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ABELIAN FOURFOLDS OF WEIL TYPE AND CERTAIN K3 DOUBLE PLANES

Abstract. Double planes branched in 6 lines give a famous example of K3 surfaces. Their moduli are well understood and related to Abelian fourfolds of Weil type. We compare these two moduli interpretations and in particular divisors on the moduli spaces. On the K3 side, this is achieved with the help of elliptic fibrations. We also study the Kuga-Satake correspondence on these special divisors.

Introduction

The wonderful geometry of double planes branched in 6 lines has been addressed extensively in the articles [10, 9]. When the lines are in general position such double planes are K3 surfaces and their moduli are well known and are determined through the period map.

The target space for the period map is defined as follows. Consider the lattice

$$(1) \qquad\qquad T = U^2 \perp \langle -1 \rangle \perp \langle -1 \rangle.$$

Here U is the hyperbolic lattice with basis $\{e, f\}$ and $\langle e, e \rangle = \langle f, f \rangle = 0$, $\langle e, f \rangle = 1$ and for $n \in \mathbb{Z}$ the notation $\langle n \rangle$ means that we have a one dimensional lattice with basis $\{g\}$ such that $\langle g, g \rangle = n$. We write $z = (z_1, z_2, z_3, z_4) \in T \otimes \mathbb{C}$ and form the associated domain

$$D(T) := \{z \in \mathbb{P}(T \otimes \mathbb{C}) \mid \langle z, z \rangle = 0, \, \langle z, z \rangle > 0\}$$

The orthogonal group $\mathrm{O}(T)$ acts on $D(T)$, a space with two connected components; the commutator subgroup of $\mathrm{O}(T)$ preserves the connected components, distinguished by the sign of $\mathrm{im}(z_3/z_1)$. Let $\mathbf{D}_4$ be the one with the positive sign. With $\mathrm{SO}^*(T)$ the group intruduced in 1.1 the target of the period map is

$$(2) \qquad\qquad \mathbf{M} = \mathbf{D}_4 / \mathrm{SO}^*(T).$$

This is the principal moduli space in this article.

The source of the period map is the configuration space U_6 of 6 ordered lines in good position.[*] The linear group $\mathrm{GL}(3; \mathbb{C})$ acts on this space and we form the quotient

$$\mathbf{X} : \quad = \quad U_6 / \mathrm{GL}(3; \mathbb{C}) : \textbf{configuration space of 6 ordered lines}$$
$$\textbf{in good position in } \mathbb{P}^2.$$

The symmetric group $\mathfrak{S}_6$ acts on this space and the quotient $\mathbf{X}$ is the configuration space of 6 non-ordered lines in good position. One of the peculiarities of the number

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[*]This means that the resulting sextic has only ordinary nodes.

6 is that there is a further (holomorphic) involution $*$ on 6 lines which comes from the correlation map with respect to a conic in the plane (see § 3.1) and which we call *the correlation involution*. Six-tuples of lines in good position related by this involution correspond to isomorphic double covering K3-surfaces. The explanation of why this is true[†] uses beautiful classical geometry related to Cremona transformations for which we gratefully acknowledge the source [4]. See Prop. 5 and Remark 7.

Furthermore, the involution commutes with the action of $\mathfrak{S}_6$. So the period map descends to this quotient and by [10] this yields a biholomorphic map

$$\mathbf{X}/H \xrightarrow{\simeq} \mathbf{M}, \quad H := \mathfrak{S}_6 \times \{*\}$$

In other words we may identify our moduli space with the quotient of the configuration space of 6 unordered lines in good position by the correlation involution.

In loc. cit. some special divisors on $\mathbf{X}$ (and hence on $\mathbf{M}$) are studied, in particular the divisor called XQ which corresponds to 6 lines all tangent to a common conic C. The associated double covers are K3 surfaces with rich geometry. Indeed this 3-dimensional variety is the moduli space $\mathbf{M}_{2,2}$ of curves of genus 2 (with a level 2 structure). To see this note that the conic C meets the ramification divisor R (the 6 lines) in 6 points and on the K3 surface this gives a genus 2 curve D lying over it. In [10, §0.19] it is explained that the Kummer surface $J(D)/\{\pm 1\}$ associated to the jacobian $J(D)$ of D is isomorphic to the double cover branched in R and that there is a natural level 2 structure on $J(D)$. Furthermore, the period map induces a biholomorphism

$$(3) \qquad \mathbf{M}_{2,2} = XQ \xrightarrow{\simeq} (\mathfrak{h}_2/\Gamma(2))^0 \hookrightarrow \mathbf{X}, \quad \Gamma = \mathrm{Sp}(2,\mathbb{Z})/\pm 1,$$

where the image is Zariski-open in $\mathfrak{h}_2/\Gamma(2)$ (in fact, in loc. cit. an explicit partial compactification $\mathbf{X}$ of X is described such that the closure of $\mathbf{M}_{2,2}$ in $\mathbf{X}$ is exactly the space $\mathfrak{h}_2/\Gamma(2)$).

In [10, 5] one also finds a relation with a certain moduli space of Abelian 4–folds. We recall that principally polarized Abelian varieties of dimension g (without any further restriction) are parametrized by the Siegel upper half space $\mathfrak{h}_g$. For a generic such Abelian variety A its rational endomorphism ring $\mathrm{End}_{\mathbb{Q}} A$ is isomorphic to $\mathbb{Q}$, but special Abelian varieties have larger endomorphism rings and are parametrised by sub-varieties of $\mathfrak{h}_g$. Here we consider 4–dimensional A with $\mathbb{Z}(i) \subset \mathrm{End}(A)$ where the endomorphisms are assumed to preserve the principal polarization. These are parametrized by the 4-dimensional domain [‡]

$$\mathbf{H}_2 := \left\{ W \in M_{2\times 2}(\mathbb{C}) \mid \frac{1}{2\mathrm{i}}(W - W^*) > 0 \right\} \simeq \mathrm{U}((2,2))/\mathrm{U}(2) \times \mathrm{U}(2)$$

which is indeed (see e.g. [5, §1.2]) a subdomain of the Siegel upper half space $\mathfrak{h}_4$ by

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[†]The referee urged us to find an explanation of this since it could not be found in the literature.

[‡]As usual, for a matrix W, we abbreviate $W^* = {}^{\mathsf{T}}\overline{W}$.

means of the embedding

$$
\begin{aligned}
\iota : \mathbf{H}_2 &\hookrightarrow \mathfrak{h}_4 \\
W &\mapsto U \begin{pmatrix} W & 0 \\ 0 & {}^{\top}W \end{pmatrix} U^*, \\
&\quad U = \tfrac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{i}\mathbf{1}_2 & -\mathrm{i}\mathbf{1}_2 \\ \mathbf{1}_2 & \mathbf{1}_2 \end{pmatrix}.
\end{aligned}
\tag{4}
$$

As for all polarized Abelian varieties, the complex structure of such Abelian fourfolds A is faithfully reflected in the polarized weight one Hodge structure on $H^1(A)$. The second cohomology carries an induced weight 2 Hodge structure. The extra complex structure on A (induced by multiplication with i) makes it possible to single out a rational sub Hodge structure $T(A) \subset H^2(A)$ of rank 6 with Hodge numbers $h^{2,0} = 1$, $h^{1,1} = 4$. Such Hodge structures are classified by points in the domain $\mathbf{D}_4$ we encountered before. Indeed, there is an isomorphism

$$
\mathbf{H}_2 \xrightarrow{\sim} \mathbf{D}_4 \simeq \mathrm{O}(2,4)/\mathrm{O}(2) \times \mathrm{O}(4)
$$

which is induced by an isomorphism between classical real Lie groups which on the level of Lie algebras gives the well known isomorphism $\mathfrak{su}(2,2) \simeq \mathfrak{so}(6)$. This isomorphism is implicit in [10]. One of the aims of this article is to review this using the explicit and classically known isomorphism between corresponding Lie groups. See § 1; the corresponding Hodge theoretic discussion is to be found in § 2. This gives an independent and coordinate free presentation of the corresponding results in [10, 9]. In particular, on the level of moduli spaces we find (see § 2.2):

$$
\mathbf{M} := \mathbf{H}_2 / \mathrm{U}^*((2,2);\mathbb{Z}[\mathrm{i}]) \xrightarrow{\sim} \mathbf{D}_4 / \mathrm{SO}^*(T).
$$

The group $\mathrm{U}^*((2,2);\mathbb{Z}[\mathrm{i}])$ is an extension of the unitary group $\mathrm{U}((2,2);\mathbb{Z}[\mathrm{i}])$ (with coefficients in the Gaussian integers) by an involution as explained below (see eqn. (9)).

One of the new results in our paper is the study of the Néron-Severi lattice using the geometry of elliptic pencils. It allows us to determine the generic Néron-Severi lattice (in § 3.2) as well as the generic transcendental lattice. And indeed, we find $T(2)$ for the latter (here the brackets mean that we multiply the form by 2; this gives an even form as it should):

THEOREM (=Theorem 1). For generic X as above we have for the Néron-Severi lattice $\mathrm{NS}(X) = U \perp D_6^2 \perp \langle -2 \rangle^2$ and for the transcendental lattice $T(X) = T(2)$.[§]

Here D_k is the lattice for the corresponding Dynkin diagram. See the notation just after the introduction.

Of course, the divisors in X parametrizing special line configuration also have a moduli interpretation on the Abelian 4-fold side. This has been studied by Hermann in [5]. To compare the results from [10] and [5] turned out to be a non trivial exercise (at least for us). This explains why we needed several details from both papers. We

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[§]See also [10, Prop. 2.3.1]

collected them in § 1 and § 2.1. We use this comparison in particular to relate (in § 3.1) Hermann's divisors D_Δ for small Δ to some of the divisors described in [10]. We need this in order to describe (in § 3.6) the geometry of the corresponding K3 surfaces in more detail. The technique here is the study of the degeneration of a carefully chosen elliptic pencil (when the surface moves to the special divisor) which reflects the corresponding lattice enhancements explained in § 3.5. This technique enables us to calculate the Néron-Severi and transcendental lattice of the generic K3 on the divisors D_Δ for $\Delta = 1, 2, 4, 6$ respectively (we use Hermann's notation).

For the full statement we refer to Theorem 3; here we want to single out the result for $\Delta = 1$:

THEOREM. *The divisor $D_1 \subset \mathbf{M}$ corresponds to the divisor $XQ \subset \mathbf{X}$ and so (see (3)) is isomorphic to a Zariski-open subset in the moduli space of genus two curves. The generic point on D_1 corresponds to a K3 surface which is a double cover of the plane branched in 6 lines tangent to a common conic. Its Néron-Severi lattice is* $NS(X_1) = U \perp D_4 \perp D_8 \perp A_3$ *and its transcendental lattice is* $U(2)^2 \perp \langle -4 \rangle$.

This requires a little further explanation beyond what is stated in Theorem 3: we have seen in (3) that the image of XQ in $\mathbf{M} = \mathbf{X}/H$ we get a variety isomorphic to a Zariski-open subset of $\mathfrak{h}_2/\Gamma$, the moduli space of principally polarized Abelian varieties of dimension 2 (=the moduli space of genus 2 curves).

A further novelty of this paper is the role of the Kuga-Satake correspondence. For the general configuration of lines this has been done by one of us in [7], building on work of Paranjape [14]. One of these earlier results reviewed in § 4 (Theorem 9) states that the Kuga-Satake construction gives back the original Abelian 4–fold up to isogeny. In the present paper we explain what this construction specializes to for the K3 surface on the generic D_Δ, this time for *all* Δ. See Theorem 5.

Notation

The bilinear form on a lattice is usually denoted by $\langle -, - \rangle$. Several standard lattices as well as standard conventions are used:

- Orthogonal direct sums of lattices is denotes by $\perp$;

- For a lattice T the orthogonal group is denoted $O(T)$ its subgroup of commutators is a subgroup of the special orthogonal group $SO(T)$ and is denoted $SO^+(T)$;

- Let $(T, \langle -, - \rangle)$ be a lattice. The dual of T is defined by

$$T^* := \{x \in T \otimes \mathbb{Q} \mid \langle x, y \rangle \in \mathbb{Z}, \quad \text{for all } y \in T\}.$$

Note that $T \subset T^*$. The discriminant group $\delta(T)$ is the finite Abelian group T^*/T. If T is even, i.e. $\langle x, x \rangle \in 2\mathbb{Z}$, the form $\langle -, - \rangle$ induces a $\mathbb{Q}/\mathbb{Z}$-valued bilinear form b_T on $\delta(T)$ with associated $\mathbb{Q}/2\mathbb{Z}$-valued quadratic form q_T.

- The group of matrices with values in a subring $R \subset \mathbb{C}$ preserving the hermitian form with Gram matrix

$$(5) \qquad \mathbf{J} := \begin{pmatrix} \mathbf{0}_2 & \mathbf{1}_2 \\ -\mathbf{1}_2 & \mathbf{0}_2 \end{pmatrix}$$

will be denoted $U((2,2); R)$ because it has signature $(2,2)$;

- For $a \in \mathbb{Z}$ the lattice $\langle a \rangle$ is the 1-dimensional lattice with basis e and $\langle e, e \rangle = a$;

- If L is a lattice, the lattice $L(a)$ is the same $\mathbb{Z}$–module, but the form gets multiplied by a;

- U is the standard hyperbolic lattice with basis $\{e, f\}$ and $\langle e, e \rangle = \langle f, f \rangle = 0$, $\langle e, f \rangle = 1$;

- A_k, D_k, E_k: the standard *negative definite* lattices associated to the Dynkin diagrams: if the diagrams have vertices $v_1, \ldots, v_k$, we put -2 on the diagonal, and in the entries ij and ji we put 1 or 0 if v_i and v_j are connected or not connected respectively:
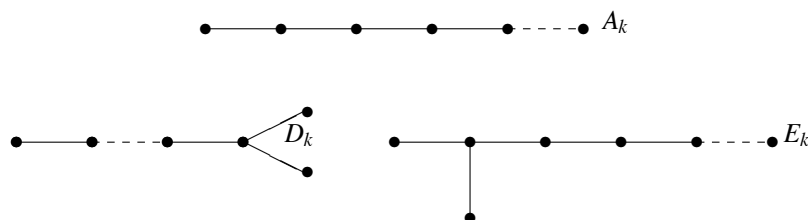
Furthermore, for a projective surface X we let $\mathrm{NS}(X) \subset H^2(X; \mathbb{Z})$ and $T(X) = \mathrm{NS}(X)^{\perp}$ be the Néron-Severi lattice, respectively the transcendental lattice equipped with the lattice structure from $H^2(X; \mathbb{Z})$, i.e. the intersection product.

Finally we recall the convention to denote congruence subgroups. Suppose R is a subring of $\mathbb{C}$, and V_R a free R-module of finite rank and G a subgroup of the group $\mathrm{Aut}\, V_R$. For any principal ideal $(\omega) \subset R$ we set

$$G(\omega) := \{g \in G \mid g \equiv \mathrm{id} \mod (\omega)\}.$$

In what follows we restrict ourselves to

$$R = \mathbb{Z}[\mathrm{i}] \subset \mathbb{C}, \quad \omega = 1 + \mathrm{i}, \quad G = \mathrm{SU}((2,2); \mathbb{Z}[\mathrm{i}]).$$

1. Two Classical Groups, Their Associated Domains and Lattices

1.1. The Groups

We summarize some classical results from [3, IV. § 8].

Let K be any field and V a 4-dimensional K–vector space. The decomposable elements in $W := \Lambda^2 V$ correspond to the 2-planes in V; the corresponding points in $\mathbb{P}(W)$ form the quadric G which is the image of the Grassmann variety of 2-planes in V under the Plücker-embedding. Concretely, choosing a basis $\{e_1, e_2, e_3, e_4\}$ and setting

$$(6) \qquad x \wedge y = q(x,y)\, e_1 \wedge e_2 \wedge e_3 \wedge e_4, \quad x, y \in W,$$

the quadric G has equation $q(x,x) = 0$.

On G we have two types of planes: the first corresponds to planes contained in a fixed hyperplane of V, the second type corresponds to planes passing through a fixed line. Both types of planes therefore correspond to 3-dimensional subspaces of W on which the bilinear form q is isotropic. Its index, being the maximal dimension of q–isotropic subspace equals 3.

Any $A \in \mathrm{GL}(V)$ induces a linear map $B = \Lambda^2 A$ of W preserving the quadric G. Conversely such a linear map B is of this form, provided it preserves the two types of planes on the quadric G. This is precisely the case if $\det(B) > 0$ and so one obtains the classical isomorphism between *simple* groups

$$\mathrm{SL}(4,K)/\mathrm{center} \quad \xrightarrow{\cong} \quad \mathrm{SO}^+(6,q,K)/\mathrm{center},$$
$$A \quad \mapsto \quad \Lambda^2 A$$

where we recall that the superscript $+$ stands for the commutator subgroup.[§]

From this isomorphism several others are deduced (loc. cit) through a process of field extensions. The idea is that if $K = k(\alpha)$, an imaginary quadratic extension of a real field k, the form q which over K has maximal index 3, over k can be made to have index 2. This is done as follows. One restricts to a subset of K-linear transformations of V which preserve a certain well-chosen anti-hermitian form f. The linear maps $\Lambda^2 A$ then preserve the *hermitian* form $g = \Lambda^2 f$ given by

$$(7) \qquad g(x \wedge y, z \wedge t) = \det \begin{pmatrix} f(x,z) & f(y,z) \\ f(x,t) & f(y,t) \end{pmatrix}$$

Suppose that in some K–basis for W the Gram matrices for g and q coincide and both have entries in k. Then the matrix of $B = \Lambda^2 A$ being at the same time q–orthogonal and g–hermitian must be real. So this yields an isomorphism

$$\mathrm{SU}(V, f, K)/\mathrm{center} \quad \xrightarrow{\cong} \quad \mathrm{SO}^+(6,q,k)/\mathrm{center}$$
$$A \quad \mapsto \quad \Lambda^2 A$$

In our situation $K = k(\mathrm{i})$ (with, as before, k a real field). We choose our basis $\{a_1, a_2, b_1, b_2\}$, for V in such a way that the anti-hermitian form f has Gram matrix $\mathrm{i}\mathbf{J}$ (see (5)).

