Rend. Sem. Mat. Univ. Politec. Torino
Vol. 71, 3–4 (2013), 473 – 492

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ON THE JOINT WORK OF ALBERTO CONTE, MARINA MARCHISIO AND JACOB MURRE

To my friend, Alberto Conte, for his 70^{th} *birthday.*

This is - a slightly expanded - written version of the lecture delivered by the author on the occasion of the seventieth birthday of Alberto Conte. It is a *survey* of joint work done by Alberto and the author starting from around 1975; later in the end of the nineties we were joined by Marina Marchisio.

The topics of our joint work are:

1. The non-rationality of the quartic threefold with a double line [9]
2. The Hodge conjecture for special fourfolds [10] and [11]
3. On Fano's theorem on threefolds whose hyperplane sections are Enriques surfaces [13]
4. On the definition and on the nature of the singularities of Fano threefolds [13]
5. On Morin's work on unirationality of hypersurfaces [14], [15], [16], [17].

The majority of these subjects have their roots in classical Italian algebraic geometry (topics 1, 3 and 5).

In the lecture I have concentrated on topics 1 and 5 and we do the same here. The purpose was to give the audience some idea of the beautiful geometry involved in these subjects. The author wants to stress however that this is *only a survey lecture*, for the details one has to go to the original papers.

The basic concepts in topic 1 and 5 are the notions of *rationality* and *unirationality*. Therefore we want to recall these notions in the precise form they will be used here.

Basic notions

Let X_n be a projective irreducible variety of dimension n defined over a field k. We assume throughout that k is of *characteristic zero* but not necessarily algebraically closed.

X is *rational over* k if there exists a *birational map* $f : \mathbb{P}_n \overset{\sim}{\dashrightarrow} X$ which is *itself also defined over* k.

X is *rational* if this is true over k, where k is the algebraic closure of k (i.e. f is defined over a finite extension k_1 of k).

X is *unirational over k* if there exists a *dominant rational map* $f : \mathbb{P}_n - - \twoheadrightarrow X$ *defined over k* which is generically finite to one (i.e. there exists a Zariski open set $U \subset X$ over which f is finite).

X is *unirational* if this is true over *k* (again f is then defined over a finite extension k_1 of k).

Acknowledgement

The author thanks the organizers for the invitation to lecture in this special conference. Very special thanks to Marina Marchisio for TeXing the manuscript.

1. Quartic threefolds with a double line

1.1. The theorem which we proved is the following.

THEOREM 1. ([9])
Let $V = V_3(4)$ *be a threefold af degree* 4 *in projective space* $\mathbb{P}_4$ *defined over a field k. Assume that V has a double line* l_0 *but is "otherwise general". Then V is unirational, but* not *rational.*

Remarks

1. We shall explain the term "otherwise general" below; we also use sometimes the expression "sufficiently general".

2. The inspiration for the above theorem came from the famous paper [6] of Clemens and Griffiths where they proved the non-rationality of a smooth cubic threefold (a fact claimed classically by Fano). The proof of Clemens-Griffiths is in characteristic zero and their basic tool is the intermediate jacobian. In the papers [37] and [38] the author studied over any field k with $char(k) \neq 2$ the group of 1-dimensional algebraic cycles on a cubic threefold and related that group to a so-called Prym variety. Based upon Mumford's theory of Prym varieties ([36]) he obtained in this way another proof of the non-rationality of the cubic threefold. The proof of the above theorem follows this more algebraic proof.

1.2. Equation for V

Let k_1 be a field. Assume that the double line $l_0 \subset V \subset \mathbb{P}_4$ is defined over k_1. We choose homogeneous coordinates $(x : y : z : u : v)$ in $\mathbb{P}_4$ such that the line l_0 is given by the equations

$$x = y = z = 0. \tag{1}$$

Then the equation of V is

$$\begin{aligned} &a_{02}(x,y,z)u^2+2a_{11}(x,y,z)uv+a_{20}(x,y,z)v^2+ \\ &+2a_{12}(x,y,z)u+2a_{21}(x,y,z)v+a_{22}(x,y,z)=0 \end{aligned} \tag{2}$$

with $a_{ij}(x,y,z)\in k[x,y,z]$ homogeneous polynomials of degree $i+j$ with coefficients in a field $k\supset k_1$ (so k is a field of definition for V). The coefficients of these polynomials can be considered as coordinates in some projective space $\mathbb{P}_N$ which parametrizes the quartic 3-folds with a double line (of course this is not the "moduli" space). "Otherwise general" means that there is a k_1-Zariski open subset $U\subset\mathbb{P}_N$ such that for V in U the theorem is true. In particular we take U such that the $V_3(4)\in U$ are smooth outside the double line $l_0\subset V_3(4)$ but there are more restrictions which could - but are cumbersome to - be made explicit.

