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ON THE EXISTENCE OF RAMIFIED ABELIAN COVERS

Dedicated to Alberto Conte on his 70th birthday.

Abstract. Given a normal complete variety Y , distinct irreducible effective Weil divisors D_1, \dots, D_n of Y and positive integers d_1, \dots, d_n , we spell out the conditions for the existence of an abelian cover $X \rightarrow Y$ branched with order d_i on D_i for $i = 1, \dots, n$.

As an application, we prove that a Galois cover of a normal complete toric variety branched on the torus-invariant divisors is itself a toric variety.

1. Introduction

Given a projective variety Y and effective divisors D_1, \dots, D_n of Y , deciding whether there exists a Galois cover branched on D_1, \dots, D_n with given multiplicities is a very complicated question, which in the complex case is essentially equivalent to describing the finite quotients of the fundamental group of $Y \setminus (D_1 \cup \dots \cup D_n)$.

In Section 2 of this paper we answer this question for a normal variety Y in the case that the Galois group of the cover is abelian (Theorem 1), using the theory developed in [3] and [1]. In particular, we prove that when the class group $\text{Cl}(Y)$ is torsion free, every abelian cover of Y branched on D_1, \dots, D_n with given multiplicities is the quotient of a maximal such cover, unique up to isomorphism.

In Section 3 we analyze the same question using toric geometry in the case when Y is a normal complete toric variety and D_1, \dots, D_n are invariant divisors and we obtain results that parallel those in Section 2 (Theorem 3). Combining the two approaches we are able to show that any cover of a normal complete toric variety branched on the invariant divisors is toric (Theorem 4).

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Notation. G always denotes a finite group, almost always abelian, and $G^* := \text{Hom}(G, \mathbb{K}^*)$ the group of characters; $o(g)$ is the order of the element $g \in G$ and $|H|$ is the cardinality of a subgroup $H < G$. We work over an algebraically closed field \mathbb{K} whose characteristic does not divide the order of the finite abelian groups we consider.

If A is an abelian group we write $A[d] := \{a \in A \mid da = 0\}$ (d an integer), $A^\vee := \text{Hom}(A, \mathbb{Z})$ and we denote by $\text{Tors}(A)$ the torsion subgroup of A .

The smooth part of a variety Y is denoted by Y_{sm} . The symbol \equiv denotes linear equivalence of divisors. If Y is a normal variety we denote by $\text{Cl}(Y)$ the group of classes, namely the group of Weil divisors up to linear equivalence.

2. Abelian covers

2.1. The fundamental relations

We quickly recall the theory of abelian covers (cf. [3], [1], and also [4]) in the most suitable form for the applications considered here.

There are slightly different definitions of abelian covers in the literature (see, for instance, [1] that treats also the non-normal case). Here we restrict our attention to the case of normal varieties, but we do not require that the covering map be flat; hence we define a cover as a finite morphism $\pi: X \rightarrow Y$ of normal varieties and we say that π is an abelian cover if it is a Galois morphism with abelian Galois group G (π is also called a “ G -cover”).

Recall that, as already stated in the Notations, throughout all the paper we assume that G has order not divisible by $\text{char } \mathbb{K}$.

To every component D of the branch locus of π we associate the pair (H, ψ) , where $H < G$ is the cyclic subgroup consisting of the elements of G that fix the preimage of D pointwise (the *inertia subgroup* of D) and ψ is the element of the character group H^* given by the natural representation of H on the normal space to the preimage of D at a general point (these definitions are well posed since G is abelian). It can be shown that ψ generates the group H^* .

If we fix a primitive d -th root ζ of 1, where d is the exponent of the group G , then a pair (H, ψ) as above is determined by the generator $g \in H$ such that $\psi(g) = \zeta^{\frac{d}{o(g)}}$. We follow this convention and attach to every component D_i of the branch locus of π a nonzero element $g_i \in G$.

If π is flat, which is always the case when Y is smooth, the sheaf $\pi_* O_X$ decomposes under the G -action as $\bigoplus_{\chi \in G^*} L_\chi^{-1}$, where the L_χ are line bundles ($L_1 = O_Y$) and G acts on L_χ^{-1} via the character χ .