The Gram matrix of $-q$ is the (integral) Gram matrix for $U \perp U \perp U$. The Gram matrix of the hermitian form $-g$ is found to be the Gram matrix for $U \perp U \perp \langle -1 \rangle \perp \langle -1 \rangle$ of signature $(2,4)$. In a different K-basis for W the Gram matrix of q is found to coincide with the Gram matrix for g:

[§]The last subgroup can also be identified with the subgroup of elements whose spinor norm is 1, but we won't use this characterization.

LEMMA 1. *Let* $\omega = \frac{1}{2}(-1+i)$ *and set*

$$g_1 = f_1, \quad g_2 = f_2, \quad g_3 = f_3, \quad g_4 = f_4, \quad g_5 = \omega f_5 - \bar{\omega} f_6, \quad g_6 = (-\bar{\omega} f_5 + \omega f_6)$$
$$T := \mathbb{Z}\text{--lattice spanned by} \quad \{g_1, \ldots, g_6\}.$$

Then the Gram matrices of $-q$ *(see (6)) and* $-g$ *(see (7)) on* T *are both equal to* $U \perp U \perp \langle -1 \rangle \perp \langle -1 \rangle$.

Indeed, in this basis we obtain the desired isomorphisms:

LEMMA 2 ([9, § 1.4]). *Set* $T_{\mathbb{Q}} := T \otimes \mathbb{Q}$. *We have an isomorphism of* $\mathbb{Q}$*--algebraic groups (see (5))*

$$\begin{array}{ccc} \mathrm{SU}((2,2);\mathbb{Q}(i))/\{\mathbf{1},-\mathbf{1}\} & \xrightarrow{\cong} & \mathrm{SO}^+(T_{\mathbb{Q}};\mathbb{Q}) \\ A & \mapsto & \Lambda^2 A|T_{\mathbb{Q}}. \end{array}$$

On the level of integral points we have

(8) $$\varphi : \mathrm{SU}((2,2);\mathbb{Z}[i])/\{\mathbf{1},-\mathbf{1}\} \xrightarrow{\sim} \mathrm{SO}^+(T).$$

This isomorphism induces an isomorphism of real Lie groups

$$\mathrm{SU}((2,2);\mathbb{C}/)\{\mathbf{1},-\mathbf{1}\} \xrightarrow{\cong} \mathrm{SO}^+((2,4);\mathbb{R}).$$

The target is the component of the identity of $\mathrm{SO}((2,4);\mathbb{R})$ *and is a simple group.*

Proof. As noted before, a matrix which is at the same time hermitian and orthogonal with respect to the same real matrix has to have real coefficients. So the map $A \mapsto \Lambda^2 A$ sends $\mathrm{SU}((2,2);\mathbb{C})$ injectively to a connected real subgroup of $\mathrm{SO}((6,q))$. A dimension count shows that we get the entire connected component of the latter group which is (isomorphic to) $\mathrm{SO}^+((2,4);\mathbb{R})$.

Assume now that A has coefficients in $\mathbb{Q}(i)$. It then also follows that $\Lambda^2 A|T_{\mathbb{Q}}$ must have rational coefficients, i.e. we have shown the first assertion of the lemma. The assertion about integral points follows since the change of basis matrix from the f-basis to the g-basis is unimodular and hence if A preserves a lattice, $\Lambda^2 A|T$ preserves the corresponding lattice. $\square$

REMARK 1. This isomorphism can be extended to $\mathrm{U}((2,2))$ modulo its center $\mathrm{U}(1)$ provided one takes the semi-direct product of the latter group with an involution τ which acts on matrices $A \in \mathrm{U}((2,2))$ by $\tau(A) = \tau A \tau = \bar{A}$. We set

(9) $$\mathrm{U}^*((2,2)) := \mathrm{U}((2,2)) \rtimes \langle \tau \rangle.$$

Now τ also induces complex conjugation on $\Lambda^2 V$ with respect to the real structure given by the real basis $\{f_1, \ldots, f_6\}$. This involution preserves $\{g_1, \ldots, g_4\}$ but interchanges g_5 and g_6. So on T the involution becomes identified with the involution

(10) $$\tilde{\tau} : T \to T, \quad \tilde{\tau}(g_k) = g_k, k = 1, \ldots, 4, \tilde{\tau}(g_5) = g_6$$

and one can extend the homomorphism φ from (8) by sending τ to $\tilde{\tau}$.

Accordingly, we define a two component subgroup of $O(2,4)$:

$$\mathrm{SO}^*(2,4) = \mathrm{SO}^+(2,4) \rtimes \langle \tilde{\tau} \rangle.$$

Note that $-\mathbf{1} \in \mathrm{SO}^*(2,4)$ so that

$$\big(\mathrm{U}((2,2))/\mathrm{U}(1)\big) \rtimes \langle \tau \rangle \simeq \mathrm{SU}((2,2))/\{\pm\mathbf{1}, \pm i\mathbf{1}\} \rtimes \langle \tau \rangle \xrightarrow{\simeq} \mathrm{SO}^*(2,4)/\{\pm\mathbf{1}\}$$
$$\mathrm{SU}((2,2))/\{\pm\mathbf{1}, \pm i\mathbf{1}\} \xrightarrow{\simeq} \mathrm{SO}^+(2,4)/\{\pm\mathbf{1}\}.$$

REMARK 2. The (symplectic) basis $\{a_1, a_2, b_1, b_2\}$ of V can be used to define a linear isomorphism

(11) $\det : \Lambda^4 V_{\mathbb{C}} \xrightarrow{\sim} \mathbb{C}, \qquad a_1 \wedge a_2 \wedge b_1 \wedge b_2 \mapsto 1.$

We use this to obtain a $\mathbb{C}$-antilinear involution t of $\Lambda^2_{\mathbb{C}} V$ as follows:

$$t : \Lambda^2 V_{\mathbb{C}} \xrightarrow{\sim} \Lambda^2 V_{\mathbb{C}} \quad \text{such that} \quad g(u,v) = -\det(t(v) \wedge u).$$

By definition of t one has $t(f_i) = f_i$, $i = 1, \ldots, 4$ and $t(f_5) = f_6$, $t(f_6) = f_5$ and since t is $\mathbb{C}$ anti-linear we see that t preserves not only the first 4 basis vectors g_i of T but also the last two g_5, g_6. So the 12-dimensional real vector space $\Lambda^2 V_{\mathbb{C}}$ splits into two real 6-dimensional t-eigenspaces, namely $T_{\mathbb{R}} = T \oplus \mathbb{R}$ for eigenvalue 1 and $iT_{\mathbb{R}}$ (for eigenvalue -1) respectively:

(12) $\Lambda^2 V_{\mathbb{C}} = T_{\mathbb{R}} \perp iT_{\mathbb{R}}.$

1.2. Congruence Subgroups

The quotient $\mathrm{SU}((2,2);R)/\mathrm{SU}((2,2);R)(\omega)$ acts naturally on $(R/(\omega R))^4 = \mathbb{F}_2^4$. Since the hermitian form if in the basis $\{a_1, a_2, b_1, b_2\}$ descends to this $\mathbb{F}_2$–vector space to give a symplectic form, we then get an isomorphism

$$\mathrm{SU}((2,2);R)/\mathrm{SU}((2,2);R)(\omega) \xrightarrow{\sim} \mathrm{Sp}(4;\mathbb{F}_2).$$

By (12) we have $T \oplus iT = \Lambda^2 V_R$. Lemma 2, states that the group $\mathrm{SU}((2,2);R)$ acts on T and so the subgroup $\mathrm{SU}((2,2);R)(\omega)$ acts on $(1+i)^2 \Lambda^2 V_R = 2\Lambda^2 V_R$. It preserves the sublattice $2T \subset T$. It follows that under the isomorphism of Lemma 2 one gets an identification

(13) $\mathrm{Sp}(4;\mathbb{F}_2) \xleftarrow{\sim} \mathrm{SU}((2,2);R)/\mathrm{SU}((2,2);R)(\omega) \xrightarrow{\sim} \mathrm{SO}^+(T)/\mathrm{SO}^+(T)(2).$

Since the involution $\tilde{\tau} \in O^+(2,4)$ (see (10)) obviously does not belong to the congruence 2 subgroup, we may define extensions as follows:

$$\mathrm{SO}^*(T)(2) := \mathrm{SO}^+(T)(2) \ltimes \langle \tilde{\tau} \rangle, \quad \mathrm{U}^*((2,2);R)(\omega) := \mathrm{U}((2,2);R)(\omega) \ltimes \langle \tau \rangle.$$

In particular we have

(14) $\mathrm{Sp}(4;\mathbb{F}_2) \xleftarrow{\sim} \mathrm{U}^*((2,2);R)/\mathrm{U}^*((2,2);R)(\omega) \xrightarrow{\sim} \mathrm{SO}^*(T)/\mathrm{SO}^*(T)(2).$

REMARK 1. 1. See also [9, § 1.5], where the result is shown by brute force. To compare, we need a dictionary. The group $\Gamma\mathcal{A}$ from loc. cit. is our $\mathrm{SO}^*(T)/\pm 1$. The group $\Gamma\mathcal{A}(2)$ is the congruence 2 subgroup which is equal to $\mathrm{O}^+(T)(2)/\pm 1$. It lacks the involution $\tilde{\tau}$ and hence has index 2 in the extended group $\mathrm{O}^+(T)(2) \ltimes \langle\tilde{\tau}\rangle$ modulo its center. This explains why $\Gamma\mathcal{A}/\Gamma\mathcal{A}(2) \simeq \mathrm{Sp}(4;\mathbb{F}_2) \times \mathbb{Z}/2\mathbb{Z}$.

2. As in [6, §3] it can be shown that the restriction homomorphism $\mathrm{O}(T(2)) \to \mathrm{O}(q_{T(2)})$ is surjective with kernel the congruence subgroup $\mathrm{O}(T(2))(2)$. The orbits of $\mathrm{O}(q_{T(2)})$ acting on $T(2)^*/T(2) \simeq \mathbb{F}_2^6$ have been described explicitly in loc. cit., using coordinates induced by the standard basis for $U(2) \perp U(2) \perp A_1 \perp A_1$. The form $q_{T(2)}$ is $\mathbb{Z}/2\mathbb{Z}$-valued on the sublattice $F = \{a = (a_1,\dots,a_6) \in \mathbb{F}_2^6 \mid a_5 + a_6 = 0\}$ and $b_{T(2)}$ restricts to zero on $F^0 = \{0, \kappa = (1,1,1,1,0,0)\}$. Hence $b_{T(2)}$ induces a symplectic form on $F/F^0 \simeq \mathbb{F}_2^4$ (this explains anew that $\Gamma\mathcal{A}/\Gamma\mathcal{A}(2) \simeq \mathrm{Sp}(4;\mathbb{F}_2) \times \mathbb{Z}/2\mathbb{Z}$). The orbits are now as follows:

1. two orbits of length 1: 0 and κ;

2. $\{a \neq 0, q_{T(2)} = 0\}$, the orbit (of length 15) of $(1,0,0,0,0,0)$;

3. $\{a \neq \kappa, q_{T(2)} = 1\}$, the orbit (of length 15) of $(1,1,0,0,0,0)$;

4. $\{a, q_{T(2)} = \frac{1}{2}\}$, the orbit (of length 12) of $(1,1,0,0,1,0)$; it splits into two equal orbits under $\mathrm{SO}^*(T(2))/\mathrm{SO}^*(T(2))(2)$ (the involution exchanging the last two coordinates act as the identity in this quotient);

5. $\{a, q_{T(2)} = -\frac{1}{2}\}$, the orbit (of length 20) of $(0,0,0,0,1,0)$; it splits also into two equal orbits under $\mathrm{SO}^*(T(2))/\mathrm{SO}^*(T(2))(2)$.

1.3. The Corresponding Symmetric Domains

Recall [9, § 1.1], that the symmetric domain associated to the group $\mathrm{U}(n,n)$, $n \geq 1$ is the n^2–dimensional domain[‖]

$$\mathbf{H}_n := \left\{W \in M_{n\times n}(\mathbb{C}) \mid \frac{1}{2\mathrm{i}}(W - W^*) > 0\right\} \simeq \mathrm{U}(n,n)/\mathrm{U}(n) \times \mathrm{U}(n).$$

Indeed, writing

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{U}(n,n), \quad A,B,C,D \in M_{n\times n}(\mathbb{C})$$

the action is given by $\gamma(W) = (AW + B)(CW + D)^{-1}$. The full automorphism group of $\mathbf{H}_n$ is the semi-direct product $[\mathrm{U}(n,n)/\mathrm{U}(1)] \rtimes \langle\tau\rangle$ where τ is the involution given by $\tau(W) = W^{-1}$. Since $\tau \circ \gamma \circ \tau = \bar{\gamma}$ this indeed corresponds to complex conjugation on $\mathrm{SU}(n,n)$ (see Remark 1).

[‖]Recall: for a matrix W, we abbreviate $W^* = {}^{\mathrm{T}}\overline{W}$.

The symmetric domain associated to a bilinear form b of signature $(2,n), n \geq 2$ is the n-dimensional connected **bounded domain of type IV**

$$\mathbf{D}_n := \{z = [(z_1 : \dots : z_{n+2})] \in \mathbb{P}^{n+1} \mid {}^{\mathsf{T}}zbz = 0, z^*bz > 0, \operatorname{im}(z_3/z_1) > 0\}.$$

Without the second defining inequality the resulting domain is no longer connected; the subgroup $\mathrm{O}^+(2,n)$ preserves each connected component.

This domain parametrizes polarized weight 2 Hodge structures (T,b) with Hodge numbers $(1,n,1)$. This can be seen as follows. The subspace $T^{2,0} \subset T \otimes \mathbb{C}$ is a line in $T \otimes \mathbb{C}$, i.e. a point $z \in \mathbb{P}(T)$. The polarizing form b is a form of signature $(2,n)$ and the two Riemann conditions translate into ${}^{\mathsf{T}}zbz = 0$ and $z^*bz > 0$. These two relations determine an open subset $D(T) \subset \mathbb{P}(T)$ and the moduli space of such Hodge structures is thus $D(T)/\mathrm{O}(T)$. Now $D(T)$ has two components, one of which is (isomorphic to) $\mathbf{D}_n$; both $\mathrm{SO}^+(T)$ and the involution $\tilde{\tau}$ preserve the components, $\mathrm{SO}^+(T)$ has index 4 in $\mathrm{O}(T)$ and hence

$$\mathrm{O}(T) = \underbrace{\mathrm{SO}^+(T) \cup \tilde{\tau}\,\mathrm{SO}^+(T)}_{\mathrm{SO}^*(T)} \cup \underbrace{\sigma\,\mathrm{SO}^+(T) \cup \sigma\tilde{\tau}\,\mathrm{SO}^+(T)}_{\sigma\,\mathrm{SO}^*(T)}$$

where σ permutes the two components of $D(T)$: our moduli space can be written as the orbit space

$$\mathbf{D}_n/\,\mathrm{SO}^*(T).$$

REMARK 3. Let us specify this to the case which interests us most, $n = 4$. Then, $\mathbf{D}_4 = \mathrm{SO}^+(2,4)/((\mathrm{O}(2) \times \mathrm{O}(4)) \cap \mathrm{SO}(6))$ and its automorphism group is $\mathrm{O}^+(2,4)/\langle -\mathbf{1} \rangle$ (acting as group of projectivities on the projective space $\mathbb{P}^5$ preserving the quadric in which $\mathbf{D}_4$ is naturally sitting).

The element $c := \operatorname{diag}(1,1,-1,-1,1,1) \in \mathrm{SO}(2,4)$ preserves the lattice T and exchanges the two components; every element of $\mathrm{SO}(2,4)$ can be written as $cg = g'c$ with $g, g' \in \mathrm{SO}^+(2,4)$. The element $a := \operatorname{diag}(1,1,1,1,U)$, $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has determinant -1 and every element in $\mathrm{O}(2,4)$ can be written as a product $acg = cag$ with $g \in \mathrm{SO}^+(2,4)$.

PROPOSITION 1 ([9, §1.1]). *There is a classical isomorphism between the two domains $\mathbf{H}_2 \xrightarrow{\sim} \mathbf{D}_4$ which is equivariant with respect to the isomorphism*

$$[\mathrm{U}((2,2))/\mathrm{U}(1)] \rtimes \langle \tau \rangle \xrightarrow{\sim} \mathrm{SO}^*(2,4)/\{\pm\mathbf{1}\}$$

of Remark 1.

