot\pi_4)=2+t.$$

From [6, 4.2], we get $t=2-D^2$, hence $\deg(\Gamma_\sigma\cdot\Delta_S)=4-D^2$.
Finally the trace $t=2-D^2$ of σ on $H^2(S,\mathbb{Q})$ is at most the rank $\rho(S)$ of $NS(S)$. By Noether's formula $c_2(S)=12-K_S^2$ because $q(S)=p_g(S)=0$, hence $\rho(S)=10-K_S^2$. Therefore $D^2\geq K_S^2-8$, with equality iff $t=\rho(S)$, i.e., iff σ acts as the identity on $H^2(S,\mathbb{Q})$. □

REMARK 5.3. Theorem 5.2 gives a simplified version of a formula that appears in [17, p. 874] and also in [7, 16.2.4], showing that in this case $\deg(\Gamma_\sigma\cdot\Delta_S)$ only depends on D^2.

EXAMPLE 5.4. (1) Let S be a minimal surface of general type with $p_g(S)=0$ and $K_S^2=8$ with an involution. By [6, 4.4] σ acts as the identity on $H^2(S,\mathbb{Q})$, $\rho(S)=t=2$ and $D^2=0$. Therefore $\deg(\Gamma_\sigma\cdot\Delta_S)=4$. If $D=0$ the number k of isolated fixed points of σ is 4, otherwise k is even and $6\leq k\leq 12$.
(2) Let S be a numerical Godeaux surface with an involution σ. S is a minimal surface of general type with $p_g(S)=q(S)=0$ and $K_S^2=1$. By [5, 4.5] σ has $k=5$ isolated fixed points and $-7\leq D^2\leq 1$. If $D^2=-7$ then σ acts as the identity on $H^2(S,\mathbb{Q})$, $t=\rho(S)=9$ and $\deg(\Gamma_\sigma\cdot\Delta_S)=11=c_2(S)$. If $D^2=1$ then $t=1$ and $\deg(\Gamma_\sigma\cdot\Delta_S)=3$.

EXAMPLE 5.5. (*K3 Surfaces with an involution*) Let S be a complex K3 surface with an involution σ. We have $p_g(S)=1$ and $q(S)=0$, hence $H^{1,0}(S)=0$ and $H^{2,0}(S)\simeq\mathbb{C}$. Also $c_2(S)=24$, $H^2(S,\mathbb{Q})=NS(X)_{\mathbb{Q}}\oplus H^2_{\mathrm{tr}}(S,\mathbb{Q})$ with $\dim H^2(S,\mathbb{Q})=22$, $\dim H^2_{\mathrm{tr}}(S,\mathbb{Q})=22-\rho(S)$ and $\rho(S)=\dim(NS(S)_{\mathbb{Q}})\leq 20$. If σ is an involution on S then $\sigma(\omega)=\pm\omega$, where ω is a generator of the vector space $H^{2,0}(S)$. Then the same argument as in [20, 3.10] shows that σ either acts as +1 or as -1 on $H^2_{tr}(S,\mathbb{Q})$. The correspondence Γ_σ induces an involution $\pi_2^{tr}\circ\Gamma_\sigma\circ\pi_2^{tr}$ on $t_2(S)$, which we will still denote by σ. Let $\pi=1/2(\pi_2^{tr}-\sigma)$. Then $\pi\in A^2(S\times S)$ is a projector of $t_2(S)$. π_* acts either as 0 or as the identity on $H^{2,0}(S)\simeq\mathbb{C}$. In the first case $\ker(\pi_*)_{|H^2_{tr}(S)}$ is a sub-Hodge structure with $(2,0)$-component equal to $H^{2,0}(S)$. Its orthogonal complement is then contained in $NS(S)_{\mathbb{Q}}$, which implies that $\pi_*=0$ on $H^2_{tr}(S,\mathbb{Q})$. Therefore σ acts as the identity on $H^2_{tr}(S,\mathbb{Q})$. In the second case π_* equals the identity on $H^2_{tr}(S,\mathbb{Q})$, so that σ acts as - 1 on $H^2_{tr}(S,\mathbb{Q})$.

