This special volume is dedicated to Alberto Conte on the occasion
of his 70th birthday for his important contributions to Algebraic Geometry.

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IBAN: IT90 Y030 6909 2171 0000 0460 182
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RENDICONTI
DEL SEMINARIO
MATEMATICO

Università e Politecnico di Torino

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Autorizzazione del Tribunale di Torino N. 2962 del 6.VI.1980

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QUESTO FASCICOLO È STAMPATO CON IL CONTRIBUTO DI:
UNIVERSITÀ DEGLI STUDI DI TORINO
POLITECNICO DI TORINO

1. Preface

This special issue of Rendiconti del Seminario Matematico dell'Università e del Politecnico di Torino is dedicated to Alberto Conte, on the occasion of his 70th birthday, for his important contributions to Algebraic Geometry. It contains research papers based on lectures delivered at the Conference "Geometria delle varietà algebriche", which took place from 21 to 23 March 2012 at University of Turin. During the Conference classical and modern themes of Algebraic Geometry, which is the favorite field of interest of Alberto Conte, were considered.

Many friends, former students and researchers, from Italy and abroad, who shared with him efforts and enthusiasm during so many years of work for Science, Culture and University attended the Conference. In particular many participants were members of the present PRIN national project 'Geometry of Algebraic Varieties' or of the past European projects AGE and EAGER. These projects have seen the participation of Alberto Conte in many relevant roles and his very strong engagement in favor of European Geometry.

List of the Speakers

Marco Andreatta (Università di Trento)

Arnaud Beauville (Université de Nice)

Ciro Ciliberto (Università di Tor Vergata)

Eduardo Esteves (IMPA Rio de Janeiro)

Gavril Farkas (Humboldt-Universitaet zu Berlin)

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Christian Peters (Università de Grenoble)

Gaetana Restuccia (Università di Messina)

Lorenzo Robbiano (Università di Genova)

Simon Salamon (King's College Londra)

Bert van Geemen (Università di Milano)

Alessandro Verra (Università Roma Tre)

2. Aknowledgements

We want to thank the Department of Mathematics, the Faculty of Science, the University of Torino and all the other Institutions whose support made possible this Conference. Furthermore we want to specially thank all the authors of this volume, and all the referees of its papers, for their precious contributions and work.

Finally, let us thank all the other members of the Organizing Committee for their strong engagement in making this Conference successful: Alberto Albano, Elisabetta Ambrogio, Cristina Bertone, Ernesto Buzano, Giuseppe Ceresa, Federica Galluzzi, Claudio Pedrini, Daniela Romagnoli.

Many of them have also been students of Alberto Conte. We wish also to remember here two other persons which were students of him and no longer are with us: Luciana Picco Botta and Giuseppe Vigna Suria. As students of Alberto, representatives of all of them from the beginning times to the last generation of students, we want to tell him our deep gratitude for spending so much of his talents and of his years in favor of our scientific growth, as an excellent advisor and professor.

We learned a lot from him in Mathematics and in Algebraic Geometry and we learned as well, as it must be for a true master, from his great views, his broad intelligence and his unexhausted resources of optimism.

Marina Marchisio and Alessandro Verra

Rend. Sem. Mat. Univ. Politec. Torino
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R. Achilles, M. Manaresi

COMPUTING THE NUMBER OF APPARENT DOUBLE POINTS OF A SURFACE

Dedicated to Alberto Conte on the occasion of his 70th birthday

Abstract. For a smooth surface $S \subset \mathbb{P}^5_K$ there are well known classical formulas giving the number $\rho(S)$ of secants of S passing through a generic point of $\mathbb{P}^5$. In this paper, for possibly singular surfaces T, a computer assisted computation of $\rho(T)$ from the defining ideal $I(T) \subset K[x_0,\dots,x_5]$ is proposed. It is based on the Stückrad-Vogel self-intersection cycle of T and requires the computation of the normal cone of the ruled join $J(T,T)$ along the diagonal. It is shown that in the case when $T \subset \mathbb{P}^5$ arises as the linear projection with center L of a surface $S \subset \mathbb{P}^N_K$ $(N > 5)$ (which satisfies some mild assumptions), the computational complexity can be reduced considerably by using the normal cone of $\operatorname{Sec} S$ along $L \cap \operatorname{Sec} S$ instead of the former normal cone. Many examples and the relative code for the computer algebra systems REDUCE, CoCoA, Macaulay2 and Singular are given.

1. Introduction

For a smooth surface $S \subset \mathbb{P}^5_K$, where K is an algebraically closed field of characteristic zero, there are well known classical formulas giving the number $\rho(S)$ of secants of S passing through a generic point of $\mathbb{P}^5$ (see, for example, the double point formula of Severi [16], p. 259 or [22], Example 4.1.3, or the secant formula of Peters-Simonis [26]). This number is called the *secant order* or the *number of apparent double points* of S, since it is the number of double points of the projection of S into $\mathbb{P}^4$ from a generic point of $\mathbb{P}^5$. These formulas are not suited if one wants to compute the number ρ starting from the equations of the surface. In this paper we provide a computational approach based on the coefficients of a certain Hilbert polynomial which comes from the Stückrad-Vogel intersection cycle and can be computed from the equations of the surface.

In [5] (see Theorem 4.3) it was shown that for a singular non-defective surface $T \subset \mathbb{P}^5_K$ the Stückrad-Vogel self-intersection cycle of T can be used to obtain a formula for $\rho(T)$. From a computational point of view, this result permits to compute $\rho(T)$ using computer algebra systems, but it requires the computation of the normal cone of the ruled join $J(T,T)$ along the diagonal, the computational complexity of which can be very high.

In this paper we propose a second method for the computation of $\rho(T)$ when the singular surface $T \subset \mathbb{P}^5$ arises as the projection of a surface $S \subset \mathbb{P}^N_K$ $(N > 5)$ along a linear subspace L. Under some mild assumption on S, this method reduces the computational complexity of the previous one, since it relies on the computation of the normal cone of $\operatorname{Sec} S$ along $L \cap \operatorname{Sec} S$ (see Theorem 3).

For some special surfaces the computational complexity can be further reduced by applying an observation of A. Verra. He observed that for certain possibly singular surfaces S (which C. Ciliberto and F. Russo called *Verra surfaces*, see Definition 2) the number $\rho(S)$ can be obtained computing the number of apparent double points of a space curve by using its self-intersection cycle (see [4], Proposition 3.7). In the case of such surfaces the computation of ρ can be done in $\mathbb{P}^3$, that is in a ring with only four variables.

We illustrate the methods through a collection of examples, where the computations have been done with REDUCE (see [1]), but they could have been done with other systems as CoCoA, Macaulay2, Singular as well.

In the last section of the paper we give the code of the procedures for the calculations in the different computer algebra systems and we show the efficiency of the new method given by Theorem 3.

2. Computational aspects of the Stückrad-Vogel intersection cycle

Let X,Y be closed (irreducible and reduced) subvarieties of the projective space $\mathbb{P}^N = \mathbb{P}^N_K$, where K is an algebraically closed field of characteristic zero. Stückrad and Vogel [30] (see also [16], Section 2.2) introduced a cycle $v(X,Y)$ called v-cycle, which is the formal sum of (algorithmically produced) subvarieties C of $X\cap Y$ (possibly defined over a pure transcendental field extension of K), each taken with an algorithmically produced positive integer coefficient $j_C = j(X,Y;C)$, the *intersection multiplicity* of X and Y along C. In order to describe the dimension range of the varieties C, denote by XY or $\mathrm{embJ}(X,Y)$ the *embedded join* of X and Y, that is, the closure of the union of all lines $\overline{xy}$, $x\in X$, $y\in Y$. Then the v-cycle can be written as

$$v(X,Y) = \sum_C j(X,Y;C)[C] = \sum_k v_k(X,Y),$$

where k runs from $\max(\dim X + \dim Y - \dim XY, 0)$ to $\dim X\cap Y$, and $v_k(X,Y)(\neq 0)$ is the k-dimensional part of $v(X,Y)$.

The part of the cycle $v_k = v_k(X,Y)$ which is defined over the base field K, the so-called *K-rational part*, will be denoted by $\mathrm{rat}(v_k)$ and the remaining part, the so-called *irrational* or *movable part*, will be denoted by $\mathrm{mov}(v_k)$, that is,

$$v_k = \mathrm{rat}(v_k) + \mathrm{mov}(v_k).$$

To prove a theorem of Bezout, Stückrad and Vogel had to take into account also the so-called empty subvariety $\emptyset \subseteq X\cap Y$ (which by definition has dimension -1 and degree 1) and its intersection number $j(X,Y,\emptyset)$.

We want to describe $j(X,Y,\emptyset)$ following van Gastel [19]. To this end, let $A_x := K[x_0,\ldots,x_N]$, $A_y := K[y_0,\ldots,y_N]$, $A := A_x \otimes_K A_y$ and denote by $I(X) \subseteq A_x$, $I(Y) \subseteq A_y$ the largest homogeneous ideals defining X and Y, respectively. Then $R := A/(I(X)A + I(Y)A)$ is the homogeneous coordinate ring of the *ruled join* $J = J(X,Y) \subseteq \mathbb{P}^{2N+1}_K = \mathrm{Proj}(A)$. The "diagonal" subspace Δ of $\mathbb{P}^{2N+1}_K$ is defined by the ideal $(x_0 - y_0,\ldots,x_N - y_N)A =: I(\Delta)$. Denoting by $\deg(J(X,Y)/XY)$ the mapping degree of the linear projection $J(X,Y) \dashrightarrow XY$ with center $\Delta\cap J(X,Y)$, van Gastel proved that

$$j(X,Y;\emptyset) = \deg(J/XY)\deg XY.$$

Then, using the above notation, the *refined Bezout theorem* (see [16], Theorem 2.2.5) can be formulated as follows.

THEOREM 1 (Stückrad-Vogel [30], van Gastel [19]). *Let X,Y be closed (irreducible and reduced) subvarieties of $\mathbb{P}^N_K$. Then*

$$\begin{aligned}\deg X \deg Y &= \deg(J/XY)\deg XY + \sum_C j(X,Y;C)\deg C\\ &= \deg(J/XY)\deg XY + \deg v(X,Y)\\ &= \deg(J/XY)\deg XY + \sum_k \deg v_k(X,Y).\end{aligned}$$

The degrees of the v_k's and $\deg(J/XY)\deg XY$ can be calculated by the Hilbert coefficients of the bigraded ideal which defines the normal cone of $J(X,Y)$ along $J(X,Y)\cap\Delta$. More precisely, let

$$\begin{aligned}&A = K[x_0,\dots,x_N,y_0,\dots,y_N], \quad R = A/I(J(X,Y)),\\ &I = I(\Delta) = (x_0-y_0,\dots,x_N-y_N)A,\end{aligned}$$

$t_0,\dots,t_N$ indeterminates, and

$$\varphi\colon A[t_0,\dots,t_N] \to G_I(R) := \bigoplus_{k\in\mathbb{N}} I^k/I^{k+1}$$

be the natural surjection which is induced by the natural homomorphism $A\to R/I$ and substituting t_i by the class of x_i-y_i in I/I^2, $i=0,\dots,N$. Define a bigrading by setting $\deg(x_i)=\deg(y_i)=(1,0)$ and $\deg(t_i)=(0,1)$. Then $\ker\varphi$ is a bigraded ideal in the bigraded ring $A[t_0,\dots,t_N]$, and

$$\text{(1)} \qquad \mathcal{R} := \mathcal{R}(X,Y) := G_I(R) \cong A[t_0,\dots,t_N]/\ker\varphi$$

is bigraded, $\mathcal{R} = \bigoplus_{j,k\in\mathbb{N}}\mathcal{R}_{j,k}$. If we set $d :=$ Krull-dim$\,R =$ Krull-dim$\,\mathcal{R}$ and write the Hilbert polynomial of $\sum_{v=0}^{k}\sum_{u=0}^{j}\dim_K(\mathcal{R}_{u,v})$ in the form

$$p(j,k) = \sum_{l=0}^{d} c_l \binom{j+l}{l}\binom{k+d-l}{d-l} + \text{ lower degree terms,}$$

then the non-negative integers $c_l =: c_l(\mathcal{R}) =: c_l(X,Y)$ are the *generalized Samuel multiplicities* of I in the sense of [3].

PROPOSITION 1 ([3], Theorem 4.1 and Proposition 1.2). *With the previous notation,*

$$c_0(\mathcal{R}) = c_0(X,Y) = j(X,Y;\emptyset) = \deg(J/XY)\deg XY$$

and for $k=1,\dots,d$,

$$c_k(\mathcal{R}) = c_k(X,Y) = \sum_{\mathcal{P}\in\mathrm{minAss}\,\mathcal{R}} \mathrm{length}(\mathcal{R}_{\mathcal{P}})\cdot c_k(\mathcal{R}/\mathcal{P}) = \deg v_k(X,Y).$$

According to (1) the minimal prime ideals $\mathcal{P}$ of $\mathcal{R}$ contract to prime ideals $\mathcal{P} \cap A$ which contain the ideals $I(\Delta)$ and $I(J(X,Y))$, hence the contraction ideals $\mathcal{P} \cap K[x_0, \dots x_N]$ (which need not all be distinct) define subvarieties of $X \cap Y \subset \mathbb{P}^N$, the so called *distinguished varieties* of the intersection of X and Y in the sense of Fulton [17]. These subvarieties are the support of the K-rational part of $v(X,Y)$. The lengths of the $\mathcal{R}_{\mathcal{P}}$'s are the geometric multiplicities of the irreducible components of the normal cone of $J(X,Y)$ along $J(X,Y) \cap \Delta$.

DEFINITION 1 (Intersection vector). *With the above notation, set* $\delta := \dim(X \cap Y) + 1$. *Then the* intersection vector $\mathbf{c}(X,Y)$ *of* X *and* Y *is defined to be the vector of non-negative integers*

$$\mathbf{c}(X,Y) = (c_0(X,Y), \dots, c_\delta(X,Y)) = (c_0(\mathcal{R}), \dots, c_\delta(\mathcal{R})) =: \mathbf{c}(\mathcal{R}),$$

and by the refined Bezout theorem

$$\deg X \deg Y = c_0(X,Y) + \cdots + c_\delta(X,Y).$$

By Proposition 1 we have

$$\mathbf{c}(X,Y) = \sum_{\mathcal{P} \in \mathrm{minAss}\, \mathcal{R}} \mathrm{length}(\mathcal{R}_{\mathcal{P}}) \cdot \mathbf{c}(\mathcal{R}/\mathcal{P}). \tag{2}$$

In particular, the self-intersection vector *of an* n*-dimensional variety* X *is defined to be*

$$\mathbf{c}(X) = \mathbf{c}(X,X) = (c_0(X,X)), \dots, c_{n+1}(X,X)),$$

and it holds

$$(\deg X)^2 = c_0(X,X) + \cdots + c_\delta(X,X).$$

The integers $c_k(X,Y) = c_k(\mathcal{R})$ can be computed by using various computer algebra systems (e.g. REDUCE, using the package SEGRE [1]), in which the calculation of the Hilbert series of a multigraded ring has been implemented, see Section 5.

For the computation of the number of apparent double points of a variety X, the coefficient

$$c_0(X,X) = j(X,X;\emptyset) = \deg(J(X,X)/\mathrm{embJ}(X,X)) \cdot \deg(\mathrm{embJ}(X,X))$$

is particularly important. Note that $\mathrm{embJ}(X,X)$ is the *secant variety*, which we denote by $\mathrm{Sec}\, X$. It is well-known that $\mathrm{Sec}\, X$ has the *expected dimension* $2 \dim X + 1$ (and is said to be *nondeficient*) if and only if for generic points $x \in X$ and $y \in X$ one has $T_{X,x} \cap T_{X,y} = \emptyset$ (see, for example, [16], Cor. 4.3.3). The non-negative integer $2 \dim X + 1 - \dim \mathrm{Sec}\, X$ is called the *deficiency* of $\mathrm{Sec}\, X$. Concerning $\deg(J(X,X)/\mathrm{Sec}\, X)$, there is the following result, which is essentially [16], Proposition 8.2.12, see also [24], Proposition 6.3.5.

PROPOSITION 2 ([5], Proposition 2.7). *Let* $X \subset \mathbb{P}^N$ *be a non-degenerate irreducible subvariety such that* $2 \dim X + 1 < N$. *Suppose that one of the following two conditions is satisfied:*

1. *X is a curve;*
2. *X is reduced and the generic tangent hyperplane to* $\operatorname{Sec} X$ *is tangent to X at only finitely many points (that is, X is not* 1*-weakly defective in the sense of [9], 2.1).*

Then $\deg(J(X,X)/\operatorname{Sec} X) = 2$ *and, in particular,* $\dim \operatorname{Sec} X = 2\dim X + 1$.

REMARK 1. If $x, y \in X$, then over the secant line xy of X there are two lines $J(x,y)$, $J(y,x)$ of $J(X,X)$, so that the rational map

$$\pi\colon J(X,X) \dashrightarrow \operatorname{Sec} X,$$

has even degree, that is,

$$\deg(J(X,X)/\operatorname{Sec} X) = 2\rho \geq 0.$$

Here ρ is the *number of secants to X passing through a general point of* $\operatorname{Sec} X$, if $\dim \operatorname{Sec} X = 2\dim X + 1$, and $\rho = 0$ otherwise. In [9] one can find a complete classification of surfaces $X \subset \mathbb{P}^N$, $N \geq 6$, for which $\rho > 1$.

3. Computing ρ by the Stückrad-Vogel intersection cycle

Let $S \subset \mathbb{P}^N$ ($N \geq 5$) be a non-degenerate surface of degree d with singular locus $\operatorname{Sing} S$. For any point $P \in \operatorname{Sm}(S)$ we denote by $T_{S,P}$ the embedded projective tangent plane to S at P. We denote by $\operatorname{Tan} S$ the tangent variety of S, that is the closure of the union of all embedded projective tangent planes to S at regular points.

It is known (see, for example, [5]) that the Stückrad-Vogel self-intersection cycle of S is

$$v(S,S) = [S] + \sum_Z j_Z[Z] + P_1(S) + \sum_P j_P[P] + \operatorname{mov} v_0(S)|_{\operatorname{Sm} S} + \operatorname{mov} v_0(S)|_{\operatorname{Sing} S}, \tag{3}$$

where Z runs through the one-dimensional irreducible components of $\operatorname{Sing} S$, $P_1(S)$ denotes the first polar locus of S, P runs through the singular points of S of embedding dimension greater or equal to 4, and $\operatorname{mov} v_0(S)|_{\operatorname{Sm} S}$ is the ramification locus of the linear projection $\pi_\Lambda\colon S \to \mathbb{P}^3$ with center a generic $(N-4)$-dimensional linear subspace $\Lambda \subset \mathbb{P}^N$ (see [15], Theorem 4.6).

Moreover,

$$\deg v_1(S) = \sum_Z j_Z \deg Z + \deg P_1(S) = c_2(S,S),$$

$$\deg v_0(S) = \sum_P j_P + \deg \operatorname{mov} v_0(S)|_{\operatorname{Sm} S} + \deg \operatorname{mov} v_0(S)|_{\operatorname{Sing} S} = c_1(S,S),$$

and, if S is non defective,

$$\deg \operatorname{mov} v_0(S)|_{\operatorname{Sm} S} = \deg(\operatorname{Tan} S)$$

(see [5], Lemma 4.1).

In [5], Theorem 4.3, the following has been proved.

THEOREM 2. *Let $S \subset \mathbb{P}^N$ ($N \geq 5$) be an irreducible and reduced possibly singular non defective surface of degree d. With the preceding notation the following formula holds:*

$$\begin{aligned} 2\rho \cdot \deg \operatorname{Sec} S &= (\deg S)^2 - \deg v(S,S) \\ &= (\deg S)^2 - c_3(S,S) - c_2(S,S) - c_1(S,S) \\ &= d^2 - d - \sum_Z j_Z \deg Z - \deg P_1(S) \\ &\quad - \sum_P j_P - \deg \operatorname{Tan} S - \deg \operatorname{mov} v_0(S)|_{\operatorname{Sing} S} \\ &= c_0(S,S), \end{aligned}$$

where Z runs through the one-dimensional irreducible components of $\operatorname{Sing} S$ *and P runs through the singular points of S of embedding dimension greater or equal to* 4*.*

For $N = 5$ one has

$$\rho(S) = \frac{1}{2} c_0(S,S).$$

REMARK 2. The self-intersection vector $v(S,S)$ of a surface $S \subset \mathbb{P}^N, N \geq 5$, encodes geometric information on S. If $S \subset \mathbb{P}^N$ ($N \geq 5$) is smooth and non defective, then the self-intersection vector of S is

$$\begin{aligned} c(S,S) &= (2\rho \deg(\operatorname{Sec} S), \deg(\operatorname{Tan} S), \deg(P_1(S)), \deg(S)) = \\ &= (2\rho \deg(\operatorname{Sec} S), 0, 0, 0) + (0, \deg(\operatorname{Tan} S), \deg(P_1(S)), \deg(S)), \end{aligned}$$

where the last line is the decomposition according to Proposition 1. In fact, for S smooth and non defective the ring of coordinates of the normal cone $\mathcal{R}$ has two minimal primes $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{P}_1 \cap K[x_0, \dots, x_N] = (x_0, \dots, x_N)$ and $\mathcal{P}_2 \cap K[x_0, \dots, x_N] = I(S)$.