1.3. The 2-planes through l_0

Take in $\mathbb{P}_4$ a 2-dimensional linear space (a *2-plane* for short) N such that $N\cap l_0=\emptyset$. To be specific let us take N with equations

$$u=v=0. \tag{3}$$

Take a point $T=(a:b:c:0:0)\in N$ and let $L_T=\langle l_0,T\rangle$ be the linear span. N parametrizes the 2-planes through l_0. Consider the intersection

$$V\cdot L_T=2l_0+K_T \tag{4}$$

with K_T a *conic* in L_T.

To make things explicit: a point $P\in L_T$ has coordinates $P=(ta:tb:tc:u:v)$ in $\mathbb{P}_4$ and (homogeneous) coordinates $(t:u:v)$ in L_T and the equation of the conic K_T in L_T is

$$\begin{aligned} &a_{02}(a,b,c)u^2+2a_{11}(a,b,c)uv+a_{20}(a,b,c)v^2+ \\ &+2a_{12}(a,b,c)ut+2a_{21}(a,b,c)vt+a_{22}(a,b,c)t^2=0. \end{aligned} \tag{5}$$

So K_T is defined over the field $k(a,b,c)$.

In L_T we have the intersection

$$K_T\cdot l_0=R_1+R_2. \tag{6}$$

Note that in general R_1 and R_2 are not rational in $k(a,b,c)$, but only conjugate over $k(a,b,c)$.

1.4. The conic bundle associated with $V_3(4)$

Starting from $V=V_3(4)\subset\mathbb{P}_4$ with the double line l_0 we have now also the variety

$$V_1=\{P\in K_T;\quad T\in N\} \tag{7}$$

with the morphism $p : V_1 \to N$ with $p(P) = T$. The fibres of p are the conics K_T and V_1 is a *conic bundle* over N. If V is general then V_1 is smooth, in fact V_1 is obtained by blowing up V along l_0 (see [9], prop. 1.17). Clearly V_1 is birational with V over k and hence *it suffices to prove the theorem for* V_1.

1.5. Two special curves in N

Consider in N the curve

(8) $$\Delta = \{T \in N; \quad K_T = l'_T + l''_T\}.$$

Over Δ the conic K_T degenerate in two lines; Δ is called the *discriminant curve* for V_1. Note that in general l'_T and l''_T are only conjugate over, and not defined in, the field $k(T)$.

Consider also the curve (see (6))

(9) $$\nabla = \{T \in N; \quad R_1 = R_2\}$$

and put

(10) $$\mathcal{B} = \Delta \cap \nabla.$$

∇ is the curve for which the K_T is tangent to l_0. From the equation (5) we deduce easily the following facts ([9], prop. 1.17 and lemma 1.5 and 1.7).
For V general the Δ and ∇ are irreducible and smooth over $k(T)$, $deg(\Delta) = 8$ hence $g(\Delta) = 21$, $deg(\nabla) = 4$ hence $g(\nabla) = 3$. For $T \in \Delta$ we have $l'_T \neq l''_T$, $l'_T \neq l_0 \neq l''_T$. $\mathcal{B}$ consists of 32 different points and for $T \in \mathcal{B}$ we have $R_1 = R_2 \in l'_T \cap l''_T \cap l_0$.

1.6. The associated Prym variety

Consider in V the lines l different from l_0 but meeting l_0, i.e.

(11) $$\Delta^* = \{l \subset V, \ l \cap l_0 \neq \emptyset, \ l \neq l_0\}.$$

Δ^* consists clearly of the lines l'_T and l''_T from (8) and is therefore a curve and

(12) $$q : \ \Delta^* \to \Delta$$

is a 2 : 1 covering. In fact since $l'_T \neq l''_T$ the q is an *étale* 2 : 1 cover and if V is general then Δ^* is irreducible; from Hurwitz we get $g(\Delta^*) = 40$. Let σ be the *involution* exchanging l'_T and l''_T

(13) $$\sigma(l'_T) = l''_T.$$

From (12) we get for the Jacobians

(14) $$q_* : \ J(\Delta^*) \to J(\Delta), \ \ q^* : \ J(\Delta) \to J(\Delta^*), \ \ q_* \cdot q^* = q.$$

Mumford has studied this situation in great detail. The involution σ gives also an involution σ on $J(\Delta^*)$ (by abuse of language denoted by the same letter). Mumford proved the following

THEOREM 2. ([36]):
This involution gives on $J(\Delta^*)$ a decomposition

$$J(\Delta^*) = J(\Delta) + Pr(\Delta^*/\Delta) \tag{15}$$

with σ operating as $+1$ on $J(\Delta)$ and as -1 on $Pr(\Delta^*/\Delta)$ and $Pr(\Delta^*/\Delta)$ is itself also an abelian variety, the so-called Prym variety *of $q: \Delta^* \to \Delta$. Furthermore $J(\Delta) \cap Pr(\Delta^*/\Delta) = \{0, a\}$ where a is a 2-torsion point. Moreover the Θ-divisor on $J(\Delta^*)$ induces a polarization 2Ξ on $Pr(\Delta^*/\Delta)$ with Ξ a principal polarization.*

Moreover Mumford studied the couple $(Pr(\Delta^*/\Delta), \Xi)$ carefully and did give a complete list of cases for which this couple is itself a jacobian variety of a curve with corresponding theta-divisor or a product of such a situation.

