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ON CREMONA CONTRACTIBILITY

Dedicated to Professor Alberto Conte on the occasion of his 70th birthday

Abstract. In this note we give a constructive proof of a classical theorem which determines irreducible plane curves that are contractible to a point by a Cremona transformation. The problem of characterizing Cremona contractible (not necessarily irreducible) hypersurfaces in a projective space is in general widely open: we report on the only known result about reducible plane curves consisting of two components, due to Iitaka, and we discuss a couple of examples concerning plane curves with more components. Finally, we prove that all varieties of codimension at least two in a projective space are Cremona contractible to a point.

Introduction

Let $\mathbb{P}^2$ be the projective plane over $\mathbb{C}$. Plane Cremona transformations are birational maps $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ and they form the Cremona group $\mathrm{Cr}(\mathbb{P}^2)$. Cremona studied properties of geometric objects which are invariants under the action of $\mathrm{Cr}(\mathbb{P}^2)$. In this setting, a general problem is the classification of plane curves and linear systems up to Cremona transformations. See [3] for a classification, in particular of Cremona *minimal degree* models of plane curves. Such minimal degree is called the *Cremona degree* of the plane curve.

In this paper we deal with reduced, not necessarily irreducible, plane curves with Cremona degree 0, i.e. curves which are contracted to a set of points by a Cremona transformation. We say that these curves are *Cremona contractible*, shortly *Cr-contractible*.

Classical tools to study a plane curve C are the *m-adjoint linear systems* to C:

$$\mathrm{ad}_m(C) = f_*(|\tilde{C} + mK_S|),$$

where $f\colon S \to \mathbb{P}^2$ is a birational morphism which resolves the singularities of the curve C and $\tilde{C}$ denotes the strict transform of C on S. Relevant invariants of C can be read off from adjoint linear systems: for example, $\dim \mathrm{ad}_1(C) + 1 = g(C)$, where $g(C) = p_a(\tilde{C})$ is the *geometric genus* of C (e.g., if C is a union of rational curves, then $\mathrm{ad}_1(C) = \emptyset$).

In [4] and in [6, III, §21, p. 188], one finds the following:

THEOREM 1. *Let C be an irreducible plane curve. There exists a plane Cremona transformation which maps C to a line (which in turn is Cr-contractible) if and only if* $\mathrm{ad}_m(C) = \emptyset$ *for each* $m \geq 1$.

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The proof in [4] is incomplete, because of a careless use of infinitely near points of order two which are *satellite*, see Section 1 for notation and definitions, while the proof in [6] lacks some details. In Section 2 we give a complete and quick proof of this theorem, which is also constructive, in the sense that it describes the steps one has to make in order to find the Cremona transformation contracting the curve.

In 1982, Kumar and Murthy proved a stronger version of Theorem 1 with different methods which however are not constructive (in modern terms, they run a log minimal model program). It is interesting to notice that Kumar and Murthy, apparently unaware of the gap in Coolidge's proof, reported it *verbatim* in [10]. Their result is as follows:

THEOREM 2 (Kumar-Murthy). *Let C be an irreducible plane curve. There exists a plane Cremona transformation which maps C to a line if and only if* $\mathrm{ad}_1(C) = \mathrm{ad}_2(C) = \emptyset$.

In general, one can consider *pairs* (D,S), where D is a smooth curve on a smooth rational surface S. This viewpoint has been introduced by Suzuki in [17] and extensively carried out by Iitaka in several papers, cf. e.g. [7]. One defines the (log) *plurigenera* of the pair (D,S) as $P_m(D,S) = h^0(S, \mathcal{O}_S(m(D+K_S)))$, for each $m \geq 1$. One shows that the plurigenera are birational invariants of the pair (D,S). Moreover, one says that the pair (D,S) has (log) *Kodaira dimension* $-\infty$, shortly $\kappa(D,S) = -\infty$, if $P_m(D,S) = 0$ for each $m \geq 1$. The study of such pairs may be considered as the basic step of the classification problem in the birational setting.