If the surface $S \subset \mathbb{P}^N, N \geq 5$, has isolated singular points $P_1, \dots, P_r$ of embedding dimension greater or equal to 4 and singular curves $Z_1, \dots, Z_s$, then the normal cone of $J(S,S)$ along $J(S,S) \cap \Delta$ has at least one component for each point P_i, $1 \leq i \leq r$ and for each curve Z_j, $1 \leq j \leq s$. The self-intersection vector of S decomposes in the following way

$$
\begin{aligned}
c(S,S) = \Bigg(& 2\rho \deg(\operatorname{Sec} S),\ \deg(\operatorname{Tan} S) + \deg \operatorname{mov} v_0(S)|_{\operatorname{Sing} S} + \sum_P j_P, \\
& \qquad \deg(P_1(S)) + \sum_Z j_Z \deg(Z),\ \deg(S) \Bigg) \\
= & \ (2\rho \deg(\operatorname{Sec} S),\ 0,\ 0,\ 0) + \\
& + \sum_{P \in \{P_i\}} \left(0, \sum_{\{\mathcal{P} | \mathcal{P} \cap A_x = I(P)\}} \operatorname{length}(\mathcal{R}_{\mathcal{P}}) \cdot c_1(\mathcal{R}/\mathcal{P}), 0, 0 \right) + \\
& + \sum_{Z \in \{Z_j\}} \sum_{\{\mathcal{P} | \mathcal{P} \cap A_x = I(Z)\}} \operatorname{length}(\mathcal{R}_{\mathcal{P}}) \cdot (0,\ c_1(\mathcal{R}/\mathcal{P}),\ c_2(\mathcal{R}/\mathcal{P}),\ 0) + \\
& + (0,\ \deg(\operatorname{Tan} S),\ \deg(P_1(S)),\ \deg(S)),
\end{aligned}
$$

where we recall that $A_x := K[x_0, \ldots, x_N]$, and we remark that for the movable points Q on the curves Z_j the coefficients $\operatorname{length}(\mathcal{R}_{\mathcal{P}})$ are equal to the intersection number j_Q and

$$
\sum_{\{\mathcal{P} | \mathcal{P} \cap A_x = I(Z)\}} \operatorname{length}(\mathcal{R}_{\mathcal{P}}) \cdot c_2(\mathcal{R}/\mathcal{P}) = j_Z \deg(Z)
$$

(see [6], Main Theorem). We also observe that

$$
\sum_{Z \in \{Z_j\}} \sum_{\{\mathcal{P} | \mathcal{P} \cap A_x = I(Z)\}} \operatorname{length}(\mathcal{R}_{\mathcal{P}}) \cdot c_1(\mathcal{R}/\mathcal{P}) = \deg \operatorname{mov} v_0(S)|_{\operatorname{Sing} S},
$$

and $\sum_{\{\mathcal{P} | \mathcal{P} \cap A_x = I(Z)\}} c_1(\mathcal{R}/\mathcal{P})$ is the number of the movable points on Z.

In order to compute the intersection number j_P of an isolated singular point of S we must compute the generalized Samuel multiplicities of the diagonal ideal in the localization of the ring R localized at the prime ideal $I(P)R + I(\Delta)R$ (see [3]). In this case we obtain three coefficients: $c_0 = j_P, c_1, c_2$, where $c_0 = 0$ if and only if the embedding dimension of S at P is smaller or equal to 3 and c_2 is the multiplicity of S at P.

PROPOSITION 3. *(C.Ciliberto) A surface $S \subset \mathbb{P}^5$ with one apparent double point ($\rho(S) = 1$) cannot have isolated singular points of embedding dimension greater or equal to* 4.

Proof. In fact, assume that $P \in S$ is a point of embedding dimension greater or equal to 4 and Π is the Zariski tangent space to S at P. If r is a generic secant line of S and $\alpha = \langle r, P \rangle \cong \mathbb{P}^2$, then $\alpha \cap \Pi$ contains at least a line ℓ through P. The line $\ell \subset \Pi$ is a limit of secants, hence it is a secant, but $r \cap \ell$ is not empty, which contradicts the genericity of r. □

This says that we cannot have surfaces in $\mathbb{P}^5$ with one apparent double point and singular points which contributes to the self-intersection cycle. We can have such examples only if $\rho \geq 2$.

By using Theorem 2, we want to compute the self-intersection vector and the number $\rho(S)$, for singular surfaces in $S \subset \mathbb{P}^5$ which are linear projections of rational normal scrolls. We also want to point out the contribution of the components of the singular locus of S to its self-intersection vector.

In the following with $S(a,b) \subset \mathbb{P}^{a+b+1}$ we denote the rational normal scroll defined by the parametric equations

$$(x_0 : \ldots : x_a : y_0 : \ldots : y_b) = (s^a u : s^{a-1}tu : \ldots : t^a u : s^b v : s^{b-1}tv : \ldots : t^b v).$$

We recall that $S(a,b)$ is a smooth surface of degree $a+b$, whose defining equations are given by the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \ldots & x_{a-1} & y_0 & \ldots & y_{b-1} \\ x_1 & x_2 & \ldots & x_a & y_1 & \ldots & y_b \end{pmatrix}$$

(for more details see, for example, [21]). The defining ideals of $S(a,b)$ can be conveniently computed by various computer algebra systems, see Section 5.

By [5], the self-intersection vector of $S(a,b)$ is

$$\begin{aligned} c(S,S) &= (2\deg(\operatorname{Sec} S),\ \deg(\operatorname{Tan} S),\ \deg(P_1(S)),\ \deg(S)) \\ &= ((a+b-2)(a+b-3),\ 2(a+b-2),\ 2(a+b-1),\ a+b). \end{aligned} \tag{4}$$

Now we are ready to present some examples, where the self-intersection vectors have been computed following procedures and codes as explained in Section 5.

EXAMPLE 1. *Surfaces in $\mathbb{P}^5$ with one or two isolated singular points which do not contribute to the self intersection cycle and one apparent double point*
Let us now consider the smooth del Pezzo surface of $S \subset \mathbb{P}^6$ given by the parametric equations

$$x_0 = uvw, x_1 = v^2w, x_2 = vw^2, x_3 = uw^2, x_4 = u^2w, x_5 = u^2v, x_6 = uv^2$$

(see [28], p. 155). The secant variety of S is the hypersurface of $\mathbb{P}^6$ defined by the equation

$$x_0^3 - x_0x_1x_4 - x_0x_2x_5 - x_0x_3x_6 + x_1x_3x_5 + x_2x_4x_6 = 0,$$

whose singular locus is S.

We project S to $\mathbb{P}^5$ from a point $P \in S$. If

$$\begin{aligned} P \in \{&[0:1:0:0:0:0:0], [0:0:1:0:0:0:0], [0:0:0:1:0:0:0], \\ &[0:0:0:0:1:0:0], [0:0:0:0:0:1:0], [0:0:0:0:0:0:1]\} \end{aligned}$$

the image of S under the projection is a surface $T \subset \mathbb{P}^5$ of degree 5 with two singular points which do not contribute to the self-intersection cycle of the surface. The decomposition of the self-intersection vector of the surface T is

$$c(T,T) = (2,8,10,5) = 2(1,0,0,0) + (0,8,10,5),$$

in particular $\rho = 1$.

If we project the surface S from the point $[0:1:1:0:0:0:0]$, we obtain a surface $T' \subset \mathbb{P}^5$ of degree 5 with one singular point which does not contribute to the self-intersection cycle of the surface. The intersection numbers of the surface T' are again

$$c(T',T') = (2,8,10,5) = 2(1,0,0,0) + (0,8,10,5),$$

in particular $\rho = 1$.

EXAMPLE 2. *Surfaces in $\mathbb{P}^5$ with an isolated singular point which contributes to the self intersection cycle and two apparent double points*
Let $S = S(3,2) \subset \mathbb{P}^6$, whose self intersection vector is $c(S(3,2)) = (6,6,8,5)$ and

$$\operatorname{Sing}\operatorname{Sec} S(3,2) = \operatorname{Sing}\operatorname{Tan} S(3,2) = S(3,2) \cap \{[x_0 : \ldots : x_6] \in \mathbb{P}^6 \mid x_0 = \cdots = x_3 = 0\}.$$

Let T_1 and T_2 be the surfaces of $\mathbb{P}^5$ obtained by projecting S from the points $P_1 = [1:0:0:1:0:0:0] \in \operatorname{Sec} S(3,2) \setminus \operatorname{Tan} S(3,2)$ and $P_2 = [0:1:0:0:0:0:0] \in \operatorname{Tan} S(3,2) \setminus \operatorname{Sing}\operatorname{Tan} S(3,2)$ respectively.

The surfaces T_1 and T_2 have one double point which contributes to the intersection cycle with intersection multiplicity $j_1 = 2$ and $j_2 = 3$ respectively. The decompositions of the intersection vectors are:

$$c(T_1,T_1) = (4,8,8,5) = (0,6,8,5) + 2(0,1,0,0) + (4,0,0,0),$$

$$c(T_2,T_2) = (4,8,8,5) = (0,5,8,5) + 3(0,1,0,0) + 4(1,0,0,0),$$

see formula (2) and Section 5.

EXAMPLE 3. *Surface in $\mathbb{P}^5$ with two isolated singular points which contribute to the self intersection cycle and four apparent double points*
Let us consider the rational normal scroll $S(3,3) \subset \mathbb{P}^7$ and we project it from the line s passing through the points $P = [0:1:\ldots:0]$ and $Q = [0:\ldots:0:1:0]$ (which are smooth points of $\operatorname{Sec} S(3,3)$) on the linear space $\{[x_0 : \ldots : x_7] \in \mathbb{P}^7 \mid x_1 = x_6 = 0\} \cong \mathbb{P}^5$.

One obtains a surface $T \subset \mathbb{P}^5$ with two isolated singular points $R_1 = [0:0:0:0:0:1]$ and $R_2 = [1:0:0:0:0:0]$, which are double points. A computer calculation as in [1, file segre4.txt] gives the self-intersection numbers of T

$$j(T,T;R_1) = j(T,T;R_2) = 3$$

and the intersection vector

$$c(T,T) = (8,12,10,6) = 8(1,0,0,0) + 3(0,1,0,0) + 3(0,1,0,0) + (0,6,10,6),$$

whereas the intersection vector of $S(3,3)$ is $c(S(3,3),S(3,3)) = (12,8,10,6)$, see (4).

We recall the definition of Verra surfaces from [11], Section 3, in a slightly modified version.

DEFINITION 2 (Verra surfaces). *Let $Y \subset \mathbb{P}^5$ be a degenerate curve, which spans a linear space V of dimension 3. Take a line $W \subset \mathbb{P}^5$ such that $V \cap W = \emptyset$. Let $C_W(Y)$ be the cone over Y with vertex W. Let $X \subset C_W(Y)$ be an irreducible, non-degenerate, not secant defective surface, which intersects the general ruling $\Pi \cong \mathbb{P}^2$ of $C_W(Y)$ along a line L. This implies that:*

(A1) the projection $p\colon \mathbb{P}^5 \dashrightarrow V$ with center W restricts to X to a dominant map $p|_X : X \dashrightarrow Y$;

(A2) if $L_i, 1 \leq i \leq 2$, are the closures of two general fibers of $p|_X$, then $L_1 \cap L_2 = \emptyset$.

Indeed, (A1) is clear, and (A2) follows, via Terracini's Lemma, from the fact that X is not secant defective. The variety X is called a Verra surface *constructed from the curve Y.*

We point out that, differently from [11], in our definition Y is not required to be a curve with one apparent double point.

PROPOSITION 4 (A. Verra). *With the previous notation we have*

$$\rho(X) = \rho(Y).$$

Proof. Let X be a Verra surface. Let $x \in \mathbb{P}^5$ be a general point, so that $y = p(x)$ is a general point of V. A secant line to X through x is a general secant line to X and projects to a general secant line to Y passing through y. Let $\rho(Y) = m$, then there are m secant lines $\ell_1, \dots, \ell_m$ through y, and let P_{i1}, P_{i2} $(i = 1, \dots, m)$ the intersection points of ℓ_i with Y. For each secant line ℓ_i of Y through y there is exactly one secant line of X through x which by p is mapped on ℓ_i. Such a line must be in the 3-dimensional linear space $Z_i = \langle \ell_i \cup W \rangle$, which intersects X along the two lines $L_{ij} \subset \langle P_{ij}, W \rangle, 1 \leq j \leq 2$, the union of which spans Z_i. The assertion follows, since there is only one secant line to $L_{i1} \cup L_{i2}$ passing through $x \in Z_i$.

□

The following two examples regard two families of Verra surfaces with a multiple line the preimage of which in the normalization are a rational normal curve and k lines, respectively.

EXAMPLE 4. *Verra surfaces in $\mathbb{P}^5$ with a multiple line and one apparent double point*

Let us consider a rational normal scroll $S(d-3,3) \subset \mathbb{P}^{d+1}$, with $d \geq 5$, and let us project $S(d-3,3)$ from the linear subspace

$$L = \{[x_0 : \dots : x_{d-3} : y_0 : \dots : y_3] \in \mathbb{P}^{d+1} \mid x_0 = x_{d-3} = y_0 = \cdots = y_3 = 0\}$$

of dimension $d-5$ contained in

$$\Pi = \{[x_0 : \dots : x_{d-3} : y_0 : \dots : y_3] \in \mathbb{P}^{d+1} \mid y_0 = \cdots = y_3 = 0\} \cong \mathbb{P}^{d-3}$$

and such that it does not intersect the rational normal curve $C(d-3) = S(d-3,3) \cap \Pi$.

The image of $S(d-3,3)$ under the linear projection π_L is a rational surface $T := T(d-3,3) \subset \mathbb{P}^5$ with a multiple line ℓ of multiplicity $d-3$ such that ${\pi_L}^{-1}(\ell) = C(d-3)$.

Clearly the restriction of the projection π_L to

$$\Pi' = \{[x_0 : \ldots : x_{d-3} : y_0 : \ldots : y_3] \in \mathbb{P}^{d+1} \mid x_0 = \cdots = x_{d-3} = 0\} \cong \mathbb{P}^3$$

gives an isomorphism between the rational normal curve $C(3) = S(d-3,3) \cap \Pi'$ and its image in $\mathbb{P}^5$, and the surface T is obtained as in Verra's construction (see Definition 2, [10], Example 5.18 and [11], Section 3).

Since T is obtained from Verra's construction, we know that $\rho(T) = \rho(C(3)) = 1$, hence $\deg(J(T,T)/\operatorname{Sec}(T)) = 2$.

We can observe that for $d = 5$,

$$\operatorname{Sing}\operatorname{Sec} S(2,3) = S(2,3) \cup \{[x_0 : \ldots : x_2 : y_0 : \ldots : y_3] \in \mathbb{P}^6 \mid y_0 = \cdots = y_3 = 0\}$$

and L is a point in $\operatorname{Sing}\operatorname{Sec} S(2,3) \setminus S(2,3)$. The surface $T(2,3)$ has degree 5 and its singular locus is a line of double points whose preimage is exactly the smooth conic

$$S(2,3) \cap \{[x_0 : \ldots : x_2 : y_0 : \ldots : y_3] \in \mathbb{P}^6 \mid y_0 = \cdots = y_3 = 0\}.$$

This example was studied in detail in [5], Section 4, and its self-intersection vector is

$$c(T(2,3), T(2,3)) = (2,6,10,5) = 2(1,0,0,0) + 2(0,2,1,0) + (0,4,8,5).$$

The self-intersection vector of $T(2,3)$ is equal to the the self-intersection vectors of the surfaces T and T' of Example 1, but their decompositions are different.

EXAMPLE 5. *A Verra surface in $\mathbb{P}^5$ with a double line and one apparent double point*
Let us now consider the rational normal scroll $S(1,4) \subset \mathbb{P}^6$ and the 3-dimensional irreducible and reduced variety (remember that embJ denotes the embedded join)

$$X := \operatorname{embJ}(S(1), S(1,4)) = \operatorname{embJ}(S(1), S(4)) = \operatorname{Sing}\operatorname{Sec} S(1,4) = \operatorname{Sing}\operatorname{Tan} S(1,4)$$

of defining ideal

$$(-x_2x_4 + {x_3}^2, -x_2x_5 + x_3x_4, -x_2x_6 + x_3x_5, -x_2x_6 + {x_4}^2, -x_3x_6 + x_4x_5, -x_4x_6 + {x_5}^2).$$

Let $P \in X \setminus S(1,4)$ and let $\pi_P\colon S(1,4) \to \mathbb{P}^5$ be the linear projection from P into $\mathbb{P}^5$. The surface $Z := Z(1,4) := \pi_P(S(1,4))$ has a singular line $\ell = \pi_P(S(1))$, the preimage of which is composed of two intersecting lines, precisely $S(1)$ and a line of the ruling. To show this we prove the following stronger claim.

CLAIM. Let $\alpha = \langle S(1), P\rangle = \langle \ell, P\rangle$ be the plane spanned by $S(1)$, or by ℓ, and the point P. Then the intersection cycle $v(\alpha, S(1,4))$ is composed by the union of $S(1)$ and a line, say r, of the ruling of $S(1,4)$ and three movable points on $S(1)$. In particular, $\pi_P^{-1}(\ell) = \alpha \cap S(1,4) = S(1) \cup r$.

Proof. To prove the claim, let $H \in \mathbb{P}^5$ be a generic hyperplane containing α. The hyperplane H intersects $S(1,4)$ in a curve which is a union of lines and having $S(1)$ as a component. In fact, H intersects $S(4)$ in four distinct points $Q_1,\dots,Q_4$ and through each of them there is a line of the ruling lying on H and intersecting $S(1)$ in a point. Denote these distinct lines by $r_1,\dots,r_4$ and let $R_i = r_i \cap S(1)$.

We observe that exactly one of the lines $r_1,\dots,r_4$ is contained in α. In fact, being $P \in X$, there exists a line $l \subset X$ passing through P. Such a line is contained in α, but it cannot be a line on $S(1,4)$ since $P \notin S(1,4)$. Let $Q \in (S(1,4)\setminus S(1)) \cap l$. Such a point Q is the only point in which the plane α intersects $S(4)$, since a plane can contain only one line of the ruling. Let $r = r_1$ be the unique line of $S(1,4)$ passing through $Q = Q_1$. This line r is contained in α since the point Q and the point $r \cap S(1)$ are in α, hence $S(1,4) \cap \alpha = S(1) \cup r$.

The intersection cycle $v(\alpha, S(1,4))$ is composed by $r \cup S(1)$ and the three embedded points R_2, R_3, R_4, which are movable on $S(1)$ when $H(\supset \alpha)$ varies. $\square$

Let $P = [0:1:1:0:0:0:0] \in X \setminus S(1,4)$ and let π_P be the linear projection from P into $\mathbb{P}^5 = \mathrm{Proj}(\mathbb{C}[x_0, x_1 - x_2, x_3, x_4, x_5, x_6])$. The surface $Z := Z(1,4) := \pi_P(S(1,4))$ is defined by the ideal

$$\begin{gathered}(-x_3x_5 + {x_4}^2, -x_3x_6 + x_4x_5, -x_4x_6 + {x_5}^2,\\ -x_0x_5 + (x_2 - x_1)x_4 + {x_3}^2, -x_0x_6 + (x_2 - x_1)x_5 + x_3x_4).\end{gathered}$$

It has degree 5 and it is singular along the line of equations $x_3 = x_4 = x_5 = x_6 = 0$, which is a line of double points, whose preimage is composed by the two intersecting lines of equations $x_2 = \cdots = x_6 = 0$ and $x_1 = x_3 = \cdots = x_6 = 0$ respectively.

The surface Z is a Verra variety since if we project the surface from ℓ, its image is a rational normal cubic curve, hence Z is given by Verra's construction.

One can observe that the surface Z has the same intersection numbers of the surface $T(2,3)$:

$$c(Z,Z) = (2,8,10,5) = 2(1,0,0,0) + 2(0,2,1,0) + (0,4,8,5).$$

In Examples 1, 4 and 5 we considered surfaces with one apparent double point. Recently Ciliberto and Russo [11] gave a complete classification of (possibly singular) surfaces $S \subset \mathbb{P}^5$ with one apparent double point, proving that S is either a smooth rational normal scroll or a (weak) del Pezzo surface of degree 5 or a Verra surface constructed from a rational normal cubic.

EXAMPLE 6. *Surface in $\mathbb{P}^5$ with a line of double points, an isolated singular point which contributes to the self intersection cycle and two apparent double points*
Let us consider the rational normal scroll

$$S(4,2) \subset \mathbb{P}^7 = \mathrm{Proj}(\mathbb{C}[x_0,\dots,x_4,y_0,y_1,y_2])\,.$$

The ideal of its tangent variety is

$$\begin{gathered}(x_0x_4-4x_1x_3+3x_2^2,\quad 4x_0x_2x_4-3x_0x_3^2-3x_1^2x_4+2x_1x_2x_3,\\ -4x_0^2x_4^2+14x_0x_1x_3x_4-9x_0x_2x_3^2-9x_1^2x_2x_4+8x_1^2x_3^2,\\ -3x_0x_2y_2+3x_0x_3y_1-x_0x_4y_0+3x_1^2y_2-3x_1x_2y_1+x_1x_3y_0,\\ -3x_0x_3y_2+4x_0x_4y_1+3x_1x_2y_2-4x_1x_3y_1-3x_1x_4y_0+3x_2x_3y_0,\\ -x_0x_4y_2+x_1x_3y_2+3x_1x_4y_1-3x_2x_3y_1-3x_2x_4y_0+3x_3^2y_0,\\ x_0y_2^2-4x_1y_1y_2+2x_2y_0y_2+4x_2y_1^2-4x_3y_0y_1+x_4y_0^2,\\ x_0x_1x_4y_2-9x_0x_2x_3y_2+12x_0x_2x_4y_1-3x_0x_3x_4y_0+8x_1^2x_3y_2-12x_1^2x_4y_1+3x_1x_2x_4y_0,\\ 3x_0^2x_4y_2-9x_0x_1x_3y_2-x_0x_1x_4y_1+9x_0x_2x_3y_1-3x_0x_2x_4y_0+6x_1^2x_2y_2-8x_1^2x_3y_1+3x_1^2x_4y_0).\end{gathered}$$

We observe that the point $Q=[0:\ldots:0:1:0]\in \operatorname{Tan} S(4,2)$ has multiplicity two for $\operatorname{Tan} S(4,2)$.