In our case $\dim J(\Delta^*) = 40$, $\dim J(\Delta) = 21$, hence $\dim Pr(\Delta^*/\Delta) = 19$. From Mumford's list we conclude

FACT. ([9], section 6, lemma 6.1 and cor. 6.2):
In our case $(Pr(\Delta^*/\Delta), \Xi)$ is as principally polarized abelian variety neither a jacobian of a curve, nor a product of such.

1.7. A double cover V_1' of the conic bundle V_1

Introduce first the double cover $f: N' \to N$ where

$$N' = \{(T,R);: T \in N, R \in K_T \cap l_0\} \tag{16}$$

and next the double cover $f_1: V_1' \to V_1$ given by

$$V_1' = V_1 \times_N N'.$$

Clearly we have a commutative diagram

$$\begin{array}{ccccc} V & \longleftarrow & V_1 & \xleftarrow{f_1} & V_1' \\ & & p\downarrow & & \downarrow p' \\ & & N & \xleftarrow[f]{} & N' \end{array}$$

Let $K_{T,R} = p'^{-1}(T,R)$, clearly $K_{T,R} = K_T$ and hence

$$V_1' = \{(T,R,P); T \in N, R \in K_T \cap l_0, P \in K_{T,R} = K_T\}. \tag{17}$$

Clearly N' is a surface, a double cover of the plane N, branched over the smooth curve ∇ of degree 4, hence it is well-known that N' is a *smooth rational surface*. Furthermore V_1' is a conic bundle over N', but now the conics $K_{T,R}$ have a rational point R. Therefore V_1' is a *rational threefold* and hence:

PROPOSITION 1. *V_1, and therefore also V itself, is* unirational.

From the 2 : 1 covering $f_1 : V_1' \to V_1$ given by $f_1(T,R,P) = (T,P)$ we get also an *involution* τ on V_1'. Namely

$$\tau(T,R_1,P) = (T,R_2,P) \tag{18}$$

where $K_{T,R_1} \cdot l_0 = K_T \cdot l_0 = R_1 + R_2$ (see 16).

Furthermore consider the curve

$$\Delta' = f^{-1}(\Delta)$$

on N', clearly this is a 2 - 1 covering of Δ in N. For $(T,R) \in \Delta'$ the fibre $p'^{-1}(T,R) = l_T' \cup l_T''$ with one of the lines, say l_T', going through R. To be explicit, let us introduce the notation $l^*_{T,R} = l_T'$ and $l^{**}_{T,R} = l_T''$ so that we can write for the fibre

$$p'^{-1}(T,R) = \{l^*_{T,R} \cup l^{**}_{T,R}, \text{ with } R \in l^*_{T,R}\}. \tag{19}$$

Finally we have clearly a *birational morphism*

$$\alpha : \Delta^* \to \Delta'.$$

Concerning α^{-1}, since Δ' is smooth outside the points $\mathcal{B}' = f^{-1}(\mathcal{B})$, the α^{-1} is defined outside this finite set of points.

1.8. Another model V_1'' birational with V_1'

Consider the variety

$$V_1'' = \{(T,R,m);\ T \in N, R \in K_T \cap l_0, m \text{ a line in } L_T \text{ with } R \in m\}. \tag{20}$$

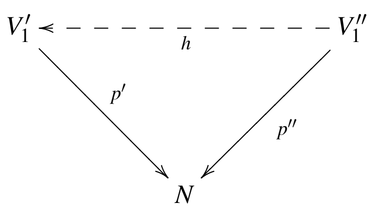
Clearly we have a morphism $p'' : V_1'' \to N'$ and also a *birational map* $h : V_1'' \dashrightarrow V_1'$ defined by

$$h(T,R,m) = (T,R,P) \tag{21}$$

where $P \in K_T$ is defined by

$$K_{T,R} \cdot m = R + P \tag{22}$$

(recall $K_{T,R} = K_T$). Clearly we have a commutative diagram



Note that V_1'' is a *projective bundle* over the rational surface N', hence rational. Let us denote the "inverse" of h by g, so

$$g(T,R,P) = (T,R,\langle RP\rangle),$$

where $\langle -,-\rangle$ means - as usual - the linear span.

Let $\mathcal{F}_h'' \subset V_1''$ be the *fundamental locus* of h, i.e. the points where h is not defined.

LEMMA 1. *$\mathcal{F}_h'' = \Lambda'' \cup \mathcal{B}_1''$, where $\Lambda'' \subset V_1''$ is the* curve

$$\Lambda'' = \{(T,R,m = l_T'),\ T \in \Delta,\ R \in l_0 \cap l_T'\} \tag{23}$$

and $\mathcal{B}_1''$ is the finite set of points $\mathcal{B}_1'' = \{(T,R,m = l_T'')\}$*, with* $(T,R) \in \mathcal{B}' = f^{-1}(\mathcal{B}) \subset N'$ *with $\mathcal{B} \subset N'$ from* (10).