Sketch of proof: We describe the isomorphism briefly as follows. To the matrix $W \in \mathbf{H}_2$ one associates the 2–plane in $\mathbb{C}^4$ spanned by the rows of the matrix $(W|\mathbf{1}_2)$. This sends $\mathbf{H}_2$ isomorphically to an open subset in the Grasmannian of 2–planes in $\mathbb{C}^4$ which by the Plücker embedding gets identified with an open subset of the Plücker quadric in $\mathbb{P}^5$. This open subset is the type IV domain $\mathbf{D}_4$. $\qquad\square$

REMARK 4. Recall we started § 1.1 with a pair (V, f), with $V = \mathbb{Q}(\mathrm{i})^4$ and f a non-degenerate skew-hermitian form on V. The above assignment $W \mapsto$ the plane $P = P_W \subset V$ spanned by the rows of the matrix $(W | \mathbf{1}_2)$ is such that the hermitian form $\mathrm{i}f|P$ is positive. In other words, the domain $\mathbf{H}_2$ parametrizes the complex 2–planes P in $V_{\mathbb{C}} = V \otimes \mathbb{C} = \mathbb{C}^4$ such $(\mathrm{i} \cdot f)|P > 0$.

1.4. Orbits in The Associated Lattice.

Recall that $T := U^2 \oplus \langle -1 \rangle^2$ is the lattice on which the group $\mathrm{O}^*(T)$ acts by isometries. To study the orbits of vectors in T we use the results from [20]. We summarize these for this example. Recall that a primitive vector x in a lattice is called **characteristic** if $\langle x, y \rangle \equiv \langle y, y \rangle$ mod 2 for all vectors y in the lattice. Other vectors are called **ordinary**. In an even lattice all primitive vectors are characteristic. In the standard basis $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ for T we let $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6)$ be the coordinates. Then $\mathbf{x}$ is characteristic if and only if $x_1, \ldots, x_4$ are even and x_5, x_6 are odd. The **type** of a primitive lattice vector is said to be 0 for ordinary vectors and 1 for characteristic vectors. Wall's result formulated for T states that two primitive vectors of the same norm squared and of the same type are in the same $\mathrm{O}(T)$–orbit. So we have:

PROPOSITION 2. *$T = U^2 \oplus \langle -1 \rangle^2$ is a unimodular odd indefinite lattice of signature $(2, 4)$, isometric to $\langle 1 \rangle^2 \perp \langle -1 \rangle^4$. Let $\mathbf{x} \in T$ be primitive with $\langle \mathbf{x}, \mathbf{x} \rangle = -(2k+1)$, respectively $-2k$, $k > 0$.*

- *In the first case $\mathbf{x}$ is always non-characteristic and $\mathrm{O}(T)$–equivalent to $(1, -k, 0, 0, 1, 0)$.*

- *In the second case, a vector $\mathbf{x}$ is $\mathrm{O}(T)$–equivalent to $(2, \frac{1}{2}(-k+1), 0, 0, 1, 1)$ if characteristic (and then $k \equiv 1$ mod 4) and to $(1, -k, 0, 0, 0, 0)$ if not.*

REMARK 5. 1) From the description of the subgroups $\mathrm{SO}(T)$ and $\mathrm{SO}^+(T)$ in Remark 3, we see that the "extra" isometries c and a do not change the two typical vectors $(1, -k, 0, 0, 0, 0)$, $(2, \frac{1}{2}(-k+1), 0, 0, 1, 1)$ while a replaces $(1, -k, 0, 0, 1, 0)$ by $(1, -k, 0, 0, 0, 1)$. This can be counteracted upon applying the map $\mathrm{diag}(-1, -1, -1, 1, U) \in \mathrm{SO}^+(T)$. In other words, the preceding Proposition remains true for orbits under the two subgroups $\mathrm{SO}(T)$ and $\mathrm{SO}^+(T)$.

2) Suppose that $-d = \langle \mathbf{x}, \mathbf{x} \rangle$ is a negative even number. It follows quite easily that in the non-characteristic case $\mathbf{x}^\perp$ is isometric to $\langle d \rangle \perp U \perp \langle -1 \rangle \perp \langle -1 \rangle$. In the characteristic case this is subtler. For instance, if $d = -2k$ and k is a sum of two squares, say $k = u^2 + v^2$, the vector $\mathbf{x}$ is in the orbit of $(0, 0, 0, 0, u + v, u - v)$ and so $\mathbf{x}^\perp \simeq U \perp U \perp \langle -d \rangle$. This is the case if $k = a^2 b$ with b square free and $b \equiv 1$ mod 4. However, in the general situation the answer is more complicated. The situation over the rational numbers is easier to explain. For later reference we introduce

$$\Delta(\mathbf{x}) := -\frac{1}{2} \langle \mathbf{x}, \mathbf{x} \rangle = \frac{1}{2}(x_5^2 + x_6^2 - 2(x_1 x_2 + x_3 x_4)) > 0.$$

Then, completing the square, one finds:

(15) $$\mathbf{x}^{\perp} \sim_{\mathbb{Q}} U \perp \langle -2 \rangle^2 \perp \langle 2\Delta(\mathbf{x}) \rangle.$$

COROLLARY 1. *Consider the set* $Y \subset T$ *of vectors of the form*

$$\mathbf{y} = (2y_1, 2y_2, 2y_3, 2y_4, y_5 + y_6, y_5 - y_6) \in T, y_i \in \mathbb{Z}, \gcd(y_1, \ldots, y_6) = 1.$$

We have

$$\Delta(\mathbf{y}) = y_5^2 + y_6^2 - 4(y_1 y_2 + y_3 y_4).$$

If $y_5 \not\equiv y_6$ mod 2, *the vector is a characteristic primitive vector,* $\Delta \equiv 1$ mod 4 *and* $\mathbf{y}$ *is in the orbit of* $(2, \frac{1}{2}(1-\Delta), 0, 0, 1, 1)$.

If $y_5 \equiv y_6$ mod 2 *the vector* $\frac{1}{2}\mathbf{y} \in T$ *is primitive and non-characteristic and either* $\Delta \equiv 0$ mod 4 *and* $\frac{1}{2}\mathbf{y}$ *is in the orbit of* $(1, -\frac{1}{4}\Delta, 0, 0, 0, 0)$, *or* $\Delta \equiv 2$ mod 4 *and* $\frac{1}{2}\mathbf{y}$ *is in the orbit of* $(1, \frac{1}{4}(2-\Delta), 0, 0, 1, 0)$.

Hence two vectors in Y *with the same* Δ–*invariant are in the same* $\mathrm{O}(T)$–*orbit. Conversely, if the* Δ–*invariants are different the vectors are in different orbits.*

To be able to make a comparison between [5] and [9] we have to realize that $T(2)$ is the transcendental lattice of the K3 surface in [9] while the lattice which plays a role in [5] is the lattice $T(-1)$. The special vectors $\mathbf{y} \in Y$ related to Hermann's paper are not necessarily primitive. A primitive vector $\mathbf{y}^*$ in the line $\mathbb{Z} \cdot \mathbf{y}$ is called a *primitive representative* for $\mathbf{y}$. This vector will be considered as a vector of $T(2)$. This makes the transition between the two papers possible.

EXAMPLE 1. As a **warning**, we should point out that it might happen that primitive vectors $\mathbf{y}^*$ with the same norm squared in $T(2)$ correspond to *different* $\Delta(\mathbf{y})$. For example $\mathbf{y} = (2, -2, 0, 0, 0, 0)$ and $(0, 0, 0, 0, 1, 1)$ correspond to $(1, -1, 0, 0, 0, 0)$, respectively $(0, 0, 0, 0, 1, 1)$. Both vectors in $T(2)$ have norm squared -4 while the first (with $y_1 = 1, y_2 = -1$) has $\Delta = 4$ and the second has $\Delta = 1$, since $y_5 = 1, y_6 = 0$. From the above it follows that the two are *not* in the same orbit under the orthogonal group of $T(2)$.

Observe now that divisors in our moduli space are cut out by hyperplanes in $\mathbb{P}(T \otimes \mathbb{C})$ orthogonal to elements $t \in T$ and any multiple of t determines the same divisor. We get therefore all possible divisors by restricting ourselves to the set $Y \subset T$. Accordingly we use the subgroup of $\mathrm{SO}^*(T)$ preserving the set Y of vectors of this form. Then it is natural to consider the basis $\{2g_1, 2g_2, 2g_3, 2g_4, g_5 + g_6, g_5 - g_6\}$ so that the new coordinates of $\mathbf{y}$ become $(y_1, y_2, y_3, y_4, y_5, y_6)$. We may identify this vector with $\mathbf{y}^*$.

REMARK 6. Suppose $\mathbf{y} \in Y$ as in Corr. 1. Set $\Delta(\mathbf{y}) = \Delta$. Let $n(\Delta)$ the number of different $\mathrm{SO}^*(T)(2)$–orbits in a given $\mathrm{SO}^*(T)$-orbit for $\mathbf{y} \in T$ when $\Delta \equiv 0$ mod 4, respectively $\frac{1}{2}\mathbf{y}$ else. Using Corr. 1 and Rem. 1.2 one sees the following:

- If $\Delta \equiv 0$ mod 4 then $n(\Delta) = 15$;

- If $\Delta \equiv 2 \bmod 8$ then $n(\Delta) = 10$;

- If $\Delta \equiv 6 \bmod 8$ then $n(\Delta) = 6$;

- If $\Delta \equiv 1 \bmod 4$ then $n(\Delta) = 1$.

Indeed, for instance a vector for which $\Delta \equiv 0 \bmod 4$ is in the $\mathrm{SO}^*(T)$-orbit of $(1, -\frac{\Delta}{4}, 0, 0, 0, 0)$ which corresponds to $(1, 0, 0, 0, 0, 0)$ or $(1, 1, 0, 0, 0, 0)$ in $\mathbb{F}_2^6$ according to whether $\Delta \equiv 0 \bmod 8$ or $\Delta \equiv 4 \bmod 8$. Both of these have orbitsize 15 under $\mathrm{O}(q_{T(2)})$. If $\Delta \equiv 2, 6 \bmod 8$, in applying Rem. 1 2., one has to take care of the extra involution explaining why the number of orbits $n(\Delta)$ is half the orbitsize under $\mathrm{O}(q_{T(2)})$.

In fact, this result is completely equivalent to [5, Prop. 2] which is stated below (Prop. 4).

2. A Moduli Interpretation: the Abelian Varieties Side

2.1. Special Abelian Varieties

We say that an even dimensional polarized Abelian variety (A, E) is of *K*-**Weil type** if $\mathrm{End}_{\mathbb{Q}}(A)$ contains an imaginary quadratic field $K = \mathbb{Q}(\alpha) \subset \mathbb{C}$ such that the action of α on the tangent space of A at 0 has half of its eigenvalues equal to α and half of its eigenvalues equal to $\overline{\alpha}$ (note that this does not depend on the embedding $K \hookrightarrow \mathbb{C}$). We say that α **has type** (k, k), where $2k = \dim A$. Moreover, we want that $E(\alpha x, \alpha y) = |\alpha|^2 E(x, y)$. If $\alpha = \mathrm{i}$ this means that A admits an automorphism M with $M^2 = -1$ which preserves the polarization. Equivalently, $R \subset \mathrm{End}(A, E)$.

As is well known (cf. [5, § 1.2], [19, §10]), the symmetric domain $\mathbf{H}_2$ parametrizes such Abelian 4–folds of $\mathbb{Q}(\mathrm{i})$–Weil type. We recall briefly how this can be seen. Consider the lattice $V_R = R^4$ equipped with the skew form $\mathbf{J}$ (see (5)). The complex vector space $V = \mathbb{C}^4$, considered as a real vector space, contains V_R as a lattice and $\mathbf{J}$ is a unimodular integral form on it. Weight -1 Hodge structures on V polarized by this form are given by complex structures J that preserve the form. They correspond to principally polarized Abelian 4-folds A, and if J commutes with multiplication by i the Abelian variety A admits an order 4 automorphism M of type $(2, 2)$ which preserves the polarization. The converse is also true.

To get the link with the domain $\mathbf{H}_2$, recall from Remark 4 that points in $\mathbf{H}_2$ correspond precisely to complex 2-planes P in V for which $(\mathrm{i} \cdot f)|P > 0$. The direct sum splitting of $V = P \perp P^{\perp}$ can be used to define a complex structure J on the 8-dimensional real vector space V as desired by imposing $J|P = \mathrm{i}\mathbf{1}$, $J|P^{\perp} = -\mathrm{i}\mathbf{1}$. This complex structure commutes with multiplication by i on V and preserves $\mathrm{i}f$ (since this is a hermitian form). The embedding (4) identifies the image with those $\tau \in \mathfrak{h}_4$ that form the fixed locus of the order 4 automorphism formed from $\mathbf{J}$ (see (5))

$$\begin{pmatrix} \mathbf{J} & \mathbf{0}_4 \\ \mathbf{0}_4 & \mathbf{J} \end{pmatrix} \in \mathrm{Sp}(4; \mathbb{Z}).$$

This automorphism corresponds to multiplication with i on the Abelian 4-fold.

The discrete group $U^*((2,2);R)$ acts naturally on $\mathbf{H}_2$. It sends an Abelian variety of the given type to an isomorphic one. This is clear for $U((2,2);R)$. Regarding τ, by [5, § 1.2] the embedding (4) is equivariant with respect to it; indeed, it acts as an integral symplectic matrix on $\mathfrak{h}_4$ and hence also τ permutes isomorphic Abelian varieties. Conversely, since $U^*((2,2))$ modulo its center is the full group of isomorphisms of $\mathbf{H}_2$ it follows that two isomorphic Abelian varieties with multiplication by R are in the same $U^*((2,2);R)$–orbit.[**] So the quotient

$$\mathbf{M} := \mathbf{H}_2 / U^*((2,2);R)$$

is the moduli space of principally polarized Abelian fourfolds with multiplication by R.

2.2. Relation With Special Weight 2 Hodge Structures

Consider the Hodge structures parametrized by the domain $\mathbf{D}_4$ introduced in § 1.3. The construction of § 2.1 relates such Hodge structures to polarized Abelian 4-folds A with multiplication by R. Indeed, $V = H_1(A;\mathbb{R})$ underlies an integral polarized Hodge structure of weight -1 and rank 8 admitting an extra automorphism M of order 4 induced by $i \in \mathrm{End}(A)$. Giving a polarized integral Hodge structure on V of weight 1 is the same as giving a complex structure J preserving the polarization; moreover, M and J commute. According to [7, § 3] this can now be rephrased as follows. Since $V = H_1(A;\mathbb{R})$ underlies a rational polarized Hodge structure of weight -1 and rank 8, the second cohomology $H^2(A) = \Lambda^2 H_1(A)^*$, inherits a polarized Hodge structure of weight 2 and rank 28. We view (V,J) as a 4-dimensional complex vector space and hence we get a complex subspace $\Lambda^2_{\mathbb{C}} V^* \subset H^2(A;\mathbb{R})$ of complex dimension 6 and hence a real Hodge structure of dimension 12. In fact it can be seen to be rational. Recall from Remark 2 that there is a further $\mathbb{C}$–anti linear involution t on $\Lambda^2_{\mathbb{C}} V^*$. Its $(+1)$–eigenspace $T(A)$ has dimension 6 and gives a polarized Hodge substructure of $H^2(A)$ of weight 2 and Hodge numbers $(1,4,1)$ as desired. So, this construction explains the isomorphism

$$\mathbf{M} := \mathbf{H}_2 / U^*((2,2);R) \xrightarrow{\cong} \mathbf{D}_4 / \mathrm{SO}^*(T)$$

Hodge theoretically as the the one induced by $A \mapsto T(A)$.

If instead we divide out by the congruence subgroup $U^*((2,2);R)(\omega)$ the quotient $\mathbf{M}^* := \mathbf{H}_2 / U^*((2,2);R)(\omega)$ under the natural morphism $\mathbf{M}^* \to \mathbf{M}$ is Galois over of $\mathbf{M}$ with group $\mathrm{Sp}(4;\mathbb{F}_2)$:

$$
\begin{array}{ccc}
\mathbf{M}^* & \xrightarrow{\cong} & \mathbf{D}_4 / \mathrm{SO}^*(T)(2) \\
\downarrow{\scriptstyle p} & & \downarrow \\
\mathbf{M} & \xrightarrow{\cong} & \mathbf{D}_4 / \mathrm{SO}^*(T).
\end{array}
$$

[**]If we would consider such Abelian varieties up to *isogeny* we would classify the isomorphism classes of Abelian 4-folds of $\mathbb{Q}(i)$–Weil type.

