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EQUIVALENT BIRATIONAL EMBEDDINGS III: CONES

Dedicated to Alberto Conte on the occasion of his 70th birthday.

Abstract. Two divisors in $\mathbb{P}^n$ are said to be Cremona equivalent if there is a Cremona modification sending one to the other. In this paper I study irreducible cones in $\mathbb{P}^n$ and prove that two cones are Cremona equivalent if their general hyperplane sections are birational. In particular I produce examples of cones in $\mathbb{P}^3$ Cremona equivalent to a plane whose plane section is not Cremona equivalent to a line in $\mathbb{P}^2$.

Introduction

Let $X \subset \mathbb{P}^n$ be an irreducible and reduced projective variety over an algebraically closed field. A classical question is to study the birational embedding of X in $\mathbb{P}^n$ up to the Cremona group of $\mathbb{P}^n$. In other words let X_1 and X_2 be two birationally equivalent projective varieties in $\mathbb{P}^n$. One wants to understand if there exists a Cremona transformation of $\mathbb{P}^n$ that maps X_1 to X_2, in this case we say that X_1 and X_2 are Cremona equivalent. This projective statement can also be interpreted in terms of log Sarkisov theory, [3], and is somewhat related to the Abhyankar–Moh problem, [1] and [10]. In the latter paper it is proved, using techniques derived form A–M problem, that over the complex field the birational embedding is unique as long as $\dim X < \frac{n}{2}$. The problem is then completely solved in [14] where it is proved that this is the case over any algebraically closed field as long as the codimension of X_i is at least 2. Examples of inequivalent embeddings of divisors are well known, see also [14], in all dimensions. The problem of Cremona equivalence is therefore reduced to study the equivalence classes of divisors. This can also be interpreted as the action of the Cremona group on the set of divisors of $\mathbb{P}^n$.

The special case of plane curves received a lot of attention both in the old times, [5], [17], [11], and in more recent times, [16], [8], [12], [4], and [15], see also [7] for a nice survey. In [4] and [15] a complete description of plane curves up to Cremona equivalence is given and in [4] a detailed study of the Cremona equivalence for linear systems is furnished. In particular it is interesting to note that the Cremona equivalence of a plane curve is dictated by its singularities and cannot be divined without a partial resolution of those, [15, Example 3.18]. Due to this it is quite hard even in the plane curve case to determine the Cremona equivalence class of a fixed curve simply by its equation.

The next case is that of surfaces in $\mathbb{P}^3$. In this set up using the $\sharp$-Minimal Model Program, [13] or minimal model program with scaling [2], a criterion for detecting surfaces Cremona equivalent to a plane is given. The criterion, inspired by the previous work of Coolidge on curves Cremona equivalent to lines [5], allows to determine all

rational surfaces that are Cremona equivalent to a plane, [15, Theorem 4.15]. Unfortu-
nately, worse than in the plane curve case, the criterion requires not only the resolution
of singularities but also a control on different log varieties attached to the pair $(\mathbb{P}^3, S)$.
As a matter of fact it is impossible to guess simply by the equation if a rational surface
in $\mathbb{P}^3$ is Cremona equivalent to a plane and it is very difficult in general to determine
such equivalences. The main difficulty comes from the condition that the sup-threshold,
see Definition 4, is positive. This is quite awkward and should be very interesting to
understand if this numerical constrain is really necessary. Via $\sharp$-MMP it is easy, see
remark 4, to reduce this problem to the study of pairs (T, S) such that T is a terminal
$\mathbb{Q}$-factorial 3-fold with a Mori fiber structure $\pi : T \to W$ onto either a rational curve
or a rational surface, and S is a smooth Cartier divisor with $S = \pi^* D$ for some divisor
$D \subset W$. The first "projective incarnation" of such pairs are cones in $\mathbb{P}^3$.

In this work I develop a strategy to study cones in arbitrary projective space. If
two cones in $\mathbb{P}^n$ are built on varieties Cremona equivalent in $\mathbb{P}^{n-1}$ then also the cones
are Cremona equivalent, see Proposition 1. The expectation for arbitrary cones built
on birational but not Cremona equivalent varieties was not clear but somewhat more
on the negative side. The result I prove is therefore quite unexpected and shows once
more the amazing power of the Cremona group of $\mathbb{P}^n$.