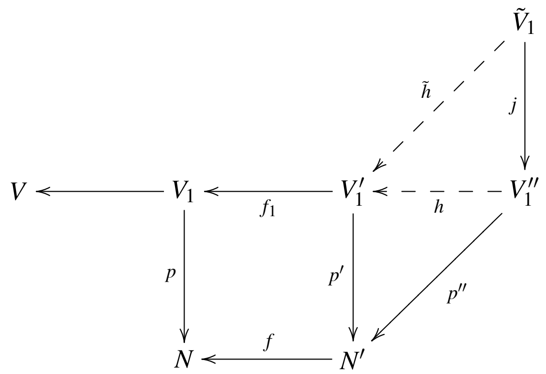
Proof. This follows immediately from the definition of h in (21). □

LEMMA 2. *The curve Λ'' is birational with Δ^* and Δ' and non-singular (hence isomorphic with Δ^*). In particular $Pr(\Delta^*/\Delta) = Pr(\Lambda''/\Delta)$.*

Proof. The birationality with Δ' (and hence with Δ^*) is immediately clear from the description in Lemma 1. For the non-singularity we refer to Lemma (3.6) of [9].

Now we blow up $\mathcal{F}_h''$ and we get $\tilde{V}_1 = B_{\mathcal{F}_h''}(V_1'')$ with maps $\tilde{h} : \tilde{V}_1 \dashrightarrow V_1'$, $j : \tilde{V}_1 \to V_1''$ all over N' and $\tilde{h}$ is a *birational* map with inverse $\tilde{g} : V_1' \dashrightarrow \tilde{V}_1$.

We have now the following big diagram



and $h^{-1} = g$ and $\tilde{h}^{-1} = \tilde{g}$.

Recall that from the $2:1$ covering f_1 we get an involution τ on V_1' and hence we have also involutions say τ_1 on V_1'' and $\tilde{\tau}$ on $\tilde{V}_1$ (all over V_1). □

1.9. The behavior of the involutions on the Chow groups

Recall that if X is a smooth, irreducible, projective variety then we have the Chow groups $CH_{d-i}(X) = CH^i(X)$ of algebraic cycles on X of codimension i modulo rational equivalence. We are interested in the subgroups $A^i(X) := CH^i_{alg}(X) \subset CH^i(X)$ of the cycle classes algebraically equivalent to zero.

Since V_1'' is a projective bundle over the rational surface N' (see 7) we have $A^2(V_1'') = 0$ and then by general theorems of blowing up (see [26],section 9)

$$A^2(\tilde{V}_1) = J(\Lambda'') = J(\Delta^*).$$

From the birational transformation $\tilde{h} : \tilde{V}_1 \dashrightarrow V_1'$ and its inverse (and the analysis of the fundamental loci) we have $A^2(V_1') = A^2(\tilde{V}_1)$ (see [9], section 3 for details). Therefore

$$A^2(V_1') = J(\Delta^*) = J(\Delta) + Pr(\Delta^*/\Delta) \tag{24}$$

on which the involutions τ and σ operate.

The *key point* is the *comparison of these two actions*. $J(\Delta)$ is the invariant part under σ and $Pr(\Delta^*/\Delta)$ the anti-invariant part (see the theorem in section 1.6).

KEY LEMMA. Let $\eta \in J(\Delta^*) \simeq A^2(V_1')$. Then $\tau(\eta) = -\sigma(\eta)$.

Proof. (outline)

We have the fibration $p' : V_1' \to N'$ and we consider the fibres $p'^{-1}(T,R)$ for the points $(T,R) \in \Delta' = f^{-1}(\Delta)$ (see sections 1.5 and 1.7). Because of the moving lemma we can restrict our attention to the points $(T,R) \in \Delta_0'$ with $\Delta_0' = \Delta' - \mathcal{B}'$ (i.e. $T \notin \mathcal{B} = \Delta \cap \nabla$, see (10)).

We have by (19)

$$p'^{-1}(T,R) = l^*_{TR} + l^{**}_{TR} = K_{TR}(= K_T) \tag{25}$$

with $R \in l^*_{TR}$. We examine now the action of τ on the 1-cycle $l^*_{TR} \in CH^2(V_1')$.

For $(T,R,P) \in l^*_{TR}$, i.e. $P \in l^*_{TR} = l'_T$ we get (see (18)):

$$\tau(T,R,P) = (T,\sigma(R),P). \tag{26}$$

Therefore as element of $CH^2(V_1')$

$$\tau(l^*_{TR}) = l^{**}_{T\sigma(R)} = K_{T\sigma(R)} - l^*_{T\sigma(R)}. \tag{27}$$

Next take a (certain) fixed point $(T_0,R_0) \in N' - \Delta'$.

Since N' is a *rational surface* we can move by *rational equivalence on* N' every point (T,R) to (T_0,R_0). Therefore

$$K_{TR} = K_{T_0R_0} \tag{28}$$

in $CH^2(V_1')$.