Let us project $S(4,2)$ from the line passing through the points

$$P=[1:0:0:0:1:0:0:0]\in \operatorname{Sec} S(4,2)\setminus \operatorname{Tan} S(4,2)$$

and

$$Q=[0:0:0:0:0:0:1:0]\in \operatorname{Tan} S(4,2),$$

that is, from the line

$$L=\{[x_0:\ldots:x_4:y_0:y_1:y_2]\in\mathbb{P}^7 \mid x_0-x_4=x_1=x_2=x_3=y_0=y_2=0\},$$

not contained in $\operatorname{Sec} S(4,2)$ and intersecting $\operatorname{Tan} S(4,2)$ only in the point Q. The projection $S=\pi_L(S(4,2))\subset\mathbb{P}^5=\{[x_0:\ldots:x_4:y_0:y_1:y_2]\in\mathbb{P}^7 \mid x_0-x_4=y_1=0\}$ is a singular surface with a line of double points and an isolated double point. After a change of coordinates (in which we eliminate x_0-x_4 and y_1), in the new coordinates we have

$$\operatorname{Sing} S=\{[x_1:\ldots:x_4:y_0:y_2]\in\mathbb{P}^5 \mid x_1=x_2=x_3=x_4=0\}\cup\{[1:0:0:0:0:0]\}=\ell\cup R$$

and

$$\begin{aligned}\operatorname{Tan} S = \quad & \{[x_1:\ldots:x_4:y_0:y_2]\in\mathbb{P}^5 \mid \\ - \quad & x_1^2x_2^2x_4^2+6x_1x_2^3x_3x_4-4x_1x_2^2x_3^3-6x_1x_2x_3x_4^3+4x_1x_3^3x_4^2-4x_2^5x_4\\ + \quad & 3x_2^4x_3^2-8x_2^3x_4^3+42x_2^2x_3^2x_4^2-48x_2x_3^4x_4-4x_2x_4^5+16x_3^6+3x_3^2x_4^4=0\}.\end{aligned}$$

The surface S is a Verra variety since if we project the surface from ℓ, its image is a quartic curve C with self-intersection vector

$$c(C,C)=(4,8,4)=4(1,0,0)+2(0,1,0)+(0,6,4),$$

hence $\rho(S)=\rho(C)=2$. This can also be confirmed by the computation of $c(S,S)=(4,14,12,6)$.

The singular point $R\in S$ contributes to the cycle with multiplicity $j=2$, since in the affine chart $x_1=1$ the self-intersection vector of S is $(2,0,2)$.

EXAMPLE 7. *Surface in $\mathbb{P}^5$ with two lines of double points and two apparent double points*
Let us consider the rational normal scroll $S(4,2) \subset \mathbb{P}^7$ and project it from the line

$$s = \{[x_0 : \ldots : x_7] \in \mathbb{P}^7 \mid x_0 = x_2 = x_3 = x_4 = x_5 = x_7 = 0\}$$

on the linear space $\{[x_0 : \ldots : x_7] \in \mathbb{P}^7 \mid x_1 = x_6 = 0\} \cong \mathbb{P}^5$. The line s is contained in $\operatorname{Tan} S(4,2)$ and intersects $\operatorname{Sing}(\operatorname{Tan} S(4,2))$ in a point.

We obtain a surface $T \subset \mathbb{P}^5$ with two lines of double points

$$r_1 : \{[x_0 : x_2 : x_3 : x_4 : x_5 : x_7] \in \mathbb{P}^5 \mid x_2 = x_3 = x_4 = x_7 = 0\},$$

$$r_2 : \{[x_0 : x_2 : x_3 : x_4 : x_5 : x_7] \in \mathbb{P}^5 \mid x_0 = x_2 = x_3 = x_4 = 0\},$$

intersecting in the point $P = r_1 \cap r_2 = [0 : \ldots : 0 : 1 : 0 : 0]$ and such that the preimage of r_1 in the projection is the line

$$\{[x_0 : \ldots : x_7] \in \mathbb{P}^7 \mid x_1 = x_2 = x_3 = x_4 = x_6 = x_7 = 0\}$$

on the scroll and the preimage of r_2 in the projection is the conic

$$\{[x_0 : \ldots : x_7] \in \mathbb{P}^7 \mid x_0 = x_1 = x_2 = x_3 = x_4 = x_6^2 - x_5 x_7 = 0\}.$$

The intersection vector of T is

$$c(T,T) = (4,12,14,6) = 4(1,0,0,0) + \{3(0,1,1,0) + 2(0,2,1,0)\} + (0,5,9,6),$$

where the contribution inside the curly braces refers to the one-dimensional part of the singular locus. Precisely $j_{r_1} = 3$ and there is a movable point of multiplicity 3 on r_1, $j_{r_2} = 2$ and there are two movable points of multiplicity 2 on r_2 .

The surface T is a Verra variety since if we project it from r_2, its image is a quartic curve C with self-intersection vector $c(C,C) = (4,8,4)$, hence $\rho(T) = \rho(C) = 2$.

We observe that $c(S(4,2), S(4,2)) = (12,8,10,6)$, and this shows that

$$\deg P_1(S(4,2)) > \deg P_1(T).$$

4. Computing ρ of a projection

In this section we propose a second method for the computation of $\rho(T)$ when the singular surface $T \subset \mathbb{P}^5$ arises as the projection of a surface $S \subset \mathbb{P}^N_K$ $(N > 5)$ along a linear subspace L. This method reduces considerably the computational cost when calculating concrete examples (see Section 5).

THEOREM 3. *Let $X \subset \mathbb{P}^N$ be a non-degenerate reduced and irreducible variety such that $2\dim X + 1 < N$ and the generic tangent hyperplane to $\operatorname{Sec} X$ is tangent to X at only finitely many points (that is, X is not 1-weakly defective in the sense of [9],* *2.1). Let $L \subset \mathbb{P}^N$ be a linear subspace such that* $\operatorname{codim} L > \dim X + 1$ *and such that the linear projection $\pi_L \colon X \to \pi_L(X) =: Y$ is generically one to one.*

Then

$$\rho(Y) \cdot \deg \operatorname{Sec} Y = c_0(L, \operatorname{Sec} X),$$

and, if L is a point and $e(\operatorname{Sec} X, L)$ denotes the multiplicity of $\operatorname{Sec} X$ *at L (which is defined to be zero if $L \notin \operatorname{Sec} X$), it holds*

$$\rho(Y) \cdot \deg \operatorname{Sec} Y = \deg \operatorname{Sec} X - e(\operatorname{Sec} X, L).$$

Proof. It is known that

$$\rho(Y) = \frac{1}{2} \deg(J(Y,Y)/\operatorname{Sec} Y)$$

and $\deg(J(X,X)/\operatorname{Sec} X) = 2$ (see Proposition 2 and Remark 1). Let us consider the following diagram of rational maps

$$\begin{array}{ccc} J(X,X) & \xrightarrow{\pi_{\Delta^N}} & \operatorname{Sec} X \\ \Big\downarrow{\scriptstyle J(\pi_L,\pi_L)} & & \Big\downarrow{\scriptstyle \pi_L} \\ J(Y,Y) & \xrightarrow{\pi_{\Delta^{N-k-1}}} & \operatorname{Sec} Y \end{array}$$

where $J(\pi_L, \pi_L)$ is the map induced by π_L, π_{Δ^N} and $\pi_{\Delta^{N-k-1}}$ are the projections along the diagonal spaces of $J(\mathbb{P}^N, \mathbb{P}^N)$ and $J(\mathbb{P}^{N-k-1}, \mathbb{P}^{N-k-1})$, respectively. By assumption $\deg J(\pi_L, \pi_L) = 1$, hence by the commutativity of the diagram it turns out that

$$\deg(J(Y,Y)/\operatorname{Sec} Y) = 2 \deg(\operatorname{Sec} X/\operatorname{Sec} Y),$$

hence

$$\rho(Y) = \deg(\operatorname{Sec} X/\operatorname{Sec} Y). \tag{5}$$

On the other hand, by van Gastel [19]

$$c_0(L, \operatorname{Sec} X) = \deg(J(L, \operatorname{Sec} X)/\mathrm{embJ}(L, \operatorname{Sec} X)) \cdot \deg(\mathrm{embJ}(L, \operatorname{Sec} X)).$$

Since one of the two intersecting varieties is a linear space, the cycle $v(L, \operatorname{Sec} X)$ can be computed without passing to the ruled join (see [16], Proposition 2.2.11), therefore

$$\begin{aligned} c_0(L, \operatorname{Sec} X) &= \deg(\operatorname{Sec} X/\pi_L(\operatorname{Sec} X)) \cdot \deg \pi_L(\operatorname{Sec} X) \\ &= \deg(\operatorname{Sec} X/\operatorname{Sec} Y) \cdot \deg \operatorname{Sec} Y, \end{aligned}$$

which, together with (5), finishes the proof of the first formula.

If L is a point, by the refined Theorem of Bezout and taking into account that $j(\operatorname{Sec} X, L; L) = e(\operatorname{Sec} X, L)$ (see e.g. [16], Lemma 5.4.7) one has

$$c_0(L, \operatorname{Sec} X) = \deg \operatorname{Sec} X - e(\operatorname{Sec} X, L),$$

which finishes the proof in this case. □

COROLLARY 1. *Let* $S \subset \mathbb{P}^N, N > 5$ *be a non-degenerate reduced and irreducible surface such that the generic tangent hyperplane to* $\operatorname{Sec} S$ *is tangent to* S *at only finitely many points (that is,* S *is not* 1*-weakly defective in the sense of [9], 2.1). Let* $L \subset \mathbb{P}^N$ *be a linear subspace of codimension* 6 *such that the linear projection* $\pi_L \colon S \to \pi_L(S) =: T \subset \mathbb{P}^5$ *is generically one to one.*

Then $\rho(T) = c_0(L, \operatorname{Sec} S)$.

Using this result we can compute the number ρ for some surfaces (of low degree) in a class of Verra surfaces in $\mathbb{P}^5$ with a multiple line and one apparent double point, which contains the surface of Example 5.

EXAMPLE 8. Let us consider the rational normal scroll $S(1, d-1)$ of degree d in $\mathbb{P}^{d+1}$, with $d \geq 5$. Let $\Lambda \subset \mathbb{P}^{d+1}$ be a linear subspace of dimension $d-5$ such that

$$\Lambda \cap S(1, d-1) = \emptyset, \quad \Lambda \cap \operatorname{embJ}(S(1), S(d-1)) = \{P_1, \ldots, P_k\},\ 1 \leq k \leq d-4.$$

We remark that $\operatorname{Sec} S(1, d-1)$ (resp. $\operatorname{Tan} S(1, d-1)$) is a cone of vertex $S(1)$ over $\operatorname{Sec} S(d-1)$ (resp. $\operatorname{Tan} S(d-1)$) and that $\operatorname{Sing} \operatorname{Sec} S(1, d-1)$ (resp. $\operatorname{Sing} \operatorname{Tan} S(1, d-1)$) is a cone of vertex $S(1)$ over $\operatorname{Sing} \operatorname{Sec} S(d-1) = S(d-1)$ (resp. $\operatorname{Sing} \operatorname{Tan} S(d-1) = S(d-1)$), hence

$$\begin{aligned} \operatorname{embJ}(S(1), S(1, d-1)) &= \operatorname{embJ}(S(1), S(d-1)) = \\ &= \operatorname{Sing} \operatorname{Sec} S(1, d-1) = \operatorname{Sing} \operatorname{Tan}(S(1, d-1)). \end{aligned}$$

Moreover, $d-4$ is the maximum number of points of $\Lambda \cap \operatorname{embJ}(S(1), S(d-1))$. In fact if Λ would intersect $\operatorname{embJ}(S(1), S(d-1))$ in $m > d-4$ points, through each of them there would be a line l_i connecting $S(1)$ with $S(d-1)$. Let $Q_i = l_i \cap S(d-1)$ and let r_i be the line of the ruling through Q_i. We observe that the point

$$Q_i \in \alpha = \langle \Lambda, S(1) \rangle \cong \mathbb{P}^{d-3},$$

hence the line r_i is contained in α. Since $S(d-1)$ is not contained in α, starting from a point $Q_{m+1} \in S(d-1)$ we can find a line r_{m+1} of the ruling which is not contained in α. The linear space

$$\langle \alpha, r_{m+1} \rangle \cong \mathbb{P}^{d-2}$$

contains $m+1 > d-3$ lines of the ruling and repeating this reasoning we could find a hyperplane $H \cong \mathbb{P}^d$ containing d lines of the ruling and the line $S(1)$ and this contradicts the theorem of Bezout, since the scroll $S(1, d-1)$ is a non degenerate surface.

The intersection cycle $v(S(1, d-1), \alpha)$ is composed by the lines $S(1), r_1, \ldots, r_k$ and $d-1-k$ movable points on $S(1)$.

Now let $Z(1, d-1) = \pi_\Lambda(S(1, d-1)) \subset \mathbb{P}^5$. Such surface has a singular line, say ℓ, of multiplicity $k+1$ whose preimage are the lines $S(1), r_1, \ldots, r_k$. We can project the surface Z from ℓ into $\mathbb{P}^3$ and we obtain an irreducible curve C of degree $d-k-1$ and Z turns out to be a Verra surface constructed from ℓ and C, hence $\rho(Z) = \rho(C)$.

If $k = d-4$ then $\rho(Z) = \rho(C) = 1$, if $k < d-4$ we can compute $\rho(C)$ by computing the self-intersection cycle of C (see [4]) or using Corollary 1.

Now we show the application of Corollary 1 to projections of $S(1,5)$ and $S(1,6)$ into $\mathbb{P}^5$.

EXAMPLE 9. Let us project the rational normal scroll $S(1,5) \subset \mathbb{P}^7$ into $\mathbb{P}^5$ from the line

$$s_1 = \{[x_0 : \ldots : x_7] \in \mathbb{P}^7 \mid x_0 = x_1 - x_2 = x_3 - x_4 = x_5 = x_6 = x_7 = 0\},$$

which intersects $\operatorname{Sec} S(1,5)$ in the point

$$P = [0 : 1 : 1 : 0 : \ldots : 0] \in \operatorname{Sing}(\operatorname{Sec} S(1,5))$$

of multiplicity 3 for $\operatorname{Sec} S(1,5)$, that is $k = 1$. The image is the surface $Z_1 \subset \mathbb{P}^5$ defined by the kernel of the map

$$\phi_1 \colon K[z_0, \ldots, z_5] \to K[x_0, \ldots, x_7]/I(S(1,5)),$$

$$z_0 \mapsto x_0,\ z_1 \mapsto x_1 - x_2,\ z_2 \mapsto x_3 - x_4,\ z_3 \mapsto x_5,\ z_4 \mapsto x_6,\ z_5 \mapsto x_7.$$

The singular locus of Z_1 is the double line

$$\ell_1 = \{[z_0 : \ldots : z_5] \in \mathbb{P}^5 \mid z_2 = z_3 = z_4 = z_5 = 0\}.$$

We have

$$c(\operatorname{Sec} S(1,5), s_1) = (2,4) = (2,0) + 2(0,1) + (0,2),$$

in particular $\rho(Z_1) = 2$. Here $2(0,1) + (0,2)$ is the contribution of P, which comes from two components of the normal cone to $s_1 \cap \operatorname{Sec} S(1,5)$ in $\operatorname{Sec} S(1,5)$, therefore $j(\operatorname{Sec} S(1,5), s_1; P) = 4$.

If we project now $S(1,5)$ into $\mathbb{P}^5$ from the line

$$s_2 = \{[x_0 : \ldots : x_7] \in \mathbb{P}^7 \mid x_0 = x_1 - x_2 = x_3 = x_4 = x_5 = x_6 - x_7 = 0\},$$

which intersects $\operatorname{Sec} S(1,5)$ in the same point

$$P = [0 : 1 : 1 : 0 : \ldots : 0] \in \operatorname{Sing}(\operatorname{Sec} S(1,5))$$

as before and in the smooth point $Q = [0 : \ldots : 0 : 1 : 1]$ (that is $k = 1$), we obtain a surface $Z_2 \subset \mathbb{P}^5$ defined by the kernel of the map

$$\phi_2 \colon K[z_0, \ldots, z_5] \to K[x_0, \ldots, x_7]/I(S(1,5)),$$

$$z_0 \mapsto x_0,\ z_1 \mapsto x_1 - x_2,\ z_2 \mapsto x_3,\ z_3 \mapsto x_4,\ z_4 \mapsto x_5,\ z_5 \mapsto x_6 - x_7.$$

The singular locus of Z_2 is composed of the double line

$$\ell_2 = \{[z_0 : \ldots : z_5] \in \mathbb{P}^5 \mid z_2 = z_3 = z_4 = z_5 = 0\}$$

and the isolated point $[0 : \ldots : 0 : 1]$. We have

$$c(\operatorname{Sec} S(1,5), s_2) = (2,4) = 2(1,0) + (0,3) + (0,1),$$

in particular $\rho(Z_2) = 2$, $j(\operatorname{Sec} S(1,5), s_2; P) = 3$ and $j(\operatorname{Sec} S(1,5), s_2; Q) = 1$.

Both Z_1 and Z_2 are Verra surfaces constructed from ℓ_i ($i = 1,2$) and the irreducible quartic curve $C_i \subset \mathbb{P}^3$ which is the projection of Z_i from ℓ_i. The curve C_i has a double point and the self-intersection vector of C_i is $c(C_i, C_i) = (4,8,4)$, in particular $\rho(C_i) = 2$.

EXAMPLE 10. Let us project the rational normal scroll $S(1,6) \subset \mathbb{P}^8$ into $\mathbb{P}^5$ from the plane

$$\pi_1 = \{[x_0 : \ldots : x_8] \in \mathbb{P}^8 \mid x_0 = x_1 - x_2 = x_3 - x_4 = x_5 = x_6 = x_7 - x_8 = 0\},$$

which intersects $\operatorname{Sec} S(1,6)$ in the point

$$P = [0:1:1:0:\ldots:0] \in \operatorname{Sing}(\operatorname{Sec} S(1,6))$$

of multiplicity 4 on $\operatorname{Sec} S(1,6)$ and in the smooth point $Q = [0:\ldots:0:1:1]$, hence $k = 1$. The image is the surface $Z_3 \subset \mathbb{P}^5$ defined by the kernel of the map

$$\phi_3 \colon K[z_0, \ldots, z_5] \to K[x_0, \ldots, x_8]/I(S(1,6)),$$

$$z_0 \mapsto x_0,\ z_1 \mapsto x_1 - x_2,\ z_2 \mapsto x_3 - x_4,\ z_3 \mapsto x_5,\ z_4 \mapsto x_6,\ z_5 \mapsto x_7 - x_8.$$

The singular locus of Z_3 is the double line

$$\ell_3 = \{[z_0 : \ldots : z_5] \in \mathbb{P}^5 \mid z_2 = z_3 = z_4 = z_5 = 0\}$$

and the point $[0:\ldots:0:1]$. We have

$$c(\operatorname{Sec} S(1,6), \pi_1) = (4,6) = 4(1,0) + 2(0,1) + (0,3) + (0,1),$$

in particular $\rho(Z_3) = 4$. Here $2(0,1) + (0,3)$ is the contribution of P and $(0,1)$ is the contribution of Q, that is, $j(\operatorname{Sec} S(1,6), \pi_1; P) = 5$ and $j(\operatorname{Sec} S(1,6), \pi_1; Q) = 1$.

Now let us project $S(1,6)$ into $\mathbb{P}^5$ from the plane

$$\pi_2 = \{[x_0 : \ldots : x_8] \in \mathbb{P}^8 \mid x_0 - x_8 = x_1 - x_2 = x_3 - x_4 = x_5 = x_6 = x_7 = 0\},$$

which intersects $\operatorname{Sec} S(1,6)$ in the line

$$\ell = \{[x_0 : \ldots : x_8] \in \mathbb{P}^8 \mid x_0 - x_8 = x_1 - x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = 0\}$$

and $\operatorname{Sing}(\operatorname{Sec} S(1,6))$ in the two points $P_1 = [0:1:1:0:\ldots:0]$ and $P_2 = [1:0:\ldots:0:1]$ of multiplicity 4 on $\operatorname{Sec} S(1,6)$, hence $k = 2$. The image is the surface $Z_4 \subset \mathbb{P}^5$ defined by the kernel of the map

$$\phi_4 \colon K[z_0, \ldots, z_5] \to K[x_0, \ldots, x_8]/I(S(1,6)),$$

$$z_0 \mapsto x_0 - x_8,\ z_1 \mapsto x_1 - x_2,\ z_2 \mapsto x_3 - x_4,\ z_3 \mapsto x_5,\ z_4 \mapsto x_6,\ z_5 \mapsto x_7.$$

The singular locus of Z_4 is the triple line

$$\ell_4 = \{[z_0 : \ldots : z_5] \in \mathbb{P}^5 \mid z_2 = z_3 = z_4 = z_5 = 0\}.$$

We have

$$c(\mathrm{Sec}\, S(1,6), \pi_2) = (2,7,1) = 2(1,0,0) + 2(0,1,0) + (0,2,0) + (0,3,0) + (0,0,1),$$

in particular $\rho(Z_4) = 2$. Here $2(0,1,0) + (0,2,0)$ is the contribution of P_1, $(0,3,0)$ is that of P_2, and $(0,0,1)$ is the contribution of ℓ, that is, $j(\mathrm{Sec}\, S(1,6), \pi_2; P_1) = 4$, $j(\mathrm{Sec}\, S(1,6), \pi_2; P_2) = 3$ and $j(\mathrm{Sec}\, S(1,6), \pi_1; \ell) = 1$.