2.3. Hypersurfaces in the Moduli Spaces

Any line $[a] \in \mathbb{P}(T \otimes \mathbb{Q})$ defines the divisor $D_{[a]} = \{x \in \mathbf{D}_4 \mid \langle a,x \rangle = 0\}$ inside the domain $\mathbf{D}_4$. As explained in § 1.4, we only consider representatives $a \in T$ which belong to the set $Y \subset T$ whose coordinates with respect to the basis $\{g_1, \ldots, g_6\}$ are of the form:

$$(16) \qquad \mathbf{y} = (2y_1, 2y_2, 2y_3, 2y_4, y_5 + y_6, y_5 - y_6) \in T.$$

The corresponding divisor $D_{[\mathbf{y}]}$ inside $\mathbf{H}_2$ can be described by means of the skew symmetric matrix

$$M(\mathbf{y}) := \begin{pmatrix} 0 & -y_2 & \frac{1}{2}(y_5 - iy_6) & -y_4 \\ y_2 & 0 & -y_3 & -\frac{1}{2}(y_5 + iy_6) \\ -\frac{1}{2}(y_5 - iy_6) & y_3 & 0 & -y_1 \\ y_4 & \frac{1}{2}(y_5 + iy_6) & y_1 & 0 \end{pmatrix}.$$

Indeed, we have [5, p. 119]

$$D_{[\mathbf{y}]} := \left\{ W \in \mathbf{H}_2 \mid \begin{pmatrix} {}^{\mathsf{T}}W & \mathbf{1}_2 \end{pmatrix} M(\mathbf{y}) \begin{pmatrix} W \\ \mathbf{1}_2 \end{pmatrix} = (\mathbf{0}_2) \right\}.$$

Then $A \in \mathrm{SU}((2,2))$ acts by sending $M = M(\mathbf{y})$ to

$$A[M] := {}^{\mathsf{T}}AMA = M(\mathbf{z}), \qquad \mathbf{z} = \Lambda^2 A(\mathbf{y}).$$

For a skew symmetric matrix M with coefficients in any field K the determinant is always a square in the field and any root is called a pfaffian of M and denoted by $\mathrm{Pf}(M)$. If $K \subset \mathbb{R}$ we take the *positive root* and call it **the pfaffian**. By Corol. 1 and Remark 5, 1) we have

PROPOSITION 3. *Given a positive integer Δ, there is precisely one $\mathrm{SO}^+(T)$ orbit of primitive vectors $\mathbf{y} \in Y$ for which $\Delta = -\frac{1}{2}\langle \mathbf{y}, \mathbf{y} \rangle$. All such vectors $\mathbf{y}$ are $\mathrm{SU}((2,2))$-equivalent and the corresponding pfaffians $\mathrm{Pf}(M_{\mathbf{y}})$ are all equal.*

Moreover, such divisors $D_{[\mathbf{y}]}$ define the same irreducible divisor D_Δ in the moduli space $\mathbf{M}$.

REMARK 2. The image of such a divisor D_Δ in the moduli space $\mathbf{M}$ can be considered as a three dimensional modular variety $\mathfrak{h}_2/\Gamma$, where Γ is a discrete subgroup of a certain modular group $\mathrm{Sp}(2;\mathbb{R})$ (depending only on Δ). For this point of view see [5]. For the special case $\Delta = 1$ see also (3) in the Introduction.

Under the congruence subgroup $\mathrm{SU}((2,2);R)(\omega)$ there are more orbits corresponding to the fact that D_Δ may split under the cover $\mathbf{M}^* \to \mathbf{M}$. By Remark 6 we have:

PROPOSITION 4 ([5, Prop. 2]). *Under the $\mathrm{Sp}(4,\mathbb{F}_2)$-cover $\pi : \mathbf{M}^* \to \mathbf{M}$ the divisor $\pi^{-1}D_\Delta$ associated to a primitive class $a \in T_{\mathbb{Z}}^*$ with $q(a) = \Delta$ splits in 15, 10, 6 or 1 components if $\Delta \equiv 0 \bmod 4, \equiv 2 \bmod 8, \equiv 6 \bmod 8$, respectively $\equiv 1 \bmod 4$.*

3. Moduli Interpretation: Special K3 surfaces

3.1. Configurations of 6 Lines in the Plane

For this section see [10] and in particular Appendix A in it.

Let $\mathbf{P} = \mathbb{P}^2$ and $\mathbf{P}^*$ the dual projective space. For any integer $n \geq 4$ a configuration of n-tuples of points in $\mathbf{P}$ corresponds to a configuration for n-tuples of lines in $\mathbf{P}^*$. The notion of *good position* is easy to describe on the dual space as follows. An n-tuple of lines $(\ell_1, \cdots, \ell_n) \in (\mathbf{P}^*)^n$ is called *in good position* if the corresponding curve $\ell_1 \cup \cdots \cup \ell_n \subset \mathbb{P}^2$ has only ordinary double points. They form a Zariski open subset

$$U_n = \{(\ell_1, \cdots, \ell_n) \text{ in good position }\} \subset (\mathbf{P}^*)^n.$$

The linear group $\mathrm{GL}(3; \mathbb{C})$ acts on this space and we form the quotient

$$X_n : \quad = \quad U_n/\mathrm{GL}(3; \mathbb{C}) : \textbf{configuration space of } n \textbf{ ordered lines}$$
$$\textbf{in good position in } \mathbb{P}^2\textbf{.}$$

This is a Zariski open subset of $\mathbb{C}^{2(n-4)}$. The symmetric group $\mathfrak{S}_n$ acts on X_n and the quotient is the configuration space of n unordered lines in general position.

If $n = 6$ there is an extra involution on X_6 induced by the correlation

$$\delta : \mathbf{P} \to \mathbf{P}^*, \quad x \mapsto \text{polar of } x \text{ with respect to a nonsingular conic } C$$

as follows: the 6 lines $\{\ell_1, \ldots, \ell_6\}$ form 2 triplets, say $\{\ell_1, \ell_2, \ell_3\}$ and $\{\ell_4, \ell_5, \ell_6\}$ each having precisely 3 intersection points. If we set $P_{ij} = \ell_i \cap \ell_j$ we thus get the triplets $\{P_{12}, P_{13}, P_{23}\}$ and $\{P_{45}, P_{46}, P_{56}\}$. A correlation δ is an involutive projective transformation: it sends the line through P and Q to the line through $\delta(P)$ and $\delta(Q)$. In particular, three distinct points which are the vertices of a triangle are sent to the three sides of some (in general different) triangle. This gives already a involution on the variety of three ordered non-aligned points which is easily seen to be holomorphic. However, it descends as the trivial involution on the space of unordered non-aligned triples since a projectivity maps any such triple to a given one.

The above procedure for six points gives a holomorphic involution on U_6

$$*_C(\ell_1, \ldots, \ell_6) = (\delta P_{12}, \delta P_{13}, \delta P_{23}, \delta P_{45}, \delta P_{46}, \delta P_{56}).$$

It descends to an involution on $\mathbf{X} = X_6$ which does not depend on the choice of C and commutes with the action of the symmetric group $\mathfrak{S}_6$. We set

$$\mathbf{Y} = \mathbf{X}/\{*\}.$$

The involution $*_C$ has as fixed point set on U_6 the 6-tuples of lines all tangent to the conic C. On X this gives a non-singular divisor $XQ \subset \mathbf{X}$, the configuration of 6-uples of lines tangent to some fixed conic. This shows in particular that $*_C$ is a non-trivial involution, in contrast to what happens for triplets of points.

We need an alternative description of this involution:

PROPOSITION 5. *The involution $*$ is, up to a projective transformation, induced by a standard Cremona transformation with fundamental points P_{14}, P_{25}, P_{36}.*

Proof. For the proof consult also Fig. 1. The points P_{14}, P_{25}, P_{36} form a triangle Δ_0. The lines ℓ_1, ℓ_2, ℓ_3 form a triangle Δ_1 and ℓ_4, ℓ_5, ℓ_6 another triangle Δ_2. In the dual plane the sides of the triangle Δ_0 correspond to three non-collinear points, say $p, q, r \in \mathbf{P}^*$. We denote the points in $\mathbf{P}^*$ corresponding to the lines ℓ_j by the same letter. The configuration of the three triangles $\Delta_0, \Delta_1, \Delta_2$ is self-dual in the obvious sense. Note that the cubic curves $\ell_1\ell_2\ell_3 = 0$ and $\ell_4\ell_5\ell_6 = 0$ span a pencil of cubics passing through the vertices of the union of the triangles $\Delta_1 \cup \Delta_2$. The same holds for the dual configuration in $\mathbf{P}^*$.

By [4, p. 118–119] this implies that the standard Cremona transformation T^* with fundamental points p, q, r transforms these "dual" 6 vertices into a so-called associated 6-uple. To see this, let Y be the 3×6 matrix whose columns are the vectors of the six points $\{\ell_1, \ldots, \ell_6\}$ (in some homogeneous coordinate system). The corresponding matrix Y^* for the associated point set $\{T^*\ell_1, \ldots, T^*\ell_6\}$ by definition satisfies $Y\Lambda^\mathsf{T}Y^* = 0$ for some diagonal matrix Λ. In our case we can take coordinates in such a way that $Y = (I_3, A)$ with A invertible and after a projective transformation we may assume that $Y^* = (I_3, -\frac{1}{\det A}A^*)$, where A^* is the matrix of cofactors of A so that $A^\mathsf{T}A^* = \det(A)I_3$. This exactly means that the point set which gives Y is related to the point set given by Y^* by the involution $*_C$ where C is the conic $x^2 + y^2 + z^2 = 0$. See the calculations in [10, Appendix A2]. $\qquad\square$

The space $\mathbf{X}$ can be compactified to $\overline{\mathbf{X}}$ by adding certain degenerate configurations to which the involution $*$ extends and the resulting compactification $\mathbf{X} = \overline{\mathbf{X}}/\{*\}$ is naturally isomorphic to $\mathbb{P}^4$. The group $\mathfrak{S}_6$ acts on both sides giving a commutative diagram

$$
\begin{array}{ccc}
\mathbf{X} & \overset{\sigma}{\to} & \mathbf{Y} \simeq \mathbb{P}^4 \\
\downarrow & & \downarrow \pi \\
\mathbf{X}/\mathfrak{S}_6 & \overset{\sigma}{\to} & \mathbf{Y}/\mathfrak{S}_6.
\end{array}
$$

Comparison with Hermann's work

We now compare the result with [5]. We need some more details of the above construction. To start with we choose coordinates in $\mathbf{Y}$ by representing first a point in $\mathbf{Y}$ by a 3×6 matrix (x_{ij}) (the 6 rows give the six lines) and let $d_{ijk}(x)$ be the minor obtained by taking columns i, j, k. Then for every permutation $\{ijk\ell mn\}$ of $\{1, \ldots, 6\}$, consider the 10 Plücker coordinates $Z_{ijk} := d_{ijk}d_{\ell mn}$ which one uses to embed $\mathbf{Y}$ in $\mathbb{P}^9$. The Plücker relations $Z_{ijk} - Z_{ij\ell} + Z_{ijm} - Z_{ijn} = 0$ then show that this embedding is a linear embedding into $\mathbb{P}^4 \subset \mathbb{P}^9$. Note that the permutation group $\mathfrak{S}_6$ interchanges the Plücker coordinates and the image 4-space is invariant under this action.

In [5, § 4] an embedding[††] of $\mathbf{Y}$ into $\mathbb{P}^5$ with homogeneous coordinates

---

[††]What Hermann calls $X(1+i)$ is in our notation Y and in Matsumoto's notation Y^*.

$(Y_0, \ldots, Y_5)$ is constructed with image the hyperplane $(Y_0 + \cdots + Y_5) = 0$. We may assume that the $Y_i, 0 \le i \le 4$ coincide with some of our Plücker coordinates which we write accordingly as $\{Y_0, \ldots, Y_9\}$. Indeed, Hermann's six coordinates are permuted as the standard permutation of $\mathfrak{S}_6$ on 6 letters and we can scale them in order that the hyperplane in which the image lies is given by the equation $(Y_0 + \cdots + Y_5) = 0$.

One of the divisors in $\overline{\mathbf{X}}$ added to $\mathbf{X}$ is needed below. It is called X_3 in [10] and has the 20 components X_3^{ijk} of configurations of 6 lines where precisely the three lines ℓ_i, ℓ_j, ℓ_k meet at one point which create one triple point. There are 12 further points where only 2 lines meet. We now show how to identify $\sigma(X_3)$ with the divisor $D_2' = \pi^{-1} D_2$ from Prop. 4. First invoke [10, Prop. 2.10.1] (see also Theorem 2 below) which makes the transition from $\mathbf{X}$ to $\mathbf{M}$ possible. Next, from [5, p. 122-123], we infer that the equation of D_2' reads

$$\prod_{abc}(Y_a + Y_b + Y_c) = 0.$$

By the Plücker relations $Y_a + Y_b + Y_c = \pm Y_d$ for some $d \in \{0, \ldots, 9\}$. Hence $Y_a + Y_b + Y_c = \pm Y_d$, say $\pm Y_d = Z_{ijk}$ and the zero locus of that factor corresponds to $D_{ijk} \cdot D_{lmn} = 0$. It follows that indeed $D_2' = X_3$. We observe that there are 20 irreducible components X_3^{ijk} but each Y_d gives two of them via the double cover σ, so we get indeed the 10 divisors of Hermann.

As a side remark, $\overline{\mathbf{X}} - \mathbf{X}$ contains further divisors, several of which parametrize K3 surfaces, namely whenever the double points in the configuration coalesce to triple points at worst.

We want to stress that the definition of *good* includes rather special configurations, one of which is needed below, namely the ones forming a divisor $\mathbf{X}_{\mathrm{coll}} \subset \mathbf{X}$ corresponding to 6 lines $\{\ell_1, \ldots, \ell_6\}$ where the intersection points $P_{12} = \ell_1 \cap \ell_2, P_{34} = \ell_3 \cap \ell_4, P_{56} = \ell_5 \cap \ell_6$ of three pairs of lines are collinear. We can identify $\sigma\mathbf{X}_{\mathrm{coll}}$ with $\pi^{-1} D_4$ as follows. From [5, p. 123], we find that the equation of D_4 is

$$\prod_{ij} Y_i - Y_j = 0.$$

The Plücker relations yield the equations

$$D_{abc} D_{efg} = D_{a'b'c'} D_{e'f'g'}$$

where in addition to $\{a, b, c, e, f, g\} = \{a', b', c', e', f', g'\} = \{1, \ldots, 6\}$ necessarily (up to commuting the factors of the products)

$$\#\{a, b, c\} \cap \{a', b', c'\} = \#\{e, f, g\} \cap \{e', f', g'\} = 2.$$

Dually looking at 6 points in $\mathbb{P}^2$, we obtain the same result if three lines (spanned by different pairs of such points) meet in one and the same point. So indeed, we get $\sigma\mathbf{X}_{\mathrm{coll}}$. Observe that there are $\frac{1}{6} 15 \times 6 = 15$ ways to make the intersection points collinear giving 15 components as it should.

Relation to recent work of Kondo

In a recent preprint [6], Kondo also studies Heegner divisors on the moduli space of 6 lines in $\mathbb{P}^2$ using Borcherd's theory of automorphic forms on bounded symmetric domains of type IV. Specifically he singles out four divisors in [6, §3] which correspond to the cases $\Delta = 1, 2, 4, 6$ studied extensively in this paper.

3.2. Double Cover Branched in 6 Lines in Good Position

We refer to [1, Ch VIII] for details of the following discussion on moduli of K3-surfaces. The second cohomology group of K3 surface X equipped with the cup product pairing is known to be isomorphic to the unimodular even lattice

$$\Lambda := U \perp U \perp U \perp E_8 \perp E_8.$$

The lattice underlies a weight 2 polarized Hodge structure with Hodge numbers $h^{2,0} = 1, h^{1,1} = 20$. The Néron-Severi lattice $\mathrm{NS}(X)$ gives a sub Hodge structure with Hodge numbers $h^{2,0} = 0$, $h^{1,1} = \rho$, the Picard number. Its orthogonal complement $T(X)$, the transcendental lattice, thus also is a polarized Hodge structure with Hodge numbers $h^{2,0} = 1, h^{1,1} = 20 - \rho$. The Néron-Severi lattice is a Tate Hodge structure, but $T(X)$ has moduli. The two Riemann bilinear relations show that these kind of Hodge structures are parametrized by a type IV domain $\mathbf{D}_{20-\rho}$ (see § 1.3). Conversely, given a sublattice $T \subset \Lambda$ of signature $(2, n), n < 20$, the polarized Hodge structures on T with Hodge numbers $h^{2,0} = 1$, $h^{1,1} = n$ are parametrized by a domain $D(T)$ of type IV whose points correspond to K3 surfaces with the property that the transcendental lattice is contained in T. For generic such points the transcendental lattice will be exactly T, but upon specialization the surfaces may acquire extra algebraic cycles which show up in T. In other words, the transcendental lattice of the specialization becomes strictly smaller than T. Note also that $\dim D(T) = n$. These surfaces, commonly called T-**lattice polarized K3 surfaces**, thus have n moduli. It is not true that all points in $D(T)$ correspond to such K3 surfaces: one has to leave out the hyperplanes H_α that are perpendicular to the roots α in T; since T is an even lattice, these are elements $\alpha \in T$ with $\langle \alpha, \alpha \rangle = -2$. We quote the following result [1, VIII, §22] which makes this precise:

THEOREM. *Let*

$$D^0(T) := D(T) - \bigcup H_\alpha, \quad \alpha \text{ a root in } T.$$

The moduli space of T-lattice polarized K3 surfaces is the quotient of $D^0(T)$ by the group $\mathrm{SO}^*(T)$.