Now let $\eta = \sum_i (T_i,R_i) \in J(\Delta^*) \subset CH^1(\Delta^*)$, with $\deg(\eta) = 0$. As element of $A^2(V_1')$ (see (24)) it corresponds with the class of the 1-cycle $\sum_i l^*_{T_iR_i}$ and by (27) we get for the action of τ in $A^2(V_1')$

$$\tau(\sum_i l^*_{T_iR_i}) = \sum_i (K_{T_i\sigma(R_i)} - l^*_{T_i\sigma(R_i)}.$$

Now using (28) we get

$$\tau(\sum_i l^*_{T_iR_i}) = \sum_i (K_{T_0R_0}) - l^*_{T_i\sigma(R_i)} = -\sum_i l^*_{T_i\sigma(R_i)}$$

because since $\deg(\eta) = 0$ the terms with $K_{T_0R_0}$ cancel. Hence finally

$$\tau(\eta) = -\text{class}(\sum_i l^*_{T_i\sigma(R_i)}) = -\sigma(\eta) \tag{29}$$

which proves the lemma. □

1.10.

Using the key lemma we obtain

MAIN THEOREM ON CHOW GROUP. ([9], thm 5.10)

$$A^2(V_1) = Pr(\Delta^*/\Delta)$$

Proof. (indication)

From the (2 : 1)-cover $f_1 : V_1' \to V_1$ (see section 1.7) we get homomorphisms

$$f_1^* : A^2(V_1) \to A^2(V_1') \qquad \text{and} \qquad f_{1*} : A^2(V_1') \to A^2(V_1) \tag{30}$$

and moreover

$$(f_1)_* \circ f_1^* = 2\, id_{A(V_1)}. \tag{31}$$

Clearly $\mathrm{Im}(f_1^*) \subset \mathrm{Inv}(\tau)$ (invariant part) and using the divisibility of the groups $A^2(-)$ we get in fact $\mathrm{Im}(f_1^*) = \mathrm{Inv}(\tau)$. Then using the key lemma of 1.9 we get

$$\mathrm{Im}(f_1^*) = \mathrm{Inv}(\tau) = Pr(\Delta^*/\Delta).$$

Using 31 we get then first $A^2(V_1) = Pr(\Delta^*/\Delta) + B$ with B some 2-torsion group but in fact (see [37] pages 201-202) we get a direct sum $A^2(V_1) = Pr(\Delta^*/\Delta) \oplus B$ and finally using divisibility again we get $B = 0$, hence the theorem. □

1.11. The non-rationality of V

It suffices clearly to consider V_1 instead of V. As already remarked in section 1.6 the $Pr(\Delta^*/\Delta)$ carries a principal polarization Ξ. From this one gets a Riemann form $l^{\Xi}(\xi,\eta)$ for $\xi,\eta \in T_l(Pr(\Delta^*/\Delta))$, with $T_l(Pr(\Delta^*/\Delta))$ the Tate group (see [35], p. 186). This Riemann form is closely related to the Riemann forms coming from $H^3(V_1) \simeq H^1_-(\Delta^*)$ where $H^1_-(\Delta^*)$ is the anti-invariant part of $H^1(\Delta^*)$, namely $l^{\Xi}(\xi,\eta) = -\xi \cup \eta$ where $\xi \cup \eta$ is the cup product of $H^3(V_1)$ (see [39], p 148 and [9], p 172).

Using this one can show (similar as in [6], Cor. 3.26) that if V_1 should be rational then $(Pr(\Delta^*/\Delta), \Xi)$ as principally polarized abelian variety is isomorphic to a jacobian variety of a curve polarized by its theta divisor or to a product of such pairs (see [38], thm 3.11).

However we have already remarked in section 1.6 that in our case this does not happen. Therefore we have

THEOREM 3. ([9], thm 6.3)
Let $V = V_3(4) \subset \mathbb{P}_4$ be a quartic 3-fold with a double line but otherwise general, then V is unirational but not *rational.*

1.12. Further developments

It turned out afterwards that in the same period (1977) Beauville in Paris wrote a beautiful thesis on "fibrés en quadriques". This paper is a standard work and fundamental on this subject and our result on the quartic 3-fold with a double line is a very special case of his results. Of course the works were done independent of each other. Beauville's method is different from ours. See [2] for his results.

2. Hodge conjecture for 4-folds covered by rational curves.

2.1. Hodge conjecture

Let us first recall the Hodge conjecture. Let X_d be a smooth projective variety defined over the complex numbers $\mathbb{C}$ and of dimension d. Consider the (Betti) cohomology groups $H^i(X,Q)$ and $H^i(X,\mathbb{C})$ $(0 \leq i \leq 2d)$. Then there is the Hodge decomposition

$$H^i(X,\mathbb{C}) = \bigoplus_{r+s=i} H^{r,s}(X) \tag{32}$$

with $H^{r,s}(X) = H^s(X,\Omega_X^r)$. On the other hand there is the *cycle map* from the group $\mathcal{Z}^p(X)$ of algebraic cycles on X of dimension $(d-p)$ (i.e. of codimension p)

$$\gamma^p : \mathcal{Z}^p(X) \otimes Q \to \{H^{2p}(X,Q) \cap H^{p,p}(X)\} \subset H^{2p}(X,\mathbb{C}). \tag{33}$$

Hodge made the *conjecture* that γ^p is *onto* $H^{2p}(X,Q) \cap H^{p,p}(X)$. This is so-called *Hodge (p,p)-conjecture*.