Both Z_3 and Z_4 are Verra surfaces constructed from ℓ_i ($i = 3,4$) and the irreducible curve $C_i \subset \mathbb{P}^3$ which is the projection of Z_i from ℓ_i. The curve C_3 has degree 5 and two double points with $j = 3$, moreover its self-intersection vector is $c(C_3, C_3) = (8,12,5)$, in particular $\rho(C_3) = 4$. The curve C_4 has degree 4 and a double point, say R, such that $j(C_4, C_4; R) = 3$. The self-intersection vector of C_4 is $c(C_4, C_4) = (4,8,4)$, in particular $\rho(C_4) = 2$.

5. Code of procedures for computing the examples

Code for computing the bidegrees $c_k(\mathcal{R})$ of a bigraded ring $\mathcal{R}$

While REDUCE (using the package SEGRE [1]) and Macaulay2 provide the functions `degs` or `multideg` and `multidegree` respectively, in CoCoA and Singular the bidegrees $c_k(\mathcal{R})$ must be computed from the numerator of the non-simplified Hilbert series of $\mathcal{R}$ according to [25], p. 167. Furthermore, in CoCoA 4.7.4 it is not possible to assign bidegrees beginning with zeros as $\deg(t_i) = (0,1)$. The trick is to pass to a $\mathbb{Z}^3$-graded ring with $\deg(x_i) = \deg(y_i) = (1,1,0)$ and $\deg(t_i) = (1,0,1)$. For the convenience of the reader we provide the details:

CODE 1. *CoCoA, version 4.7.4*:

```
Define BiDegree(I,A,B)

  F := Flatten([[1 | X In 1..A],[0 | X In 1..B]]);

  S := Flatten([[0 | X In 1..A],[1 | X In 1..B]]);

  G:= Mat([[1 | X In 1..(A+B)], F, S]);
```

This creates the matrix

$$G = \begin{pmatrix} 1\ldots\ldots 1 & 1\ldots\ldots 1 \\ 1\ldots\ldots 1 & 0\ldots\ldots 0 \\ \underbrace{0\ldots\ldots 0}_{A \text{ columns}} & \underbrace{1\ldots\ldots 1}_{B \text{ columns}} \end{pmatrix}$$

the columns of which are the degrees of the $A+B$ ring variables for the subsequent Hilbert series computation. Note that since CoCoA-4 does not allow zero-entries in the first row of the matrix which defines the degrees, we have added a first row of ones.

```
  H := HilbertSeriesMultiDeg(CurrentRing()/I, G);
```

Now we extract the numerator of the non-simplified Hilbert series:

```
  Num := Sum([X[1]*LogToTerm(X[2]) | X In @H[1]]);
```

According to [25], p. 167, the normalized leading coefficients of the Hilbert polynomial are obtained from the numerator `Num` by substituting each variable t by $1-t$ and then collecting the coefficients of the terms having total degree $\mathrm{codim}(\mathrm{CurrentRing}/I)$, i.e., the coefficients of the lowest degree terms. To get rid of the artificially introduced first variable which is due to the first row of G, this variable must be substituted by one. Doing this, we obtain from the numerator the polynomial N, which we write as a polynomial in the first two variables of the current ring.

```
  N := Eval(Num, [1, 1-Indet(1), 1-Indet(2)]);

  M := Min([Deg(X) | X In Monomials(N)]);

  P := Sum([X In Monomials(N) | Deg(X) = M]);
```

The polynomial P, written in the first two variables of the current ring, is the bidegree in the sense of [25], p. 167. For better readability, the coefficients of P are printed, but P is returned:

```
  PrintLn [CoeffOfTerm(X,P) | X In Support((Indet(1)+Indet(2))^M)];

Return P;

EndDefine;
```

Singular, version 3.1.6:

We give the code of a procedure which computes the bidegrees of an ideal I.

```
LIB "multigrading.lib";

proc bidegree(ideal I, int a, int b)

{

  ideal SI = std(I);

  def currentring = basering;

  int n = nvars(basering);
```

```
intmat m[2][a+b] = 1:a,0:b,0:a,1:b;
```

This creates the matrix

$$m = \begin{pmatrix} 1\ldots\ldots1 & 0\ldots\ldots0 \\ \underbrace{0\ldots\ldots0}_{a\text{ columns}} & \underbrace{1\ldots\ldots1}_{b\text{ columns}} \end{pmatrix}$$

the columns of which are the degrees of the $a+b$ ring variables.

```
setBaseMultigrading(m);

def h = hilbertSeries(SI);

setring h;

poly f = substitute(numerator1,t_(1),1-t_(1),t_(2),1-t_(2));
```

Here `numerator1` is the numerator of the non-simplified Hilbert series, which is called the *first Hilbert series* in the Singular Manual.

```
poly g = jet(f,mindeg(f));
```

The polynomial g, that is, the homogeneous part of lowest degree of f, is by [25] the bidegree of I. It will be returned as a polynomial of the base ring written in the first two variables of the base ring.

```
  setring currentring;

  return(fetch(h,g));

};
```

Code for computing the defining ideals of rational normal scrolls $S(a,b)$

The defining ideals of $S(a,b)$ can be conveniently computed by various computer algebra systems, e.g. using the following functions:

CODE 2. *Macaulay2, version 1.4:*

```
scroll = (a,b,K) ->

(

R := K[x_0 .. x_a, y_0 .. y_b];
```

```
M := map(R^2, a, (i,j)->x_(i+j));
N := map(R^2, b, (i,j)->y_(i+j));
I := minors(2, M|N)
)
```

CoCoA, version 4.7.4:

```
Define Scroll(A,B)
  ScrollRing ::=  Q[x[0..A],y[0..B]];
  Using ScrollRing Do M := Mat([Concat(x[0]..x[A-1],y[0]..y[B-1]),
  Concat(x[1]..x[A],y[1]..y[B])]);
  Return Ideal(Minors(2,M));
  EndUsing;
EndDefine;
```

Singular, version 3.1.6:

```
proc scroll(int a, int b, int ch)
  {
  ring scrollring = ch,(x(0..a),y(0..b)),dp;
  matrix M[2][a+b] = x(0..a-1),y(0..b-1),x(1..a),y(1..b);
  ideal scrollideal = minor(M,2);
  export(scrollring,scrollideal);
  }
```

Code for computing the intersection vector

We refer to Example 3 in order to explain the code we used for the computer aided calculations in our examples.

CODE 3. REDUCE:

```
Reduce (Free PSL version), 30-Nov-11 ...

1: load_package segre;

SEGRE 1999/2012-07-11 with package CALI, for help type: help(help);

2: setideal(s33, scroll{3,3})$

3: t := eliminate(s33, {x1,x6})$

4: setring({x0,x2,x3,x4,x5,x7},{},lex)$

5: setideal(nc, int_ncone{t,t})$

6: degs(nc, {6,6});

{8,12,10,6,0,0,0}

7: on time;

Time: 17284 ms  plus GC time: 579 ms
```

Macaulay2, version 1.4:

```
i1 : load "scroll.m2"

i2 : t1 = cpuTime();

i3 : S33 = scroll(3,3,QQ);

o3 : Ideal of QQ[x , x , x , x , y , y , y , y ]
                  0   1   2   3   0   1   2   3

i4 : ringP7 = ring(S33);

i5 : ringP5 = QQ[z_0 .. z_5];

i6 : center = {x_0, x_2, x_3, y_0, y_1, y_3}

o6 = {x , x , x , y , y , y }
       0   2   3   0   1   3

o6 : List

i7 : T = trim kernel map(ringP7/S33, ringP5, center);

o7 : Ideal of ringP5
```

```
i8 : idealNormalCone = intNcone(T,T);

o8 : Ideal of QQ[z , z , z , z , z , z , w , w , w , w , w , w ]
                  0   1   2   3   4   5   0   1   2   3   4   5

i9 : multidegree idealNormalCone

       6      5        4 2     3 3
o9 = 8T  + 12T T  + 10T T  + 6T T
       0      0 1      0 1     0 1

o9 : ZZ[T , T ]
         0   1

i10 : cpuTime() - t1  --time in ms, CPU Intel(R) Core(TM) i5-2410M

o10 = 395.618

o10 : RR (of precision 53)
```

CoCoA, version 4.7.4:

```
Source "scroll.coc";
Set Timer;
Null
-------------------------------
S33:=Scroll(3,3);

Cpu time = 0.31, User time = 0
-------------------------------
Use ScrollRing;

Cpu time = 0.00, User time = 0
-------------------------------
T:=Elim([x[1],y[2]],S33);

Cpu time = 0.62, User time = 0
-------------------------------
Use RingP5::=Q[z[0..5]];

Cpu time = 0.00, User time = 0
-------------------------------
F:=RMap(z[0],0,z[1],z[2],z[3],z[4],0,z[5]);

Cpu time = 0.00, User time = 0
-------------------------------
T:=Ideal(Image(Gens(T),F));
```

```
Cpu time = 0.47, User time = 0
-------------------------------
J:=RuledJoin(T,T);

Cpu time = 0.31, User time = 0
-------------------------------
Use JoinRing;

Cpu time = 0.00, User time = 0
-------------------------------
B:=BlowUp(J[1],J[2]);

Cpu time = 31.52, User time = 3
-------------------------------
N:=NumIndets(BlowUpRing)/3;

Cpu time = 0.00, User time = 0
-------------------------------
Use CoeffRing[x[1..N],t[1..N]];

Cpu time = 0.00, User time = 0
-------------------------------
G:=RMap(Concat(x[1]..x[N],x[1]..x[N],t[1]..t[N]));

Cpu time = 0.00, User time = 0
-------------------------------
NormalCone:=Image(B[2],G);

Cpu time = 1.09, User time = 0
-------------------------------
BiDegree(NormalCone,6,6);
[8, 12, 10, 6, 0, 0, 0]
8x[1]^6 + 12x[1]^5x[2] + 10x[1]^4x[2]^2 + 6x[1]^3x[2]^3
-------------------------------

Cpu time = 1.25, User time = 0
-------------------------------
```

Singular, version 3.1.6, input file:

```
< "scroll.s";

timer = 0;

system("--ticks-per-sec",1000);

int t1 = timer;
```

```
scroll(3,3,0);

ideal s33 = scrollideal;

ideal t = eliminate(s33,x(1)*y(2));

ring ringP5 = 0, (x(0),x(2),x(3),y(0),y(1),y(3)), dp;

ideal t = imap(scrollring, t);

rjoin(t,t);

setring joinring;

formring(joinideal, diagonalideal);

int n = nvars(form_r)/3;

ring R = char(form_r),(x(1..n),t(1..n)),dp;

setring R;

map f = form_r, x(1..n),x(1..n),t(1..n);

bidegree(f(form_i),6,6);

"time in ms = ", timer-t1;

quit;
```

Output file (running Singular in quite mode):

```
Singular -q < inputfile > outputfile
```

```
 ------------
This proc returns a ring with polynomials called 'numerator1/2'
and 'denominator1/2'!
They represent the first and the second Hilbert Series.
The s_(i)-variables are defined to be the inverse of the
t_(i)-variables.
 ------------
8*x(1)^6+12*x(1)^5*x(2)+10*x(1)^4*x(2)^2+6*x(1)^3*x(2)^3
time in ms =  10300
```

Code for computing the secant variety

If $T = \pi_L(S)$ is a linear projection of S from L, then Theorem 3 and Corollary 1 can be applied to compute $\rho(T)$ by computing $c_0(\mathrm{Sec}\, S, L)$. Hence the defining ideal of the secant variety $\mathrm{Sec}\, S$ has to be computed. Nevertheless this method reduces the computation time of $T = \pi_L(S)$ considerably.

For example, the times required for the computation of Example 3 (see the previous subsection) reduces from ca. 18 seconds to less than 1 second (REDUCE), from 4 seconds to less than 1 second (CoCoA), from more than 6 minutes to 2 seconds (Macaulay2), and from ca. 11 seconds to less than 1 second (Singular). The computations have been performed using a Cygwin installation under Microsoft Windows 7 with CPU Intel(R) Core(TM) i5-2410M.

REDUCE (with the package SEGRE) has the built-in facilities `ej(I,J)` and `ejoin({I,J})` which permit the calculation of the ideal of the embedded join of the projective varieties defined by the homogeneous ideals I and J. If $I = J$, then this is the ideal of the secant variety of the projective variety defined by I.

Here we propose analogue procedures for CoCoA, Macaulay2, and Singular.

CODE 4. *Macaulay2, version 1.4:*

```
embJoin = (I,J) ->

(

R := ring(I);

K := coefficientRing(R);

n := numgens(R);

T := tensor(R/I,R/J);

G := gens(T);

x := take(G,{0,n-1});

y := take(G,{n,2*n-1});

F := map(T,R,x-y);

ker F

)
```

Singular, version 3.1.6:

```
// Author: Peter Schenzel, schenzel@informatik.uni-halle.de
```

```
proc join(ideal I, ideal J)
{
  def rj = basering;
  int n = nvars(rj);
  def sj = extendring(n,"v(","c,dp",1,rj);
  setring sj;
  ideal I1 = imap(rj,I);
  ideal J1 = imap(rj,J);
  int j;
  for(j = 1; j <= n; j++)
    {
      I1 = subst(I1,var(j),v(j));
      J1 = subst(J1,var(j),var(j)-v(j));
    }
  ideal K = I1+J1;
  ideal join = elim(K,n+1..2*n);
  setring rj;
  ideal join = imap(sj,join);
  return(join);
};
```

Further procedures needed include those for associated graded rings of quotient rings, which can be obtained by standard elimination theory. They are built-in functions in SEGRE and Macaulay2 but not in CoCoA and Singular. We shall not reproduce here our code (which is certainly not optimal), but make it available at http://www.dm.unibo.it/∼achilles/code.

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AMS Subject Classification: 14Q10, 14J17

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Lavoro pervenuto in redazione il 19.06.2013.

Rend. Sem. Mat. Univ. Politec. Torino
Vol. 71, 3–4 (2013), 307 – 315

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ON THE EXISTENCE OF RAMIFIED ABELIAN COVERS

Dedicated to Alberto Conte on his 70th birthday.

Abstract. Given a normal complete variety Y, distinct irreducible effective Weil divisors $D_1,\dots D_n$ of Y and positive integers $d_1,\dots d_n$, we spell out the conditions for the existence of an abelian cover $X \to Y$ branched with order d_i on D_i for $i = 1,\dots n$.

As an application, we prove that a Galois cover of a normal complete toric variety branched on the torus-invariant divisors is itself a toric variety.

1. Introduction

Given a projective variety Y and effective divisors $D_1,\dots D_n$ of Y, deciding whether there exists a Galois cover branched on $D_1,\dots D_n$ with given multiplicities is a very complicated question, which in the complex case is essentially equivalent to describing the finite quotients of the fundamental group of $Y \setminus (D_1 \cup \dots \cup D_n)$.

In Section 2 of this paper we answer this question for a normal variety Y in the case that the Galois group of the cover is abelian (Theorem 1), using the theory developed in [3] and [1]. In particular, we prove that when the class group $\mathrm{Cl}(Y)$ is torsion free, every abelian cover of Y branched on $D_1,\dots D_n$ with given multiplicities is the quotient of a maximal such cover, unique up to isomorphism.

In Section 3 we analyze the same question using toric geometry in the case when Y is a normal complete toric variety and $D_1,\dots D_n$ are invariant divisors and we obtain results that parallel those in Section 2 (Theorem 3). Combining the two approaches we are able to show that any cover of a normal complete toric variety branched on the invariant divisors is toric (Theorem 4).

Acknowledgments. We wish to thank Angelo Vistoli for useful discussions on the topic of this paper (cf. Remark 2).

Notation. G always denotes a finite group, almost always abelian, and $G^* := \mathrm{Hom}(G, \mathbb{K}^*)$ the group of characters; $o(g)$ is the order of the element $g \in G$ and $|H|$ is the cardinality of a subgroup $H < G$. We work over an algebraically closed field $\mathbb{K}$ whose characteristic does not divide the order of the finite abelian groups we consider.
If A is an abelian group we write $A[d] := \{a \in A \mid da = 0\}$ (d an integer), $A^\vee := \mathrm{Hom}(A, \mathbb{Z})$ and we denote by $\mathrm{Tors}(A)$ the torsion subgroup of A.
The smooth part of a variety Y is denoted by Y_{sm}. The symbol $\equiv$ denotes linear equivalence of divisors. If Y is a normal variety we denote by $\mathrm{Cl}(Y)$ the group of classes, namely the group of Weil divisors up to linear equivalence.

2. Abelian covers

2.1. The fundamental relations

We quickly recall the theory of abelian covers (cf. [3], [1], and also [4]) in the most suitable form for the applications considered here.

There are slightly different definitions of abelian covers in the literature (see, for instance, [1] that treats also the non-normal case). Here we restrict our attention to the case of normal varieties, but we do not require that the covering map be flat; hence we define a cover as a finite morphism $\pi\colon X \to Y$ of normal varieties and we say that π is an abelian cover if it is a Galois morphism with abelian Galois group G (π is also called a "G-cover").

Recall that, as already stated in the Notations, throughout all the paper we assume that G has order not divisible by $\operatorname{char}\mathbb{K}$.

To every component D of the branch locus of π we associate the pair (H,ψ), where $H < G$ is the cyclic subgroup consisting of the elements of G that fix the preimage of D pointwise (the *inertia subgroup* of D) and ψ is the element of the character group H^* given by the natural representation of H on the normal space to the preimage of D at a general point (these definitions are well posed since G is abelian). It can be shown that ψ generates the group H^*.

If we fix a primitive d-th root ζ of 1, where d is the exponent of the group G, then a pair (H,ψ) as above is determined by the generator $g \in H$ such that $\psi(g) = \zeta^{\frac{d}{o(g)}}$. We follow this convention and attach to every component D_i of the branch locus of π a nonzero element $g_i \in G$.

If π is flat, which is always the case when Y is smooth, the sheaf $\pi_* \mathcal{O}_X$ decomposes under the G-action as $\oplus_{\chi\in G^*} L_\chi^{-1}$, where the L_χ are line bundles ($L_1 = \mathcal{O}_Y$) and G acts on L_χ^{-1} via the character χ.

Given $\chi \in G^*$ and $g \in G$, we denote by $\overline{\chi}(g)$ the smallest non-negative integer a such that $\chi(g) = \zeta^{\frac{ad}{o(g)}}$. The main result of [3] is that the L_χ, D_i (the *building data* of π) satisfy the following *fundamental relations*:

$$L_\chi + L_{\chi'} \equiv L_{\chi+\chi'} + \sum_{i=1}^{n} \varepsilon^i_{\chi,\chi'} D_i \qquad \forall \chi,\chi' \in G^* \tag{1}$$

where $\varepsilon^i_{\chi,\chi'} = \lfloor \frac{\overline{\chi}(g_i)+\overline{\chi'}(g_i)}{o(g_i)} \rfloor$. (Notice that the coefficients $\varepsilon^i_{\chi,\chi'}$ are equal either to 0 or to 1). Conversely, distinct irreducible divisors D_i and line bundles L_χ satisfying (1) are the building data of a flat (normal) G-cover $X \to Y$; in addition, if $h^0(\mathcal{O}_Y) = 1$ then $X \to Y$ is uniquely determined up to isomorphism of G-covers.

If we fix characters $\chi_1, \dots \chi_r \in G^*$ such that G^* is the direct sum of the subgroups generated by the χ_j, and we set $L_j := L_{\chi_j}$, $m_j := o(\chi_j)$, then the solutions of the fundamental relations (1) are in one-one correspondence with the solutions of the following *reduced fundamental relations*:

$$m_j L_j \equiv \sum_{i=1}^{n} \frac{m_j \overline{\chi_j}(g_i)}{d_i} D_i, \qquad j = 1, \dots r \tag{2}$$

As before, denote by d the exponent of G; notice that if $\mathrm{Pic}(Y)[d] = 0$, then for fixed (D_i, g_i), $i = 1, \dots n$, the solution of (2) is unique, hence the *branch data* (D_i, g_i) determine the cover.

In order to deal with the case when Y is normal but not smooth, we observe first that the cover $X \to Y$ can be recovered from its restriction $X' \to Y_{\mathrm{sm}}$ to the smooth locus by taking the integral closure of Y in the extension $\mathbb{K}(X') \supset \mathbb{K}(Y)$. Observe then that, since the complement $Y \setminus Y_{\mathrm{sm}}$ of the smooth part has codimension > 1, we have $h^0(\mathcal{O}_{Y_{\mathrm{sm}}}) = h^0(\mathcal{O}_Y) = 1$, and thus the cover $X' \to Y_{\mathrm{sm}}$ is determined by the building data L_χ, D_i. Using the identification $\mathrm{Pic}(Y_{\mathrm{sm}}) = \mathrm{Cl}(Y_{\mathrm{sm}}) = \mathrm{Cl}(Y)$, we can regard the L_χ as elements of $\mathrm{Cl}(Y)$ and, taking the closure, the D_i as Weil divisors on Y, and we can interpret the fundamental relations as equalities in $\mathrm{Cl}(Y)$. In this sense, if Y is normal variety with $h^0(\mathcal{O}_Y) = 1$, then the G-covers $X \to Y$ are determined by the building data up to isomorphism.

We say that an abelian cover $\pi\colon X \to Y$ is *totally ramified* if the inertia subgroups of the divisorial components of the branch locus of π generate G, or, equivalently, if π does not factorize through a cover $X' \to Y$ that is étale over Y_{sm}. We observe that a totally ramified cover is necessarily connected; conversely, equations (2) imply that if G is an abelian group of exponent d and Y is a variety such that $\mathrm{Cl}(Y)[d] = 0$, then any connected G-cover of Y is totally ramified.

2.2. The maximal cover

Let Y be a complete normal variety, let $D_1, \dots D_n$ be distinct irreducible effective divisors of Y and let $d_1, \dots d_n$ be positive integers (it is convenient to allow the possibility that $d_i = 1$ for some i). We set $d := \mathrm{lcm}(d_1, \dots d_n)$.