THEOREM 1. *Let S_1 and S_2 be two cones in $\mathbb{P}^n$. Let X_1 and X_2 be corresponding
general hyperplane sections. If X_1 and X_2 are birational then S_1 is Cremona equivalent
to S_2.*

The main ingredient in the proof is the reduction to subvarieties of codimen-
sion 2 to be able to apply the main result in [14]. To do this I produce a special log
resolution of the pair $(\mathbb{P}^n, S)$ that allows me to blow down the strict transform of S
to a codimension 2 subvariety. Despite the fact that this step cannot produce a Cre-
mona equivalence for S it allows me to work out Cremona equivalence on the lower
dimensional subvariety and then lift the Cremona equivalence.

In the special case of cones in $\mathbb{P}^3$ the statement can be improved to characterize
the Cremona equivalence of cones with the geometric genus of the plane section, see
Corollary 1. In particular this shows that any rational cone in $\mathbb{P}^3$ has positive sup-
threshold, see Definition 4. From the $\sharp$-MMP point of view we may easily translate it
as follows.

THEOREM 2. *Let $S \subset \mathbb{P}^3$ be a rational surface. Assume that there is a $\sharp$-minimal
model of the pair (T, S_T) such that T has a scroll structure $\pi : T \to W$ onto a rational
surface W and $S_T = \pi^* C$, for some rational curve $C \subset W$. Then $\overline{\rho}(T, S_T) = \overline{\rho}(\mathbb{P}^3, S) > 0$.*

The next candidate for the sup-threshold problem are pairs whose $\sharp$-minimal
model is a conic bundle and the surface is trivial with respect to the conic bundle
structure. With the technique developed in this paper I am only able to treat a special
class of these, see Corollary 2.

1. Notations and preliminaries

I work over an algebraically closed field of characteristic zero. I am interested in birational transformations of log pairs. For this I introduce the following definition.

DEFINITION 1. *Let $D \subset X$ be an irreducible and reduced divisor on a normal variety X. We say that (X,D) is birational to (X',D'), if there exists a birational map $\varphi : X \xrightarrow{\quad} X'$ with $\varphi_*(D) = D'$. Let $D,D' \subset \mathbb{P}^n$ be irreducible reduced divisors then we say that D is Cremona equivalent to D' if $(\mathbb{P}^n,D)$ is birational to $(\mathbb{P}^n,D')$.*

Let us proceed recalling a well known class of singularities.

DEFINITION 2. *Let X be a normal variety and $D = \sum d_i D_i$ a $\mathbb{Q}$-Weil divisor, with $d_i \leq 1$. Assume that $(K_X + D)$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a log resolution of the pair (X,D) with*

$$K_Y = f^*(K_X + D) + \sum a(E_i, X, D) E_i$$

We call

$$disc(X,D) := \min_{E_i}\{a(E_i,X,D)|$$

$$E_i \text{ is an } f\text{-exceptional divisor for some log resolution}\}$$

Then we say that (X,D) is

$$\left. \begin{array}{c} terminal \\ canonical \end{array} \right\} \quad if\, disc(X,D) \quad \left\{ \begin{array}{c} > 0 \\ \geq 0 \end{array} \right.$$

REMARK 1. Terminal surfaces are smooth, this is essentially the celebrated Castelnuovo theorem. Any log resolution of a smooth surface can be obtained via blow up of smooth points. Hence a pair (S,D), with S a smooth surface has canonical singularities if and only if $\text{mult}_p D \leq 1$ for any point $p \in S$.

Note further that one direction is true in any dimension. Assume that X is smooth and $\text{mult}_p D \leq 1$ for any point $p \in X$. Let $f : Y \to X$ be a smooth blow up, with exceptional divisor E. Then $K_Y = f^*(K_X) + aE$ for some positive integer a and $a(E,X,D) \geq a - 1 \geq 0$. This proves that (X,D) has canonical singularities if X is smooth and $\text{mult}_p D \leq 1$ for any $p \in X$. This simple observation allows to produce many inequivalent embeddings of divisors, see [14, §3]

For future reference we recall a technical result on pseudoeffective divisors, i.e. the closure of effective divisors.