Given $\chi \in G^*$ and $g \in G$, we denote by $\bar{\chi}(g)$ the smallest non-negative integer a such that $\chi(g) = \zeta^{\frac{ad}{o(g)}}$. The main result of [3] is that the L_χ, D_i (the *building data* of π) satisfy the following *fundamental relations*:

$$(1) \quad L_\chi + L_{\chi'} \equiv L_{\chi+\chi'} + \sum_{i=1}^n \varepsilon_{\chi,\chi'}^i D_i \quad \forall \chi, \chi' \in G^*$$

where $\varepsilon_{\chi,\chi'}^i = \lfloor \frac{\bar{\chi}(g_i) + \bar{\chi}'(g_i)}{o(g_i)} \rfloor$. (Notice that the coefficients $\varepsilon_{\chi,\chi'}^i$ are equal either to 0 or to 1). Conversely, distinct irreducible divisors D_i and line bundles L_χ satisfying (1) are the building data of a flat (normal) G -cover $X \rightarrow Y$; in addition, if $h^0(O_Y) = 1$ then $X \rightarrow Y$ is uniquely determined up to isomorphism of G -covers.

If we fix characters $\chi_1, \dots, \chi_r \in G^*$ such that G^* is the direct sum of the subgroups generated by the χ_j , and we set $L_j := L_{\chi_j}, m_j := o(\chi_j)$, then the solutions of the fundamental relations (1) are in one-one correspondence with the solutions of the

following *reduced fundamental relations*:

$$(2) \quad m_j L_j \equiv \sum_{i=1}^n \frac{m_j \overline{\chi_j}(g_i)}{d_i} D_i, \quad j = 1, \dots, r$$

As before, denote by d the exponent of G ; notice that if $\text{Pic}(Y)[d] = 0$, then for fixed (D_i, g_i) , $i = 1, \dots, n$, the solution of (2) is unique, hence the *branch data* (D_i, g_i) determine the cover.

In order to deal with the case when Y is normal but not smooth, we observe first that the cover $X \rightarrow Y$ can be recovered from its restriction $X' \rightarrow Y_{\text{sm}}$ to the smooth locus by taking the integral closure of Y in the extension $\mathbb{K}(X') \supset \mathbb{K}(Y)$. Observe then that, since the complement $Y \setminus Y_{\text{sm}}$ of the smooth part has codimension > 1 , we have $h^0(\mathcal{O}_{Y_{\text{sm}}}) = h^0(\mathcal{O}_Y) = 1$, and thus the cover $X' \rightarrow Y_{\text{sm}}$ is determined by the building data L_χ, D_i . Using the identification $\text{Pic}(Y_{\text{sm}}) = \text{Cl}(Y_{\text{sm}}) = \text{Cl}(Y)$, we can regard the L_χ as elements of $\text{Cl}(Y)$ and, taking the closure, the D_i as Weil divisors on Y , and we can interpret the fundamental relations as equalities in $\text{Cl}(Y)$. In this sense, if Y is normal variety with $h^0(\mathcal{O}_Y) = 1$, then the G -covers $X \rightarrow Y$ are determined by the building data up to isomorphism.

We say that an abelian cover $\pi: X \rightarrow Y$ is *totally ramified* if the inertia subgroups of the divisorial components of the branch locus of π generate G , or, equivalently, if π does not factorize through a cover $X' \rightarrow Y$ that is étale over Y_{sm} . We observe that a totally ramified cover is necessarily connected; conversely, equations (2) imply that if G is an abelian group of exponent d and Y is a variety such that $\text{Cl}(Y)[d] = 0$, then any connected G -cover of Y is totally ramified.

2.2. The maximal cover

Let Y be a complete normal variety, let D_1, \dots, D_n be distinct irreducible effective divisors of Y and let d_1, \dots, d_n be positive integers (it is convenient to allow the possibility that $d_i = 1$ for some i). We set $d := \text{lcm}(d_1, \dots, d_n)$.

We say that a Galois cover $\pi: X \rightarrow Y$ is *branched on D_1, \dots, D_n with orders d_1, \dots, d_n* if:

- the divisorial part of the branch locus of π is contained in $\sum_i D_i$;
- the ramification order of π over D_i is equal to d_i .