We say that a Galois cover $\pi\colon X \to Y$ is *branched on $D_1, \dots D_n$ with orders $d_1, \dots d_n$* if:

- the divisorial part of the branch locus of π is contained in $\sum_i D_i$;
- the ramification order of π over D_i is equal to d_i.

Let $\eta\colon \widetilde{Y} \to Y$ be a resolution of the singularities and set $\mathrm{N}(Y) := \mathrm{Cl}(Y)/\eta_*\mathrm{Pic}^0(\widetilde{Y})$. Since the map $\eta_*\colon \mathrm{Pic}(\widetilde{Y}) = \mathrm{Cl}(\widetilde{Y}) \to \mathrm{Cl}(Y)$ is surjective, $\mathrm{N}(Y)$ is a quotient of the Néron-Severi group $\mathrm{NS}(\widetilde{Y})$, hence it is finitely generated. It follows that $\eta_*\mathrm{Pic}^0(\widetilde{Y})$ is the largest divisible subgroup of $\mathrm{Cl}(Y)$ and therefore $\mathrm{N}(Y)$ does not depend on the choice of the resolution of Y (this is easily checked also by a geometrical argument).

The group $\mathrm{Cl}(Y)^{\vee}$ coincides with $\mathrm{N}(Y)^{\vee}$, hence it is a finitely generated free abelian group of rank equal to the rank of $\mathrm{N}(Y)$.

Consider the map $\mathbb{Z}^n \to \mathrm{Cl}(Y)$ that maps the i-th canonical generator to the class of D_i, let $\phi\colon \mathrm{Cl}(Y)^{\vee} \to \oplus_{i=1}^n \mathbb{Z}_{d_i}$ be the map obtained by composing the dual map $\mathrm{Cl}(Y)^{\vee} \to (\mathbb{Z}^n)^{\vee}$ with $(\mathbb{Z}^n)^{\vee} = \mathbb{Z}^n \to \oplus_{i=1}^n \mathbb{Z}_{d_i}$ and let $K_{\min}$ be the image of ϕ. Let $G_{\max}$ be the abelian group defined by the exact sequence:

$$0 \to K_{\min} \to \oplus_{i=1}^n \mathbb{Z}_{d_i} \to G_{\max} \to 0. \tag{3}$$

Then we have the following:

THEOREM 1. *Let Y be a normal variety with $h^0(\mathcal{O}_Y) = 1$, let $D_1, \dots D_n$ be distinct irreducible effective divisors, let $d_1, \dots d_n$ be positive integers and set $d := \mathrm{lcm}(d_1, \dots d_n)$. Then:*

1. *If $X \to Y$ is a totally ramified G-cover branched on $D_1, \dots D_n$ with orders $d_1, \dots d_n$, then:*
   (a) the map $\oplus_{i=1}^n \mathbb{Z}_{d_i} \to G$ that maps $1 \in \mathbb{Z}_{d_i}$ to g_i descends to a surjection $G_{\max} \twoheadrightarrow G$;
   (b) the map $\mathbb{Z}_{d_i} \to G_{\max}$ is injective for every $i = 1, \dots n$.
2. *If the map $\mathbb{Z}_{d_i} \to G_{\max}$ is injective for $i = 1, \dots n$ and $\mathrm{N}(Y)[d] = 0$, then there exists a maximal totally ramified abelian cover $X_{\max} \to Y$ branched on $D_1, \dots D_n$ with orders $d_1, \dots d_n$; the Galois group of $X_{\max} \to Y$ is equal to $G_{\max}$.*
3. *If the map $\mathbb{Z}_{d_i} \to G_{\max}$ is injective for $i = 1, \dots n$ and $\mathrm{Cl}(Y)[d] = 0$, then the cover $X_{\max} \to Y$ is unique up to isomorphism of $G_{\max}$-covers and every totally ramified abelian cover $X \to Y$ branched on $D_1, \dots D_n$ with orders $d_1, \dots d_n$ is a quotient of $X_{\max}$ by a subgroup of $G_{\max}$.*

Proof. Let $H_1, \dots H_t \in \mathrm{N}(Y)$ be elements whose classes are free generators of the abelian group $\mathrm{N}(Y)/\mathrm{Tors}(\mathrm{N}(Y))$, and write:

$$D_i = \sum_{j=1}^{t} a_{ij} H_j \mod \mathrm{Tors}(\mathrm{N}(Y)), \qquad j = 1, \dots t \tag{4}$$

Hence, the subgroup $K_{\min}$ of $\oplus_{i=1}^n \mathbb{Z}_{d_i}$ is generated by the elements $z_j := (a_{1j}, \dots a_{nj})$, for $j = 1, \dots t$.

Let $X \to Y$ be a totally ramified G-cover branched on $D_1, \dots D_n$ with orders $d_1, \dots d_n$ and let (D_i, g_i) be its branch data. Consider the map $\oplus_{i=1}^n \mathbb{Z}_{d_i} \to G$ that maps $1 \in \mathbb{Z}_{d_i}$ to g_i: this map is surjective, by the assumption that $X \to Y$ is totally ramified, and its restriction to $\mathbb{Z}_{d_i}$ is injective for $i = 1, \dots n$, since the cover is branched on D_i with order d_i. If we denote by K the kernel of $\oplus_{i=1}^n \mathbb{Z}_{d_i} \to G$, to prove (1) it suffices to show that $K \supseteq K_{\min}$. Dually, this is equivalent to showing that $G^* \subseteq K_{\min}^{\perp} \subset \oplus_{i=1}^n (\mathbb{Z}_{d_i})^*$. Let $\psi_i \in (\mathbb{Z}_{d_i})^*$ be the generator that maps $1 \in \mathbb{Z}_{d_i}$ to $\zeta^{\frac{d}{d_i}}$ and write $\chi \in G^*$

as $(\psi_1^{b_1},\dots\psi_n^{b_n})$, with $0\le b_i<d_i$; if $o(\chi)=m$ then (2) gives $mL_\chi\equiv\sum_{i=1}^n\frac{mb_i}{d_i}D_i$. Plugging (4) in this equation we obtain that $\sum_{i=1}^n\frac{b_ia_{ij}}{d_i}$ is an integer for $j=1,\dots t$, namely $\chi\in K_{\min}^\perp$.

(2) Let $\chi_1,\dots\chi_r$ be a basis of $G^*_{\max}$ and, as above, for $s=1,\dots r$ write $\chi_s=(\psi_1^{b_{s1}},\dots\psi_n^{b_{sn}})$, with $0\le b_{si}<d_i$. Since by assumption $\mathrm{N}(Y)[d]=0$, by the proof of (1) the elements $\sum_{j=1}^t(\sum_{i=1}^n\frac{b_{si}a_{ij}}{d_i})H_j$, $s=1,\dots r$, can be lifted to solutions $\overline{L_s}\in\mathrm{N}(Y)$ of the reduced fundamental relations (2) for a $G_{\max}$-cover with branch data (D_i,g_i), where $g_i\in G$ is the image of $1\in\mathbb{Z}_{d_i}$. Since the kernel of $\mathrm{Cl}(Y)\to N(Y)$ is a divisible group, it is possible to lift the $\overline{L_s}$ to solutions $L_s\in\mathrm{Cl}(Y)$. We let $X_{\max}\to Y$ be the $G_{\max}$-cover determined by these solutions. It is a totally ramified cover since the map $\oplus_{i=1}^n\mathbb{Z}_{d_i}\to G_{\max}$ is surjective by the definition of $G_{\max}$.

(3) Since $\mathrm{Cl}(Y)[d]=0$, any G-cover such that the exponent of G is a divisor of d is determined uniquely by the branch data; in particular, this holds for the cover $X_{\max}\to Y$ in (2) and for every intermediate cover $X_{\max}/H\to Y$, where $H<G_{\max}$. The claim now follows by (1). $\square$

EXAMPLE 1. Take $Y=\mathbb{P}^{n-1}$ and let $D_1,\dots D_n$ be the coordinate hyperplanes. In this case the group $K_{\min}$ is generated by $(1,\dots 1)\in\oplus_{i=1}^n\mathbb{Z}_{d_i}$. Since any connected abelian cover of $\mathbb{P}^{n-1}$ is totally ramified, by Theorem 1 there exists a abelian cover of $\mathbb{P}^{n-1}$ branched over $D_1,\dots D_n$ with orders $d_1,\dots d_n$ iff d_i divides $lcm(d_1,\dots,\widehat{d_i},\dots d_n)$ for every $i=1,\dots n$. For $d_1=\dots=d_n=d$, then $G_{\max}=\mathbb{Z}_d^n/<(1,\dots 1)>$ and $X_{\max}\to\mathbb{P}^{n-1}$ is the cover $\mathbb{P}^{n-1}\to\mathbb{P}^{n-1}$ defined by $[x_1,\dots x_n]\mapsto[x_1^d,\dots x_n^d]$.

In general, $X_{\max}$ is a weighted projective space $\mathbb{P}(\frac{d}{d_1},\dots\frac{d}{d_n})$ and the cover is given by $[x_1,\dots x_n]\mapsto[x_1^{d_1},\dots x_n^{d_n}]$.

3. Toric covers

NOTATIONS 2. Here, we fix the notations which are standard in toric geometry. A (complete normal) toric variety Y corresponds to a fan Σ living in the vector space $N\otimes\mathbb{R}$, where $N\cong\mathbb{Z}^s$. The dual lattice is $M=N^\vee$. The torus is $T=N\otimes\mathbb{C}^*=\mathrm{Hom}(M,\mathbb{C}^*)$.

The integral vectors $r_i\in N$ will denote the integral generators of the rays $\sigma_i\in\Sigma(1)$ of the fan Σ. They are in a bijection with the T-invariant Weil divisors D_i ($i=1,\dots,n$) on Y.

DEFINITION 1. *A* toric cover $f\colon X\to Y$ *is a finite morphism of toric varieties corresponding to the map of fans* $F\colon(N',\Sigma')\to(N,\Sigma)$ *such that:*

1. $N'\subseteq N$ *is a sublattice of finite index, so that* $N'\otimes\mathbb{R}=N\otimes\mathbb{R}$.
2. $\Sigma'=\Sigma$.

The proof of the following lemma is immediate.

LEMMA 1. *The morphism f has the following properties:*

1. *It is equivariant with respect to the homomorphism of tori $T' \to T$.*
2. *It is an abelian cover with Galois group $G = \ker(T' \to T) = N/N'$.*
3. *It is ramified only along the boundary divisors D_i, with multiplicities $d_i \geq 1$ defined by the condition that the integral generator of $N' \cap \mathbb{R}_{\geq 0} r_i$ is $d_i r_i$.*

PROPOSITION 1. *Let Y be a complete toric variety such that $\operatorname{Cl}(Y)$ is torsion free, and $X \to Y$ be a toric cover. Then, with notations as above, there exists the following commutative diagram with exact rows and columns.*

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \operatorname{Cl}(Y)^\vee \arrow[r] \arrow[d] & K \arrow[d] & \\
0 \arrow[r] & \oplus_{i=1}^n \mathbb{Z} d_i D_i^* \arrow[r] \arrow[d, "p'"] & \oplus_{i=1}^n \mathbb{Z} D_i^* \arrow[r] \arrow[d, "p"] & \oplus_{i=1}^n \mathbb{Z}_{d_i} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N' \arrow[r] & N \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

(Here the D_i^* are formal symbols denoting a basis of $\mathbb{Z}^n$). Moreover, each of the homomorphisms $\mathbb{Z}_{d_i} \to G$ is an embedding.

Proof. The third row appeared in Lemma 1, and the second row is the obvious one.

It is well known that the boundary divisors on a complete normal toric variety span the group $\operatorname{Cl}(Y)$, and that there exists the following short exact sequence of lattices:

$$0 \to M \longrightarrow \oplus_{i=1}^n \mathbb{Z} D_i \longrightarrow \operatorname{Cl}(Y) \to 0.$$

Since $\operatorname{Cl}(Y)$ is torsion free by assumption, this sequence is split and dualizing it one obtains the central column. Since $\oplus_{i=1}^n \mathbb{Z} D_i^* \to N$ is surjective, then so is $\oplus_{i=1}^n \mathbb{Z}_{d_i} \to G$. The group K is defined as the kernel of this map.

Finally, the condition that $\mathbb{Z}_{d_i} \to G$ is injective is equivalent to the condition that the integral generator of $N' \cap \mathbb{R}_{\geq 0} r_i$ is $d_i r_i$, which holds by Lemma 1. □

THEOREM 3. *Let Y be a complete toric variety such that $\operatorname{Cl}(Y)$ is torsion free, let $d_1, \ldots d_n$ be positive integers and let $K_{\min}$ and $G_{\max}$ be defined as in sequence* (3). *Then:*

1. *There exists a toric cover branched on D_i of order d_i, $i = 1, \dots n$, iff the map $\mathbb{Z}_{d_i} \to G_{\max}$ is injective for $i = 1, \dots n$.*
2. *If condition (1) is satisfied, then among all the toric covers of Y ramified over the divisors D_i with multiplicities d_i there exists a maximal one $X_{\mathrm{Tmax}} \to Y$, with Galois group $G_{\max}$, such that any other toric cover $X \to Y$ with the same branching orders is a quotient $X = X_{\mathrm{Tmax}}/H$ by a subgroup $H < G_{\max}$.*

Proof. Let $X \to Y$ be a toric cover branched on $D_1, \dots D_n$ with orders $d_1, \dots d_n$, let N' be the corresponding sublattice of N and $G = N/N'$ the Galois group. Let $N'_{\min}$ be the subgroup of N generated by $d_i r_i$, $i = 1, \dots n$. By Lemma 1 one must have $N'_{\min} \subseteq N'$, hence the map $\mathbb{Z}_{d_i} \to N/N'_{\min}$ is injective since $\mathbb{Z}_{d_i} \to G = N/N'$ is injective by Proposition 1. We set $X_{\mathrm{Tmax}} \to Y$ to be the cover for $N'_{\min}$. Clearly, the cover for the lattice N' is a quotient of the cover for the lattice $N'_{\min}$ by the group $H = N'/N'_{\min}$.

Consider the second and third rows of the diagram of Proposition 1 as a short exact sequence of 2-step complexes $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$. The associated long exact sequence of cohomologies gives

$$\mathrm{Cl}(Y)^\vee \longrightarrow K \longrightarrow \mathrm{coker}(p') \longrightarrow 0$$

For $N' = N'_{\min}$, the map p' is surjective, hence $\mathrm{Cl}(Y)^\vee \to K$ is surjective too, and $K = K_{\min}$, $N/N'_{\min} = G_{\max}$.

Vice versa, suppose that in the following commutative diagram with exact row and columns each of the maps $\mathbb{Z}_{d_i} \to G_{\max}$ is injective.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathrm{Cl}(Y)^\vee \arrow[r, "q"] \arrow[d] & K_{\min} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \oplus_{i=1}^n \mathbb{Z} d_i D_i^* \arrow[r] & \oplus_{i=1}^n \mathbb{Z} D_i^* \arrow[r] \arrow[d, "p"] & \oplus_{i=1}^n \mathbb{Z}_{d_i} \arrow[r] \arrow[d] & 0 \\
 & & N \arrow[d] & G_{\max} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

We complete the first row on the left by adding $\ker(q)$. We have an induced homomorphism $\ker(q) \to \oplus \mathbb{Z} d_i D_i^*$, and we define N' to be its cokernel.

Now consider the completed first and second rows as a short exact sequence of 2-step complexes $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$. The associated long exact sequence of cohomologies says that $\ker(q) \to \oplus_{i=1}^n \mathbb{Z} d_i D_i^*$ is injective, and the sequence

$$0 \longrightarrow N' \longrightarrow N \longrightarrow G_{\max} \longrightarrow 0$$

is exact. It follows that $N' = N'_{\min}$ and the toric morphism $(N'_{\min}, \Sigma) \to (N, \Sigma)$ is then the searched-for maximal abelian toric cover. $\square$

REMARK 1. Condition (1) in the statement of Theorem 3 can also be expressed by saying that for $i = 1, ..., n$ the element $d_i r_i \in N'_{\min}$ is primitive, where $N'_{\min} \subseteq N$ is the subgroup generated by all the $d_i r_i$.

We now combine the results of this section with those of §2 to obtain a structure result for Galois covers of toric varieties.

THEOREM 4. *Let Y be a normal complete toric variety and let $f\colon X \to Y$ be a connected cover such that the divisorial part of the branch locus of f is contained in the union of the invariant divisors $D_1, \dots D_n$.*
Then $\deg f$ *is not divisible by* $\operatorname{char} \mathbb{K}$ *and* $f\colon X \to Y$ *is a toric cover.*

Proof. Let $U \subset Y$ be the open orbit and let $X' \to U$ be the cover obtained by restricting f. Since U is smooth, by the assumptions and by purity of the branch locus, $X' \to U$ is étale. Let $X'' \to U$ be the Galois closure of $X' \to U$: the cover $X'' \to U$ is also étale, hence by [2, Prop. 1] it is, up to isomorphism, a homomorphism of tori. Since the kernel of an étale homomorphism of tori is reduced, it follows that $X'' \to U$ is an abelian cover such that $\operatorname{char} \mathbb{K}$ does not divide the order of the Galois group.

Moreover, the intermediate cover $X' \to U$ is also abelian (actually $X' = X''$). The cover $f\colon X \to Y$ is abelian, too, since X is the integral closure of Y in $\mathbb{K}(X')$. We denote by G the Galois group of f and by $d_1, \dots d_n$ the orders of ramification of $X \to Y$ on $D_1, \dots D_n$.
Assume first that $\operatorname{Cl}(Y)$ has no torsion, so that every connected abelian cover of Y is totally ramified (cf. §2). Then by Theorem 1 every connected abelian cover branched on $D_1, \dots D_n$ with orders $d_1, \dots d_n$ is a quotient of the maximal abelian cover $X_{\max} \to Y$ by a subgroup $H < G_{\max}$. In particular, this is true for the cover $X_{\mathrm{Tmax}} \to Y$ of Theorem 3. Since $X_{\max}$ and X_{Tmax} have the same Galois group it follows that $X_{\max} = X_{\mathrm{Tmax}}$. Hence $X \to Y$, being a quotient of X_{Tmax}, is a toric cover.

Consider now the general case. Recall that the group $\operatorname{Tors} \operatorname{Cl}(Y)$ is finite, isomorphic to $N/\langle r_i \rangle$, and the cover $Y' \to Y$ corresponding to $\operatorname{Tors} \operatorname{Cl}(Y)$ is toric, and one has $\operatorname{Tors} \operatorname{Cl}(Y') = 0$. Indeed, on a toric variety the group $\operatorname{Cl}(Y)$ is generated by the T-invariant Weil divisors D_i. Thus, $\operatorname{Cl}(Y)$ is the quotient of the free abelian group $\oplus \mathbb{Z} D_i$ of all T-invariant divisors modulo the subgroup M of principal T-invariant divisors. Thus, $\operatorname{Tors} \operatorname{Cl}(Y) \simeq M'/M$, where $M' \subset \oplus \mathbb{Q} D_i$ is the subgroup of $\mathbb{Q}$-linear functions on N taking integral values on the vectors r_i. Then $N' := M'^{\vee}$ is the subgroup of N generated by the r_i, and the cover $Y' \to Y$ is the cover corresponding to the map of fans $(N', \Sigma) \to (N, \Sigma)$. On Y' one has $N' = \langle r_i \rangle$, so $\operatorname{Tors} \operatorname{Cl}(Y') = 0$.

Let $X' \to Y'$ be a connected component of the pull back of $X \to Y$: it is an abelian cover branched on the invariant divisors of Y', hence by the first part of the proof it is toric. The map $X' \to Y$ is toric, since it is a composition of toric morphisms, hence the intermediate cover $X \to Y$ is also toric. $\square$

REMARK 2. The argument that shows that the map f is an abelian cover in the proof of Theorem 4 was suggested to us by Angelo Vistoli. He also remarked that it is possible to prove Theorem 3.6 in a more conceptual way by showing that the torus action on the cover $X' \to U$ of the open orbit of Y extends to X, in view of the properties of the integral closure. However our approach has the advantage of describing explicitly the fan/building data associated with the cover.

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AMS Subject Classification: 14E20, 14L32

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Lavoro pervenuto in redazione il 18.06.2013.

Rend. Sem. Mat. Univ. Politec. Torino
Vol. 71, 3–4 (2013), 317 – 338

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TWISTOR LINES ON CUBIC SURFACES*

Abstract. It is shown that there exist non-singular cubic surfaces in $\mathbb{CP}^3$ containing 5 twistor lines. This is the maximum number of twistor fibres that a non-singular cubic can contain. Cubic surfaces in $\mathbb{CP}^3$ with 5 twistor lines are classified up to transformations preserving the conformal structure of S^4.

Introduction

The twistor space, Z, of an oriented Riemannian 4-manifold M is the bundle of almost-complex structures on M compatible with the metric and orientation. The 6-dimensional total space of the twistor space admits a canonical almost-complex structure which is integrable whenever the 4-manifold is half conformally flat.

The definition of the twistor space does not require the full Riemannian metric; it only depends upon the conformal structure of the manifold. The idea of studying the twistor space is that, on a half conformally flat manifold, the conformal geometry of M is encoded into the complex geometry of Z.

As an example, the condition that an almost-complex structure on M compatible with the conformal structure is *integrable* can be interpreted as saying that the corresponding section of Z defines a *holomorphic* submanifold of Z.

The basic example of twistor space is that of the 4-sphere S^4, which itself may be identified with the quaternionic projective line $\mathbb{HP}^1$, and is topologically $\mathbb{R}^4 \cup \infty$. The twistor space in this case is biholomorphic to $\mathbb{CP}^3$, and the associated bundle structure $\mathbb{CP}^3 \to \mathbb{HP}^1$ is the Hopf fibration. Following the work of Penrose, Ward and Atiyah, it was used to great effect in classifying instanton bundles on S^4 [3, 2].