If C is a reduced plane curve, we consider the pair $(\tilde{C}, S)$ as above. By abusing notation, we also denote such a pair by $(C, \mathbb{P}^2)$.

In this setting, Kumar-Murthy's Theorem 2 says that the pair $(C, \mathbb{P}^2)$, with C irreducible, has Kodaira dimension $-\infty$ if and only if $P_2(C, \mathbb{P}^2) = 0$, which is a log-analogue of Castelnuovo's rationality criterion for regular surfaces.

Since C is effective, if $(C, \mathbb{P}^2)$ has Kodaira dimension $-\infty$, then the adjoint linear systems $\mathrm{ad}_m(C)$ are empty, for each $m \geq 1$. Hence Theorem 2 says that being all the adjoint systems to C are empty is equivalent to $\kappa(C, \mathbb{P}^2) = -\infty$.

An interesting problem is the extension of Kumar-Murthy's Theorem 2 to the case of reducible curves. The only result in this direction is due to Iitaka, see [8, 9]:

THEOREM 3 (Iitaka). *Let* $C = C_1 + C_2$ *be a reduced plane curve with two irreducible components. There exists a plane Cremona transformation which maps C to two lines (which are Cr-contractible) if and only if* $\mathrm{ad}_1(C) = \mathrm{ad}_2(C) = \emptyset$*, or equivalently* $P_2(C, \mathbb{P}^2) = 0$.

Again Cr-contractibility is equivalent to $\kappa(C, \mathbb{P}^2) = -\infty$ in this case.

After Iitaka's extension of Kumar-Murthy's result, an optimistic and naif conjecture would be the equivalence between Cr-contractibility and $P_2(C, \mathbb{P}^2) = 0$, or equivalently $\mathrm{ad}_1(C) = \mathrm{ad}_2(C) = \emptyset$.

Interestingly enough, things are more complicated. An old example of Pompilj in [16] shows that Iitaka's Theorem 3 cannot be extended to the case of three components and moreover that the implication

$$\mathrm{ad}_1(C) = \mathrm{ad}_2(C) = \emptyset \Longrightarrow \mathrm{ad}_m(C) = \emptyset, \text{ for each } m \geq 1,$$

is not true if C has more than two components, see Example 1 in Section 3.

Note, however, the following result:

THEOREM 4 (Kojima-Takahashi). *Let C be a reduced plane curve with at most four irreducible components. Then, $\kappa(C,\mathbb{P}^2) = -\infty$ if and only if $P_6(C,\mathbb{P}^2) = 0$.*

Kojima and Takahashi do not consider the Cr-contractibility of C.

A further remark is that, in general, for a reducible plane curve, the property that $\kappa(C,\mathbb{P}^2) = -\infty$ is not equivalent to $\mathrm{ad}_m(C) = \emptyset$, for each $m \geq 1$, as it happens in the irreducible case, see Example 2 in Section 3. Example 2 is also a counterexample to the assertion in [6, III, §21, p. 190] that the irreducibility assumption in Theorem 1 can be removed.

All this given, one can consider the following:

PROBLEM 1. Is it true that a reduced plane curve C is Cr-contractible if and only if $\kappa(C,\mathbb{P}^2) = -\infty$?

In a work in progress, we deal with this problem for a reduced union of lines.

Concerning higher dimensional analogues, in Section 4 we show that any Zariski closed subset of $\mathbb{P}^r$ of codimension at least two is contractible to a point by a Cremona transformation $\mathbb{P}^r \dashrightarrow \mathbb{P}^r$. Thus the analogue to Problem 1 is meaningful only for hypersurfaces in $\mathbb{P}^r$: for example, which reduced and irreducible hypersurfaces in $\mathbb{P}^r$ can be contracted to a point by a Cremona transformation? When $r = 3$, Mella and Polastri in [13] gave a criterion, which is difficult to use, because one needs information on infinitely many birational models of the pair given by the surface and $\mathbb{P}^3$. The case of cones has been solved by Mella in this volume, see [12].