Let $\eta: \tilde{Y} \rightarrow Y$ be a resolution of the singularities and set $N(Y) := \text{Cl}(Y) / \eta_* \text{Pic}^0(\tilde{Y})$. Since the map $\eta_*: \text{Pic}(\tilde{Y}) = \text{Cl}(\tilde{Y}) \rightarrow \text{Cl}(Y)$ is surjective, $N(Y)$ is a quotient of the Néron-Severi group $\text{NS}(\tilde{Y})$, hence it is finitely generated. It follows that $\eta_* \text{Pic}^0(\tilde{Y})$ is the largest divisible subgroup of $\text{Cl}(Y)$ and therefore $N(Y)$ does not depend on the choice of the resolution of Y (this is easily checked also by a geometrical argument).

The group $\text{Cl}(Y)^\vee$ coincides with $\text{N}(Y)^\vee$, hence it is a finitely generated free abelian group of rank equal to the rank of $\text{N}(Y)$.

Consider the map $\mathbb{Z}^n \rightarrow \text{Cl}(Y)$ that maps the i -th canonical generator to the class of D_i , let $\phi: \text{Cl}(Y)^\vee \rightarrow \bigoplus_{i=1}^n \mathbb{Z}_{d_i}$ be the map obtained by composing the dual map $\text{Cl}(Y)^\vee \rightarrow (\mathbb{Z}^n)^\vee$ with $(\mathbb{Z}^n)^\vee = \mathbb{Z}^n \rightarrow \bigoplus_{i=1}^n \mathbb{Z}_{d_i}$ and let K_{\min} be the image of ϕ . Let G_{\max} be the abelian group defined by the exact sequence:

$$(3) \quad 0 \rightarrow K_{\min} \rightarrow \bigoplus_{i=1}^n \mathbb{Z}_{d_i} \rightarrow G_{\max} \rightarrow 0.$$

Then we have the following:

THEOREM 1. *Let Y be a normal variety with $h^0(\mathcal{O}_Y) = 1$, let D_1, \dots, D_n be distinct irreducible effective divisors, let d_1, \dots, d_n be positive integers and set $d := \text{lcm}(d_1, \dots, d_n)$. Then:*

1. *If $X \rightarrow Y$ is a totally ramified G -cover branched on D_1, \dots, D_n with orders d_1, \dots, d_n , then:*
 - (a) *the map $\bigoplus_{i=1}^n \mathbb{Z}_{d_i} \rightarrow G$ that maps $1 \in \mathbb{Z}_{d_i}$ to g_i descends to a surjection $G_{\max} \twoheadrightarrow G$;*
 - (b) *the map $\mathbb{Z}_{d_i} \rightarrow G_{\max}$ is injective for every $i = 1, \dots, n$.*
2. *If the map $\mathbb{Z}_{d_i} \rightarrow G_{\max}$ is injective for $i = 1, \dots, n$ and $\text{N}(Y)[d] = 0$, then there exists a maximal totally ramified abelian cover $X_{\max} \rightarrow Y$ branched on D_1, \dots, D_n with orders d_1, \dots, d_n ; the Galois group of $X_{\max} \rightarrow Y$ is equal to G_{\max} .*
3. *If the map $\mathbb{Z}_{d_i} \rightarrow G_{\max}$ is injective for $i = 1, \dots, n$ and $\text{Cl}(Y)[d] = 0$, then the cover $X_{\max} \rightarrow Y$ is unique up to isomorphism of G_{\max} -covers and every totally ramified abelian cover $X \rightarrow Y$ branched on D_1, \dots, D_n with orders d_1, \dots, d_n is a quotient of X_{\max} by a subgroup of G_{\max} .*

Proof. Let $H_1, \dots, H_t \in \text{N}(Y)$ be elements whose classes are free generators of the abelian group $\text{N}(Y)/\text{Tors}(\text{N}(Y))$, and write:

$$(4) \quad D_i = \sum_{j=1}^t a_{ij} H_j \pmod{\text{Tors}(\text{N}(Y))}, \quad j = 1, \dots, t$$

Hence, the subgroup K_{\min} of $\bigoplus_{i=1}^n \mathbb{Z}_{d_i}$ is generated by the elements $z_j := (a_{1j}, \dots, a_{nj})$, for $j = 1, \dots, t$.