Combining these two facts, we see that complex hypersurfaces in $\mathbb{CP}^3$ locally give rise to integrable complex structures on S^4 compatible with the metric. For topological reasons there are no global almost-complex structures on S^4, so no hypersurface in $\mathbb{CP}^3$ can intersect every fibre of the Hopf fibration in exactly one point.

One can try to investigate the algebraic geometry of surfaces in $\mathbb{CP}^3$ from this twistor perspective. In this paper, we take the opportunity to revisit some of the beautiful results on cubic surfaces discovered by geometers in the nineteenth century. A brief history of their discoveries can be found in [9].

A natural question when studying complex surfaces from this point of view is to classify surfaces in $\mathbb{CP}^3$ of degree d up to a conformal transformation of the base space S^4. Various conformal invariants of a surface can be defined immediately. The fibres of the Hopf fibration are complex projective lines in $\mathbb{CP}^3$, and the number of fibres that

*Based on the third author's talk on 23 March 2012 at the conference *Geometria delle Varietà Algebriche* in Turin, dedicated to Alberto Conte lie entirely in the suface is an invariant of the surface up to conformal transformation.

Closely related invariants arise from the topology of the discriminant locus. A generic fibre, intersecting the surface transversely, will contain d points. This is simply because the defining polynomial of the surface, when restricted to the fibre, gives a polynomial of degree d. The set of points where the discriminant of this polynomial vanishes is called the discriminant locus. It can be thought of as the set of fibres that are not transverse to the surface at each point of intersection, or as the set of points of S^4 where we cannot locally define a complex structure corresponding to the surface.

In [18], quadric surfaces are classified up to conformal transformation in considerable detail. The table below shows all the possible topologies of the discriminant locus in this case and how they correspond to the number of twistor lines. A preliminary question to ask when trying to study the conformal geometry of surfaces of higher degree is: what is the maximum number of twistor lines on surfaces of that degree?

No of twistor lines	Topology of discriminant locus
0	Torus
1	Torus with two points pinched together
2	Torus with two pairs of points pinched together
∞	Circle

Before restricting to twistor lines, it is worth reviewing pure algebro-geometric results on the number of projective lines on a surface of given degree. Since twistor lines are fibres of a fibration they must be skew (i.e., mutually disjoint), so we will also review the maximum number of skew lines on a surface of degree d.

Dimension counting alone leads one to expect that a quadric surface will contain an infinite number of lines, a cubic surface a finite number of lines and a higher degree surface will generically contain no lines at all.

A startling result is the celebrated Cayley–Salmon theorem: all non-singular cubic surfaces contain precisely 27 lines. Moreover a non-singular cubic surface contains precisely 72 sets of 6 skew lines.

The situation for higher degree curves is less well understood. The state of knowledge about the number of lines on surfaces of degree d was both reviewed and advanced in [4]. We summarize these findings next.

Define N_d to be the maximum number of lines on a smooth projective surface of degree d. Then:

- there are always 27 lines on a cubic,
- $N_4 = 64$ (see [20]),
- $N_d \leqslant (d-2)(11d-6)$ (see [20]),
- $N_d \geqslant 3d^2$ (see [6]),
- $N_6 \geqslant 180$, $N_8 \geqslant 352$, $N_{12} \geqslant 864$, $N_{20} \geqslant 1600$ (see [6, 4]).

Here are the bounds on S_d, the maximum number of skew lines:

- there are always 6 skew lines on a cubic,
- $S_4 = 16$ (see [15]),
- $S_d \leqslant 2d(d-2)$ when $d \geqslant 4$ (see [13]),
- $S_d \geqslant d(d-2)+2$ (see [17]),
- $S_d \geqslant d(d-2)+4$ when $d \geqslant 7$ is odd (see [4]).

Specializing to the case of twistor lines, it was noted in the first version of [18] that the number of twistor lines is at most d^2 when $d \geqslant 3$ and that there exists a quartic surface containing exactly 8 twistor lines.

In this paper, we determine the maximum number of twistor lines on a smooth cubic surface. We shall show that in fact there are at most 5 twistor lines, and we shall give a detailed classification of all cubic surfaces with 5 twistor lines. In particular we shall prove the

THEOREM. Any set of 5 points on a 2-sphere, no 4 of which lie on a circle, determines a one-parameter family of non-singular cubic surfaces with 5 twistor lines. All cubics in the family are projectively, but not conformally, equivalent. Two such cubic surfaces are projectively equivalent if and only if the sets of 5 points on the 2-sphere are conformally equivalent. All cubic surfaces with 5 twistor lines arise in this way.

One would like explicit examples of such surfaces. We provide the necessary formulae and find the most symmetrical examples. In particular, we shall show that the cubic surface with 5 twistor lines which has the largest conformal symmetry group is projectively, but not conformally, equivalent to the Fermat cubic. There are various choices one can make for a twistor structure on $\mathbb{CP}^3$ that give the Fermat cubic 5 twistor lines, and the set of such structures has 54 connected components.

The paper begins with a brief review of the twistor fibration of $\mathbb{CP}^3$ and then moves on to discuss cubic surfaces. We review the classical results on cubic surfaces and demonstrate how the same ideas can be used to prove results about the twistor geometry.

1. The twistor fibration

To identify S^4 with $\mathbb{HP}^1$, we define two equivalence relations on $\mathbb{H}\times\mathbb{H}$:

$$[q_1,q_2] \sim_{\mathbb{H}} [\lambda q_1, \lambda q_2], \qquad \lambda \in \mathbb{H}^*,$$

$$[q_1,q_2] \sim_{\mathbb{C}} [\lambda q_1, \lambda q_2], \qquad \lambda \in \mathbb{C}^*.$$

By definition, the quotient of $\mathbb{H}\times\mathbb{H}$ by the first equivalence relation is the quaternionic projective line. Since $\mathbb{H}\times\mathbb{H} \cong \mathbb{C}^4$, the quotient by the second relation is isomorphic to the complex projective space $\mathbb{CP}^3$.

Thus we can define a map $\pi : \mathbb{CP}^3 \to S^4$ by sending a complex 1-dimensional subspace of $\mathbb{C}^4$ to the quaternionic 1-dimensional subspace of $\mathbb{H}^2$ that it spans. The map π is equivalent to the more general twistor fibration defined on an arbitrary oriented Riemannian 4-manifold as the total space of the bundle of almost-complex structures compatible with the metric and orientation.

On any twistor fibration one can define a map j which sends an almost-complex structure J to $-J$. In our case, applying j can be thought of as the action of multiplying a 1-dimensional complex subspace of $\mathbb{C}^4$ by the quaternion j in order to get a new 1-dimensional subspace.

The map j is an anti-holomorphic involution of the twistor space to itself with no fixed points. Starting with such a map j, one can recover the twistor fibration: given a point z in $\mathbb{CP}^3$ there is a unique projective line connecting z and $j(z)$. These lines form the fibres. We will call an anti-holomorphic involution on $\mathbb{CP}^3$ obtained by conjugating j by a projective transformation a *twistor structure*. The standard twistor structure on $\mathbb{CP}^3$ is given by

$$[z_1, z_2, z_3, z_4] \longmapsto [-\overline{z_2}, \overline{z_1}, -\overline{z_4}, \overline{z_3}].$$

The conformal symmetries of S^4 can be represented by quaternionic Möbius transformations

$$q \longmapsto (qc+d)^{-1}(qa+b), \qquad q \in \mathbb{H}$$

(see, for example, [10]). These correspond to the projective transformations of $\mathbb{CP}^3$ that preserve j. Thus we will say that two complex submanifolds of $\mathbb{CP}^3$ are *conformally equivalent* if they are projectively equivalent by a transformation that preserves j.

As an example, consider lines in $\mathbb{CP}^3$. If both lines are fibres of π then they are conformally equivalent by an isometry of S^4 sending the image of one line under π to the image of the other line. If a line is not a fibre of π then its image will be a round 2-sphere in S^4 (corresponding to a 2-sphere or a 2-plane in $\mathbb{R}^4$). Given such a 2-sphere in S^4, there are in fact two projective lines lying above it in $\mathbb{CP}^3$. Therefore, a line in $\mathbb{CP}^3$ is given by either an oriented 2-sphere or a point in S^4. Moreover, any two such 2-spheres are conformally equivalent. This geometrical correspondence is described in detail by Shapiro [21].

As another example, consider planes in $\mathbb{CP}^3$. A plane in $\mathbb{CP}^3$ cannot be transverse to every fibre of π because it would then define a complex structure on the whole of S^4, which is a topological impossibility. Thus a plane always contains at least one twistor fibre. Twistor fibres are always skew, whereas two lines in a plane always meet. Therefore a plane always contains exactly one twistor fibre. If one picks another line in the plane transverse to the fibre, its image under π will be a 2-sphere. We can find a conformal transformation of S^4 mapping any 2-sphere with a marked point to any other 2-sphere with a marked point. We deduce that any two planes in $\mathbb{CP}^3$ are conformally equivalent.

The case of quadric surfaces is considered in detail in [18] and is much more complicated. The aim of this paper is to make a start on the analogous question for cubic surfaces.

2. The Schläfli graph

Before looking at the twistor geometry of cubic surfaces. Let us review the classical results about the lines on twistor surfaces.

The Cayley–Salmon theorem states that every non-singular cubic surface contains exactly 27 lines [7]. Schläfli discovered that the intersection properties of these 27 lines are the same for all cubics [19]. This means that we can define the Schläfli graph of a cubic surface to be a graph with 27 vertices corresponding to each line on the cubic and and with an edge between the two vertices whenever the corresponding lines *do not* intersect. This graph will be independent of the choice of non-singular cubic surface. This definition is the standard one used by graph theorists, but from our point of view the complement of the Schläfli graph showing which lines *do* intersect is more natural. It is shown in Figure 1.

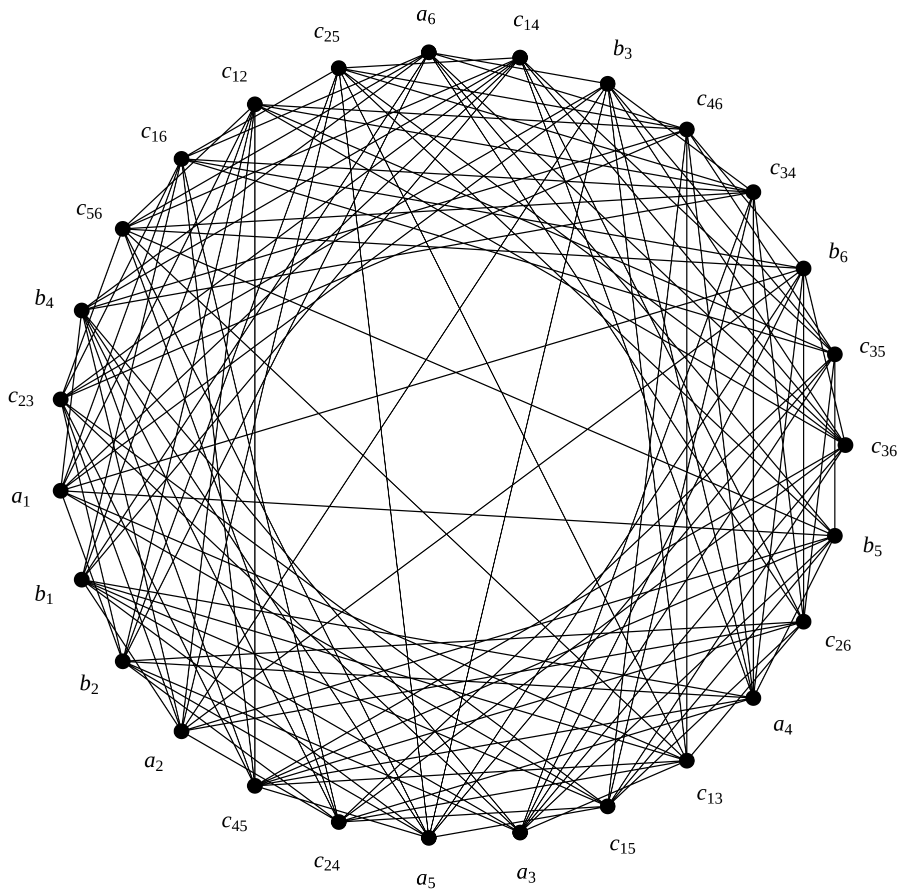

Figure 1: The complement of the Schlafli graph emphasizing a symmetry of order 9

Understanding the Schläfli graph provides a good deal of insight into the geometry of cubic surfaces. It is interesting from a purely graph theoretic point of view. Among its many properties, one particularly nice one is that it is 4-*ultrahomogeneous*. A graph is said to be k-ultrahomogeneous if every isomorphism between subgraphs with at most k vertices extends to an automorphism of the entire graph. If a graph is 5-ultrahomogenous it is k-ultrahomogeneous for any k. It turns out that the Schläfli graph and its complement are the only 4-ultrahomogeneous connected graphs that are not 5-ultrahomogeneous [5].

Although our picture of the Schläfli graph is pretty, it is not very practical. Schläfli devised a notation that allows one to understand the graph more directly, and we shall now describe this.

Among the 27 lines one can always find a set of 6 skew lines. We label these a_1, a_2, a_3, a_4, a_5 and a_6. Having chosen these labels, there will now be 6 more skew lines b_i ($1 \leqslant i \leqslant 6$) with each b_i intersecting all of the a lines except for a_i. We thereby obtain the configuration shown in Figure 2.

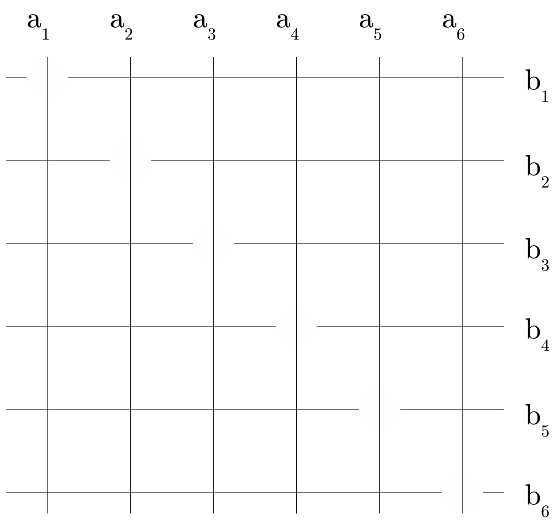

Figure 2: A "double six"

The remaining 15 lines are labelled c_{ij} with ($1 \leqslant i < j \leqslant 6$). The line c_{ij} is defined by the property that it intersects a_i and a_j, but no other a lines. The full intersection rules for distinct lines on the cubic surface are:

- a_i never intersects a_j.
- a_i intersects b_j iff $i \neq j$.
- a_i intersects c_{jk} iff $i \in \{j,k\}$.

- b_i never intersects b_j.
- b_i intersects c_{jk} iff $i \in \{j,k\}$.
- c_{ij} intersects c_{kl} iff $\{i,j\} \cap \{k,l\} = \varnothing$.

Another graphical representation of these properties was given in [16] and is reproduced in Figure 3. This is intended to be used rather like the tables of distances between towns that are used to be found in road atlases. A red/darker square indicates that the two lines intersect and a cyan/lighter ones indicates that the lines are skew. In this case we have chosen the ordering of the lines to show that this figure can be constructed using only a small number of different types of tile of size 3×3. The grouping is indicated with black lines.

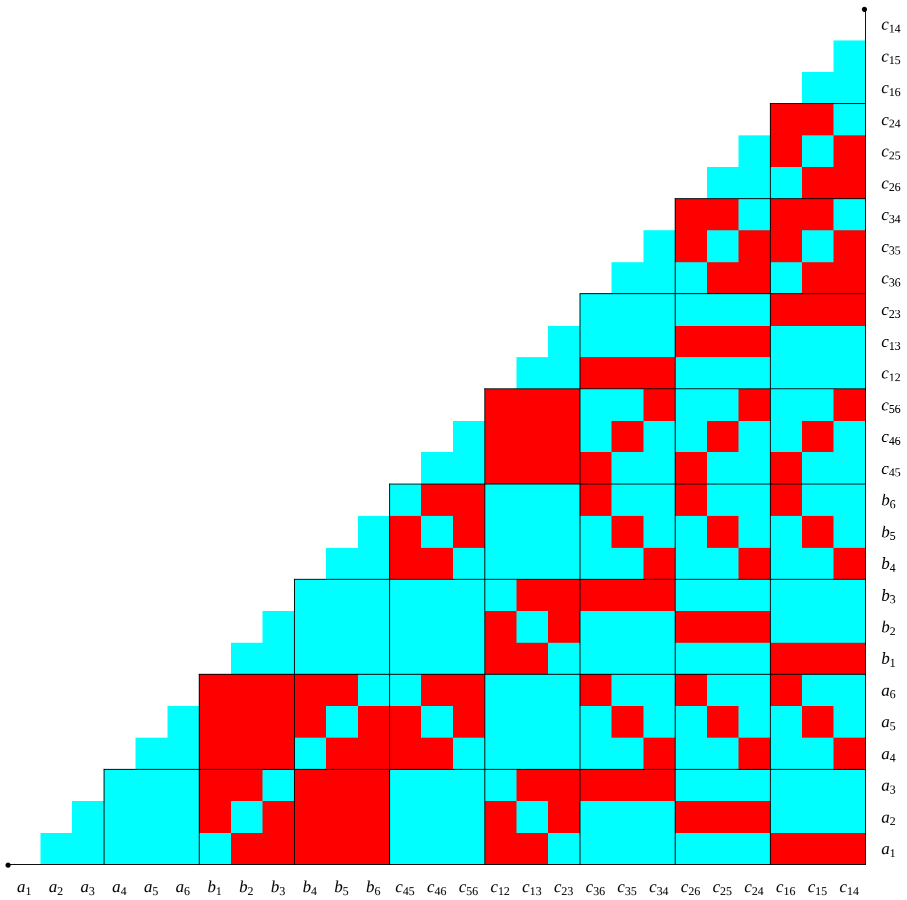

Figure 3: A tabular representation of the Schäfli graph

Being 4-ultrahomogeneous, it is no surprise that the Schläfli graph has a large group of automorphisms. Each choice of 6 skew lines from the 27 will give us a different way of labelling the lines as a_i, b_i and c_{ij}. Schläfli used the following notation for each choice:

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{pmatrix}$$

A set of 12 lines with these intersection properties is called a "double-six". In this notation here are the forms of the other double sixes:

$$\begin{pmatrix} a_1 & a_2 & a_3 & c_{56} & c_{46} & c_{45} \\ c_{23} & c_{13} & c_{12} & b_4 & b_5 & b_6 \end{pmatrix}$$

$$\begin{pmatrix} a_1 & b_1 & c_{23} & c_{24} & c_{25} & c_{26} \\ a_2 & b_2 & c_{13} & c_{14} & c_{15} & c_{16} \end{pmatrix}$$

By varying the indices this gives a total of 36 double sixes on any cubic surface — hence 72 choices of a set of six disjoint lines and $72 \cdot 6! = 51840$ automorphisms of the Schläfli graph.

This large automorphism group is in fact the Weyl group of the exceptional Lie group E_6, and 27 is the dimension of the smallest non-trivial irreducible representation of E_6. Relative to the isotropy subgroup $SU(6) \times_{\mathbb{Z}_2} SU(2)$ of the corresponding Wolf space [22], this representation decomposes into irreducible subspaces of dimension 12 and 15, namely $\mathbb{C}^2 \otimes \mathbb{C}^6$ and $\bigwedge^2 \overline{\mathbb{C}^6}$. This is the algebraic interpretation of a double six.

As Schläfli discovered, consideration of the arrangement of the 27 lines on a non-singular cubic surface rapidly leads to a classification of cubic surfaces up to projective transformation [19]. By applying the same ideas, we can find a similar classification of cubic surfaces with sufficiently many twistor lines.

3. Classifying cubic surfaces with 5 twistor lines

As a first application of the Schläfli graph to the study of twistor lines on cubic surfaces we prove

LEMMA 1. *If a non-singular cubic surface in $\mathbb{CP}^3$ contains four twistor lines a_1, a_2, a_3, a_4 then, in Schläfli's notation, $jb_5 = b_6$.*

Proof. Since the Schläfli graph is 4 ultrahomogenoeus and twistor lines are always skew, we can assume that the first four lines are indeed those of a double six.

In Schläfli's notation, the line b_5 intersects a_1, a_2, a_3 and a_4. Therefore jb_5 intersects $ja_1 = a_1$, $ja_2 = a_2$, $ja_3 = a_3$ and $ja_4 = a_4$. Since it jb_5 is a line and since it intersects the cubic surface in 4 points, it must lie in the cubic surface. Since j has no fixed points, the points of intersection of b_5 and jb_5 with the line a_1 must be distinct. So $jb_5 \neq b_5$, Given this and the fact that it intersects a_1, a_2, a_3 and a_4 we deduce that $jb_5 = b_6$. $\square$

Corollary 1. *If a non-singular cubic surface contains five twistor lines, and we label the first four a_1, a_2, a_3, a_4 in Schläfli's notation then the fifth line is c_{56}.*

Proof. Since the fifth twistor line must be skew to a_1, a_2, a_3 and a_4, it must be one of a_5, a_6, c_{56}. Suppose that the fifth line is a_6. This means it intersects b_5 so $ja_6 = a_6$ intersects $jb_5 = b_6$, which is a contradiction. The same argument shows that the fifth line cannot be a_5. □

The arrangement of lines described by Corollary 1 is illustrated in Figure 4. The twistor lines are shown as roughly vertical.