1. Notation and preliminaries

We will use standard notation in surface theory, e.g. $K = K_S$ will denote a canonical divisor, the linear equivalence of divisors will be denoted by $\equiv$, etc.

1.1. Infinitely near, proximate and satellite points (cf., e.g., [1, 3, 6])

Let S be a rational smooth irreducible projective surface. Any birational morphism $\sigma\colon S \to \mathbb{P}^2$ is the composition of blowing-ups $\sigma_i\colon S_i \to S_{i-1}$ at a point $p_i \in S_{i-1}$, $i = 1,\dots,n$:

$$\sigma\colon S = S_n \xrightarrow{\sigma_n} S_{n-1} \xrightarrow{\sigma_{n-1}} \cdots \xrightarrow{\sigma_2} S_1 \xrightarrow{\sigma_1} S_0 = \mathbb{P}^2. \tag{1}$$

Let $p \in \mathbb{P}^2$ be a point. One says that q is an *infinitely near point to p of order n*, and we write $q >^n p$, if there exists a birational morphism $\sigma \colon S \to \mathbb{P}^2$ as in (1), such that $p_1 = p$, $\sigma_i(p_{i+1}) = p_i$, $i = 1, \dots, n-1$, and $q \in Z_n = \sigma_n^{-1}(p_{n-1})$. For each $i = 1, \dots, n$, let $E_i = \sigma_i^{-1}(p_i) \subset S_i$ be the exceptional curve of σ_i. If $i > j$, let $\sigma_{i,j}$ be the morphism $S_i \to S_j$. For each $i = 1, \dots, n-1$, set $Z_i = \sigma_{n,i+1}^*(E_i)$ and let E_i' be the strict transform of E_i on S. Recall that $Z_1, \dots, Z_{n-1}, Z_n = E_n$ generate the Picard group $\mathrm{Pic}(S)$ of S over $\mathrm{Pic}(\mathbb{P}^2)$.

One says that q is *proximate to p*, and we write $q \to p$, if either $q >^1 p$ or $q >^n p$ with $n > 1$ and q lies on the strict transform E_1' of E_1 on S. In the latter case, one says that q is *satellite to p*, and we write $q \odot p$. This may happen only if p_i lies on the strict transform of E_1 on S_{i-1}, for each $i = 2, \dots, n$.

We will refer to *points on* the plane $\mathbb{P}^2$ including infinitely near ones. We will say that a point p is *proper*, and we will write $p \in \mathbb{P}^2$, if p is not infinitely near to any point of $\mathbb{P}^2$.

Let now C' be a curve on S and $C = \sigma_*(C')$. Then $C' = \sigma^*(C) - \sum_{i=1}^n m_i Z_i$, where $m_1, \dots, m_n$ are integers. If C is a curve, i.e. if C' is not contracted by σ, one says that m_i, $i = 1, \dots, n$, is the *(virtual) multiplicity* of C at the point p_i. If no component of C' is contracted by σ, then, for each $i = 1, \dots, n$, one has $C' \cdot E_i' \geqslant 0$, equivalently

$$m_i \geqslant \sum_{j\colon p_j \to p_i} m_j. \tag{2}$$

which is the *proximity inequality at p_i*.

A complete linear system $\mathcal{L}$ on S has the form $\mathcal{L} = |dH - \sum_{i=1}^n m_i Z_i|$ where d and the m_i are integers and $H = \sigma^*(L)$, with L a general line in $\mathbb{P}^2$. Abusing notation, we also write $\mathcal{L}$ as $\mathcal{L} = |dL - \sum_{i=1}^n m_i p_i|$.