Let $X \rightarrow Y$ be a totally ramified G -cover branched on D_1, \dots, D_n with orders d_1, \dots, d_n and let (D_i, g_i) be its branch data. Consider the map $\bigoplus_{i=1}^n \mathbb{Z}_{d_i} \rightarrow G$ that maps $1 \in \mathbb{Z}_{d_i}$ to g_i : this map is surjective, by the assumption that $X \rightarrow Y$ is totally ramified, and its restriction to \mathbb{Z}_{d_i} is injective for $i = 1, \dots, n$, since the cover is branched on D_i with order d_i . If we denote by K the kernel of $\bigoplus_{i=1}^n \mathbb{Z}_{d_i} \rightarrow G$, to prove (1) it suffices to show that $K \supseteq K_{\min}$. Dually, this is equivalent to showing that $G^* \subseteq K_{\min}^\perp \subseteq \bigoplus_{i=1}^n (\mathbb{Z}_{d_i})^*$. Let $\psi_i \in (\mathbb{Z}_{d_i})^*$ be the generator that maps $1 \in \mathbb{Z}_{d_i}$ to $\zeta_{d_i}^{\frac{d}{d_i}}$ and write $\chi \in G^*$

as $(\psi_1^{b_1}, \dots, \psi_n^{b_n})$, with $0 \leq b_i < d_i$; if $o(\chi) = m$ then (2) gives $mL_\chi \equiv \sum_{i=1}^n \frac{mb_i}{d_i} D_i$. Plugging (4) in this equation we obtain that $\sum_{i=1}^n \frac{b_i a_{ij}}{d_i}$ is an integer for $j = 1, \dots, t$, namely $\chi \in K_{\min}^\perp$.

(2) Let χ_1, \dots, χ_r be a basis of G_{\max}^* and, as above, for $s = 1, \dots, r$ write $\chi_s = (\psi_1^{b_{s1}}, \dots, \psi_n^{b_{sn}})$, with $0 \leq b_{si} < d_i$. Since by assumption $N(Y)[d] = 0$, by the proof of (1) the elements $\sum_{j=1}^r (\sum_{i=1}^n \frac{b_{si} a_{ij}}{d_i}) H_j$, $s = 1, \dots, r$, can be lifted to solutions $\bar{L}_s \in N(Y)$ of the reduced fundamental relations (2) for a G_{\max} -cover with branch data (D_i, g_i) , where $g_i \in G$ is the image of $1 \in \mathbb{Z}_{d_i}$. Since the kernel of $\text{Cl}(Y) \rightarrow N(Y)$ is a divisible group, it is possible to lift the \bar{L}_s to solutions $L_s \in \text{Cl}(Y)$. We let $X_{\max} \rightarrow Y$ be the G_{\max} -cover determined by these solutions. It is a totally ramified cover since the map $\oplus_{i=1}^n \mathbb{Z}_{d_i} \rightarrow G_{\max}$ is surjective by the definition of G_{\max} .

(3) Since $\text{Cl}(Y)[d] = 0$, any G -cover such that the exponent of G is a divisor of d is determined uniquely by the branch data; in particular, this holds for the cover $X_{\max} \rightarrow Y$ in (2) and for every intermediate cover $X_{\max}/H \rightarrow Y$, where $H < G_{\max}$. The claim now follows by (1). \square

EXAMPLE 1. Take $Y = \mathbb{P}^{n-1}$ and let D_1, \dots, D_n be the coordinate hyperplanes. In this case the group K_{\min} is generated by $(1, \dots, 1) \in \oplus_{i=1}^n \mathbb{Z}_{d_i}$. Since any connected abelian cover of \mathbb{P}^{n-1} is totally ramified, by Theorem 1 there exists a abelian cover of \mathbb{P}^{n-1} branched over D_1, \dots, D_n with orders d_1, \dots, d_n iff d_i divides $\text{lcm}(d_1, \dots, \hat{d}_i, \dots, d_n)$ for every $i = 1, \dots, n$. For $d_1 = \dots = d_n = d$, then $G_{\max} = \mathbb{Z}_d^n / \langle (1, \dots, 1) \rangle$ and $X_{\max} \rightarrow \mathbb{P}^{n-1}$ is the cover $\mathbb{P}^{n-1} \rightarrow \mathbb{P}^{n-1}$ defined by $[x_1, \dots, x_n] \mapsto [x_1^d, \dots, x_n^d]$.