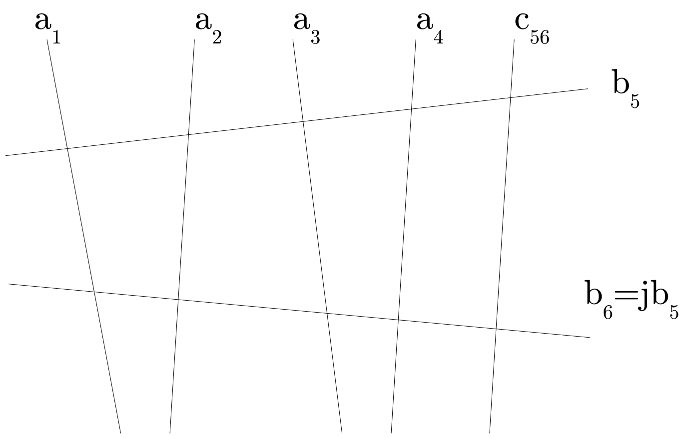

Figure 4: Seven lines

In particular we have proved:

Theorem 1. *A non-singular cubic surface contains at most five twistor lines.*

This raises the question of whether or not we can find cubic surfaces containing 5 twistor lines. Simple dimension counting suggests it should be easy to find cubic surfaces which contain 4 twistor lines. Simply select any four twistor lines and apply the well-known

Proposition 1. *Four lines in $\mathbb{CP}^3$ always lie on a (possibly singular) cubic surface.*

Proof. Pick 4 points on each line to get a total of 16 points. If a cubic surface has 4 points in common with a line, then it contains the entire line. So if we can find a cubic containing all 16 points, it will contain all 4 lines.

The general equation for a cubic surface has 20 coefficients, since this is the dimension of $S^3(\mathbb{C}^4)$. Putting the coordinates of these 16 points into the equation for the cubic surface gives us 16 linear equations in the 20 unknown coefficients, so non-trivial solutions exist. □

It will become clear later that if we choose everything generically, the cubic surface will be non-singular. A corollary of this is that *four generic lines in* $\mathbb{CP}^3$ *have two lines intersecting all four of them*. This observation can be proved easily enough without appealing to the theory of cubics — for example one can use 2-forms to represent points of the Klein quadric, or Schubert calculus. Indeed, in [12] this observation is used as the starting point to establish the existence of the 27 lines on a cubic!

The same dimension counting argument tells us that 5 lines (let alone twistor ones) do not generically all lie on a cubic surface. Let us understand geometrically when 5 lines *do* all lie on a cubic surface.

PROPOSITION 2. *Five lines in* $\mathbb{CP}^3$ *lie on a (possibly singular) cubic surface if they are collinear, that is, there exists a fifth line intersecting all four.*

Proof. Let $\ell_1, \ldots, \ell_5$ be the lines and let k be another line intersecting all five in the points p_i. Choose 3 other points on each of the lines to get a set P of 20 points.

The condition on the coefficients of a cubic surface for it to contain all the points in P except for p_5 is represented by 19 linear equations in 20 unknowns. So we can find a cubic surface passing through all the points marked in black in Figure 5. This cubic surface has 4 points in common with k so it contains k. In particular it contains p_5. So it actually contains all 5 of the ℓ lines. □

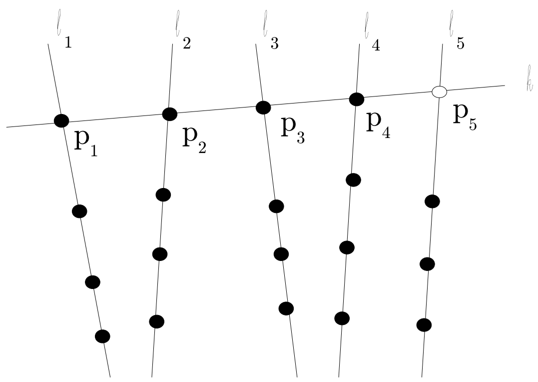

Figure 5: Nineteen points plus one

As a partial converse to Proposition 2, we remark that if 5 *skew* lines lie on a cubic surface then they are necessarily collinear. For example, if the cubic is non-singular then the Schläfli graph guarantees that the 5 lines are collinear; we can label them $a_1, \ldots, a_5$ (all intersecting just a_6) or $a_1, a_2, a_3, a_4, c_{56}$ (all intersecting b_5 and b_6).

The situation that is of interest for us is the second, in which we have 5 lines that are collinear in two different ways. The dimension counting argument above now shows that if one has 5 lines that are collinear in two ways, there will be a one-parameter family of (possibly singular) cubic surfaces containing all the lines.

Another way of seeing why there is a one-parameter family of cubics through such a configuration of lines is to observe that there is a one-parameter family of projective transformations that fixes all the lines. To see this observe that you can choose coordinates such that b_5 is given by the equations $z_1 = z_3 = 0$ and b_6 is given by the equations $z_2 = z_4 = 0$. Since the lines a_1, a_2, a_3, a_4, c_{56} are all skew, their intersections with b_5 are distinct. So we can make a Möbius transformation of z_2 and z_4 so that in the inhomogenous coordinate z_2/z_4 the intersection points of a_1, a_2, c_{56} with b_5 are respectivly 0, 1, ∞. Similarly we can choose our coordinates such that the intersections of a_1, a_2, c_{56} with b_6 correspond to $z_1/z_3 = 0$, 1, ∞. With these specifications, one can choose to independently rescale the coordinate pairs (z_1, z_3), (z_2, z_4) and one will still have coordinates with these properties. This choice of coordinates gives us a one-parameter family of projective transformations that fix all the lines.

Given two non-singular cubics $(\mathscr{C}_1, \mathscr{C}_2)$ that each contain all 7 lines, we can construct the projective transformation mapping $\mathscr{C}_1$ to $\mathscr{C}_2$ directly from the geometry of the cubics.

Indeed, given a point $p \in \mathbb{CP}^3$ away from b_5 and b_6 there is a unique line ℓ_p passing through b_5, b_6 and p. This line intersects each $\mathscr{C}_i$ in 3 points, so for generic p there is a unique projective transformation of ℓ_p fixing the points where ℓ_p intersects b_5 and b_6, and mapping the remaining point of $\ell_p \cap \mathscr{C}_1$ to that of $\ell_p \cap \mathscr{C}_2$. If we define Φ to map p to the image of p under this projective transformation, then we see that, so long as it is defined, Φ maps $\mathscr{C}_1$ to $\mathscr{C}_2$. If we can show that Φ extends to a biholomorphism, then we will have shown that Φ is a projective transformation. This is not too difficult to prove directly, but we will postpone the proof to the next section when it falls out from general theory.

We have just shown that any two non-singular cubics that contain all 7 lines will be projectively equivalent by a projective transformation that fixes all 7 lines.

We have already seen that there is only a one-parameter family of projective transformations that fix all the lines, so there is at most a parameter family of non-singular cubics containing all 7 lines. Since non-singularity is an open condition on the space of cubic surfaces, there is at most a one-parameter family of cubics containing all 7 lines if there are any non-singular cubics containing all 7 lines.

Putting all of this information together, we end up with a classification of non-singular cubic surfaces. To make things explicit, write $(l_1, l_2; l_3, l_4)_k$ for the cross ratio of the intersection points of four lines l_i meeting on a fifth line k. We can then define four invariants associated with the configuration of lines as follows:

$$\begin{aligned} \alpha &= (c_{56}, a_1; a_2, a_3)_{b_5} \\ \alpha' &= (c_{56}, a_1; a_2, a_3)_{b_6} \\ \beta &= (c_{56}, a_1; a_2, a_4)_{b_5} \\ \beta' &= (c_{56}, a_1; a_2, a_4)_{b_6} \end{aligned} \tag{1}$$

With this notation, we state

THEOREM 2. *If b_5 and b_6 are skew lines in $\mathbb{CP}^3$ and a_1, a_2, a_3, a_4, c_{56} are five other skew lines each passing through b_5 and b_6 then consider the pencil of cubics spanned by the following two polynomials in z_1, z_2, z_3, z_4:*

$$\begin{aligned}\mathscr{C}_1 \quad = \quad & \left[(-\alpha\beta\beta' + \beta^2\beta' + \beta\alpha'\beta' - \beta^2\alpha'\beta' - \beta(\beta')^2 + \alpha\beta(\beta')^2)z_3\right] z_2^2 \\ + \quad & \left[\left(\alpha\beta\alpha' - \beta^2\alpha' - \alpha\alpha'\beta' + \beta^2\alpha'\beta' + \alpha(\beta')^2 - \alpha\beta(\beta')^2\right) z_1 \right. \\ & \left. + \left(-\beta\alpha' + \beta^2\alpha' + \alpha\beta' - \beta^2\beta' - \alpha(\beta')^2 + \beta(\beta')^2\right) z_3\right] z_2 z_4 \\ + \quad & \left[(\beta\alpha' - \alpha\beta\alpha' - \alpha\beta' + \alpha\beta\beta' + \alpha\alpha'\beta' - \beta\alpha'\beta')z_1\right] z_4^2\end{aligned} \tag{2}$$

$$\begin{aligned}\mathscr{C}_2 \quad = \quad & [(-\beta\alpha' + \alpha\beta\alpha' + \alpha\beta' - \alpha\beta\beta' - \alpha\alpha'\beta' + \beta\alpha'\beta')z_2] z_3^2 \\ + \quad & \left[\left(\beta(\alpha')^2 - \alpha\beta(\alpha')^2 - \alpha^2\beta' + \alpha\beta\beta' + \alpha^2\alpha'\beta' - \beta\alpha'\beta'\right) z_2 \right. \\ & \left. + \left(-\alpha^2\alpha' + \beta\alpha' + \alpha(\alpha')^2 - \beta(\alpha')^2 - \alpha\beta' + \alpha^2\beta'\right) z_4\right] z_1 z_3 \\ + \quad & \left[\left(\alpha^2\alpha' - \alpha\beta\alpha' - \alpha(\alpha')^2 + \alpha\beta(\alpha')^2 + \alpha\alpha'\beta' - \alpha^2\alpha'\beta'\right) z_4\right] z_1^2\end{aligned} \tag{3}$$

All the cubics in this pencil contain all 7 lines. If there is a non-singular cubic containing all 7 lines, then all cubics containing the 7 lines lie in the pencil. All the non-singular surfaces in the pencil are projectively isomorphic.

All non-singular cubics arise in this way.

Proof. We have proved all of this already, except the explicit formulae.

One approach to proving this is brute force. Write down the 18×20 matrix corresponding to the 18 equations in 20 unknowns. One can then compute its kernel in order to find the two equations. This is not as tedious as one might expect; it can be done by hand, and is the work of moments for a computer algebra system. We have included the formulae for completeness, but will not use them directly, so we will omit the details.

It is interesting, however, to understand the general form of these equations, and this is something we will take advantage of.

We have seen that the projective transformations

$$\phi(u,v) : [z_1, z_2, z_3, z_4] \longrightarrow [uz_1, vz_2, uz_3, vz_4] \tag{4}$$

will preserve the 7 lines. We therefore look for cubic surfaces which are linear in z_1, z_3 and quadratic in z_2, z_4. In other words cubics of the form:

$$(az_1 + bz_3)z_2^2 + (cz_1 + dz_3)z_2z_4 + (ez_1 + fz_3)z_4^2 = 0 \tag{5}$$

The justification for considering such surfaces is that they will always contain b_5 and b_6 and will be invariant under $\phi(u,v)$.

Suppose that $(0, w_2, 0, w_4)$ and $(w_1, 0, w_3, 0)$ are points on b_5 and b_6. The general point on the line between these points is:

$$[\lambda w_1, \mu w_2, \lambda w_3, \mu w_4]$$

with λ, μ in $\mathbb{C}$. When we put the coordinates of this point into equation (5), we get a common factor of $\lambda\mu^2$. Hence the line lies on this cubic surface if and only if a single point on the line away from b_5 and b_6 does.

If we choose a generic plane transverse to b_5 and b_6 it will intersect the lines a_1, a_2, a_3, a_4, c_{56} in 5 points. Plugging the coordinates of these intersection points into equation (5) we get 5 linear equations in the 6 unknowns $a, b, \dots, f$. So there is a cubic surface of the given form that contains all 7 lines. $\square$

We have chosen this presentation of the classification of cubic surfaces because it yields the following classification for twistor lines on cubic surfaces.

THEOREM 3. *For a generic set of* 5 *points lying on a* 2*-sphere in* S^4*, there exists a one-parameter family of projectively isomorphic but conformally non-isomorphic non-singular cubic surfaces with* 5 *twistor lines corresponding to the* 5 *points.*

All cubic surfaces with 5 *twistor lines arise in this way. Given such a surface, one can label the twistor lines* a_1, a_2, a_3, a_4, c_{56} *and the two transversals* b_5, b_6.

One can associate a real invariant ξ *to a labelled cubic surface with five twistor lines in such a way that labelled cubic surfaces containing* 5 *twistor lines are conformally isomorphic if and only if the points on the sphere are conformally isomorphic and the values for* ξ *are equal.*

Proof. A 2-sphere in S^4 lifts to two projective lines b and jb in $\mathbb{CP}^3$. The choice of 5 points on the sphere determines 5 collinear lines in $\mathbb{CP}^3$. It follows from above that there exists a one-parameter family of cubics containing all 7 lines. We shall show later that if the 5 points are chosen generically then the general cubic in this family is non-singular. This being the case, we can label the twistor fibres a_1, a_2, a_4, a_4, c_{56}, and the transversals $b = b_5$, $jb = b_6$.

The bijection $b_6 \to b_5$ is determined by its action on 3 points, so in the coordinates used in our study we have $j[z_1, 0, z_3, 0] = [0, \overline{z_1}, 0, \overline{z_3}]$. The action of j on all of $\mathbb{CP}^3$ follows by anti-linearity.

Since j maps the intersection of a_i and b_5 to the intersection of a_i with b_6, it follows that $\alpha = \overline{\alpha'}$ and $\beta = \overline{\beta'}$ in (1).

In general, a projective transformation (4) of $\mathbb{CP}^3$ which fixes all 7 labelled lines will not correspond to a conformal transformation of S^4. It will do so if and only if it preserves j. This will be the case if and only if $|u| = |v|$.

The general cubic surface containing all 7 lines is given by a linear combination of $\mathscr{C}_1$ and $\mathscr{C}_2$ as defined in equation (2) and (3). Given such a cubic, define $M \in \mathbb{R}$ to be the coefficient of $z_1 z_4^2$ and $N \in \mathbb{R}$ to be the coefficient of $z_3 z_2^2$. Define $\xi = |M/N|$.

We need to check that neither M nor N is zero. We know that M is a non-zero multiple of the corresponding coefficient in the polynomial (3). Suppose that this coefficient were equal to zero. This would mean that any cubic surface containing the 7 lines would depend only linearly upon z_1 since this is the only non-linear term in z_1 in either (3) or (2). This would mean that the cubic was ruled and hence singular. Similarly, we see that N is non-zero.

By construction, ξ is invariant under transformations $\phi(u,v)$ with $|u| = |v|$. Thus it is well defined solely in terms of the cubic surface and the labelling. Since ξ changes in proportion to u/v, ξ will always distinguish conformally inequivalent surfaces. $\square$

COROLLARY 2. *Given the* 27 *lines on a non-singular cubic surface there is an algorithm to determine whether it has a twistor structure such that* 5 *of the* 27 *are twistor fibres.*

Proof. Run through all pairs of skew lines, compute the cross ratios of the intersection points and check whether $\alpha = \overline{\alpha'}$ and $\beta = \overline{\beta'}$ $\square$

We can summarize by saying that a cubic surface depends up to projective transformation upon a choice of 4 complex parameters $\alpha, \beta, \alpha', \beta'$ determined by the cross ratios of the line intersection points. These 4 complex parameters do depend upon a labelling of the lines in the cubic — so we have an action of the graph isomorphism group of the Schläfli graph on the space of such parameters. This is the Weyl group $W(E_6)$ of the exceptional Lie group E_6. Thus the moduli space of cubic surfaces up to projective isomorphism is given by an open subset of $\mathbb{C}^4$ quotiented by $W(E_6)$. For more details, we refer the reader to [14] and [1].

In the case of conformal isomorphism classes of cubic surfaces with 5 twistor lines we have a choice of 2 complex parameters α, β and one real parameter ξ. In addition we have a choice of labelling for the 5 twistor lines and a labelling of the two lines collinear to all the twistor lines. So the moduli space of cubic surfaces with 5 twistor lines up to conformal isomorphism is given by an open subset of $\mathbb{C}^2 \times \mathbb{R}$ quotiented by $S_5 \times \mathbb{Z}_2$.

In both cases we can write down an explicit equation for a cubic surface with given values for the parameters by choosing appropriate multiples of polynomials (2) and (3).

4. Identifying non-singular cubic surfaces with 5 twistor lines

Modern treatments of the classification of cubic surfaces usually state that non-singular cubic surfaces are given by blowing up 6 points in $\mathbb{CP}^2$ in general position, the latter meaning that no 3 points are collinear and that the 6 points do not all lie on a conic.

This perspective highlights the intrinsic complex geometry of the cubic surfaces — it ostensibly describes cubic surfaces up to biholomorphism rather than up to projective transformation. However, these two classifications are equivalent. This is guaranteed by the fact that any automorphism of a smooth hypersurface of $\mathbb{CP}^n$ ($n \geqslant 3$) of degree $d \neq n+1$ is induced by a projective transformation. This in turn follows from the general correspondence between maps to projective space and sections of complex line bundles, see [11].

Because we are interested in the classification up to conformal transformation, we have emphasized the embedding of the cubic surface om $\mathbb{CP}^3$. Let us review the connection between this and the intrinsic geometry.

Given a cubic surface $\mathscr{C}$, let ψ_1 be the biholomorphism $\mathbb{CP}^1 \to b_5$ mapping 0, 1, ∞ to the intersection points of b_5 with a_1, a_2, c_{56} respectively. Similarly define $\psi_2 : \mathbb{CP}^1 \to b_6$ by sending 0, 1, ∞ to points on a_1, a_2, c_{56}. One can now define a rational map $\psi : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathscr{C}$ by defining $\psi(z_1, z_2)$ to be the intersection point of the line containing $\psi_1(z_1)$, $\psi_2(z_2)$ with the surface $\mathscr{C}$.

The map ψ will be well defined for general points (z_1, z_2) in $\mathbb{CP}^1 \times \mathbb{CP}^1$. If we incorporate the multiplicity of the intersection into our definition, it is clear that we have a map ψ well defined at all points except: $(0,0)$, $(1,1)$, (α, α'), (β, β'), (∞, ∞) where the α's and β's are the cross-ratio invariants defined earlier. These points correspond to the lines a_1, a_2, a_3, a_4, c_{56} respectively, and are indicated in Figure 6. It turns out that ψ extends to a biholomorphism from $\mathbb{CP}^1 \times \mathbb{CP}^1$ blown up at these five points to the cubic surface $\mathscr{C}$.

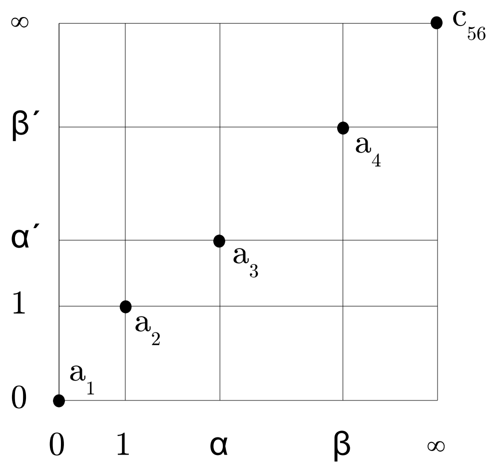

Figure 6: The five points to blow up on $\mathbb{CP}^1 \times \mathbb{CP}^1$

Now, $\mathbb{CP}^1 \times \mathbb{CP}^1$ can be thought of as $\mathbb{CP}^2$ with two points at infinity blown up and then the line at infinity blown down. This allows us to think of the blow up of $\mathbb{CP}^1 \times \mathbb{CP}^1$ at 5 points as being the blow up of $\mathbb{CP}^2$ at 6 points corresponding to a_1, a_2, a_3, a_4, a_5, a_6. Now, c_{56} corresponds to the line at infinity.

To be very concrete, blowing up the two points $[1,0,0]$ and $[0,1,0]$ at infinity and then blowing down the proper transform of the line at infinity is given by the rational map $(z_1, z_2) \to (z_1, z_2)$. The left hand side should be viewed as giving inhomogeneous coordinates for $\mathbb{CP}^2$, the right hand side as giving inhomogeneous coordinates for each factor of $\mathbb{CP}^1 \times \mathbb{CP}^1$. We define a rational map $\tilde{\psi}$ from $\mathbb{CP}^2$ to our cubic by $\tilde{\psi}(z_1, z_2) = \psi(z_1, z_2)$.

Since any four points in general position in $\mathbb{CP}^3$ are projectively equivalent, we see that a choice of 6 points to blow up corresponds to the 4 cross ratios α, α', β, β'.

Notice that the lines on the cubic surface are easily understood in terms of the blow-up picture. The a lines correspond to the points that have been blown up. The line c_{ij} corresponds to the straight line in $\mathbb{CP}^2$ passing through the points a_i and a_j. The line b_i corresponds to the conic passing through all the blown-up points except a_i. One can immediately read off the intersection properties of all the lines when they are thought of in terms of this picture. This gives a particularly nice way of remembering the structure of the Schläfli graph.

This intrinsic view of cubic surfaces allows us to tie up a loose end we left dangling in the previous section. Recall that given two cubics $\mathscr{C}_1$ and $\mathscr{C}_2$ containing the lines a_1, a_2, a_3, a_4, c_{56} we constructed a rational map $\Phi : \mathbb{CP}^3 \to \mathbb{CP}^3$ sending $\mathscr{C}_1$ to $\mathscr{C}_2$. We claimed that this map could was in fact a biholomorphism. Identifying each of $\mathscr{C}_1$ and $\mathscr{C}_2$ with $\mathbb{CP}^2$ blown up at six points, we see that Φ restricted to $\mathscr{C}_1$ is essentially the identity — hence it is certainly a biholomorphism.