Examples of K3 surfaces with $\rho = 1$ (having 19 moduli) are the double covers of the plane branched in a generic smooth curve of degree 6 parametrized by a 19-dimensional type IV domain. If we let this curve acquire more and more singularities we get deeper and deeper into this domain. We are especially interested in those

sextics that are the union of six lines in good position (see Fig. 1) and certain of their degenerations which were treated in § 3.1.
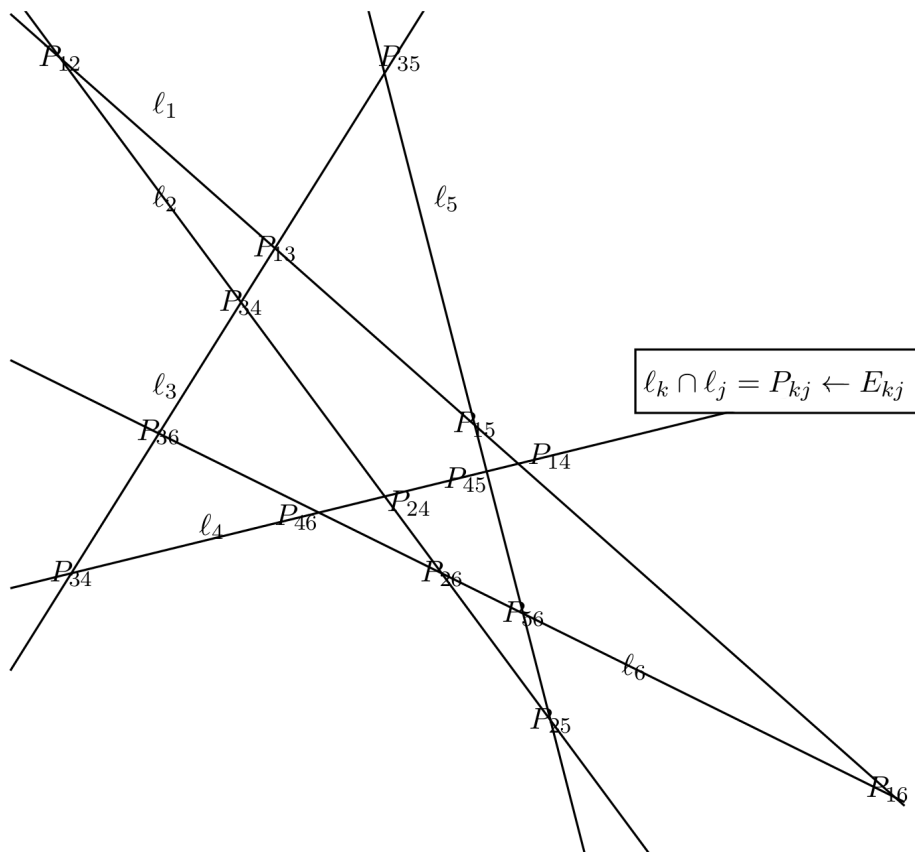

Figure 1: The 6 branch lines and the 15 exceptional curves E_{kj}

PROPOSITION 6. *Let X be the minimal resolution of the double of cover of the plane given by an equation:*

$$(17) \qquad\qquad w^2 = \ell_1(x,y,z)\cdots\ell_6(x,y,z).$$

Assume that the 6 lines ℓ_i, $i = 1,\ldots,6$ are in general (and, in particular, in good) position. Then the Picard number $\rho(X)$ equals $\rho(X) = 16$.

Proof. The 15 ordinary double points P_{ij} ($1 \le i < j \le 6$) in the configuration give 15 E_{ij} disjoint exceptional divisors on X; these are (-2)-curves, i.e. rational curves with self intersection (-2). The generic line gives one further divisor ℓ with $\ell^2 = 2$ and

which is orthogonal to the E_{ij}. The divisors $\{\ell, E_{12}, \ldots, E_{56}\}$ thus form a sublattice N of rank 16 within the Néron-Severi lattice $NS(X)$. So for the Picard number we have $\rho \geq 16$. As explained above, we have $20 - \rho$ moduli where ρ is the Picard number of a generic member of the family. So $\rho = 16$ and hence for generic choices of lines ℓ_i the lattice $NS(X)$ contains N as a sublattice of finite index. $\qquad \square$

We need a simple consequence of the proof. To explain it we need a few notions from lattice theory. Recall that the dual of a lattice L is defined by

$$L^* := \{x \in L \otimes \mathbb{Q} \mid \langle x, y \rangle \in \mathbb{Z}, \quad \text{for all } y \in L\}$$

and that the discriminant group $\delta(L)$ is the finite Abelian group L^*/L. We say that L is *p*–**elementary**, if this is so for $\delta(L)$, i.e.

$$L^*/L \cong (\mathbb{Z}/p\mathbb{Z})^\ell, \quad \ell := \text{\textbf{length of} } L \leq \text{rank}(L).$$

From the above proof we see that

$$\langle 2 \rangle \perp \langle -2 \rangle^{15} \subset NS(X).$$

and so

COROLLARY 2. *The Néron-Severi lattice $NS(X)$ is 2–elementary.*

In what follows we shall first of all determine both the Néron-Severi and the transcendental lattice for such a generic K3 surface X. We shall prove:

THEOREM 1. *For generic X as above we have $NS(X) = U \perp D_6^2 \perp A_1^2$ and $T(X) = U(2)^2 \perp A_1^2 = T(2)$.*

This gives an interpretation of the previous results in terms of the moduli of K3 surfaces. Indeed we have $D(T(2)) = \mathbf{D}_4$ and we note that $T(X) = T(2)$ and T have the same orthogonal group. So the above result shows that our moduli space equals

$$D^0(T)/SO^*(T) = \mathbf{D}_4^0/SO^*(T).$$

Moreover, by the results of [10] we can now identify this moduli space with the configuration spaces from § 3.1.

THEOREM 2 ([10, Prop. 2.10.1]). *There is a commutative diagram*

$$
\begin{array}{ccc}
\mathbf{X}/\{*\} = \mathbf{Y} & \xrightarrow[\simeq]{\tilde{p}} & \mathbf{D}_4^0/SO^*(T)(2) \subset \mathbf{M}^* \\
\downarrow{\scriptstyle \pi} & & \downarrow \\
\mathbf{X}/[\{*\} \times \mathfrak{S}_6] = \mathbf{Y}/\mathfrak{S}_6 & \xrightarrow[p]{\simeq} & \mathbf{D}_4^0/SO^*(T) \subset \mathbf{M}.
\end{array}
$$

The holomorphic maps $\tilde{p}$ and p are biholomorphisms. They are the period maps.

In other words, the moduli space of T–lattice polarized K3's can be identified with the quotient of the configuration space of unordered 6-tuples of lines in $\mathbb{P}^2$ by the correlation involution $*$. Note also that the group $\mathfrak{S}_6$ on the left is indeed isomorphic to the quotient $\mathrm{SO}^*(T)/\mathrm{SO}^*(T)(2) = \mathrm{Sp}(4;\mathbb{F}_2)$ (see (13)).

REMARK 7. Recall that $*$ sends the corresponding double covering K3 surface X to a K3 surface $*X$. The involution $*$ sends the 6 branch lines defining X to the three branch lines defining $*X$. By Prop. 5 there is a standard Cremona transformation with fundamental points the three points P_{14}, P_{25}, P_{36} which induces this involution on the level of the plane and hence X and $*X$ are isomorphic K3-surfaces, the isomorphism being induced by the Cremona transformation. It follows that the quasi-polarization (given by the class of a line ℓ on the plane) is not preserved under this isomorphism: it is sent to $2\ell - e_{14} - e_{25} - e_{36}$ where e_{ij} is the class of the exceptional curve E_{ij}.

By [9, §1.4.] the involution $*$ corresponds to the involution j which on $T = U^2 \perp \langle -1 \rangle^2$ fixes the first 4 basis vectors and sends the fifth to minus the sixth. Since T as well as its orthogonal complement $S = T^\perp$ is 2-elementary, by [11, Theorem 3.6.2] the restriction $\mathrm{O}(S) \to \mathrm{O}(q_S)$ is surjective. Any lift of the image of j under the homomorphism $\mathrm{O}(T) \to \mathrm{O}(q_T)$ to S then can be glued together with j to obtain an isometry of the K3-lattice Λ. Such an isometry sends the period of X to the period of an isomorphic K3-surface which must be $*X$ by the Torelli theorem.

Next, we study what happens when the line configuraton degenerates.

THEOREM 3. *Put*

$$X_\Lambda : \text{the generic K3 surface on } D_\Lambda \subset \mathbf{M}$$

We have

1. $\mathrm{NS}(X_2) = U \perp D_4^2 \perp E_7,\, T(X_2) = U(2)^2 \perp A_1$;

2. $\mathrm{NS}(X_4) = U \perp D_6^2 \perp A_3,\, T(X_4) = U(2) \perp \langle 4 \rangle \perp A_1^2$.

3. $\mathrm{NS}(X_1) = U \perp D_4 \perp D_8 \perp A_3,\, T(X_1) = U(2)^2 \perp \langle -4 \rangle$.

4. $\mathrm{NS}(X_6) = U \perp D_6^2 \perp A_1 \perp A_2,\, T(X_6) = U(2) \perp A_1^2 \perp \langle 6 \rangle$.

To prove this we will make substantial use of elliptic fibrations. We should point out that most, if not all computations can be carried out with explicit divisor classes on the K3 surfaces; elliptic fibrations have the advantage of easing the lattice computations as well as providing geometric insights, since the root lattices in the above decomposition of NS appear naturally as singular fibers of the fibration (conf. for instance [2], [13]).

After reviewing the basics on elliptic fibrations needed, we will first prove Theorem 1 in 3.4. Then using lattice enhancements the three cases of Theorem 3 will be covered in 3.6, 3.6 and 3.6.

3.3. Elliptic fibrations and the Mordell-Weil lattice

We start by reviewing basic facts on elliptic fibrations, and in particular on the Néron-Severi lattice for an elliptic fibration with section following Shioda as summarized in [17].

Let $S \to C$ be an elliptic fibration of a surface S, with a section s and general fiber f. These two span a rank 2 sublattice U of the Néron-Severi lattice $\mathrm{NS} = (\mathrm{NS}(S), \langle , \rangle)$, isomorphic to the hyperbolic plane. We call s the *zero section*; it meets every singular fiber in a point which figures as the neutral element in a group G_v whose structure is given in the table below.

Fiber type	F_v	e_v	G_v	$\mathrm{discr}(F_v)$	discr. gr.
I_n	A_{n-1}	n	$\mathbb{C}^* \times \mathbb{Z}/n\mathbb{Z}$	$(-1)^n(n+1)$	$\mathbb{Z}/n\mathbb{Z}$
II	$-$	1	$\mathbb{C}^*$	1	$\{1\}$
III	A_1	1	$\mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$	-2	$\mathbb{Z}/2\mathbb{Z}$
IV	A_2	2	$\mathbb{C} \times \mathbb{Z}/3\mathbb{Z}$	-3	$\mathbb{Z}/3\mathbb{Z}$
I_{2n}^*	D_{2n+4}	$2n+5$	$\mathbb{C} \times (\mathbb{Z}/2\mathbb{Z})^2$	4	$(\mathbb{Z}/2\mathbb{Z})^2$
I_{2n+1}^*	D_{2n+5}	$2n+6$	$\mathbb{C} \times \mathbb{Z}/4\mathbb{Z}$	-4	$\mathbb{Z}/4\mathbb{Z}$
II^*	E_8	9	$\mathbb{C}$	1	$\{1\}$
III^*	E_7	8	$\mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$	-2	$\mathbb{Z}/2\mathbb{Z}$
IV^*	E_6	7	$\mathbb{C} \times \mathbb{Z}/3\mathbb{Z}$	3	$\mathbb{Z}/3\mathbb{Z}$

In the table we enumerate Kodaira's list of singular fibers. The components of a singular fiber f_v not met by the zero section define mutually orthogonal negative-definite sublattices F_v of the Néron-Severi lattice, all orthogonal to the hyperbolic plane U. The Euler number of the fiber f_v is abbreviated by e_v in the table. The last two entries are the discriminant and the discriminant group of the lattice F_v.

The lattice

$$T := U \perp \bigoplus_v F_v$$

is called the **trivial lattice** of the elliptic surface X. It is a sublattice of NS, but *not necessarily primitive*. Its orthogonal complement (inside $\mathrm{NS}(X)$)

$$L := T_{\mathrm{NS}}^{\perp}$$

is called the **essential lattice**. The group of sections forms the **Mordell Weil group** E. Its torsion part can be calculated as follows:

$$T' := \text{primitive closure of } T \text{ in NS}; \quad T'/T \simeq E_{\mathrm{tors}}.$$

It is one of the main results of the theory of elliptic surfaces that

$$(18) \qquad\qquad E \cong \mathrm{NS}/T.$$

The most famous incarnation of this fact is often referred to as Shioda-Tate formula:

$$(19) \qquad\qquad \mathrm{rank}(\mathrm{NS}) = 2 + \sum_v \mathrm{rank}(T_v) + \mathrm{rank}(E).$$

The main idea now is to endow E/E_{tors} with the structure of a positive definite lattice, the **Mordell-Weil lattice** $\text{MWL} = \text{MWL}(S)$. This can be achieved as follows. Since $L_{\mathbb{Q}} \perp T_{\mathbb{Q}} = \text{NS}_{\mathbb{Q}}$, the restriction to E

$$\pi_E : \text{NS}_{\mathbb{Q}}|_E \to L_{\mathbb{Q}}$$

of the orthogonal projection is well-defined with kernel E_{tors}. The **height pairing** on $\text{MWL} = E/E_{\text{tors}}$ by definition is induced from the pairing on the Néron-Severi group:

$$\langle P, Q \rangle := -\langle \pi_E(P), \pi_E(Q) \rangle, \quad P, Q \in E.$$

Note that by definition, this pairing need not be integral. Shioda has shown that the height pairing can be calculated directly from the way the sections P, Q meet each other, the zero section, and in particular the singular fibers f_v. In the sequel we will need this only for the height $\langle P, P \rangle$ of an individual section $P \in E$. The only components of f_v possibly met by a section are the multiplicity 1 components not met by the zero section. For I_n, this gives $n-1$ components that one enumerates successively, starting from the first component next to the one meeting the zero section (upto changing the orientation). For I_n^*, $n > 0$, there are 3 components: the *near* one (the first component next to the one meeting the zero section) and two far ones (for I_0^* fibers the three simple non-identity components are indistinguishable). In the end, the height formula reads

$$(20) \qquad h(P) := \langle P, P \rangle = 2\chi(O_S) + 2\langle P, s \rangle - \sum_v c_v,$$

where the local contribution c_v for f_v can be found in the following table and is determined by the component which the section P meets (numbered $i = 0, 1, \ldots, n-1$ as above for fibers of type I_n):

Fiber type	root lattice	c_v
I_n $(n > 1)$	A_{n-1}	$\frac{i(n-i)}{n}$
III	A_1	$\frac{1}{2}$
IV	A_2	$\frac{2}{3}$
I_n^* $(n \geq 0)$	D_{n+4}	1 (near), $1 + \frac{n}{4}$ (far)
II^*	E_8	$-$
III^*	E_7	$\frac{3}{2}$
IV^*	E_6	$\frac{4}{3}$

The following formula for the discriminant of $\text{NS} = \text{NS}(S)$, the Néron-Severi lattice can be shown to follow from the above observations:

$$(21) \qquad \text{discr}(N) = \frac{(-1)^{\text{rank} E}}{|E_{\text{tors}}|^2} \text{discr}(T) \cdot \text{discr}(\text{MWL}).$$

3.4. Generic NS(X) and the Proof of Theorem 1

For later use, we start by computing the Néron-Severi lattice NS(X) and the transcendental lattice $T(X)$ with the help of elliptic fibrations with a section, the so-called **jacobian elliptic fibrations**. It is a special feature of K3 surfaces that they may admit several jacobian elliptic fibrations. For instance, we can multiply any three linear forms from to the RHS to the LHS of (17) such as

$$(22) \qquad\qquad X: \quad \ell_1 \cdots \ell_3 w^2 = \ell_4 \cdots \ell_5.$$

Here a fibration is simply given by projection onto $\mathbb{P}^1_w$; it is the quadratic base change $v = w^2$ of a cubic pencil with 9 base points $P_{14}, \ldots, P_{36}$ as sections. However, this plentitude of sections (forming a Mordell-Weil lattice of rank 4) makes the lattice computations quite complicated, so we will rather work with two other elliptic fibrations on X.

Note that due to the freedom in arranging the lines in (22), the K3 surface X admits indeed several different fibrations of the above shape. This ambiguity will persist for all elliptic fibrations throughout this note.

Standard elliptic fibration

We shall now derive an elliptic fibration on X which will serve as our main object in the following. For this purpose we specify the elliptic parameter u giving the fibration by

$$u = \ell_1 / \ell_2.$$

One easily computes the divisor of u as

$$(u) = 2\ell_1 + E_{13} + \ldots + E_{16} - 2\ell_2 - (E_{23} + \ldots + E_{26}).$$

Both zero and pole divisor encode divisors of Kodaira type I_0^*, hence the morphism

$$u : X \to \mathbb{P}^1$$

defines an elliptic fibration on X with sections $\ell_3, \ldots, \ell_6$. Note that the exceptional divisors $E_{ij}(3 \leq i < j \leq 6)$ are orthogonal to both fibers; hence they comprise components of other fibers. We sketch some of these curves in the following figure:

There is an immediate sublattice N of NS(X) generated by the zero section and fiber components. Here this amounts to

$$N = U \perp D_4^2 \perp A_1^6.$$

Since the rank of N equals the Picard number $\rho = 16$ of X, the sublattice N has finite index in NS(X). We note two consequences. First, the (-2) curves $E_{ij}(3 \leq i < j \leq 6)$ generically sit on 6 fibers of type I_2 (because otherwise there would be an additional fiber component contributing to NS(X)). For later reference, we denote the other component of the respective fiber by E'_{ij}; this gives another -2-curve on X. Secondly, we
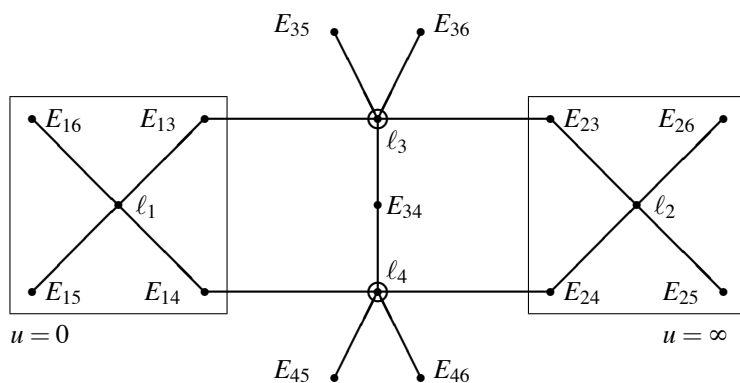
Figure 2: Some sections and fiber components of the standard fibration

deduce from (19) that generically the Mordell-Weil rank is zero. Since I_0^* fibers can only accommodate torsion section of order 2, the given four sections give the full 2-torsion. Alternatively, this can be computed with the height pairing as sketched in 3.3 or it can be derived from the actual equations which we give in 3.6. From (21) we deduce that $NS(X)$ has discriminant

$$(23) \qquad\qquad \mathrm{discr}\, NS(X) = -2^{10}/2^4 = -2^6.$$

Since by Corollary 2 $NS(X)$ is 2–elementary we find that (23) implies the length of the Néron–Severi lattice to equal 6.