2.2. Known cases

For $p = 1$ the Hodge (1,1)-conjecture is known by the so-called (1,1) theorem of Lefschetz. By another theorem of Lefschetz, the strong Lefschetz theorem, it then follows that also the Hodge $(d-1, d-1)$-conjecture is true. The Hodge conjecture is therefore true for 3-folds (and for surfaces and curves).

The first open case is in dimension 4, namely the Hodge $(2,2)$-conjecture.

Here we have the following result:

2.3. Theorem. [10]

Hodge (2,2) is true for fourfolds which are covered by rational curves.

For the proof, which is not difficult, we refer to the original paper.

2.4. Applications.

Hodge (2,2) is true for unirational fourfolds ([40]), in particular for the cubic fourfold (this was also proven by different methods by Zucker [48]). In the paper [11] we did give many more examples, including in particular $V_4(4) \subset \mathbb{P}_5$ (covered by lines), $V_4(5) \subset \mathbb{P}_5$, (covered by conics), $V_4(2,4)$ and $V_4(3,3)$ in $\mathbb{P}_6$ and many more cases. Also the theorem applies to *all uniruled fourfolds*.

2.5. Some further developments.

In 1992 Campana [4] and Kollar-Miyaoka-Mori [27] independently proved that Fano varieties are rationally connected, therefore our theorem from 2.3 above applies *to all Fano fourfolds*. From this result of rationally connectedness follows also a completely different proof of Hodge (2,2) for Fano 4-folds by theorem of Bloch-Srinivas in 1983 ([3]) (see also section 9 of [42]).

3. On Fano's theorem on threefolds which have as hyperplane sections Enriques surfaces.

3.1.

Fano threefolds of the so-called principal series, i.e. 3-folds V for which $-K_V$ is *very* ample, have, when they are embedded by this anticanonical system, K_3-surfaces as hyperplane sections (this is a classical fact, but see for instance [41], p.43). Therefore it is natural to ask whether there exist threefolds having as hyperplane sections Enriques surfaces. Indeed such threefolds exist and they were studied in 1933 by Godeaux and in much deeper detail and depth by Fano in 1938. In his paper [22] Fano obtained the following remarkable result.

THEOREM (FANO). Let V a threefold in $\mathbb{P}_N$ such that a general hyperplane section $S = V \cap H$ is a *smooth Enriques surface*. If V is "otherwise general" then V has exactly 8 singular points $P_1, P_2, \dots, P_8$. Each P_i is a quadruple point on V, and the tangent cone to V in such a point P_i is isomorphic to the cone over the Veronese surface.

In the above paper [22] Fano claimed also a classification of such threefolds and he described - essentially - 4 different types (see below section 3.3).

3.2.

Fascinated by this striking result we (Conte and the author) started around 1980 to study Fano's paper. His "proof" contains many gaps. For many steps Fano did not give an argument at all, apparently he did – by his fabulous geometric insight and knowledge - simply "see" that this should be true! By an often case by case examination with long calculations we filled in these steps in our paper [13] and it turned out that Fano was almost always right at the end!

At the Varenna conference on Algebraic Threefolds in 1981 Conte did give a survey of our proof in [7] and in [8] he discussed two types of the four types mentioned by Fano; these examples are full of beautiful classical geometry.

3.3. Further developments

The classification problem was taken up again from a modern point of view by T. Sano [45] and by L. Bayle [1]. The point of view of Bayle is as follows. First he shows that such a threefold V with Enriques hyperplane section can be obtained from a pair (W,τ) consisting of a Fano threefold W together with an involution τ on W with a finite number of fixed points. Then using the classification of Fano 3-folds W by Iskovskikh and Mori-Mukai he studied which W admit such an involution. In this way he gets a complete classification. Bayle found that there are 6 different types; hence Fano missed two cases. We refer for these results to the nice paper of Bayle ([1]).

4. On the definition and on the nature of the singularities of Fano threefolds.

The original point of view of Fano of what nowadays are called *Fano threefolds* is that these are three dimensional varieties V which are embedded in projective space $\mathbb{P}_N$ with the property that the general linear section $C = V \cap H_1 \cap H_2$ is a *canonically embedded curve* (here H_1 and H_2 are sufficiently general hyperplanes). If V is smooth and embedded by $|-K_V|$ then this is equivalent with the modern definition (Iskovskikh) namely that the anticanonical class is ample [24].

In the paper [13] Conte and the author studied threefolds V with the property that $C = V \cap H_1 \cap H_2$ is canonically embedded but without the assumption that V is smooth. It turned out that the above condition puts severe restrictions on the nature of the singularities. In particular the singularities are Gorenstein singularities. Furthermore if there are only isolated singularities then there can be at most one "non-rational singularity" and in fact such a singularity must be necessarily "elliptic".