1.2. Simplicity of a plane curve

Let C be a reduced plane curve. Let $d = \deg(C)$ and let $m_1 \geq m_2 \geq \cdots \geq m_n$ be the respective multiplicities of the singular points $p_1, p_2, \dots, p_n$ of C. If C is smooth, we assume $m_1 = 1$ and p_1 is a general point of C. By the proximity inequalities, we may and will assume that $p_i > p_j$ implies $i > j$. Therefore $p_1 \in \mathbb{P}^2$ and either $p_2 \in \mathbb{P}^2$ or $p_2 >^1 p_1$.

Set $h = (d - m_1)/2$, let t be the number of points p_i of multiplicity $m_i > h$ and let s be the number of *satellite* points among them.

The triplet (h, t, s) is called the *simplicity* of the curve C. A curve C' is said to be *simpler* than C if the simplicity of C' is lexicographically lower than the simplicity of C.

1.3. Cremona transformations

A linear system $\mathcal{L}$ of plane curves, with no divisorial fixed component, is called a *net* if $\dim(\mathcal{L}) = 2$. If, in addition, $\mathcal{L}$ defines a birational map $\gamma \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, i.e. a

Cremona transformation, the net is called *homaloidal*. In that case, the general curve C of $\mathcal{L}$ is irreducible and rational. If d is the degree of the members of $\mathcal{L}$, one says that γ has *degree* d. Cremona transformations of degree 1 are *projective* or *linear* transformations. When $d = 2$, one says that the Cremona transformation γ is *quadratic*. In that case, the homaloidal net defining γ has three simple base points p_1, p_2, p_3 and one says that γ is *centered* at p_1, p_2, p_3 and that p_1, p_2, p_3 are the fundamental points of γ. When γ has degree $d \geq 2$ and the homaloidal net $\mathcal{L}$ has a base point p_1 of multiplicity $d-1$, then $\mathcal{L}$ also has $2d-2$ simple base points $p_2, \dots, p_{2d-1}$ and one says that γ is a *de Jonquières* transformation centered at $p_1, p_2, \dots, p_{2d-1}$.

The famous Noether-Castelnuovo Theorem states that each Cremona transformation is the composition of finitely many linear and quadratic transformations (see [3] for a short proof that uses the simplicity).

The next lemmas show that quadratic transformations can make a curve simpler.

LEMMA 1. *Let (h,t,s) be the simplicity of an irreducible plane curve C. Assume there exists the quadratic Cremona transformation γ centered at points p_1, p_i, p_j where C has respective multiplicities m_1, m_i, m_j with $m_1 = d-2h \geq m_i \geq m_j > h$. Then $\gamma_*(C)$ is simpler than C.*

Proof. Let $\tilde{p}_k$, $k = 1, i, j$, be the fundamental points of γ^{-1}. Setting $\delta = m_1 + m_i + m_j - d > m_1 + 2h - d = 0$, the curve $\gamma_*(C)$ has degree $\tilde{d} = d - \delta$ and multiplicity $\tilde{m}_k = m_k - \delta$ at $\tilde{p}_k$, $k = 1, i, j$, while all the other multiplicities of $\gamma_*(C)$ are the same as the corresponding ones of C. Denote by $(\tilde{h}, \tilde{t}, \tilde{s})$ the simplicity of $\gamma_*(C)$. Setting $\tilde{m}$ the maximal multiplicity of points of $\gamma_*(C)$, one has that $2\tilde{h} = \tilde{d} - \tilde{m} \leq \tilde{d} - \tilde{m}_1 = d - m_1 = 2h$. If $\tilde{h} < h$, then $\gamma_*(C)$ is simpler than C. Otherwise $\tilde{h} = h$ but $\tilde{m}_i = m_i - \delta = 2h - m_j < h$ and $\tilde{m}_j = m_j - \delta = 2h - m_i < h$, hence $\tilde{t} = t - 2$ and $\gamma_*(C)$ is simpler than C. □

LEMMA 2. *Let (h,t,s) be the simplicity of an irreducible plane curve C. Suppose that C has multiplicity m_1, m_i, m_j at the points p_1, p_i, p_j with $m_1 = d - 2h \geq m_i \geq m_j > h$, $p_j >^1 p_i >^1 p_1$ and p_j is satellite to p_1. Then, a quadratic Cremona transformation γ centered at p_1, p_i, p, where p is a general point in $\mathbb{P}^2$, is such that $\gamma_*(C)$ is simpler than C.*