LEMMA 1 ([15, Lemma 1.5]). *Let (X,D_X) and (Y,D_Y) be birational pairs with canonical singularities. Then $K_X + D_X$ is pseudoeffective if and only if $K_Y + D_Y$ is pseudoeffective.*

The main difficulty to study Cremona equivalence in $\mathbb{P}^r$ with $r \geq 3$ is the poor knowledge of the Cremona group. The case of surfaces in $\mathbb{P}^3$ is already quite mysterious. It is easy to show that Quadrics and rational cubics are Cremona equivalent to a plane. Rational quartics with either 3-ple or 4-uple points are again easily seen to be Cremona equivalent to planes, the latter are cones over rational curves Cremona equivalent to lines. It has been expected that Noether quartic should be the first example of a rational surface not Cremona equivalent to a plane, but this is not the case as proved in [15, Example 4.3]. Having in mind these examples and the ♯-MMP developed in [13] for linear systems on uniruled 3-folds, I recall the definition of (effective) threshold.

DEFINITION 3. *Let (T,H) be a terminal $\mathbb{Q}$-factorial uniruled variety and H an irreducible and reduced Weil divisor on T. Let*

$$\rho(T,H) =: \sup \{m \in \mathbb{Q} | H + mK_T \text{ is an effective } \mathbb{Q}\text{-divisor} \} \geq 0,$$

be the (effective) threshold of the pair (T,H).

REMARK 2. The threshold is not a birational invariant of the pair and it is not preserved by blowing up. Consider a Quadric cone $Q \subset \mathbb{P}^n$ and let $Y \to \mathbb{P}^n$ be the blow up of the vertex then $\rho(Y,Q_Y) = 0$, while $\rho(\mathbb{P}^n,Q) > 0$.

To study Cremona equivalence, unfortunately, we have to take into account almost all possible thresholds.

DEFINITION 4. *Let (Y,S_Y) be a pair birational to a pair (T,S). We say that (Y,S_Y) is a good birational model if Y has terminal $\mathbb{Q}$-factorial singularities and S_Y is a Cartier divisor with terminal singularities. The sup-threshold of the pair (T,S) is*

$$\overline{\rho}(T,S) := \sup\{\rho(Y,S_Y)\},$$

where the sup is taken on good birational models.

REMARK 3. It is clear that any pair $(\mathbb{P}^n,S)$ Cremona equivalent to a hyperplane satisfies $\overline{\rho}(\mathbb{P}^n,S) > 0$. The pair $(\mathbb{P}^n,H)$, where H is a hyperplane, is a good model with positive threshold.

Considering birationally super-rigid MfS's one can produce examples of pairs, say (T,S), with $\overline{\rho}(T,S) = 0$. It is not clear to me if such examples can exist also on varieties with bigger pliability, see [6] for the relevant definition.

We are ready to state the characterization of surfaces Cremona equivalent to a plane.

THEOREM 1 ([15, Theorem 4.15]). *Let $S \subset \mathbb{P}^3$ be an irreducible and reduced surface. The following are equivalent:*

a) *S is Cremona equivalent to a plane,*

b) *$\overline{\rho}(T,S) > 0$ and there is a good model (T,S_T) with $K_T + S_T$ not pseudoeffective.*

REMARK 4. As remarked in the introduction the main drawback of the above criterion is the bound on the sup-threshold. It is very difficult to compute it. While the requirement that $K_T + S_T$ is not pseudoeffective is natural and justified also by Lemma 1, it is not clear if pairs with vanishing sup threshold may exist on a rational 3-fold. This naturally leads to study good models with vanishing threshold.

Let (X, S) be a good pair with X rational and $\rho(X, S) = 0$. Then the $\sharp$-MMP applied to this pair may lead to a Mori fiber space $\pi : T \to W$ such that S_T is trivial with respect to π and it is a smooth surface, see [13, Theorem 3.2] and the proof of [15, Theorem 4.9]. In particular $S_T = \pi^* D$ for some irreducible divisor $D \subset W$. Then if W is a curve S_T is a smooth fiber of π, that is a del Pezzo surface. If W is a surface then S_T is a (not necessarily minimally) ruled surface and π is a conic bundle structure. In the latter case if π has a section it is easy, see for instance the proof of Corollary 2, to prove that (T, S_T) is birational to a cone in $\mathbb{P}^3$.

2. Cremona equivalence for cones

Here I am interested in cones in $\mathbb{P}^n$. Let $S \subset \mathbb{P}^n$ be an irreducible reduced divisor of degree d with a point p of multiplicity d. Let H be a hyperplane in $\mathbb{P}^n \setminus \{p\}$ and $C = S \cap H$. Then S can be viewed as the cone over the variety C. It is easy to see that if $C_1, C_2 \subset \mathbb{P}^{n-1}$ are Cremona equivalent divisors then the cones over them are Cremona equivalent.