Transcendental lattice

We want to compute the transcendental lattice $T(X)$. Again we need some general facts from lattice theory which we collect at this place for the reader's convenience.

For an even non-degenerate integral lattice $(L, \langle -, - \rangle)$ recall (see "Notation") the discriminant group $\delta(L) = L^*/L$ and the $\mathbb{Q}/2\mathbb{Z}$–valued **discriminant form** induced by $\langle -, - \rangle$ denoted q_L. The importance of this invariant stems from the following result of Nikulin [11]: two even lattices with the same signature and discriminant form are in the same genus, i.e. are isomorphic over the rationals.

Below we need a more precise result in a special situation:

PROPOSITION 7 ([12, Prop. 4.3.2]). *Any indefinite 2-elementary lattice is determined up to isometry by signature, length, and the property whether the discriminant form takes only integer values or not.*

We return to our double sextics X in the generic situation. Since $T(X)$ and $NS(X)$ are orthogonal complements embedded primitively into the unimodular lattice

Λ, we find by construction an isomorphism of discriminant forms

(24) $$q_{T(X)} \cong -q_{\mathrm{NS}(X)}.$$

In particular, $T(X)$ is again 2-elementary of length $l = 6$, and of signature $(2,4)$. So we may apply the above Prop. 7. To do so we need to be able to determine properties of the discriminant form. To decide this without going through explicit computations with divisor classes on X, we switch to another elliptic fibration on X.

Alternative elliptic fibration

In order to exhibit another elliptic fibration on X, we start by identifying two perpendicular divisors of Kodaira type I_2^*:

$$\begin{aligned} D_1 &= E_{15} + E_{16} + 2(\ell_1 + E_{13} + \ell_3) + E_{35} + E_{46}' \\ D_2 &= E_{25} + E_{26} + 2(\ell_2 + E_{24} + \ell_4) + E_{45} + E_{36}' \end{aligned}$$

Their linear systems induce an elliptic fibration with section induced by ℓ_6, since $\ell_6.D_i = 1$. In addition to the two fibers of type I_2^*, there are 2 further reducible fibers with identity component E_{34}' on the one hand and E_{56} on the other hand. Rank considerations imply that their type is generically I_2, so that the given fibers and the zero section generate the sublattice $U + D_6^2 + A_1^2$ of $\mathrm{NS}(X)$. In fact, since ranks and discriminants agree, we find the generic equality

(25) $$\mathrm{NS}(X) = U \perp D_6^2 \perp A_1^2.$$

In particular, this singles out A_1^*/A_1 as an orthogonal summand of the discriminant group $\delta(\mathrm{NS}(X)) = \mathrm{NS}(X)^*/\mathrm{NS}(X)$. Its quadratic form thus takes non-integer values in $\frac{1}{2}\mathbb{Z}/2\mathbb{Z}$. As $T(X)$ has the same invariants as $U(2)^2 \perp A_1^2$, by Prop. 7 these must be isomorphic:

$$T(X) = U(2)^2 \perp A_1^2.$$

This concludes the proof of Theorem 1.

The representation of $\mathrm{NS}(X)$ in (25) is especially useful for the concept of lattice enhancements as it allows for writing an abstract isomorphism of discriminant forms as in (24). We will make this isomorphism explicit in 3.5 and exploit it on the level of elliptic fibrations.

3.5. Lattice Enhancements

General Theory

The theory of lattice polarised K3 surfaces as sketched in 3.2 predicts for a given even lattice L of signature $(1, r - 1)$ that K3 surfaces admitting a primitive embedding

$$L \hookrightarrow \mathrm{NS}$$

come in $(20 - r)$-dimensional families (if L admits a primitive embedding into the K3 lattice $\Lambda = U^3 \perp E_8^2$ at all). Equivalently, on the level of transcendental lattices, the primitive embedding has to be reversed for M the orthogonal complement of L in Λ:

$$T \hookrightarrow M.$$

Lattice enhancements provide an easy concept of specifying subfamilies of lattice polarised K3 surfaces. Namely one picks a vector $v \in M$ of negative square $v^2 < 0$ and enhances NS by postulating v to correspond to an algebraic class. Generically this leads to a codimension one subfamily of K3 surfaces with transcendental lattice

$$T = v^\perp \subset M.$$

The generic Néron-Severi lattice arises as primitive closure (or saturation)

$$\text{NS} = (L + \mathbb{Z}v)' \subset \Lambda.$$

Explicitly NS can be computed with the discriminant form. Namely v induces a unique primitive element in the dual lattice M^*. The resulting equivalence class $v \in M^*/M$ corresponds via the isomorphism of discriminant forms $q_M \cong -q_L$ as in (24) with an equivalence class $w \in L^*/L$. Enhancing $L + \mathbb{Z}v$ by $v + w$ results in a well-defined even saturated lattice which exactly gives NS.

Note that presently $M = U(2)^2 \perp A_1^2$ has rank equalling its length, so any primitive vector $v \neq 0$ induces an order 2 element v in M^*/M. In other words, $(L + \mathbb{Z}v)$ has index 2 in its primitive closure. If v is assumed to be primitive in M, then we find the generic discriminant of the lattice enhancement

(26) $$\text{discr NS} = (\text{discr} L) \cdot v^2/4 = -16v^2.$$

In the following we want to study the K3 surfaces corresponding to the divisors D_Λ in the moduli space and relate them to Hermann's work [5]. To this end, we shall enhance NS by a primitive representative $\mathbf{y}^*$ as explained in Corollary 1 and the following paragraph.

Example: lattice enhancements by a -4 vector

We return to our double sextics branched along 6 lines. Following 3.5 we will enhance the Néron-Severi lattice by a -4-vector from $M = U(2)^2 \perp A_1^2$. We consider two ways to do so which we will soon see to be inequivalent and exhaustive. Following up on Example 1 we shall take either

$$v_1 = (0,0,0,0,1,1) \quad \text{or} \quad v_2 = (1,-1,0,0,0,0).$$

Computing their orthogonal complements in M, we find the generic transcendental lattices of the enhanced lattice polarised K3 surfaces:

(27) $$T_1 = v_1^\perp \quad = \quad U(2)^2 \perp \langle -4 \rangle,$$

(28) $$T_2 = v_2^\perp \quad = \quad U(2) \perp A_1^2 \perp \langle 4 \rangle.$$

Note that the first lattice (which corresponds to $\Delta = 1$ by Example 1, see also 3.6) is 2-divisible as an even lattice while the second lattice (corresponding to $\Delta = 4$, see also 3.6) certainly is not. This confirms that these two cases are indeed inequivalent. In what follows, we will interpret the enhancements in terms of elliptic fibrations. Along the way, we will verify that any other lattice enhancement by a -4-vector is equivalent to one of the above.

Interpretation in terms of elliptic fibrations

In view of the Picard number, a jacobian elliptic fibration can be enhanced in only 2 ways: either by a degeneration of singular fibers (changing the configuration of ADE-types) or by an additional section (which any multisection can be reduced to by (18)). For the second alternative, there are usually many possibilities, distinguished by the height of the section, but also by precise intersection numbers, for instance encoding the fiber components met. However, once we fix the discriminant which we are aiming at, this leaves only a finite number of possibilities.

As an illustration, consider the fibration from 3.4 exhibiting the representation

$$\text{NS}(X) = U \perp D_6^2 \perp A_1^2.$$

Enhancing NS as in 3.5, we reach a subfamily of lattice polarised K3 surfaces of Picard number $\rho = 17$ and discriminant 64 by (26). As the discriminant stays the same as before up to sign, there are only 3 possibilities of enhancement to start with:

- 2 fibers of type I_2 degenerate to I_4,

- I_1 and I_2^* degenerate to I_3^*,

- or a section P of height $h(P) = 1$.

Using the theory of Mordell-Weil lattices from 3.3, the third case can be broken down into another 3 subcases, depending on the precise fiber components met. Recall that the non-identity components of I_n^* fibers ($n > 0$) are divided into the near component (only one component away from the identity component) and the two far components as visible in the corresponding root diagram of Dynkin type D_{n+4}:
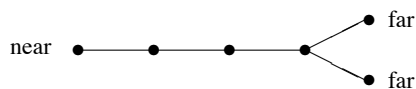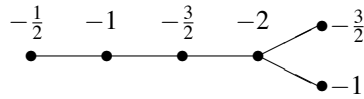

An easy enumeration of the possible configurations reveals the following possibilities for a section P of height $h(P) = 1$; all of them have P perpendicular to the zero section.

alternative	I_2^*'s	I_2's
(1)	far, far	id, id
(2)	far, near	id, non-id
(3)	near, near	non-id, non-id

Comparison of enhancements

We shall now compare our investigation of the above elliptic fibration with the concept of lattice enhancements by making the isomorphism (24) explicit. We start by calculating the discriminant form of D_6. The discriminant group is $(\mathbb{Z}/2\mathbb{Z})^2$ with generators $a^\pm \in D_6 \otimes \mathbb{Q}$ represented by elements meeting each one of the two far nodes $r^\pm$ precisely once and none of the remaining roots $r_1, \ldots, r_4$ (enumerated from left to right).

$$-\tfrac{1}{2} \quad -1 \quad -\tfrac{3}{2} \quad -2 \qquad \bullet\, -\tfrac{3}{2}$$
$$\bullet \quad\quad \bullet \quad\quad \bullet \quad\quad \bullet \quad\quad \bullet\, -1$$

The correct rational linear combination $a^+ = -(\frac{1}{2}r + \frac{3}{2}r^+ + r^-)$, $r = r_1 + 2r_2 + 3r_3 + 4r_4$ a root, is shown in the figure. Of course the combination for a^- is similar and so we find $(a^\pm)^2 = -\frac{3}{2}$ and $a^+ \cdot a^- = -1$ so that

$$q_{D_6} = \begin{pmatrix} -\tfrac{3}{2} & -1 \\ -1 & -\tfrac{3}{2} \end{pmatrix}.$$

The discriminant group of $U(2)$ is also $(\mathbb{Z}/2\mathbb{Z})^2$ with basis e, f induced from the standard basis of $U(2)$. It follows that

$$q_{U(2)} = \begin{pmatrix} 0 & \tfrac{1}{2} \\ \tfrac{1}{2} & 0 \end{pmatrix}.$$

Thus one easily verifies the isomorphism in the standard basis:

$$\begin{array}{ccc} q_{U(2)\perp A_1} & \xrightarrow{\;\cong\;} & -q_{D_6 \perp A_1} \\ (e, f, g) & \mapsto & (g+e, g+f, e+f+g) \end{array}$$

Duplicated this directly extends to the isomorphism (24). We continue by computing the impact of the enhancing vectors v_i from 3.5.

Starting out with v_1, this vector induces the element $(0, 0, 1)$ in either copy of $q_{U(2)\perp A_1} \hookrightarrow q_{T(X)}$. In each $q_{D_6 \perp A_1}$ this corresponds to the class $(1, 1, 1)$. Thus we obtain an algebraic class meeting each reducible fiber. A priori this would be a multisection, but using the group structure it induces a section, necessarily of height 1, of the third alternative in 3.5.

Next we turn to v_2. We have

$$v_2 = ((1, 1, 0), (0, 0, 0)) \in (q_{U(2)\perp A_1})^2 \cong q_{T(X)}.$$

Hence v_2 induces the same class in $(q_{D_6 \perp A_1})^2$. This corresponds to an algebraic class meeting only one I_2^* fiber non-trivially. By inspection of the alternatives in 3.5, this class cannot be a section of height 1 (which always meets both I_2^* fibers non-trivially), but it fits in with the degeneration of I_1 and I_2^* to I_3^*.

Connection with other enhancements

Before returning to the arrangement of the 6 lines, we comment on the other three possible enhancements of the elliptic fibration in 3.5. In fact, the freedom of choosing some lines out of the 6 carries over to these elliptic fibrations endowing X with several different ones of the same shape. We leave it to the reader to follow the degeneration of singular fibers on the given fibration through the other elliptic fibrations. Without too much effort, this enables us to identify all remaining degenerations with the one which was shown in 3.5 to correspond to the lattice enhancement by v_2.

3.6. Special Arrangements of the 6 **Lines; Proof of Theorem 3**

We are now in the position to investigate the subfamilies of our double sextics corresponding to the first few divisors on the moduli space of Abelian fourfolds of Weil type. In each case, we start from the special arrangement of lines to fill out the geometric and lattice theoretic details.

$\Delta = 2$

Recall from § 3.1 that D_2 corresponds to X_3. We now consider the component X_3^{345}, that is, when the lines ℓ_3, ℓ_4, ℓ_5 meet in a single point. On the double covering K3 surface, this results in a triple point whose resolution requires an additional blow-up. On the degenerate K3 surface, the original exceptional divisors can still be regarded as perpendicular (with notation adjusted, see the figure below); with the lines, how-ever, they do not connect to a hexagon anymore, but to a star through the additional exceptional component D (Kodaira type IV^*):
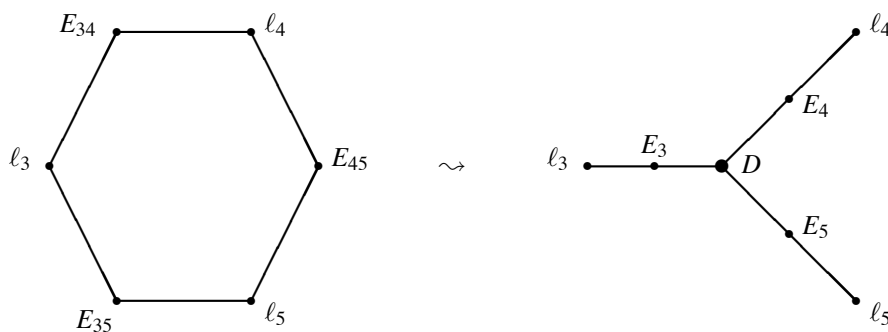
On the standard elliptic fibration from 3.4, this degeneration causes three I_2 fibers to merge to a single additional fiber of type I_0^* (with a 'new' rational curve E as 4th simple component; compare Figure 2 where also some rational curves such as ℓ_5, ℓ_6 have been omitted) as indicated in the figure on the next page.

Thus NS has the index 4 sublattice $U \perp D_4^3 \perp A_1^3$ – which is again 2-elementary. The remaining generators of NS(X) can be given by the 2-torsion sections. To decide
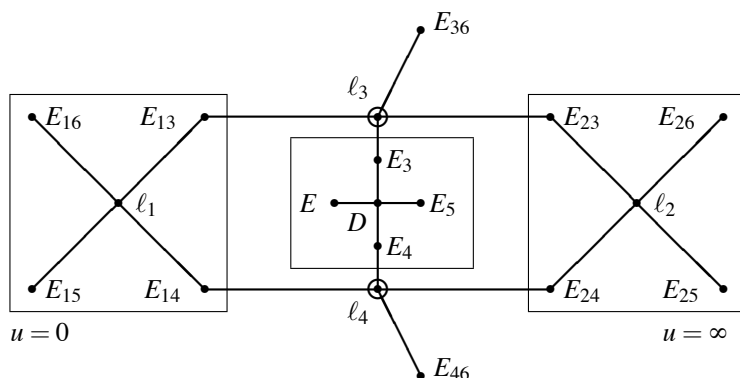
on the discriminant form, we once more switch to the alternative elliptic fibration from 3.4. The fibration degenerates as follows: in the notation from 3.4 we have to replace E_{35}, E_{45} by E_3, E_4 as components of the I_2^* fibers, and E_{34}' by E_5 as component of one I_2 fiber. Then E_{56} still sits on a second I_2 fiber while E gives yet another one. Here D induces a 2-torsion section: visibly it meets both I_2^* fibers at far components. As for the I_2 fibers, it meets the one containing E_{56} at the other component (i.e. non-identity) and the one containing E_5 in this very component (non-identity again). Since the height of a section is non-negative, this already implies that the section D has height 0; then the fiber types predict that D can only be 2-torsion.