Proof. Let $\tilde{p}_1, \tilde{p}_i, \tilde{p}$ be the fundamental points of γ^{-1}. Setting $\lambda = d - m_1 - m_i$, one has $0 \leq \lambda = 2h - m_i < h$. The curve $\gamma_*(C)$ has degree $\tilde{d} = d + \lambda \geq d$ and multiplicity $\tilde{m}_1 = m_1 + \lambda \geq m_1$ at $\tilde{p}_1$, multiplicity $\tilde{m}_i = m_i + \lambda$ at $\tilde{p}_i$ and multiplicity λ at $\tilde{p}$, while the remaining multiplicities are not affected by γ. Denote by $(\tilde{h}, \tilde{t}, \tilde{s})$ the simplicity of $\gamma_*(C)$. Then, $2\tilde{h} = \tilde{d} - \tilde{m}_1 = d - m_1 = 2h$ and furthermore $\tilde{t} = t$, but $\tilde{s} = s - 1$ because the point $\tilde{p}_j$ corresponding to p_j via γ is no longer satellite, in fact $\tilde{p}_j >^1 \tilde{p}_1$. □

2. A proof of Theorem 1

The proof is by induction on the simplicity.

One has that $h \geq 0$ and that $h = 0$ if and only if C is a line, that is the assertion.

If $h = 1/2$, then C is a curve of degree $d \geq 2$ with a point p_1 of multiplicity $m_1 = d-1$ and no other singular point, otherwise the line through it and p_1 would be a component of C, contradicting the irreducibility assumption. The curve C is mapped to a line by a de Jonquières transformation of degree d centered at p_1, with multiplicity $d-1$, and $2d-2$ general points of C.

Let now $h \geq 1$. The proof of Theorem 1 will be concluded by the following

PROPOSITION 1. *Let C be an irreducible plane curve such that its maximal multiplicity is $m_1 \leq d-2$, where d is the degree of C. If*

$$\mathrm{ad}_m(C) = \emptyset, \quad \text{for each } m \geq 1, \tag{3}$$

then there exists a quadratic transformation which maps C to a simpler curve.

First we need some lemmas.

LEMMA 3. *In the above setting, one has $m_1 > h$, or equivalently $m_1 > d/3$.*

Proof. The equivalence between $m_1 > h$ and $m_1 > d/3$ is clear. The assertion is trivial if $d < 3$, so we assume $d \geq 3$.
Suppose by contradiction that $m_1 < d/3$. Then the $[d/3]$-adjoint to C, where $[x] = \max\{m \in \mathbb{N} \mid m \leq x\}$, would be non-empty:

$$|\tilde{C} + [d/3]K| = \left| \left(d - 3\left[\frac{d}{3}\right]\right) L - \sum_{i=1}^{n} \left(m_i - \left[\frac{d}{3}\right]\right) p_i \right| \supseteq |(d \bmod 3)L| \neq \emptyset,$$

contradicting (3). □

Set $h = [h] \in \mathbb{Z}$ and $\varepsilon = 2(h-h) \in \{0,1\}$. Lemma 3 reads $m_1 \geq h+1$.

LEMMA 4. *In the above setting, one has $m_2 \geq m_3 \geq h+1$.*

Proof. Suppose by contradiction that $m_3 \leq h$. Then

$$\begin{aligned} |\tilde{C} + hK| &\supseteq \left|\varepsilon L + (m_1 - h)(L - p_1) - (m_2 - h)p_2\right| = \\ &= (m_2 - h)L_{12} + |\varepsilon L + (m_1 - m_2)(L - p_1)| \neq \emptyset, \end{aligned}$$

where L_{12} is (the strict transform of) the line passing through p_1 and p_2, against (3). □

