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NON-RATIONALITY OF THE $\mathfrak{S}_6$-SYMMETRIC QUARTIC THREEFOLDS

Abstract. We prove that the quartic hypersurfaces defined by $\sum x_i = t\sum x_i^4 - (\sum x_i^2)^2 = 0$ in $\mathbb{P}^5$ are not rational for $t \neq 0,2,4,6,\frac{10}{7}\cdot$

Pour Alberto, à l'occasion de son 70^e *anniversaire*

1. Introduction

Let V be the standard representation of $\mathfrak{S}_6$ (that is, V is the hyperplane $\sum x_i = 0$ in $\mathbb{C}^6$, with $\mathfrak{S}_6$ acting by permutation of the basis vectors). The quartic hypersurfaces in $\mathbb{P}(V)$ $(\cong \mathbb{P}^4)$ invariant under $\mathfrak{S}_6$ form the pencil

$$X_t :\ t\sum x_i^4 - (\sum x_i^2)^2 = 0\ , \quad t \in \mathbb{P}^1\ .$$

This pencil contains two classical quartic hypersurfaces, the Burkhardt quartic X_2 and the Igusa quartic X_4 (see for instance [6]); they are both rational.

For $t \neq 0,2,4,6$ and $\frac{10}{17}$, the quartic X_t has exactly 30 nodes; the set of nodes $\mathcal{N}$ is the orbit under $\mathfrak{S}_6$ of $(1,1,\rho,\rho,\rho^2,\rho^2)$, with $\rho = e^{\frac{2\pi i}{3}}$ ([7], §4). We will prove:

THEOREM. *For $t \neq 0,2,4,6,\frac{10}{7}$, X_t is not rational.*

The method is that of [1] : we show that the intermediate Jacobian of a desingularization of X_t is 5-dimensional and that the action of $\mathfrak{S}_6$ on its tangent space at 0 is irreducible. From this one sees easily that this intermediate Jacobian cannot be a Jacobian or a product of Jacobians, hence X_t is not rational by the Clemens-Griffiths criterion. We do not know whether X_t is unirational.

I am indebted to A. Bondal and Y. Prokhorov for suggesting the problem, to A. Dimca for explaining to me how to compute explicitly the defect of a nodal hypersurface, and to I. Cheltsov for pointing out the rationality of $X_{\frac{10}{7}}$.

2. The action of $\mathfrak{S}_6$ on $T_0(JX)$

We fix $t \neq 0,2,4,6,\frac{10}{7}$, and denote by X the desingularization of X_t obtained by blowing up the nodes. The main ingredient of the proof is the fact that the action of $\mathfrak{S}_6$ on JX is non-trivial. To prove this we consider the action of $\mathfrak{S}_6$ on the tangent space $T_0(JX)$, which is by definition $H^2(X,\Omega^1_X)$.

LEMMA 1. *Let $\mathcal{C}$ be the space of cubic forms on $\mathbb{P}(V)$ vanishing along $\mathcal{N}$. We have an isomorphism of $\mathfrak{S}_6$-modules $\mathcal{C} \cong V \oplus H^2(X, \Omega^1_X)$.*

Proof : The proof is essentially contained in [2]; we explain how to adapt the arguments there to our situation. Let $b : P \to \mathbb{P}(V)$ be the blowing-up of $\mathbb{P}(V)$ along $\mathcal{N}$. The threefold X is the strict transform of X_t in P. The exact sequence

$$0 \to N^*_{X/P} \longrightarrow \Omega^1_{P|X} \longrightarrow \Omega^1_X \to 0$$

gives rise to an exact sequence

$$0 \to H^2(X, \Omega^1_X) \longrightarrow H^3(X, N^*_{X/P}) \longrightarrow H^3(X, \Omega^1_{P|X}) \to 0$$

([2], proof of theorem 1), which is $\mathfrak{S}_6$-equivariant. We will compute the two last terms.

The exact sequence

$$0 \to \Omega^1_P(-X) \longrightarrow \Omega^1_P \longrightarrow \Omega^1_{P|X} \to 0$$

provides an isomorphism $H^3(X, \Omega^1_{P|X}) \xrightarrow{\sim} H^4(P, \Omega^1_P(-X))$, and the latter space is isomorphic to $H^4(\mathbb{P}(V), \Omega^1_{\mathbb{P}(V)}(-4))$ ([2], proof of Lemma 3). By Serre duality $H^4(\mathbb{P}(V), \Omega^1_{\mathbb{P}(V)}(-4))$ is dual to $H^0(\mathbb{P}(V), T_{\mathbb{P}(V)}(-1)) \cong V$. Thus the $\mathfrak{S}_6$-module $H^3(X, \Omega^1_{P|X})$ is isomorphic to V^*, hence also to V.