For completeness we study the fiber containing E in detail. Since ℓ_6 is also a section for the standard fibration, it meets some simple component of the degenerate I_0^* fiber. Obviously ℓ_6 does not meet any of E_3, E_4, E_5. Hence ℓ_6 has to meet E. In conclusion E is the identity component of the degenerate I_2 fiber of the alternative fibration. As D meets this fiber trivially, we find the orthogonal decomposition

$$\mathrm{NS}(X) = U \perp \langle D_6^2, A_1^2, D \rangle \perp A_1.$$

As before, we deduce from the orthogonal summand A_1 that the discriminant form takes non-integer values. Hence by Proposition 7 we deduce that

(29) $T(X) = U(2)^2 \perp A_1$ corresponding to $\Delta = 2$.

In the language of lattice enhancements, the subfamily thus arises from a generator of either A_1 summand in the generic transcendental lattice. Recall that geometrically, this vector corresponds to the extra rational curve D involved in the resolution of the triple point where three lines come together. Conversely, we can derive from Proposition 7 again that the Néron-Severi lattice admits several representations purely in terms of U and root lattices such as

$$\mathrm{NS}(X) = U \perp D_4^2 \perp E_7.$$

This concludes the proof of Theorem 3 1.   □

$\Delta = 4$

Recall from § 3.1 that X_4 comes from the divisor $\mathbf{X}_{\mathrm{coll}}$. Let ℓ be the line which contains the collinear points. Then ℓ splits on X as $\pi^*\ell = \ell' + \ell''$. Let $D = \ell' - \ell''$. Since D is anti-invariant for the covering involution, it defines an algebraic divisor on X which is orthogonal to the classes specialising from the generic member. By construction, $\ell'.\ell'' = 0$ so that $D^2 = -4$. In particular, D is primitive in $\mathrm{NS}(X)$, and X arises from a lattice enhancement by the -4-vector D as in 3.5. Presently we can even give a $\mathbb{Z}$-basis of $\mathrm{NS}(X)$ by complementing the generic basis by ℓ', say. To compute $\mathrm{NS}(X)$ and $T(X)$ without writing out intersection matrices etc, we make use of elliptic fibrations again.

For the standard fibration, it is convenient to choose the collinear points as P_{12}, P_{34}, P_{56}. From the obvious -2-curves, each ℓ' and ℓ'' then meets exactly the corresponding exceptional divisors E_{12}, E_{34}, E_{56} on X. On the standard fibration from 3.4, the two singular fibers of type I_2 at E_{34} and E_{56} are thus connected by ℓ', ℓ'', merging to a fiber of type I_4. Note that this indeed preserves the discriminant up to sign while raising the rank by one.
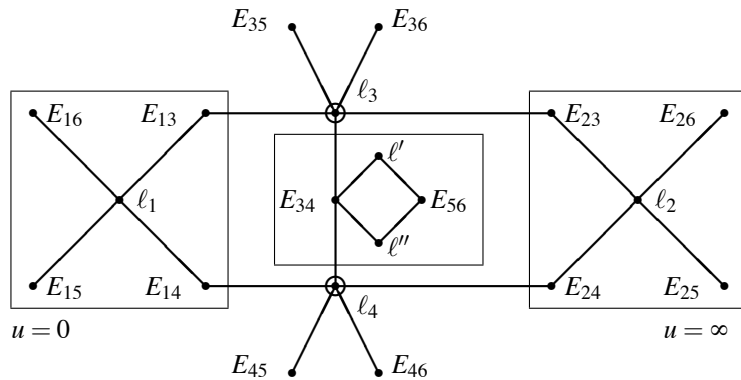
Switching to the alternative fibration from 3.4, the classes ℓ', E_{56}, ℓ'' which correspond to the root lattice A_3 remain orthogonal to the fibers D_1, D_2. Generically they are therefore contained in a fiber of type I_4, merging the I_2 fibers generically at E_{56} and at E'_{34}. That is, $\mathrm{NS}(X) = U \perp D_6^2 \perp A_3$. By 3.5, 3.5 we can thus verify that X arises from the lattice enhancement by v_2 with transcendental lattice $T(X) = U(2) \perp \langle 4 \rangle \perp A_1^2$ as stated in Theorem 3 2.

$\Delta = 1$

As the key part of this subsection, we now come to the case $\Delta = 1$ which will cover almost the rest of this section up to 3.6. Recall from §3.1 that we have the divisor $XQ \subset X$ of 6-uples of lines tangent to a fixed conic. This divisor can be identified with D_1 as follows from [9, Prop. 2.13.4]. Indeed, that hyperplane XQ is exactly the hyperplane orthogonal to our v_1 (see § 3.5).

This can also be read off directly from the fact that X is a Kummer surface. Indeed, by [10, § 0.19] the surface X arises from the jacobian of the genus 2 curve which is the double cover of the conic branched along the six intersection points with the lines. This gives $T(X) = U(2)^2 + \langle -4 \rangle$ in agreement with the lattice enhancement by v_1. We shall confirm this from our methods using elliptic fibrations.

The above argument, however, gives no information about the extra algebraic class needed to generate $\mathrm{NS}(X)$ over $\mathbb{Z}$. To overcome this lack of a generator, we shall work geometrically with elliptic fibrations, starting with the alternative fibration. Going backwards in our constructions, we first develop the corresponding section on the standard fibration and then interpret this in terms of a certain conic in $\mathbb{P}^2$ which splits on X. Finally we confirm our geometric arguments by providing explicit equations. Throughout we do not use any information about the Kummer surface structure.

First, denoting the conic by C, we have a splitting $C = C_1 + C_2$ on X. As in 3.6, the divisor $D = C_1 - C_2$ is anti-invariant for the covering involution and therefore orthogonal to the rank 16 sublattice of $\mathrm{NS}(X)$ generated by the classes of the lines and the exceptional divisors. The subtle difference, though, is that D is in fact 2-divisible in $\mathrm{NS}(X)$ since $C \sim 2H$:

$$\frac{1}{2}D = H - C_2 \in \mathrm{NS}(X).$$

Since $D^2 = -16$, we find that the latter class has square -4, hence we are indeed confronted with a lattice enhancement as in 3.5.

From alternative to standard fibration

In 3.5 we interpreted lattices enhancements in terms of the alternative elliptic fibration from 3.4. By 3.5, 3.5 it is alternative (3) which corresponds to the lattice enhancement by v_1. In detail, the alternative fibration admits a section P intersecting the following fiber components (see Figure 3 for the resulting diagram of -2-curves):

singular fiber	$D_1 = I_2^*$	$D_2 = I_2^*$	I_2	I_2
component met by ℓ'	E_{15}	E_{25}	opposite E'_{34}	opposite E_{56}

On the standard fibration, P defines a multisection whose degree is not immediate. Here we develop a backwards engineering argument to prove that the degree is actually 1, i.e. P is a section for both fibrations.

A priori the degree d of the multisection P need not be 1 on the standard fibration, but P always induces a section P' of height 1. The essential point of our argument is that we can read off from the alternative fibration which fiber components are not met by the multisection on the standard fibration. For each singular fiber this leaves only one fiber component with intersection multiplicity depending on the degree d. But then we can use the group structure to determine which fiber component will be met by the induced section P'. Thanks to the specific singular fibers, the argument only depends on the parity of d:

singular fiber	I_0^*	I_0^*	I_2	I_2	I_2	I_2	I_2	I_2
comp's met by P	$E_{15},$ $(d-1)E_{14}$	$E_{25},$ $(d-1)E_{23}$	dE_{34}	dE_{35}'	dE_{36}	dE_{45}'	dE_{46}	dE_{56}'
comp met by P' for even d	E_{16} non-id	E_{25} non-id	E_{34} id	E_{35} id	E_{36} id	E_{45}' id	E_{46}' id	E_{56}' id
comp met by P' for odd d	E_{15} non-id	E_{25} non-id	E_{34} id	E_{35}' non-id	E_{36} id	E_{45}' id	E_{46} non-id	E_{56}' id

Note that for even d the section P' would have even height by inspection of the fiber components met, contradicting $h(P') = h(P) = 1$. Hence d is odd, and the induced section P' meets the fiber components indicated in the last two rows of the table. With this section at hand, we can complete the circle: namely P' defines a section for both fibrations, of exactly the same shape as P, hence $P' = P$ (and $d = 1$).

For later reference, we point out the symmetry in the fiber components met by P on the standard fibration: on the I_0^* fibers, it is exactly those met by the section ℓ_5, while on the I_2 fibers it is exactly those not met by ℓ_5. This symmetry is essential for the section to be well-defined as it ensures that adding a 2-torsion section to P will always result in a section of height 1 (compare 3.6).

From standard fibration to double sextic

On the double sextic model, the section P arises from a curve Q in $\mathbb{P}^2$ which splits into 2 rational curves Q_1, Q_2 on X. Here we give an abstract description of Q and its components on X based on the geometry of the elliptic fibrations.

From the alternative fibration we know that P is perpendicular to the lines ℓ_i for $i \neq 5$. On the other hand, the standard fibration reveals by inspection of the above table which exceptional curves intersect P:

$$\text{exactly} \quad E_{15}, E_{25}, E_{34}, E_{36}, E_{46} \quad \text{plus possibly} \quad E_{12}.$$

The latter is the only exceptional curve which is not visible as section or fiber component on the standard fibration. The remaining two intersection numbers can be computed as follows: regarding ℓ_5 as a 2-torsion section of the standard fibration, the height pairing $\langle P, \ell_5 \rangle = 0$ implies by virtue of the fiber components met that $P.\ell_5 = 0$. As for E_{12}, consider the auxiliary standard fibration defined by $u' = \ell_1/\ell_5$. Then P defines a section for this fibration as well, as it meets the fiber

$$(u')^{-1}(\infty) = 2\ell_5 + E_{25} + E_{35} + E_{45} + E_{56}$$

exactly in E_{25} (transversally) by the above considerations. Looking at the fiber

$$(u')^{-1}(0) = 2\ell_1 + E_{12} + E_{13} + E_{14} + E_{16}$$

we deduce $P.E_{12} = 1$ from the fact that P does not intersect the other fiber components.

Turning to the double sextic model of X, P defines a curve meeting E_{12}, E_{15}, E_{25}, E_{34}, E_{36}, E_{46} transversally, but no other exceptional curves E_{ij} nor any of the ℓ_i. On the base $\mathbb{P}^2$, this curve necessarily corresponds to a conic Q through the six underlying nodes. On X, this conic splits into two disjoint rational curves Q_1, Q_2 where $Q_1 = P$, say, and Q_2 corresponds to the section $-P$ on the elliptic fibrations.

Explicit equations

We start out with the general equation of a jacobian elliptic K3 surface X with singular fibers of type I_0^* twice and 6 times I_2 over some field k of characteristic $\neq 2$. Necessarily this comes with full 2-torsion. Locating the I_0^* fibers at $t = 0, \infty$, we can write

$$(30) \qquad\qquad X: \quad tw^2 = x(x - p(t))(x - q(t))$$

where $p, q \in k[t]$ have degree 2. Here the I_2 fibers are located at $p = 0, q = 0$ $p = q$. Up to symmetries, there are only two ways to endow the above fibration with a section of height 1. Abstractly, restrictions are imposed by the compatibility with the 2-torsion sections. On the one hand, there is the symmetric arrangement encountered in 3.6. This will be investigated below. On the other hand, an asymmetric arrangement can be encoded, for instance, in terms of the standard fibration by a section with the following intersection behaviour:

singular fiber	I_0^*	I_0^*	I_2	I_2	I_2	I_2	I_2	I_2
component met	E_{15}	E_{26}	E_{34}	E_{35}	E_{36}	E_{45}	E_{46}	E_{56}'

To see that the arrangements do indeed generically describe different K3 surfaces, we switch to the alternative fibration from 3.4 for one final time. Here the section P with intersection pattern as in the above table induces a bisection meeting far and near component of D_1 and far and identity component of D_2. Using the group structure the induced section intersects both I_2^* fibers in a far component. In terms of 3.5 this corresponds to alternative (1) which was shown in 3.5, 3.5 to differ from alternative (3) which underlies the section arrangement in 3.6.

We shall now continue by deriving equations admitting a section of height 1 as encountered in 3.6. In agreement with this, we model the section P to intersect the same components of the I_0^* fibers as the 2-torsion section $(0,0)$. In terms of the RHS of (30), these correpond to the factor x. The section P therefore takes the shape

$$P = (at, \ldots) \quad \text{for some constant} \quad a \in k.$$

Upon subsituting into (30), we now require the other two factors on the RHS to produce the same quadratic polynomial up to a constant. Concretely this polynomial can be given by

$$(31) \qquad\qquad g = at - p.$$

Then we consider the codimension 1 subfamily of all X such that there exists $b \in k, b \neq 1$ such that

$$(32) \qquad q = at - bg.$$

By construction, these elliptic K3 surfaces admit the section

$$P = (at, \sqrt{abg}).$$

This section has height 1: not only does it meet both I_0^* fibers non-trivially thanks to our set-up, but also the I_2 fibers at $p - q = (b-1)g = 0$ while being perpendicular to the zero section. As a whole, the family of K3 surfaces can be given by letting g, a, b vary and p, q depend on them as above. There are still normalisations in t and in (x, w) left which bring us down to the 3 moduli dimensions indeed.

Standard fibration reflecting the 6 lines

We shall now translate the above considerations to the standard fibration as it comes from the 6 lines in 3.4. With $u = \ell_1/\ell_2$, we naturally have an equation

$$(33) \qquad uw^2 = \ell_3 \ell_4 \ell_5 \ell_6.$$

Here we can use u to eliminate x, say, so that after clearing denominators the expressions ℓ_i are separately linear in both y and u. In particular, the family of elliptic curves over $\mathbb{P}_u^1$ becomes evident, with 2-torsion sections given by $\ell_i = 0$ ($i = 3, 4, 5, 6$).

For ease of explicit computations, we normalise the lines to be

$$\ell_1 = x, \ell_2 = y, \ell_3 = x + y + z, \ell_4 = a_1 x + a_2 y + a_3 z, \ell_5 = z, \ell_6 = b_1 x + b_2 y + b_3 z.$$

Working affinely in the chart $z = 1$, equation (33) readily takes the shape of a twisted Weierstrass form

$$uw^2 = ((u+1)y + 1)((a_1 u + a_2)y + a_3)((b_1 u + b_2)y + b_3).$$

Standard variable transformations take this to:

$$uw^2 = (y + (a_1 u + a_2)(b_1 u + b_2))(y + a_3(u+1)(b_1 u + b_2))(y + b_3(u+1)(a_1 u + a_2)).$$

Translating to the shape of (30) and solving for (31), (32), we find

$$b_1 = ba_1 b_3/(a_3 + (b-1)a_1), \quad b_2 = ba_2 b_3/(a_3 + (b-1)a_2)$$

with $a = -ba_3 b_3(a_1 - a_2)^2/[(a_3 + (b-1)a_1)(a_3 + (b-1)a_2)]$.

Conics on the double sextic

Finally we can trace back the section P to the double sextic model. Step by step, it leads to the following conic in the affine chart $z = 1$:

$$
\begin{aligned}
Q \;=\; & -a_1^2 x^2 a_2 + a_1^2 x^2 a_3 + a_1^2 x^2 b a_2 - a_1^2 x a_2 y + x a_1^2 a_2 b + a_1^2 x a_2 y b \\
& -a_1 x a_2^2 y + 2 a_3 x y a_1 a_2 - x a_1^2 a_2 + x a_3 a_1^2 + a_1 x a_2^2 y b + a_1 a_2^2 y b \\
& -a_1 a_2^2 y + a_3 y a_2^2 - a_2^2 y^2 a_1 + a_2^2 y^2 a_3 + a_2^2 y^2 b a_1.
\end{aligned}
$$

One directly verifies that Q indeed passes through the nodes $P_{12}, P_{15}, P_{25}, P_{34}, P_{36}, P_{46}$ in $\mathbb{P}^2$ as in 3.6. Hence Q splits into two components on the double sextic X one of which is P.

We conclude this paragraph by verifying that the subfamily of double sextics constructed in 3.6 does in fact admit a conic which is tangent to each of the 6 lines. For this purpose, set

$$
\alpha = a_1(a_2 - a_3), \beta = a_2(a_3 - a_1), \gamma = a_3(a_1 - a_2).
$$

Then the conic in $\mathbb{P}^2$ given by

$$
\alpha^2 x^2 + \beta^2 y^2 + \gamma^2 z^2 - 2(\alpha\beta xy + \alpha\gamma xz + \beta\gamma yz) = 0
$$

meets each of the 6 lines ℓ_i tangentially.

Conclusion

By comparison of moduli dimensions, it follows conversely that the K3 surfaces X_1 with a conic tangent to each of the 6 lines of the branch locus also admits a conic through a selection of 6 nodes as above. From 3.6 we therefore deduce that X_1 generically arises from X via the lattice enhancement by the vector v_1; that is, by 3.5

$$
T(X_1) = U(2)^2 \perp \langle -4 \rangle.
$$

The Néron-Severi lattice $\mathrm{NS}(X_1)$ is thus generically generated by the sublattice $U \perp D_6^2 \perp A_1^2$ coming from X enhanced by the section P from 3.6. The simple representation of $\mathrm{NS}(X_1) = U \perp D_4 \perp D_8 \perp A_3$ in Theorem 3 is derived from the above fibration by switching to yet another jacobian elliptic fibration as depicted below.