Either $p_2 \in \mathbb{P}^2$ or $p_2 >^1 p_1$. Moreover, either $p_3 \in \mathbb{P}^2$, or $p_3 >^1 p_1$, or $p_3 >^1 p_2$. In any case, there is no line passing through p_1, p_2, p_3, because $m_1 + m_2 + m_3 > m_1 + 2h = d$ and C is irreducible of degree $d \geq 3$. By Lemma 1, the curve C is mapped to a simpler curve by a quadratic transformation γ centered at p_1, p_2, p_3, unless either

1. $p_3 >^1 p_2 >^1 p_1$ and p_3 is satellite to p_1, or
2. $p_2 >^1 p_1$ and $p_3 >^1 p_1$,

that are the cases when there is no quadratic transformation centered at p_1, p_2, p_3.

In case (1), the curve C is mapped to a simpler curve by the quadratic transformation centered at p_1, p_2 and at a general point in $\mathbb{P}^2$ by Lemma 2.

We are left with case (2). Let j be the largest integer such that $p_i >^1 p_1$ for each $2 \le i \le j$. By construction, one has $j \ge 3$.

LEMMA 5. *In the above setting, one has* $j+1 \le n$ *and* $m_{j+1} \ge h+1$.

Proof. The proximity inequality at p_1 implies that $m_1 \ge m_2 + m_3 + \cdots + m_j$, therefore

$$m_1 - h \ge m_2 + m_3 + \cdots + m_j - (j-1)h = (m_2 - h) + (m_3 - h) + \cdots + (m_j - h).$$

Suppose by contradiction that either $j = n$ or $m_{j+1} \le h$. Then

$$|\tilde{C} + hK| \supseteq \left| \varepsilon L + (m_1 - h)(L - p_1) - \sum_{i=2}^{j} \left(m_i - h\right) p_i \right| \supseteq$$

$$\supseteq \sum_{i=2}^{j} \left(m_i - h\right) L_{1i} + \left| \varepsilon L + \left(m_1 - h - \sum_{i=2}^{j} (m_i - h) \right) (L - p_1) \right| \ne \emptyset,$$

where L_{1i} is (the strict transform of) the line passing through p_1 and p_i, contradicting (3). □

Then, either $p_{j+1} \in \mathbb{P}^2$ or $p_{j+1} >^1 p_i >^1 p_1$, with $1 < i \le j$. In the latter case, either p_{j+1} is satellite or it is not. If p_{j+1} is satellite, then we get a simpler curve by using Lemma 2. Otherwise, there is no line passing through p_1, p_i, p_{j+1}, because of the irreducibility of C and we get a simpler curve by using Lemma 1.

This ends the proof of Proposition 1 and hence of Theorem 1.

3. Examples and a remark

We recall an interesting example due to Pompilj in [16].

EXAMPLE 1 (Pompilj). Let C_1, C_2 be two irreducible rational plane quartic curves and let C_3 be a line such that $C = C_1 + C_2 + C_3$ is a reduced curve of degree 9 with 10 triple points, where the multiplicities of the C_i, $i = 1, 2, 3$, are as follows:

	deg	p_0	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
C_1	4	2	2	2	1	1	1	1	1	1	1
C_2	4	1	1	1	2	2	2	1	1	1	1
C_3	1	0	0	0	0	0	0	1	1	1	1
C	9	3	3	3	3	3	3	3	3	3	3

One checks that $\mathrm{ad}_1(C) = \mathrm{ad}_2(C) = \emptyset$, but $\mathrm{ad}_3(C) \ne \emptyset$, therefore C is not Cr-contractible. The existence of C can be proved by giving equations: for example we take $C_1 : x^2y^2 + 2x^2 + 3y^2 + 6xy(x+y+1) = 0$, $C_3 : x+y+1 = 0$, and C_2 the orthogonal symmetric of C_1 with respect to C_3.