Similarly the exact sequence $0 \to \mathcal{O}_P(-2X) \longrightarrow \mathcal{O}_P(-X) \longrightarrow N^*_{X/P} \to 0$ and the vanishing of $H^i(P, \mathcal{O}_P(-X))$ ([2], Corollary 2) provide an isomorphism of $H^3(X, N^*_{X/P})$ onto $H^4(P, \mathcal{O}_P(-2X))$, which is naturally isomorphic to the dual of $\mathcal{C}$ ([2], proof of Proposition 2). The lemma follows. □

LEMMA 2. *The dimension of $\mathcal{C}$ is* 10.

Proof : Recall that the *defect* of X_t is the difference between the dimension of $\mathcal{C}$ and its expected dimension, namely :

$$\operatorname{def}(X_t) := \dim \mathcal{C} - (\dim H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(3)) - \#\mathcal{N}) \, .$$

Thus our assertion is equivalent to $\operatorname{def}(X_t) = 5$.

To compute this defect we use the formula of [5], Theorem 1.5. Let $F = 0$ be an equation of X_t in $\mathbb{P}^4$; let $R := \mathbb{C}[X_0, \ldots, X_4]/(F'_{X_0}, \ldots, F'_{X_4})$ be the Jacobian ring of F, and let R^{sm} be the Jacobian ring of a *smooth* quartic hypersurface in $\mathbb{P}^4$. The formula is

$$\operatorname{def}(X_t) = \dim R_7 - \dim R^{sm}_7 \, .$$

In our case we have $\dim R^{sm}_7 = \dim R^{sm}_3 = 35 - 5 = 30$; a simple computation with Singular (for instance) gives $\dim R_7 = 35$. This implies the lemma. □

PROPOSITION 1. *The $\mathfrak{S}_6$-module $H^2(X, \Omega^1_X)$ is isomorphic to V.*

Proof : Consider the homomorphisms a and b of $\mathbb{C}^6$ into $H^0(\mathbb{P}(V),\mathcal{O}_{\mathbb{P}(V)}(3))$ given by $a(e_i)=x_i^3$, $b(e_i)=x_i\sum x_j^2$. They are both $\mathfrak{S}_6$-equivariant and map V into $\mathcal{C}$; the subspaces $a(V)$ and $b(V)$ of $\mathcal{C}$ do not coincide, so we have $a(V)\cap b(V)=0$. By Lemma 2 this implies $\mathcal{C}=a(V)\oplus b(V)$, so $H^2(X,\Omega^1_X)$ is isomorphic to V by Lemma 1. □

REMARK 1. Suppose $t=2,6$ or $\frac{10}{7}$. Then the singular locus of X_t is $\mathcal{N}\cup\mathcal{N}'$, where $\mathcal{N}'$ is the $\mathfrak{S}_6$-orbit of the point $(1,-1,0,0,0,0)$ for $t=2$, $(1,-1,1,-1,1,-1)$ for $t=6$, $(-5,1,1,1,1,1)$ for $t=\frac{10}{7}$ [7]. Since $x_1^3-x_0^3$ does not vanish on $\mathcal{N}'$, the space of cubics vanishing along $\mathcal{N}\cup\mathcal{N}'$ is strictly contained in $\mathcal{C}$. By Lemma 1 it contains a copy of V, hence it is isomorphic to V; therefore $H^2(X,\Omega^1_X)$ and JX are zero in these cases. We have already mentioned that X_2 and X_4 are rational. The quartic $X_{\frac{10}{7}}$ is rational: it is the image of the anticanonical map of $\mathbb{P}^3$ blown up along 6 lines which are permuted by $\mathfrak{S}_6$ (see [4], proof of Lemma 4.5, and the references given there). We do not know whether this is the case for X_6.

3. Proof of the theorem

To prove that X is not rational, we apply the Clemens-Griffiths criterion ([3], Cor. 3.26): it suffices to prove that JX is not a Jacobian or a product of Jacobians.

Suppose $JX\cong JC$ for some curve C of genus 5. By the Proposition $\mathfrak{S}_6$ embeds into the group of automorphisms of JC preserving the principal polarization; by the Torelli theorem this group is isomorphic to $\mathrm{Aut}(C)$ if C is hyperelliptic and $\mathrm{Aut}(C)\times\mathbb{Z}/2$ otherwise. Thus we find $\#\mathrm{Aut}(C)\geq\frac{1}{2}6!=360$. But this contradicts the Hurwitz bound $\#\mathrm{Aut}(C)\leq 84(5-1)=336$.

Now suppose that JX is isomorphic to a product of Jacobians $J_1\times\ldots\times J_p$, with $p\geq 2$. Recall that such a decomposition is *unique* up to the order of the factors: it corresponds to the decomposition of the Theta divisor into irreducible components ([3], Cor. 3.23). Thus the group $\mathfrak{S}_6$ permutes the factors J_i, and therefore acts on $[1,p]$; by the Proposition this action must be transitive. But we have $p\leq\dim JX=5$, so this is impossible. □

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