In general, X_{\max} is a weighted projective space $\mathbb{P}(\frac{d}{d_1}, \dots, \frac{d}{d_n})$ and the cover is given by $[x_1, \dots, x_n] \mapsto [x_1^{\frac{d}{d_1}}, \dots, x_n^{\frac{d}{d_n}}]$.

3. Toric covers

NOTATIONS 2. Here, we fix the notations which are standard in toric geometry. A (complete normal) toric variety Y corresponds to a fan Σ living in the vector space $N \otimes \mathbb{R}$, where $N \cong \mathbb{Z}^s$. The dual lattice is $M = N^\vee$. The torus is $T = N \otimes \mathbb{C}^* = \text{Hom}(M, \mathbb{C}^*)$.

The integral vectors $r_i \in N$ will denote the integral generators of the rays $\sigma_i \in \Sigma(1)$ of the fan Σ . They are in a bijection with the T -invariant Weil divisors D_i ($i = 1, \dots, n$) on Y .

DEFINITION 1. A toric cover $f: X \rightarrow Y$ is a finite morphism of toric varieties corresponding to the map of fans $F: (N', \Sigma') \rightarrow (N, \Sigma)$ such that:

1. $N' \subseteq N$ is a sublattice of finite index, so that $N' \otimes \mathbb{R} = N \otimes \mathbb{R}$.
2. $\Sigma' = \Sigma$.

The proof of the following lemma is immediate.

LEMMA 1. *The morphism f has the following properties:*

1. *It is equivariant with respect to the homomorphism of tori $T' \rightarrow T$.*
2. *It is an abelian cover with Galois group $G = \ker(T' \rightarrow T) = N/N'$.*
3. *It is ramified only along the boundary divisors D_i , with multiplicities $d_i \geq 1$ defined by the condition that the integral generator of $N' \cap \mathbb{R}_{\geq 0} r_i$ is $d_i r_i$.*

PROPOSITION 1. *Let Y be a complete toric variety such that $\text{Cl}(Y)$ is torsion free, and $X \rightarrow Y$ be a toric cover. Then, with notations as above, there exists the following commutative diagram with exact rows and columns.*

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & & & \text{Cl}(Y)^\vee & \longrightarrow & K & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & \bigoplus_{i=1}^n \mathbb{Z} d_i D_i^* & \longrightarrow & \bigoplus_{i=1}^n \mathbb{Z} D_i^* & \longrightarrow & \bigoplus_{i=1}^n \mathbb{Z} d_i \longrightarrow 0 \\
 & & \downarrow p' & & \downarrow p & & \downarrow \\
 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & G \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

(Here the D_i^* are formal symbols denoting a basis of \mathbb{Z}^n). Moreover, each of the homomorphisms $\mathbb{Z} d_i \rightarrow G$ is an embedding.

Proof. The third row appeared in Lemma 1, and the second row is the obvious one.

It is well known that the boundary divisors on a complete normal toric variety span the group $\text{Cl}(Y)$, and that there exists the following short exact sequence of lattices:

$$0 \rightarrow M \rightarrow \bigoplus_{i=1}^n \mathbb{Z} D_i \rightarrow \text{Cl}(Y) \rightarrow 0.$$

Since $\text{Cl}(Y)$ is torsion free by assumption, this sequence is split and dualizing it one obtains the central column. Since $\bigoplus_{i=1}^n \mathbb{Z} D_i^* \rightarrow N$ is surjective, then so is $\bigoplus_{i=1}^n \mathbb{Z} d_i \rightarrow G$. The group K is defined as the kernel of this map.