REMARK 1. Let C be a reduced union of lines in the plane. By definition, $(C,\mathbb{P}^2)$ has Kodaira dimension $-\infty$ if and only if $|m(C+K)|=\emptyset$, for each $m\geqslant 1$. On the other hand, since $C+K$ intersects negatively all components of C, then $|mC+mK|\neq\emptyset$ if and only if $|(m-1)C+mK|\neq\emptyset$. When $m=2$, this means that

$$P_2(C,\mathbb{P}^2)=\dim(\mathrm{ad}_2(C))+1.$$

When $m\geq 3$, as the next example shows, it is possible that

$$P_m(C,\mathbb{P}^2)>\dim(\mathrm{ad}_m(C))+1.$$

EXAMPLE 2. Let C be the union of $d\geq 9$ distinct lines $L_1,\dots,L_d$ with the first $d-3$ passing through a point p_0, while the remaining three in general position. Thus C has an ordinary point p_0 of multiplicity $d-3$ and $3(d-2)$ nodes $p_1,\dots,p_{3(d-2)}$. One checks that $\mathrm{ad}_m(C)=\emptyset$ for each $m\geq 1$. Since

$$\tilde{C}=L_1+L_2+\cdots+L_d\equiv dL-(d-3)p_0-2(p_1+p_2+\cdots+p_{3(d-2)}),$$

where we denote by L_i, $i=1,\dots,d$, also the strict transform of L_i in S, one has

$$\begin{aligned}|2\tilde{C}+3K| &= |(2d-9)L-(2d-9)p_0-1(p_1+\cdots+p_{3(d-2)})|=\\ &=\{L_1+\cdots+L_{d-3}+L_1'+L_2'+L_3'\}\neq\emptyset,\end{aligned}$$

where L_i', $i=1,2,3$, is the line passing through p_0 and a vertex of the triangle whose sides are the three general lines L_{d-2},L_{d-1},L_d.

4. Cremona contractibility in codimension at least 2

4.1. Monoids.

A *monoid* in $\mathbb{P}^r$ is a hypersurface of degree d with a point of multiplicity $d-1$, called the *vertex* of the monoid. A monoid will be called a *true* monoid if it is irreducible and the vertex has multiplicity exactly one less than the degree. Such a monoid is rational, because its *stereographic projection* from the vertex to a hyperplane not containing the vertex is birational.

REMARK 2. The linear system of monoids of degree d and fixed vertex in $\mathbb{P}^r$ has dimension

$$\binom{d+r}{r}-1-\binom{d+r-2}{r}=\frac{2}{(r-1)!}d^{r-1}+\frac{r-1}{(r-2)!}d^{r-2}+O(d^{r-3}).$$

LEMMA 6. *Consider a Zariski closed subset Z of $\mathbb{P}^r$, no component of which is a hypersurface. For a general point $p\in\mathbb{P}^r$, Z is contained in a true monoid with vertex p.*

Proof. Let $n \leq r-2$ be the dimension of Z, i.e. the maximal dimension of the components of Z.

Let $\pi\colon Z \dashrightarrow Z' \subset \Pi$ be the projection of Z to a hyperplane Π not containing p. Let us take affine coordinates so that p is the point at infinity of the x_r-axis and $\Pi : x_r = 0$.

Assume first that $n < r-2$. Then we may consider two polynomials $f, g \in k[x_1, \ldots, x_{r-1}]$ of degree d and $d-1$, respectively, with no common factor, vanishing on Z'. They clearly exist for $d \gg 0$. Then the monoid with equation $f + x_r g = 0$ is a true monoid with vertex p containing Z.

Suppose now $n = r-2$ and let δ be the degree of the codimension 2 part Z_1 of Z. The linear system $\mathcal{L}$ of monoids with vertex p containing Z contains the linear system $\mathcal{L}'$ of cones with vertex p over Z_1 plus a monoid of degree $d-\delta$ with vertex p containing $Z_2 = \overline{Z \setminus Z_1}$.