In our paper we used heavily the works of Du Val, Merindol and Epema on surfaces which have as hyperplane sections canonical curves (see [20]).

5. On Morin's work on the unirationality of hypersurfaces.

5.1. Introduction

The question of unirationality for hypersurfaces was studied by U. Morin (University of Padova) in a remarkable series of papers rangings over the period 1936-55 (see [34]). In 1940 he proved the following nice theorem

THEOREM 4. ([32])
Let $V = V_{n-1}(d) \subset \mathbb{P}_n$ *be a "general" hypersurface of degree d and dimension* $(n-1)$. *There exists a constant* $\rho(d)$ *such that if* $n > \rho(d)$ *then V is unirational.*

"General" means as usual that in the parameter space of the $V_{n-1}(d)$ there exists a Zariski open set such that the theorem holds for the V in that set.

Morin's theorem was extended to complete intersections by Predonzan in 1949 [43]. Modern proofs for Morin's theorem were given by Ciliberto [5] in 1980, Ramero (1990) and Paranjape - Srinivas in 1992. In 1998 Harris - Mazur - Pandharipande relaxed the condition "general" to "smooth".

In all these proofs the $\rho(d)$ is *very* large and certainly far from the best bound.

In a paper in 1938 (resp. in 1952) Morin did give a much sharper bound for the case $d = 5$ [31] (resp. $d = 4$ [33]). These papers are for "modern" readers difficult to read, firstly because they are written in a classical style and secondly - more important - Morin's exposition is very concise. On the other hand Morin's ideas are very geometrical and nice and therefore Conte and myself, later also joined by Marina Marchisio, have given a modern treatment of these papers ([14], [17]). Although we have at some places somewhat simplified the proofs and also slightly generalized the result for $d = 5$, I want to stress that we have followed Morin's ideas and, by looking carefully, Morin's original proofs are certainly correct. See section 5.3 and 5.4 below for the theorems.

5.2. Generalization to double coverings

Following an idea of Ciliberto from his above mentioned paper [5] from 1980 we extended Morin's 1940 result for hypersurfaces to double covers on $\mathbb{P}_N$.

THEOREM 5. ([15])
Let $B = B_{n-1}(2d) \subset \mathbb{P}_n$ *be a hypersurface of degree 2d in* $\mathbb{P}_n$ *and let* $\pi : W = W_n[2d, B] \to \mathbb{P}_n$ *be a double covering branched over B. Then there exists a constant* $\rho(d)$ *such that if* $n > \rho(d)$ *and B is general then W is unirational.*

For the proof see [15].

5.3. Quartic hypersurfaces

THEOREM 6. ([33])
Let $V = V_{n-1}(4) \subset \mathbb{P}_n$ be a general quartic (in particular smooth) hypersurface. Then V is unirational if $n > 5$.

In [14] we did give a "modern" proof of this. We followed the basic idea of Morin but we could give some simplifications based on results of S. Lang [28] and B. Segre [46] which were not available to Morin in 1952.

Note that the problem of the *unirationality* of the *general quartic* hypersurface remains open for dimension 3 and 4.

Examples of *"special"* smooth quartic 3-folds which are unirational are given by B. Segre [47], Predonzan [44] and Marina Marchisio [29], [30]; Marina Marchisio constructed in fact families of such 3-folds in her thesis. Also note that a smooth quartic 3-fold is always *non-rational* as was shown by Iskovskikh-Manin in 1971 [25].

5.4. Quintic hypersurfaces

THEOREM 7. (Morin 1938 [31])
Let $V = V_{n-1}(5) \subset \mathbb{P}_n$ be a general quintic hypersurface. Then V is unirational if $n > 17$.

In 2007 we (CMM) did give a modern proof of this theorem (again based on Morin's ideas). In fact we obtained a more precise result which included Morin's theorem.

THEOREM 8. ([17])
Let $V = V_{n-1}(5) \subset \mathbb{P}_n$ be defined over a field k (always of characteristic zero for simplicity). Assume that there exists a 3-dimensional linear space $\Sigma \subset V$ and defined *also over k. Let V be otherwise general- Then V is unirational if $n \geq 7$.*

This generalizes indeed the above theorem of Morin because for $n > 17$ there exists such a linear 3-dimensional space Σ inside a general $V_{n-1}(5)$ (as is shown by standard arguments).

Proof. (Very rough outline in order to give the idea). We assume for simplicity that $n = 7$ (otherwise we need to pay some extra care).

The idea of the proof - due to Morin - is to use a famous theorem of Enriques ([19]), namely the theorem that the variety $V_3(2,3) \subset \mathbb{P}_5$ (i.e. the intersection of a quadric and a cubic in $\mathbb{P}_5$) is unirational. However it is necessary to use this theorem in a refined form (see details below)!