PROPOSITION 1. *Let C_1 and C_2 be Cremona equivalent divisors in $\mathbb{P}^{n-1}$ and S_1, S_2 cones over them in $\mathbb{P}^n$. Then S_1 is Cremona equivalent to S_2.*

Proof. Without loss of generality I may assume that S_1 and S_2 have the same vertex in the point $[0, \ldots, 0, 1]$ and $C_1 \cup C_2 \subset (x_n = 0)$. Let $\mathcal{H} \subset |\mathcal{O}_{\mathbb{P}^{n-1}}(h)|$ be a linear system realizing the Cremona equivalence between C_1 and C_2. Hence I have a Cremona map $\psi : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$ given by a n-tuple $\{f_0, \ldots, f_{n-1}\}$, with $f_i \in k[x_0, \ldots, x_{n-1}]_h$. This allows me to produce a map $\Psi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ considering the linear system

$$\mathcal{H}' := \{f_0, \ldots, f_{n-1}, x_n x_0^{h-1}\}.$$

Note that the general element in $\mathcal{H}'$ has multiplicity $h - 1$ at the point p and, in the chosen base, there is only one element of multiplicity exactly $h - 1$. This shows that lines through p are sent to lines through a fixed point. Moreover the restriction $\Psi_{|(x_n=0)}$ is the original map ψ. Hence the map Ψ is birational and realizes the required Cremona equivalence. $\qquad \square$

Next I want to understand what happens if $C_1, C_2 \subset \mathbb{P}^{n-1}$ are simply birational as abstract varieties. To do this I need to produce "nice" good models of the pairs $(\mathbb{P}^n, S_1)$ and $(\mathbb{P}^n, S_2)$.

Let $C \subset H \subset \mathbb{P}^n$ be a codimension two subvariety and S be a cone with vertex $p \in \mathbb{P}^n \setminus H$ over C. I produce a good model of the pair $(\mathbb{P}^n, S)$ as follows. First I blow up p producing a morphism $\varepsilon : Y \to \mathbb{P}^n$ with exceptional divisor E. Note that Y has a

scroll structure $\pi : Y \to \mathbb{P}^{n-1}$ given by lines through p, and S_Y, the strict transform, is just $\pi^*(C)$. Let $\nu : W \to \mathbb{P}^{n-1}$ be a resolution of the singularities of C and take the fiber product

$$
\begin{array}{ccc}
Z & \xrightarrow{\ \nu_Y\ } & Y \\
{\scriptstyle \pi_W}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
W & \xrightarrow{\ \nu\ } & \mathbb{P}^{n-1}.
\end{array}
$$

Then the strict transform S_Z is a smooth divisor and (Z, S_Z) is a good model of $(\mathbb{P}^n, S)$. Note that the threshold $\rho(Z, S_Z)$ vanishes. According to the $\sharp$-MMP philosophy this forces us to produce different good models.

The following Lemma is probably well known, but I prefer to state it, and prove it, to help the reader.

LEMMA 2. *Let $C \subset X$ be an irreducible and reduced subvariety. Assume that there exists a birational map $\chi : X \dashrightarrow Y$ such that χ is an isomorphism on the generic point of C. Let $D := \chi(C)$ be the image and X_C, respectively Y_D the blow up of C and D with morphism f_C, f_D, and exceptional divisors E_C, E_D respectively. Then there is a birational map $\chi_C : X_C \dashrightarrow Y_D$ mapping E_C onto E_D. In other words (X_C, E_C) is birational to (Y_D, E_D).*

Proof. Let $U \subset X$ be an open and dense subset intersecting C such that $\chi_{|U}$ is an isomorphism. Then considering the fiber product
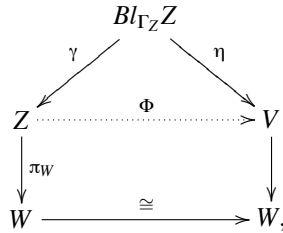
$$
\begin{array}{ccc}
X_C \supset U_C & \xrightarrow{\ \chi_C\ } & Y_D \\
{\scriptstyle f_U}\big\downarrow & & \big\downarrow{\scriptstyle f_D} \\
X \supset U & \xrightarrow{\ \chi_{|U}\ } & Y,
\end{array}
$$

I conclude, by the Universal Property of Blowing Up, the existence of the morphism χ_C with the required properties. $\qquad\square$