By Remark 2 and by the Riemann-Roch Theorem, we have

$$\dim \mathcal{L} \geq \frac{2}{(r-1)!} d^{r-1} + \frac{r-1-\delta}{(r-2)!} d^{r-2} + O(d^{r-3}).$$

On the other hand, by the same argument,

$$\dim \mathcal{L}' = \frac{2}{(r-1)!} d^{r-1} + \frac{r-1-2\delta}{(r-2)!} d^{r-2} + O(d^{r-3}),$$

therefore $\dim \mathcal{L}' < \dim \mathcal{L}$, so that $\mathcal{L}'$ is strictly contained in $\mathcal{L}$.

If Z_1 is irreducible, the assertion follows by Bertini's Theorem. If Z_1 is reducible, one has to iterate this argument using Bertini's Theorem by stepwise eliminating all cones over the components of Z_1. □

REMARK 3. In the hypothesis of the previous lemma, the generality assumption on p can be relaxed: it suffices to assume that the projection from p preserves the degree of the components of Z of codimension 2, if it preserves the dimension of Z.

4.2. Monoidal transformations.

We call a Cremona transformation $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ *monoidal* if it is defined by a homaloidal linear system $\mathcal{L}$ whose general member is a monoid. The classification of monoidal linear systems is an interesting problem which, to the best of our knowledge, has never been considered in its generality: the case of quadrics is classical but a complete classification in $\mathbb{P}^3$ has been obtained only recently [14]; contributions to special quadratic ones is due to [2] and [15]. See also [5] as a general reference.

A *special* case is the one in which the vertex p of the monoids of the system is fixed.

EXAMPLE 3 (De Jonquières transformations). Let X be a true monoid of degree d with vertex p. Let us consider a true monoid Y of degree $d-1$ with vertex p and let C be the intersection scheme of X and Y. Let $\mathcal{L}$ be the linear system of monoids of degree d with vertex p containing C. Then $\mathcal{L}$ is a special homaloidal linear system.

Indeed, the restriction sequence of $\mathcal{L}$ to X (or, for all that matters, to the general monoid of $\mathcal{L}$) shows that $\dim \mathcal{L} = r$ and that the rational map defined by $\mathcal{L}$ restricts to X to the stereographic projection from the vertex.

The examples implies:

PROPOSITION 2. *Every true monoid is Cr-contractible, i.e. it is contained in a homaloidal linear system.*

4.3. Cr-contractibility in codimension at least 2.

THEOREM 5. *Let Z be a Zariski closed subset of $\mathbb{P}^r$, no component of which is a hypersurface. For a general point $p \in \mathbb{P}^r$, there exists a Cremona transformation $\gamma\colon \mathbb{P}^r \dashrightarrow \mathbb{P}^r$ such that $\gamma(Z)$ is the projection of Z from p to a hyperplane not containing p.*

Proof. It follows by Lemma 6 and Example 3. □

REMARK 4. The generality of the point p is not necessary in the statement of the previous theorem. It suffices to take the point off the 0-dimensional components of Z.

Let $n \leq r-2$ be the dimension of Z. By iterating Theorem 5, there exists a Cremona transformation $\gamma\colon \mathbb{P}^r \to \mathbb{P}^r$ such that $\gamma_{|Z}\colon Z \to \mathbb{P}^n$ is a general projection, hence it is a finite morphism.

LEMMA 7. *Fix a linear subspace $\mathbb{P}^n$ in $\mathbb{P}^r$, $r > n$. Then there exists a Cremona transformation $\mathbb{P}^r \dashrightarrow \mathbb{P}^r$ which contracts $\mathbb{P}^n$ to a point.*

Proof. Follows from Theorem 5 and Remark 4. □

COROLLARY 1. *Let Z be a Zariski closed subset of $\mathbb{P}^r$, no component of which is a hypersurface. Then there exists a Cremona transformation $\gamma\colon \mathbb{P}^r \dashrightarrow \mathbb{P}^r$, which is defined at the general point of each component of Z, such that $\gamma(Z)$ is a point.*

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