Finally, the condition that $\mathbb{Z} d_i \rightarrow G$ is injective is equivalent to the condition that the integral generator of $N' \cap \mathbb{R}_{\geq 0} r_i$ is $d_i r_i$, which holds by Lemma 1. \square

THEOREM 3. *Let Y be a complete toric variety such that $\text{Cl}(Y)$ is torsion free, let d_1, \dots, d_n be positive integers and let K_{\min} and G_{\max} be defined as in sequence (3). Then:*

1. There exists a toric cover branched on D_i of order d_i , $i = 1, \dots, n$, iff the map $\mathbb{Z}_{d_i} \rightarrow G_{\max}$ is injective for $i = 1, \dots, n$.
2. If condition (1) is satisfied, then among all the toric covers of Y ramified over the divisors D_i with multiplicities d_i there exists a maximal one $X_{\Gamma_{\max}} \rightarrow Y$, with Galois group G_{\max} , such that any other toric cover $X \rightarrow Y$ with the same branching orders is a quotient $X = X_{\Gamma_{\max}}/H$ by a subgroup $H < G_{\max}$.

Proof. Let $X \rightarrow Y$ be a toric cover branched on D_1, \dots, D_n with orders d_1, \dots, d_n , let N' be the corresponding sublattice of N and $G = N/N'$ the Galois group. Let N'_{\min} be the subgroup of N generated by $d_i r_i$, $i = 1, \dots, n$. By Lemma 1 one must have $N'_{\min} \subseteq N'$, hence the map $\mathbb{Z}_{d_i} \rightarrow N/N'_{\min}$ is injective since $\mathbb{Z}_{d_i} \rightarrow G = N/N'$ is injective by Proposition 1. We set $X_{\Gamma_{\max}} \rightarrow Y$ to be the cover for N'_{\min} . Clearly, the cover for the lattice N' is a quotient of the cover for the lattice N'_{\min} by the group $H = N'/N'_{\min}$.

Consider the second and third rows of the diagram of Proposition 1 as a short exact sequence of 2-step complexes $0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0$. The associated long exact sequence of cohomologies gives

$$\text{Cl}(Y)^\vee \longrightarrow K \longrightarrow \text{coker}(p') \longrightarrow 0$$

For $N' = N'_{\min}$, the map p' is surjective, hence $\text{Cl}(Y)^\vee \rightarrow K$ is surjective too, and $K = K_{\min}$, $N/N'_{\min} = G_{\max}$.

Vice versa, suppose that in the following commutative diagram with exact row and columns each of the maps $\mathbb{Z}_{d_i} \rightarrow G_{\max}$ is injective.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \text{Cl}(Y)^\vee & \xrightarrow{q} & K_{\min} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \bigoplus_{i=1}^n \mathbb{Z}d_i D_i^* & \longrightarrow & \bigoplus_{i=1}^n \mathbb{Z}D_i^* & \longrightarrow & \bigoplus_{i=1}^n \mathbb{Z}d_i \longrightarrow 0 \\
 & & \downarrow p & & \downarrow & & \\
 & & N & & G_{\max} & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

We complete the first row on the left by adding $\ker(q)$. We have an induced homomorphism $\ker(q) \rightarrow \bigoplus \mathbb{Z}d_i D_i^*$, and we define N' to be its cokernel.

Now consider the completed first and second rows as a short exact sequence of 2-step complexes $0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0$. The associated long exact sequence of cohomologies says that $\ker(q) \rightarrow \bigoplus_{i=1}^n \mathbb{Z}d_i D_i^*$ is injective, and the sequence

$$0 \longrightarrow N' \longrightarrow N \longrightarrow G_{\max} \longrightarrow 0$$

is exact. It follows that $N' = N'_{\min}$ and the toric morphism $(N'_{\min}, \Sigma) \rightarrow (N, \Sigma)$ is then the searched-for maximal abelian toric cover. \square

REMARK 1. Condition (1) in the statement of Theorem 3 can also be expressed by saying that for $i = 1, \dots, n$ the element $d_i r_i \in N'_{\min}$ is primitive, where $N'_{\min} \subseteq N$ is the subgroup generated by all the $d_i r_i$.

We now combine the results of this section with those of §2 to obtain a structure result for Galois covers of toric varieties.

THEOREM 4. *Let Y be a normal complete toric variety and let $f: X \rightarrow Y$ be a connected cover such that the divisorial part of the branch locus of f is contained in the union of the invariant divisors D_1, \dots, D_n .*

Then $\deg f$ is not divisible by $\text{char } \mathbb{K}$ and $f: X \rightarrow Y$ is a toric cover.