REMARK 5. Let me stress that the above result is in general not true with the weaker assumption that χ is a morphism on the general point of W. On the other hand if Y is $\mathbb{P}^n$ the statement can be rephrased also in this weaker form.
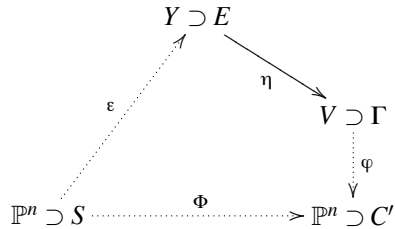
Let us go back to the pair (Z, S_Z). Let Γ_Z be the strict transform of a general hyperplane section of S. Then I may consider the following elementary transformation

of the scroll structure π_W

$$
\begin{array}{ccc}
 & Bl_{\Gamma_Z}Z & \\
{}^{\gamma}\swarrow & & \searrow^{\eta} \\
Z & \overset{\Phi}{\dashrightarrow} & V \\
\downarrow^{\pi_W} & & \downarrow \\
W & \underset{\cong}{\longrightarrow} & W,
\end{array}
$$

where γ is the blow up of Γ_Z and η is the blow down of the strict transform of S_Z to a codimension 2 subvariety, say Γ_S. Then there is a birational map $\varphi : V \dashrightarrow \mathbb{P}^n$ sending Γ_S to a codimension 2 subvariety, say Γ', and such that φ is an isomorphism on the generic point of Γ_S. The map φ can be easily constructed again via an elementary transformation of the scroll structure followed by blow downs of exceptional divisors. We may summarize the above construction in the following proposition and diagram.

PROPOSITION 2. *Let $S \subset \mathbb{P}^n$ be a cone and C a general hyperplane section. Then there are birational maps $\varepsilon : \mathbb{P}^n \dashrightarrow Y$ and $\eta : Y \dashrightarrow V$ such that $\varepsilon_* S =: E$ and η is the blow down of E to a codimension 2 subvariety Γ. In particular S is the valuation associated to the ideal I_Γ. Moreover there is a third birational map $\varphi : V \dashrightarrow \mathbb{P}^n$ sending Γ to a codimension 2 subvariety C' and such that again S is the valuation associated to $I_{C'}$*

$$
\begin{array}{ccc}
 & Y \supset E & \\
{}^{\varepsilon}\nearrow & & \searrow^{\eta} \\
 & & V \supset \Gamma \\
 & & \downarrow^{\varphi} \\
\mathbb{P}^n \supset S & \overset{\Phi}{\dashrightarrow} & \mathbb{P}^n \supset C'
\end{array}
\qquad .
$$

The composition $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is a birational map sending S to the codimension 2 subvariety C' and such that S is the valuation associated to the ideal $I_{C'}$.

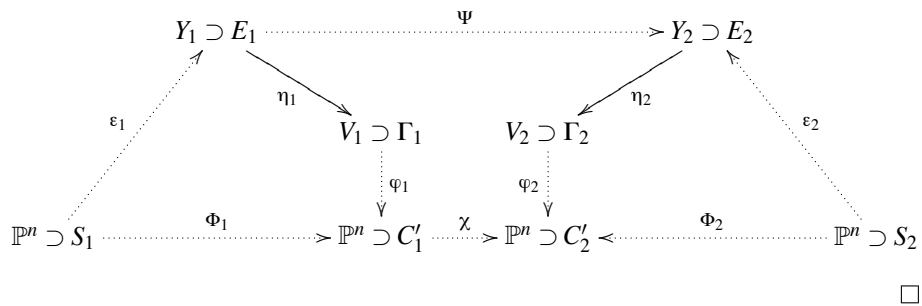We are now ready to prove the main result on cones in $\mathbb{P}^n$.

THEOREM 2. *Let $S_1, S_2 \subset \mathbb{P}^n$ be cones and C_1, C_2 general hyperplane sections. If C_1 is birational to C_2 then S_1 is Cremona equivalent to S_2. In particular all divisorial cones over a rational variety are Cremona equivalent to a hyperplane.*

REMARK 6. I doubt the other direction is true. Let $X \subset \mathbb{P}^n$ be a non rational but stably rational variety. Assume that $X \times \mathbb{P}^a$ is rational but $X \times \mathbb{P}^{a-1}$ is not rational for some $a \geq 1$. Then $X \times \mathbb{P}^a$ can be birationally embedded as a cone, say S, with hyperplane section birational to $X \times \mathbb{P}^{a-1}$. In principle S could be Cremona equivalent to a hyperplane but its hyperplane section cannot be rational. This cannot occur for surfaces, see Corollary 1.