So taking $n = 7$, let $V = V_6(5) \subset \mathbb{P}_7$ be a variety containing a 3-space $\Sigma \subset V$ (so $\Sigma \simeq \mathbb{P}_3$) but otherwise general and let k be a field over which both V and Σ are defined. Since Σ is defined over k we can take homogeneous coordinates $(t_0 : t_1 : t_2 : \ldots : t_7)$ in

$\mathbb{P}_7$ such that Σ has equations

$$t_4 = t_5 = t_6 = t_7 = 0. \tag{34}$$

Let $Y = (1, y_1, y_2, y_3, 0, 0, 0, 0)$ be the generic point of Σ over k, i.e., y_1, y_2, y_3 are independent transcendental over k; let $k(y) = k(y_1, y_2, y_3)$.

Consider now in $\mathbb{P}_7$ the variety K_Y defined by

$$K_Y = K_Y(V) := \{P' \in \mathbb{P}_7;\ P' \in l \subset V \text{ with } l \text{ a line in } \mathbb{P}_7 \text{ such that } l \cap V = 4Y + P\} \tag{35}$$

i.e. K_Y is a cone with vertex Y consisting of lines through Y which intersect V in Y with multiplicity at least four. Clearly the variety K_Y is defined over the field $k(y)$ and

$$K_Y(V) = T_Y(V) \cap Q_Y(V) \cap C_Y(V) \tag{36}$$

when $T_Y(V)$ is the tangent space in Y to V (given by a linear equation), $Q_Y(V)$ the (usual) tangent cone in Y to V (given by a quadratic equation) and $C_Y(V)$ the cone with the vertex Y of the lines with contact higher than three to V (given by a cubic equation); all these equations have their coefficients in the field $k(y)$.

Consider the variety

$$W_Y = W_Y(V) := K_Y(V) \cap V. \tag{37}$$

Clearly P from (35) is on W_Y. If $F(t) = 0$ is the equation of V then P satisfies the equations (symbolically):

$$\begin{cases} F(P) = 0 \\ \dfrac{\partial F}{\partial t}(Y)(P) = 0 \\ \dfrac{\partial^2 F}{\partial t^2}(Y)(P) = 0 \\ \dfrac{\partial^3 F}{\partial t^3}(Y)(P) = 0 \end{cases} \tag{38}$$

Note also that starting with a point P on V we find 24 such points Y.

LEMMA 3. *If P is generic on W_Y over $k(y)$ then P is generic on V over k.*

Proof. Easy dimension count. □

Hence we have

$$k(V) \cong k(P) \subset k(Y, P) = k(y)(P).$$

Hence: it suffices to prove that $W_Y(V)$ is *unirational over* $k(y)$ (i.e. not merely unirational, but *unirational over the field* $k(y)$ *itself*).

Let $H_\infty = \{t_0 = 0\} \subset \mathbb{P}_7$ be the hyperplane "at infinity" and consider in H_∞ the variety

(39) $$W'_Y = W'_Y(V) := H_\infty \cap T_Y(V) \cap Q_Y(V) \cap C_Y(V).$$

LEMMA 4. *There is a birational transformation*

$$f: \quad W_Y \overset{\sim}{\dashrightarrow} W'_Y$$

given by $f(P) = P_\infty$ where $P_\infty = l \cap H_\infty$ and f is defined over the field $k(y)$.

Proof. This is immediately clear from the way the variety K_Y has been defined in (5.2), i.e., W'_Y is the projection to H_∞ of W_Y from the point Y and f is birational since on the "general" line l there is only one such point P (see (35)). □

Hence: it is sufficient to prove that $W'_Y(V)$ is *unirational over the field* $k(y)$.

Now $W'_Y(V) \subset H_\infty \cap T_Y(V)$ and $H_\infty \cap T_Y(V)$ is a projective space $\mathbb{P}_{5,Y}$ (if $n = 7$), all over the field $k(y)$ and in $\mathbb{P}_{5,Y}$ the $W'_Y(V)$ is the intersection

(40) $$W'_Y(V) = Q'_Y(V) \cap C'_Y(V)$$

with $Q'_Y(V) = Q_Y(V) \cap \mathbb{P}_{5,Y}$ and $C'_Y(V) = C_Y(V) \cap \mathbb{P}_{5,Y}$, i.e., $W'_Y(V)$ is a variety of type $V_3(2,3) \subset \mathbb{P}_5$ defined over the field $k(y)$, By the theorem of Enriques ([19]) it is indeed unirational. However we need the more precise result that it is *unirational over the field* $k(y)$ itself. Although Morin is very short and not explicit it seems clear (by reading between the lines!) that he was aware of this fact and also knew that in his case the Enriques theorem holds in this refined form due to his linear space Σ in V.

In our paper [16] we have checked all this carefully and proved the Enriques theorem in the following precise form.

THEOREM 9. *Let $W = Q \cap C \subset \mathbb{P}_5$ with Q (resp. C) a quadric (resp. cubic) hypersurface containing a 2-plane Δ, but otherwise "general". Let Q, C* and *Δ be defined over a field K. Then W is* unirational over K.

For the proof see [17].

By this theorem it follows from the above that the proof of the theorem for the quintic is complete if we take $W = W_Y$ and $\Delta = \Sigma \cap H_\infty$ and $K = k(y)$. □

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AMS Subject Classification: 14J30, 14J45

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Lavoro pervenuto in redazione il 02.07.2013.