$\Delta = 6$

Our aim for the final bit of this section is to understand the geometry of the K3 surfaces for D_6, i.e. the case $\Delta = 6$ (the final Heegner divisor singled out in [6]). By Corollary 1 and the discussion succeeding it, this corresponds to a lattice enhancement by a vector $v \in T(2)$ of square $v^2 = -6$. Here we choose the primitive representative

$$
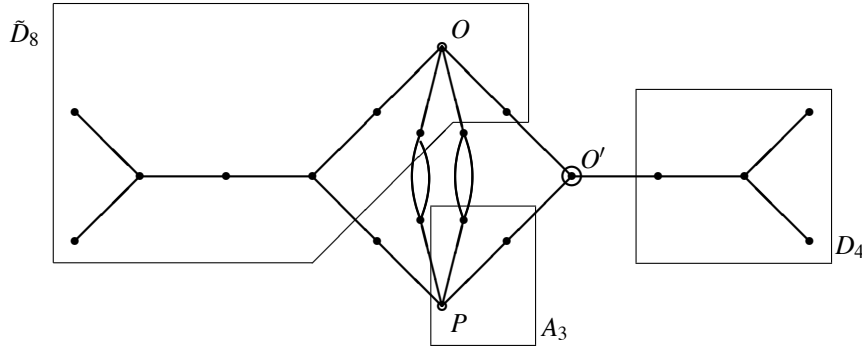v = (1, -1, 1) \text{ in one copy of } U(2) \perp A_1 \subset T(2),
$$

Figure 3: From alternative fibration to $U \perp D_4 \perp D_8 \perp A_3$ on X_1

augmented by zeroes in $T(2)$. Then $v/2$ defines a class in the discriminant group $T(2)^*/T(2)$ which via the isomorphism in 3.5 maps to the class

$$(0,0,1) \in (D_6 \perp A_1)^*/(D_6 \perp A_1) \hookrightarrow \mathrm{NS}^*/\mathrm{NS},$$

augmented by zeroes in $\mathrm{NS}^*/\mathrm{NS}$. The Néron-Severi lattice NS is thus enhanced by a divisor which only meets one reducible fiber of the alternative fibration in a non-identity component (corresponding to A_1). If this divisor were a section, then it would have height $h \geq 4 - 1/2 = 7/2$ by 3.3, but certainly not $3/2$. Hence the lattice enhancement can only result in a fiber degeneration

$$A_1 \rightsquigarrow A_2$$

on the alternative fibration. Thus we find the enhanced Néron-Severi lattice

(34) $$\mathrm{NS}' = U \perp 2D_6 \perp A_1 \perp A_2$$

in agreement with the generic transcendental lattice of the enhanced subfamily,

$$T(X) = U(2) \perp A_1^2 \perp \langle 6 \rangle \quad \text{corresponding to} \quad \Delta = 6.$$

In order to determine the corresponding special curve in $\mathbb{P}^2$ explicitly, we assume without loss of generality that the I_2 fiber with E_{56} as non-identity component degenerates to Kodaira type I_3. That is, there are two other smooth rational curves D_1, D_2 as fiber components. Both give sections of the standard fibration, meeting exactly the following fiber components:

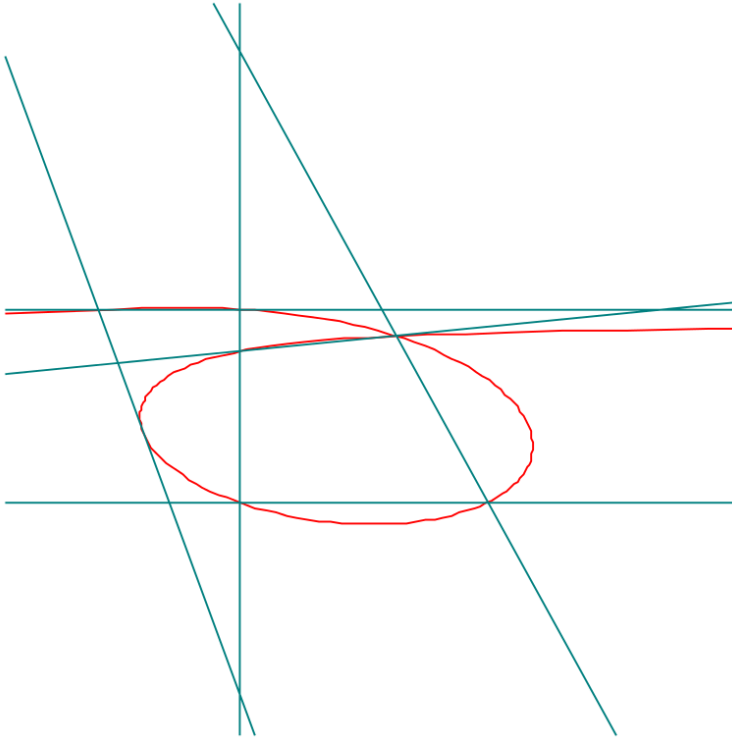| singular fiber	I_0^*	I_0^*	I_2	I_2	I_2	I_2	I_2	I_2	I_2
component met	E_{14}	E_{23}	E_{34}	E'_{35}	E_{36}	E'_{45}	E_{46}	E'_{56}	
	non-id	id	id	non-id	id	id	non-id	non-id	

Figure 4: The 6 lines and the singular cubic

One directly checks that the height pairing from 3.3 gives $h(D_1) = h(D_2) = 3/2$; in fact the sections are inverse to each other, since they meet the same fiber components and $\langle D_1, D_2 \rangle = -3/2$. The intersection numbers with all other rational curves from the line arrangement are zero except for E_{12} and ℓ_5 which are neither visible on the alternative fibration nor fiber components of the standard fibration. Here, since ℓ_5 defines a 2-torsion section of the standard fibration, the height pairing

$$0 = \langle D_1, \ell_5 \rangle = 2 - D_1.\ell_5 - \underbrace{1/2}_{I_0^* \text{ at } E_{14}, E_{15}} - \underbrace{1/2}_{I_2 \text{ at } E_{46}} \quad \text{gives} \quad D_1.\ell_5 = 1,$$

and likewise for D_2. As for E_{12}, arguing with an auxiliary standard fibration such as the one induced by ℓ_1/ℓ_4, we find

$$D_1.E_{12} = D_2.E_{12} = 2.$$

It follows that D_1 and D_2 correspond to a cubic curve $C \subset \mathbb{P}^2$ of the following shape:

- with a singularity at the node underlying E_{12},

- through the nodes underlying $E_{14}, E_{23}, E_{34}, E_{36}, E_{46}, E_{56}$,

- meeting ℓ_5 tangentially in a smooth point.

4. The Kuga-Satake Construction

4.1. Clifford Algebras

Let V be a finite dimensional k-vector space equipped with a non-degenerate bilinear form q. Its tensor algebra is $\mathsf{T}V = \oplus_{p \geq 0} V^{\otimes p}$, where the convention is that $V^{(0)} = k$. Recall that the **Clifford algebra** is the following quotient algebra of this algebra:

$$\mathrm{Cl}(V) = \mathrm{Cl}(V, q) := \mathsf{T}V / \text{ideal generated by } \{x \otimes x - q(x, x) \cdot 1 \mid x \in V\}.$$

Then we have $xy + yx = 2q(x, y)$ in the algebra $\mathrm{Cl}(V)$; in particular x and y anticommute whenever they are orthogonal.

The Clifford algebra has dimension 2^n where $n = \dim_k V$. Let us make this explicit for $k = \mathbb{Q}$. Then Q can be diagonalised in some basis, say $\{e_1, \ldots, e_n\}$. Consider $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{F}_2^n$. Taking all 2^n possibilities, we find a basis for $\mathrm{Cl}(V)$:

$$e^{\mathbf{a}} := e_1^{a_1} \cdots e_n^{a_n}.$$

The even Clifford algebra $\mathrm{Cl}^+(V)$ is generated by those $e^{\mathbf{a}}$ for which $\sum a_j$ is even.

To describe Clifford algebras certain quaternion algebras play a role. Let F be a field and $a, b \in F^\times$. The quaternion algebra $(a, b)_F$ over a field F has an F–basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ such that $\mathbf{i}^2 = a, \mathbf{j}^2 = b, \mathbf{ij} = -\mathbf{ji} = \mathbf{k}$. The Clifford algebra $\mathrm{Cl}(\langle a \rangle \perp \langle b \rangle)$, $a, b \in \mathbb{Q}^\times$ is isomorphic to $(a, b)_\mathbb{Q}$ while $\mathrm{Cl}^+(\langle a \rangle \perp \langle b \rangle) = \mathbb{Q}(\sqrt{-ab})$. One can see (cf. [16]) that for rank $3, 4$ and 5 the results are:

LEMMA 3. *Suppose* $Q = \mathrm{diag}(a_1, \ldots, a_m)$. *Put* $d = (-)^m a_1 \cdots a_m$. *Then*

1. *For* $m = 3$ *we have* $\mathrm{Cl}^+(Q) = (-a_1 a_2, -a_2 a_3)_\mathbb{Q}$;

2. *For* $m = 4$ *we have* $\mathrm{Cl}^+(Q) = (-a_1 a_2, -a_2 a_4)_\mathbb{Q} \otimes F$ *with* $F = \mathbb{Q}\sqrt{d}$;

3. *For* $m = 5$ *we have* $\mathrm{Cl}^+(Q) = (-a_1 a_2, -a_2 a_3)_\mathbb{Q} \otimes_\mathbb{Q} (a_1 a_2 a_3 a_4, -a_4 a_5)_\mathbb{Q}$.

4.2. From Certain Weight 2 Hodge Structures to Abelian Varieties

Next suppose that (V, q) carries a weight 2 Hodge structure polarized by q with $h^{2,0} = 1$. Then $V^{2,0} \oplus V^{0,2}$ is the complexification of a real plane $W \subset V$ carrying a Hodge substructure and q polarizes it. Then $b(x, y) := -q(x, y)$ is a metric on this plane since $C = -1$ is the Weil-operator [‡‡] of this Hodge structure. A choice of orientation for W then defines a unique almost complex structure which is the rotation over $\pi/2$

[‡‡] Recall that C is defined by $C|H^{p,q} = \mathrm{i}^{p-q}$.

in the positive direction. Equivalently, this almost complex structure is determined by any positively oriented orthonormal basis $\{f_1, f_2\}$ for W. Such a choice also defines an almost complex structure $J = f_1 f_2$ on $\mathrm{Cl}^+(V)$ since $f_1 f_2 f_1 f_2 = -f_1^2 f_2^2 = -1$. Then J defines a weight 1 Hodge structure: the eigenspaces of J for the eigenvalues $\pm i$ are the Hodge summands $H^{1,0}$, respectively $H^{0,1}$.

It turns out that the Hodge structure is polarized by a very natural skew form

$$E : \mathrm{Cl}^+(V) \times \mathrm{Cl}^+(V) \to \mathbb{Q}, \quad (x,y) \mapsto \mathrm{tr}(\varepsilon \iota(x) y)$$

built out of the canonical involution

$$\iota : \mathrm{Cl}^+(V) \to \mathrm{Cl}^+(V), \quad e_1^{a_1} \cdots e_n^{a_n} \mapsto e_1^{a_n} \cdots e_1^{a_1},$$

the trace map

$$\mathrm{tr} : \mathrm{Cl}^+(V) \to \mathbb{Q}, \quad c \mapsto \mathrm{tr}(R_c), R_c : x \mapsto cx, \text{ (left multiplication by } c)$$

and $\varepsilon \in \mathrm{Cl}^+(V)$ any element with $\iota \circ \varepsilon = -\varepsilon$, for instance $\varepsilon = e_1 e_2$. If we set

$$U := \text{ vector space dual to } \mathrm{Cl}^+(V),$$

any choice of a free $\mathbb{Z}$-module $U_{\mathbb{Z}}$ makes $U/U_{\mathbb{Z}}$ into a complex torus which is polarized by E. This Abelian variety by definition is the **Kuga Satake variety** and is denoted by $A(V,q)$. It is an Abelian variety of dimension 2^{n-2}, half of the real dimension of $\mathrm{Cl}^+(V)$.

Recall (e.g. [15]):

THEOREM 4. *One has* $\mathrm{Cl}^+(V) \subset \mathrm{End}_{\mathbb{Q}}(A(V,q))$. *If*

$$\mathrm{Cl}^+(V) = M_{n_1}(D_1) \times \cdots \times M_{n_d}(D_d), D_j \text{ division algebra, } j = 1, \dots, d,$$

then we have a decomposition into simple polarized Abelian varieties (here $\sim$ denotes isogeny):

$$(35) \qquad A(V,q) \sim A_1^{n_1} \times \cdots \times A_d^{n_d}, A_j \text{ with } D_j \subset \mathrm{End}_{\mathbb{Q}}(A_j), j = 1, \dots, d.$$

4.3. Abelian Varieties of Weil Type

In this note we are mainly interested in the following two examples which arise as certain Kuga-Satake varieties. The first class of examples is related to Abelian varieties in the full moduli space $\tilde{\mathbf{M}}$, and they are described by the following proposition.

PROPOSITION 8 ([7, Theorem 6.2]). *Let V be a rational vector space of dimension 6 and let $q = \langle 2 \rangle \oplus \langle 2 \rangle \oplus \langle -2 \rangle \oplus \langle -2 \rangle \oplus \langle -a \rangle \oplus \langle -b \rangle$, $a, b \in \mathbb{Q}^+$. We put $\alpha = i\sqrt{ab}$.*

Suppose that (V,q) is a polarized weight 2 Hodge structure with $h^{2,0} = 1$. Then the Kuga-Satake variety $A(V,q)$ is a 16-dimensional Abelian variety of $\mathbb{Q}(\alpha)$-Weil-type. For a generic such Hodge structure $\mathrm{End}_{\mathbb{Q}}(A(V,q)) = M_4(\mathbb{Q}(\alpha))$.

In the situation of Theorem 4 we have $d = 1$ and $n_1 = 4$, i.e. :

$$A(V,q) \sim B^4, \; B \text{ simple 4-dim. Abelian variety with } \mathbb{Q}(\alpha) \subset \text{End}_{\mathbb{Q}}(B).$$

On the other hand, the procedure of § 2.2 describes how one may associate to any Abelian variety A of dimension 4 of $\mathbb{Q}(i)$–Weil-type a polarized Hodge substructure $T(A)$ of $H^2(A)$ with $h^{2,0} = 1$. We have:

PROPOSITION 9 ([7, Theorem 6.5]). *For an Abelian variety A of dimension 4 which is of $\mathbb{Q}(i)$–Weil type we have*

$$A(T(A)) \sim A^4,$$

in other words, the Kuga-Satake procedure applied to $T(A)$ gives back the original Abelian variety A up to isogeny.

The second class relates to the Abelian varieties in the hypersurfaces D_Δ of the moduli space **M**. We recall equation (15) where we found the generic transcendental subspace $(T_\Delta)_{\mathbb{Q}} \subset (T(A_\Delta))_{\mathbb{Q}}$ of such an Abelian variety A_Δ.

We have:

THEOREM 5. *Let $T_\Delta = \langle 2\Delta \rangle \perp U \perp \langle -2 \rangle^2$ be a polarized Hodge structure of type $(1,3,1)$ and let $A(T_\Delta)$ be its associated Kuga-Satake variety. If A_Δ is any Abelian variety with moduli point in the hypersurface D_Δ of the moduli space **M** such that $T(A_\Delta) = T_\Delta$ as polarized $\mathbb{Q}$–Hodge structures, then we have an isogeny*

$$A(T_\Delta) \sim A_\Delta^2, \quad (-1,\Delta)_{\mathbb{Q}} \subset \text{End}_{\mathbb{Q}}(A_\Delta).$$

Proof. We consider the Clifford algebra associated to the Hodge structure T_Δ, since $U \cong_{\mathbb{Q}} \langle 2 \rangle \oplus \langle -2 \rangle$ the quadratic form is $Q = \text{diag}(2, -2, -2, -2, 2\Delta)$. For the corresponding Clifford algebra we deduce from Lemma 3 that

$$
\begin{aligned}
\text{Cl}^+(Q) & \cong (4, -4)_{\mathbb{Q}} \otimes (-16, 4\Delta)_{\mathbb{Q}} \\
& \cong (1, -1)_{\mathbb{Q}} \otimes (-1, \Delta)_{\mathbb{Q}} \cong M_2((-1, \Delta)_{\mathbb{Q}}).
\end{aligned}
$$

From Thm. 4 the Kuga-Satake variety can be decomposed as $A(T_\Delta) \sim B^2$ where B is an Abelian fourfold with $(-1,\Delta)_{\mathbb{Q}}$ contained in $\text{End}_{\mathbb{Q}}(B)$. Since T_Δ is a sub Hodge structure of $T(A_\Delta))$, the corresponding Kuga-Satake variety is a factor of $A(T(A_\Delta)) \sim A_\Delta^4$, i.e. $B \sim A_\Delta$ so that $A(T_\Delta) \sim A_\Delta^2$. $\qquad\square$

REMARK 3. The quaternion algebra $(-1,\Delta)_{\mathbb{Q}}$ has zero divisors (i.e. $(-1,\Delta)_{\mathbb{Q}} \simeq M_2(\mathbb{Q})$) precisely when Δ is a sum of two squares in $\mathbb{Z}$ which is the case if and only if all primes $p \equiv 3 \mod 4$ divide Δ with even power. In these cases the Abelian fourfold is isogeneous to a product B^2 with B an Abelian surface. It is an interesting open question if and how the Kummer surface of B and the K3 double plane are related. This occurs for instance if $\Delta = 1, 2, 4$ and for $\Delta = 1$ we have a candidate for B.

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