Proof. Let S_1 and S_2 be two cones and C_1, respectively, C_2 general hyperplane sections. Then by Proposition 2 there are birational maps $\Phi_i : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ sending S_i to a codimension 2 subvariety C_i' and such that:

- S_i is the valuation associated to $I_{C_i'}$,

- C_i' is birational to C_i.

I am assuming that C_1 is birational to C_2. Then the main result and its proof [14, p. 92] states that there is a Cremona map $\chi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ sending C_1' to C_2' and such that χ is an isomorphism on the generic point of C_1'. Then by Lemma 2 I may extend χ to the blow up of the C_i' to produce the required Cremona equivalence Ψ

$$
\begin{array}{ccccc}
Y_1 \supset E_1 & \xrightarrow{\;\;\;\;\Psi\;\;\;\;} & & & Y_2 \supset E_2 \\
{\scriptstyle\varepsilon_1}\nearrow & \searrow{\scriptstyle\eta_1} & & \swarrow{\scriptstyle\eta_2} & \nwarrow{\scriptstyle\varepsilon_2} \\
& V_1 \supset \Gamma_1 & & V_2 \supset \Gamma_2 & \\
& \downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_2} & \\
\mathbb{P}^n \supset S_1 & \xrightarrow{\;\;\Phi_1\;\;} \mathbb{P}^n \supset C_1' & \xrightarrow{\;\chi\;} \mathbb{P}^n \supset C_2' & \xleftarrow{\;\;\Phi_2\;\;} & \mathbb{P}^n \supset S_2
\end{array}
$$

$\square$

As observed before the result can be strengthened in lower dimension.

COROLLARY 1. *Let $S_1, S_2 \subset \mathbb{P}^3$ be cones and C_1, C_2 general hyperplane sections. Then C_1 is birational to C_2 if and only if S_1 is Cremona equivalent to S_2. In particular rational cones are Cremona equivalent to a plane.*

Proof. I have to prove that if S_1 and S_2 are Cremona equivalent then also C_1 and C_2 are birational. Note that the irregularity of a resolution of S_i is a birational invariant and it is the geometric genus of the curve C_i. This yields $g(C_1) = g(C_2)$ and concludes the proof. $\square$

REMARK 7. As observed, in the special case of rational surfaces in $\mathbb{P}^3$ this gives the Cremona equivalence to a plane for any rational cone. In particular any rational cone has positive sup-threshold.

It remains to translate the statement in $\sharp$-MMP dictionary for conic bundles.

COROLLARY 2. *Let $S \subset \mathbb{P}^3$ be a rational surface. Assume that there is:*

a) *a $\sharp$-minimal model of the pair (T, S_T) such that T has a conic bundle structure $\pi : T \to W$ onto a rational surface W, $S_T = \pi^* C$, for a curve $C \subset W$,*

b) *a birational map $\chi : T \dashrightarrow \mathbb{P}^3$ that contracts S_T to a curve, say Γ, such that S_T is the valuation associated to I_Γ.*

Then $\overline{\rho}(T, S_T) = \overline{\rho}(\mathbb{P}^3, S) > 0$. Assumption b) is always satisfied if π has a section, i.e. if π is a scroll structure.

Proof. Let (T, S_T) be a good model as in assumption a). By hypothesis there is a birational map $\chi : T \dashrightarrow \mathbb{P}^3$ that contracts S_T onto a curve Γ and such that S_T is the valuation associated to I_Γ. The curve Γ is dominated by a rational surface, and it is therefore rational. The extension trick used in Theorem 2 yields that (T, S_T) is birationally equivalent to a plane in $\mathbb{P}^3$. This is enough to prove that $\overline{\rho}(\mathbb{P}^3, S) > 0$.

If π has a section then all fibers are irreducible. Let $\phi : T \dashrightarrow Y$ be an elementary transformation that blows down S_T to a curve Γ. Then Y has a scroll structure onto W and I may run a $\sharp$-MMP on the base surface W, as described in [13, p. 700], that is an isomorphism in a neighborhood of Γ. This yields a new 3-fold model Z with a Mori fiber space structure onto either $\mathbb{P}^2$ or a ruled surface and then via elementary transformation of the scroll structure I produce the required map χ. $\square$

REMARK 8. Unfortunately the birational geometry of rational conic bundles without sections is very poorly understood and it is difficult to understand whether condition b) is always satisfied or not, even assuming the standard conjectures [9].

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