Now suppose that $\sigma(\omega)=-\omega$, so that σ acts as -1 on $H^2_{\mathrm{tr}}(S)$. By [22, 1.2], the quotient surface S/σ is an Enriques surface iff $S^\sigma=\emptyset$, while S/σ is a rational surface if $S^\sigma\neq\emptyset$. In either case $t_2(S/\sigma))=0$, by [2], hence $t_2(S)\neq t_2(S/\sigma))$ because $p_g(S)\neq 0$. Let Ψ_S be the map defined in (3.2). Then, by [13, Prop. 1(iv)], $\Psi_S(\Gamma_\sigma)=-id_{t_2(S)}=\Psi_S(-\Delta_S)$. Since $q(S)=0$ the kernel of Ψ_S is $\mathcal{J}(S)$ by Lemma 3.3. Hence $\Gamma_\sigma+\Delta_S\in\mathcal{J}(S)$, i.e., Γ_σ has valence 1. Since $p_g(S)\neq 0$, Δ_S cannot have valence 0 by

Theorem 4.1. The correspondence Γ_σ has indices $\alpha(\Gamma_\sigma) = \beta(\Gamma_\sigma) = 1$. By Theorem 4.7, if $t = \mathrm{trace}_{NS(S)}(\Gamma_\sigma)$ we get

$$\deg(\Gamma_\sigma \cdot \Delta_S) = 2 + t + \rho(S) - 22 = 2 + t - \dim\, H^2_{\mathrm{tr}}(S, \mathbb{Q}).$$

By [22, Th.3], σ has no isolated fixed points and a 1-dimensional (possibly empty) fixed locus D. Let τ be the trace of σ on $H^2(S, \mathbb{C})$. By the topological fixed point formula (see [6, 4]) $\tau + 2 = e(D)$, where $e(D) = -D^2 - D \cdot K_S$ is the topological Euler characteristic of D. By the holomorphic fixed point formula, see [6, 4]

$$-D \cdot K_S = 4(\mathrm{trace}_{H^0(S,O_S)}(\sigma) + \mathrm{trace}_{H^2(S,O_S)}(\sigma)) = 0,$$

because σ acts as -1 on $H^2(S, O_S)$. Therefore $e(D) = -D^2$ and

$$t = \tau - \mathrm{trace}_{H^2_{\mathrm{tr}}(S)}(\Gamma_\sigma) = \tau + \dim\, H^2_{\mathrm{tr}}(S) = -D^2 - 2 + \dim\, H^2_{\mathrm{tr}}(S),$$

because σ acts as -1 on $H^2_{\mathrm{tr}}(S)$, so that

$$\deg(\Gamma_\sigma \cdot \Delta_S) = -D^2.$$

If $S^\sigma = \emptyset$, in which case $S \to S/\sigma$ is the canonical unramified cover of an Enriques surface, then $D = 0$ and $(\Gamma_\sigma \cdot \Delta_S) = 0$.

If $S^\sigma \neq \emptyset$ then, by [22, Th. 3], the fixed locus D of σ is the disjoint union of m smooth curves C_i, with $1 \le m \le 10$, so that $D^2 = \sum_{1 \le i \le m}(2g(C_i) - 2)$, by the adjunction formula. If $m = 10$ and the curves C_i are rational then $D^2 = -20$ and $\rho(S) = 20$, see [22, Th. 3′(1)]. Therefore $\deg(\Gamma_\sigma \cdot \Delta_S) = 20 = \rho(S)$, $\dim\, H^2_{\mathrm{tr}}(S) = 2$ and $t = 20 = \rho(S)$. Hence σ acts as the identity on $NS(S)_{\mathbb{Q}}$. S is the unique (up to isomorphism) complex K3 surface described in [18, Th. 1].

If $\sigma^*(\omega) = \omega$, with $\omega \in H^{2,0}(S)$, then σ is a *Nikulin involution*. A Nikulin involution has 8 isolated fixed points, no 1-dimensional fixed locus and the desingularization Y of the quotient surface S/σ is a K3 surface, see [12, 5.2]. Therefore $c_2(S) = c_2(Y) = 24$, and $\rho(S) = \rho(Y)$ because σ acts as the identity on $H^2_{tr}(S)$. From the Lefschetz fixed point formula (see Remark 4.5) we get

$$\deg(\Gamma_\sigma \cdot \Delta_S) = \tau + 2 = 8$$

where τ is the trace of σ on $H^2(S, \mathbb{C})$. In order to compute $\deg(\Gamma_\sigma \cdot \Delta_S)$ using Theorem 4.7 we should know that the correspondence Γ_σ has a a valence v. By [13, Th.4] this is equivalent to $t_2(S) \simeq t_2(Y)$ in which case $v(\Gamma_\sigma) = -1$. Then we get

$$\deg(\Gamma_\sigma \cdot \Delta_S) = 2 + t - (\rho(S) - 22)$$

where

$$t = \mathrm{trace}_{NS}(\Gamma_\sigma) = \tau - \dim H^2_{tr}(S) = \tau - (22 - \rho(S)) = \rho(S) - 16,$$

so that $\deg(\Gamma_\sigma \cdot \Delta_S) = 8$. The condition $t_2(S) \simeq t_2(Y)$ is satisfied if the K3 surface S has a finite dimensional motive, which is in particular the case if $\rho(S) = 19$ or $\rho(S) = 20$, see [13, Thms. 2 and 3].

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