Proof. Let $U \subset Y$ be the open orbit and let $X' \rightarrow U$ be the cover obtained by restricting f . Since U is smooth, by the assumptions and by purity of the branch locus, $X' \rightarrow U$ is étale. Let $X'' \rightarrow U$ be the Galois closure of $X' \rightarrow U$: the cover $X'' \rightarrow U$ is also étale, hence by [2, Prop. 1] it is, up to isomorphism, a homomorphism of tori. Since the kernel of an étale homomorphism of tori is reduced, it follows that $X'' \rightarrow U$ is an abelian cover such that $\text{char } \mathbb{K}$ does not divide the order of the Galois group.

Moreover, the intermediate cover $X' \rightarrow U$ is also abelian (actually $X' = X''$). The cover $f: X \rightarrow Y$ is abelian, too, since X is the integral closure of Y in $\mathbb{K}(X')$. We denote by G the Galois group of f and by d_1, \dots, d_n the orders of ramification of $X \rightarrow Y$ on D_1, \dots, D_n .

Assume first that $\text{Cl}(Y)$ has no torsion, so that every connected abelian cover of Y is totally ramified (cf. §2). Then by Theorem 1 every connected abelian cover branched on D_1, \dots, D_n with orders d_1, \dots, d_n is a quotient of the maximal abelian cover $X_{\max} \rightarrow Y$ by a subgroup $H < G_{\max}$. In particular, this is true for the cover $X_{T_{\max}} \rightarrow Y$ of Theorem 3. Since X_{\max} and $X_{T_{\max}}$ have the same Galois group it follows that $X_{\max} = X_{T_{\max}}$. Hence $X \rightarrow Y$, being a quotient of $X_{T_{\max}}$, is a toric cover.

Consider now the general case. Recall that the group $\text{TorsCl}(Y)$ is finite, isomorphic to $N/\langle r_i \rangle$, and the cover $Y' \rightarrow Y$ corresponding to $\text{TorsCl}(Y)$ is toric, and one has $\text{TorsCl}(Y') = 0$. Indeed, on a toric variety the group $\text{Cl}(Y)$ is generated by the T -invariant Weil divisors D_i . Thus, $\text{Cl}(Y)$ is the quotient of the free abelian group $\oplus \mathbb{Z}D_i$ of all T -invariant divisors modulo the subgroup M of principal T -invariant divisors. Thus, $\text{TorsCl}(Y) \simeq M'/M$, where $M' \subset \oplus \mathbb{Q}D_i$ is the subgroup of \mathbb{Q} -linear functions on N taking integral values on the vectors r_i . Then $N' := M'^{\vee}$ is the subgroup of N generated by the r_i , and the cover $Y' \rightarrow Y$ is the cover corresponding to the map of fans $(N', \Sigma) \rightarrow (N, \Sigma)$. On Y' one has $N' = \langle r_i \rangle$, so $\text{TorsCl}(Y') = 0$.

Let $X' \rightarrow Y'$ be a connected component of the pull back of $X \rightarrow Y$: it is an abelian cover branched on the invariant divisors of Y' , hence by the first part of the proof it is toric. The map $X' \rightarrow Y$ is toric, since it is a composition of toric morphisms, hence the intermediate cover $X \rightarrow Y$ is also toric. \square

REMARK 2. The argument that shows that the map f is an abelian cover in the proof of Theorem 4 was suggested to us by Angelo Vistoli. He also remarked that it is possible to prove Theorem 3.6 in a more conceptual way by showing that the torus action on the cover $X' \rightarrow U$ of the open orbit of Y extends to X , in view of the properties of the integral closure. However our approach has the advantage of describing explicitly the fan/building data associated with the cover.

References

- [1] V. ALEXEEV, R. PARDINI, *Non-normal abelian covers*, *Compositio Math.* **148** (2012), no. 04, 1051–1084.
- [2] M. MIYANISHI, *On the algebraic fundamental group of an algebraic group*, *J. Math. Kyoto Univ.* **12** (1972), 351–367.
- [3] R. PARDINI, *Abelian covers of algebraic varieties*, *J. Reine Angew. Math.* **417** (1991), 191–213.
- [4] R. PARDINI, F. TOVENA, *On the fundamental group of an abelian cover*, *International J. Math.* **6** no. 5 (1995), 767–789.

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