The most important feature of this intrinsic view of cubic surfaces from the perspective of the twistor geometry is the criteria it gives for determining whether a cubic surface is non-singular. The blow-up of $\mathbb{CP}^2$ at 6 points is obviously smooth, so if we can find a cubic surface corresponding to the 6 points, that cubic surface will be non-singular. To construct a cubic surface given 6 points in general position, one considers the vector space $\mathcal{C}$ of cubic *curves* in $\mathbb{CP}^2$ that pass through all 6 points. This space will be 4-dimensional so long as no 6 points lie on a conic and no 3 points lie on a line. One then defines a rational map sending a point $z \in \mathbb{CP}^2$ to the projectized dual space $\mathbb{P}(\mathcal{C}^*)$ by mapping a cubic polynomial to its value at z. This rational map is a biholomorphism of the blow up at 6 points to a cubic surface in $\mathbb{P}(\mathcal{C}^*)$. This result was first discoved by Clebsch in [8]. Details of the proof can be found in [11].

We saw in the previous section that given 5 points lying on a 2-sphere in S^4 we can find a family of cubic surfaces with 5 twistor lines corresponding to these five points. It follows from the discussion above that if the 5 points on the 2-sphere are chosen in general position then the cubic surfaces will be non-singular. We would like to identify more clearly what "in general position" actually means in this case.

THEOREM 4. *Given* 5 *points lying on a 2-sphere in* S^4*, there is a non-singular cubic surface with* 5 *twistor lines corresponding to these points if and only if no* 4 *of the points lie on a circle.*

Proof. There are two lines in $\mathbb{CP}^3$ lying above S^4 under the twistor correspondence. Label one of them b_5 and the other b_6.

To each of the five points on b_5, there is a unique twistor line over that point. We label these lines arbitrarily as a_1, a_2, a_3, a_4 and c_{56}.

Three distinct points on a 2-sphere are conformally equivalent, and so always in general position. We choose an inhomogeneous coordinate z_1 for b_5 and z_2 for b_6 by requiring that the intersections of a_1, a_2, a_3, a_4 and c_{56} with b_5 are given by 0, 1, α, β and ∞. Similarly we choose an inhomogeneous coordinate z_2 for b_6 such that the intersection points are 0, 1, $\overline{\alpha}$, $\overline{\beta}$ and ∞.

We now have an unambiguously defined rational map ϕ from $\mathbb{CP}^2$ to $b_5 \times b_6$ given in inhomogenous coordinates by $\phi(z_1, z_2) = (z_1, z_2)$. Corresponding to each of the lines a_1, a_2, a_3, a_4 and c_{56} we can define points a_1', a_2', a_3', a_4' and c_{56}' in $b_5 \times b_6$ given by sending a line to its intersection point with each b_i. We can then define six points in $\mathbb{CP}^2$ as follows:

$$\begin{aligned}
A_1 &= \phi^{-1}(a_1') = [0,0,1] \\
A_2 &= \phi^{-1}(a_2') = [1,1,1] \\
A_3 &= \phi^{-1}(a_3') = [\alpha,\overline{\alpha},1] \\
A_4 &= \phi^{-1}(a_4') = [\beta,\overline{\beta},1] \\
A_5 &= [1,0,0] \\
A_6 &= [0,1,0]
\end{aligned}$$

Note that c_{56} corresponds to the line at infinity in $\mathbb{CP}^2$. A_5 and A_6 correspond to the two lines at infinity in $b_5 \times b_6$. The setup is precisely as summarized in Figure 6.

The point we are making is that these points in $\mathbb{CP}^2$ are determined entirely by the 5 points on the sphere and the choice of labelling: we do not need there to be a non-singular cubic through the five twistor lines in order to construct the A_i.

The blow up of $\mathbb{CP}^2$ at the A_i corresponds to a smooth cubic if and only if these A_i are in general position (meaning no three collinear and no conic through all 6). This cubic must then be biholomorphic to one of the cubic curves in the pencil generated by (2) and (3). We deduce that there is a smooth cubic with 5 twistor lines corresponding to the five points on S^2 if and only if these six points A_i in $\mathbb{CP}^2$ are in general position.

A_1, A_2 and A_3 are collinear if and only if $\alpha = \overline{\alpha}$. This is equivalent to saying that 0, 1, α and ∞ all lie on the real line. In invariant terms this is equivalent to requiring that a_1, a_2, a_3 and c_{56} all lie on a circle in S^2.

We deduce that there is a smooth cubic corresponding to the five points on S^2 only if no four of the points lie on a circle.

It is a simple calculation to check that the condition that no four points lie on a circle implies that no three of the A_i lie on a line. We also need to confirm that the same condition implies that there is no conic through all 6 of the A_i.

Suppose for a contradiction that there is such a conic and so the A_i form an "inscribed hexagon". Pascal's theorem implies the intersection points

$$A_1A_2 \cap A_4A_5, \quad A_2A_3 \cap A_5A_6, \quad A_3A_4 \cap A_6A_1$$

are collinear. These points can be computed using the vector cross product; the first is $(A_1 \times A_2) \times (A_4 \times A_5)$ with a slight abuse of notation. The collinearity condition is then

$$\det \begin{pmatrix} \overline{\beta} & \overline{\beta} & 1 \\ \alpha - 1 & \overline{\alpha} - 1 & 0 \\ 0 & \beta\overline{\alpha} - \alpha\overline{\beta} & \beta - \alpha \end{pmatrix} = 0,$$

which gives

$$|\alpha|^2(\beta-\overline{\beta})-|\beta|^2(\alpha-\overline{\alpha})+\alpha\overline{\beta}-\overline{\alpha}\beta=0.$$

But this is easily seen to be exactly the condition that $0,1,\alpha,\beta$ lie on a circle in $\mathbb{C}$. □

On first reading this proof one may wonder where the asymmetry between the a_i and c_{56} arises. It can be traced directly to the choice to associate c_{56} with the points $z_1=\infty$ and $z_2=\infty$.

For a coordinate-free explanation of why the cubic must be singular if four of the 5 points in S^2 lie in a circle Γ, recall that by by [18, Theorem 3.10] that there is a quadric surface $\mathscr{Q}$ in $\mathbb{CP}^3$ containing $\pi^{-1}(\Gamma)$ (where π is the twistor projection). Suppose that there is also non-singular cubic $\mathscr{C}$ for which the twistor fibres a_1,a_2,a_3,a_4 from part of a double six (a_i,b_j). In this case, $\mathscr{C}$ must be the only cubic containing the double six. But each b_j intersects at least three of a_1,a_2,a_3,a_4 and therefore lies in $\mathscr{Q}$. The latter must now contain a_5,a_6 as well. But then the union of $\mathscr{Q}$ and any plane is a cubic containing the double six, which is a contradiction.

As an application of our theorem, we observe that the well known Clebsch diagonal surface does not have 5 twistor lines irrespective of the twistor structure j one places on $\mathbb{CP}^3$. The Clebsch diagonal surface is the complex surface in $\mathbb{CP}^4$ defined by the two equations

$$\begin{gathered} z_1^3+z_2^3+z_3^3+z_4^3+z_5^3=0 \\ z_1+z_2+z_3+z_4+z_5=0. \end{gathered}$$

It is biholomorphic to the surface in $\mathbb{CP}^3$ given by the single equation:

$$z_1^3+z_2^3+z_3^3+z_4^3=(z_1+z_2+z_3+z_4)^3,$$

and is the only cubic surface with symmetry group S_5. It has the nice property that all 27 lines on the cubic surface are real lines. This immediately means that it admits no twistor structure j such that it has five twistor lines. Simply note that the cross ratios of all the intersection points on the lines must be real. Therefore any four points on one of its lines must lie on a circle when that line is viewed as the Riemann sphere.

5. The Fermat cubic

Having found a large family of cubic surfaces with 5 twistor lines we would like to ask if there are any particularly nice examples. In particular what is the most symmetrical cubic surface with 5 twistor lines?

A conformal transformation of S^4 that induces a symmetry of a cubic surface with 5 twistor lines must leave the 2-sphere image of b_5 and b_6 fixed. If the conformal transformation leaves the image of the 5 twistor lines fixed, then the associated projective transformation must swap b_5 and b_6. Otherwise the conformal transformation must permute the 5 points on the 2-sphere.

Therefore let us first choose the most symmetrical arrangement of 5 points on a 2-sphere no four of which lie on a circle. If we have a rotation of the sphere that permutes n of the points then those points must all lie on a circle. So $n \leqslant 3$. So any rotation fixes at least two points. Either those two points lie on the axis of rotation, or the rotation is a rotation through 180 degrees and the fixed points all lie on a circle. We deduce that the largest possible symmetry group for the five points is $\mathbb{Z}_3 \times \mathbb{Z}_2$ and, up to conformal transformation of the 2-sphere, we can assume that our five points are 0, ∞ and the three cube roots of unity. The $\mathbb{Z}_3$ action rotates the three cube roots into each other. The $\mathbb{Z}_2$ action swaps 0 and ∞.

Setting α and β to be complex cube roots of unity and α' and β' to be their conjugates, polynomials (2) and (3) simplify to:

$$-3i\sqrt{3}\left(z_2 z_3^2 - z_1^2 z_4\right),$$
$$3i\sqrt{3}\left(-z_2^2 z_3 + z_1 z_4^2\right).$$

We now wish to choose a linear combination of these polynomials that will remain fixed under the transformation that swaps the lines b_5 and b_6. This corresponds to the projective transformation $z_1 \mapsto z_2$, $z_2 \mapsto z_1$, $z_3 \mapsto z_4$, $z_4 \mapsto z_3$.

Hence there is, up to conformal transformation, a unique non-singular cubic surface with 5 twistor lines and symmetry group $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. It is defined by

$$z_1 z_4^2 + z_2 z_3^2 - z_3 z_2^2 - z_4 z_1^2 = 0. \tag{6}$$

One can make a conformal transformation (corresponding to using the cube roots of -1 rather than those of 1) to replace the two minus signs with plus signs. In any case, it is projectively, but not conformally, equivalent to a familar example: the Fermat cubic, which is defined by the equation

$$z_1^3 + z_2^3 + z_3^3 + z_4^3 = 0.$$

One can prove that these surfaces are projectively equivalent by calculating the cross ratio invariants we defined earlier. This approach allows one to write down an explicit linear transformation sending the Fermat cubic to the surface (6).

A more pleasing approach is to use the symmetries of the Fermat cubic to deduce that there must be some twistor structure that gives it five twistor lines. To see how this is done, first choose a complex cube root of unity ω and label 7 of the lines on the Fermat cubic as follows:

Label	Line
b_5	$z_1 + \omega z_2 = z_3 + \omega^2 z_4 = 0$
b_6	$z_1 + \omega^2 z_2 = z_3 + \omega z_4 = 0$
a_1	$z_1 + \omega z_2 = z_3 + \omega z_4 = 0$
a_2	$z_1 + z_4 = z_2 + z_3 = 0$
a_3	$z_1 + \omega z_4 = z_2 + \omega z_3 = 0$
a_4	$z_1 + \omega^2 z_4 = z_2 + \omega^2 z_3 = 0$
c_{56}	$z_1 + \omega^2 z_2 = z_3 + \omega^2 z_4 = 0$

Consider the symmetry of the cubic given by

$$(z_1, z_2, z_3, z_4) \longmapsto (z_1, z_2, \omega\, z_3, \omega\, z_4).$$

This generates a $\mathbb{Z}_3$ action that fixing the lines b_5, b_6, a_1 and c_{56} and permuting a_2, a_3, a_4. Thus we have a $\mathbb{Z}_3$ symmetry of b_5 fixing the intersection points with a_1 and c_{56} and permuting the intersection points with a_2, a_3 and a_4. Therefore these 5 points on b_5 are conformally equivalent to the points 0, ∞, 1, ω and $\overline{\omega}$ on the Riemann sphere. The same applies to the 5 intersection points with b_6. This implies that the invariants (1) satisfy $\alpha = \overline{\beta}$ and $\alpha' = \overline{\beta'}$

Now consider the symmetry:

$$(z_1, z_2, z_3, z_4) \longmapsto (z_3, z_4, z_1, z_2).$$

This swaps b_5 and b_6, swaps a_3 and a_4 and fixes a_1, a_2 and c_{56}. We deduce that:

$$\alpha = (c_{56}, a_1; a_2, a_4)_{b_5} = (c_{56}, a_1; a_2, a_3)_{b_6} = \beta' = \overline{\alpha'}.$$

Thus the cross ratios of the intersection points on b_5 and b_6 are the same as the cross ratios for the intersection points on the cubic surface (6). We conclude that the cubic given by (6) is projectively isomorphic to the Fermat cubic.

THEOREM 5. *The set of twistor structures on $\mathbb{CP}^3$ for which the Fermat cubic has* 5 *twistor lines is a real* 1*-manifold with 54 components.*

Proof. The surface (6) has 12 conformal symmetries. The Fermat cubic has symmetry group $S_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ given by permutations of the coordinates and by multiplying various coordinates by cube roots of unity. Thus there are $4! \cdot 27/12 = 54$ twistor structures on $\mathbb{CP}^3$ such that the Fermat cubic is isomorphic to surface (6) with the standard twistor structure. We can then vary the invariant ξ to get a one-parameter family of conformally inequivalent twistor structures.

We need to check that there are no other twistor structures that give the Fermat cubic five twistor lines. We we gave an algorithm to do this eariler: run through all pairs of skew lines and compute cross ratios. We can speed this up significantly using the symmetries of the Fermat cubic. The general line on the Fermat cubic is

$$z_i + \eta_1 z_j = z_k + \eta_2 z_l = 0$$

where $\{i, j, k, l\}$ is a permutation of $\{1, 2, 3, 4\}$ and η_1 and η_2 are cube roots of unity. So given two skew lines on the Fermat cubic, using the cubic's symmetries we can assume that the first line is:

$$z_1 + z_2 = z_3 + z_4 = 0$$

and the second line is one of:

$$z_1 + \eta_1 z_2 = z_3 + \eta_2 z_4 = 0,$$
$$z_1 + \eta_1 z_3 = z_2 + \eta_2 z_4 = 0.$$

In the first case there is a $\mathbb{Z}_3$ symmetry preserving both lines — so if we have 5 twistor lines it will be one of the cases already considered. In the second case we can further assume that $\eta_1 = 1$ and, since the lines are skew, $\eta_2 \neq 1$. Therefore we just need to show that the cross ratios of the intersection points of the 5 lines on the Fermat cubic meeting $z_1+z_2 = z_3+z_4 = 0$ and $z_1+z_3 = z_2+\omega z_4 = 0$ do not form complex conjugate pairs. This is easily done. □

There is a lot more one could ask about the twistor geometry of the Fermat cubic. For example: what is the topology of its discriminant locus? How does this vary as one varies the choice of twistor structure? We will consider these questions in another paper.

Acknowledgment. This paper develops material from the second author's PhD thesis [16]. A major debt of gratitute is due to the Politecnico di Torino and to colleagues in its Department of Mathematical Sciences.

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AMS Subject Classification: 53C28, 14N10, 53A30

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Lavoro pervenuto in redazione il 02.07.2013.

Rend. Sem. Mat. Univ. Politec. Torino
Vol. 71, 3–4 (2013), 339 – 383

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ABELIAN FOURFOLDS OF WEIL TYPE AND CERTAIN K3 DOUBLE PLANES

Abstract. Double planes branched in 6 lines give a famous example of K3 surfaces. Their moduli are well understood and related to Abelian fourfolds of Weil type. We compare these two moduli interpretations and in particular divisors on the moduli spaces. On the K3 side, this is achieved with the help of elliptic fibrations. We also study the Kuga-Satake correspondence on these special divisors.

Introduction

The wonderful geometry of double planes branched in 6 lines has been addressed extensively in the articles [10, 9]. When the lines are in general position such double planes are K3 surfaces and their moduli are well known and are determined through the period map.

The target space for the period map is defined as follows. Consider the lattice

$$T = U^2 \perp \langle -1 \rangle \perp \langle -1 \rangle. \tag{1}$$

Here U is the hyperbolic lattice with basis $\{e, f\}$ and $\langle e,e \rangle = \langle f, f \rangle = 0$, $\langle e, f \rangle = 1$ and for $n \in \mathbb{Z}$ the notation $\langle n \rangle$ means that we have a one dimensional lattice with basis $\{g\}$ such that $\langle g,g \rangle = n$. We write $z = (z_1,z_2,z_3,z_4) \in T \otimes \mathbb{C}$ and form the associated domain

$$D(T) := \{z \in \mathbb{P}(T \otimes \mathbb{C}) \mid \langle z,z \rangle = 0,\ \langle z,\bar{z} \rangle > 0\}$$

The orthogonal group $\mathrm{O}(T)$ acts on $D(T)$, a space with two connected components; the commutator subgroup of $\mathrm{O}(T)$ preserves the connected components, distinguished by the sign of $\mathrm{im}(z_3/z_1)$. Let $\mathbf{D}_4$ be the one with the positive sign. With $\mathrm{SO}^*(T)$ the group intruduced in 1.1 the target of the period map is

$$\mathbf{M} = \mathbf{D}_4 / \mathrm{SO}^*(T). \tag{2}$$

This is the principal moduli space in this article.

The source of the period map is the configuration space U_6 of 6 ordered lines in good position.* The linear group $\mathrm{GL}(3;\mathbb{C})$ acts on this space and we form the quotient

$$\mathbf{X} : \;=\; U_6/\mathrm{GL}(3;\mathbb{C}) : \textbf{configuration space of 6 ordered lines in good position in } \mathbb{P}^2.$$

The symmetric group $\mathfrak{S}_6$ acts on this space and the quotient $\mathbf{X}$ is the configuration space of 6 non-ordered lines in good position. One of the peculiarities of the number

*This means that the resulting sextic has only ordinary nodes.

6 is that there is a further (holomorphic) involution $*$ on 6 lines which comes from the correlation map with respect to a conic in the plane (see § 3.1) and which we call *the correlation involution*. Six-tuples of lines in good position related by this involution correspond to isomorphic double covering K3-surfaces. The explanation of why this is true† uses beautiful classical geometry related to Cremona transformations for which we gratefully acknowledge the source [4]. See Prop. 5 and Remark 7.

Furthermore, the involution commutes with the action of $\mathfrak{S}_6$. So the period map descends to this quotient and by [10] this yields a biholomorphic map

$$\mathbf{X}/H \xrightarrow{\simeq} \mathbf{M}, \quad H := \mathfrak{S}_6 \times \{*\}$$

In other words we may identify our moduli space with the quotient of the configuration space of 6 unordered lines in good position by the correlation involution.

In loc. cit. some special divisors on $\mathbf{X}$ (and hence on $\mathbf{M}$) are studied, in particular the divisor called XQ which corresponds to 6 lines all tangent to a common conic C. The associated double covers are K3 surfaces with rich geometry. Indeed this 3-dimensional variety is the moduli space $\mathbf{M}_{2,2}$ of curves of genus 2 (with a level 2 structure). To see this note that the conic C meets the ramification divisor R (the 6 lines) in 6 points and on the K3 surface this gives a genus 2 curve D lying over it. In [10, §0.19] it is explained that the Kummer surface $J(D)/\{\pm 1\}$ associated to the jacobian $J(D)$ of D is isomorphic to the double cover branched in R and that there is a natural level 2 structure on $J(D)$. Furthermore, the period map induces a biholomorphism

$$\mathbf{M}_{2,2} = XQ \xrightarrow{\simeq} (\mathfrak{h}_2/\Gamma(2))^0 \hookrightarrow \mathbf{X}, \quad \Gamma = \mathrm{Sp}(2,\mathbb{Z})/\pm 1, \tag{3}$$

where the image is Zariski-open in $\mathfrak{h}_2/\Gamma(2)$ (in fact, in loc. cit. an explicit partial compactification $\bar{\mathbf{X}}$ of X is described such that the closure of $\mathbf{M}_{2,2}$ in $\bar{\mathbf{X}}$ is exactly the space $\mathfrak{h}_2/\Gamma(2)$).

In [10, 5] one also finds a relation with a certain moduli space of Abelian 4–folds. We recall that principally polarized Abelian varieties of dimension g (without any further restriction) are parametrized by the Siegel upper half space $\mathfrak{h}_g$. For a generic such Abelian variety A its rational endomorphism ring $\mathrm{End}_{\mathbb{Q}} A$ is isomorphic to $\mathbb{Q}$, but special Abelian varieties have larger endomorphism rings and are parametrised by subvarieties of $\mathfrak{h}_g$. Here we consider 4–dimensional A with $\mathbb{Z}(\mathrm{i}) \subset \mathrm{End}(A)$ where the endomorphisms are assumed to preserve the principal polarization. These are parametrized by the 4-dimensional domain ‡

$$\mathbf{H}_2 := \left\{ W \in M_{2\times 2}(\mathbb{C}) \mid \frac{1}{2\mathrm{i}}(W - W^*) > 0 \right\} \simeq \mathrm{U}((2,2))/\mathrm{U}(2) \times \mathrm{U}(2)$$

which is indeed (see e.g. [5, §1.2]) a subdomain of the Siegel upper half space $\mathfrak{h}_4$ by

†The referee urged us to find an explanation of this since it could not be found in the literature.
‡As usual, for a matrix W, we abbreviate $W^* = {}^{\mathsf{T}}\bar{W}$.

means of the embedding

$$\left.\begin{array}{lll}\iota:\mathbf{H}_2 & \hookrightarrow & \mathfrak{h}_4\\ W & \mapsto & U\begin{pmatrix}W & 0\\ 0 & {}^{\mathsf{T}}W\end{pmatrix}U^*,\\ & & U=\frac{1}{\sqrt{2}}\begin{pmatrix}\mathrm{i}\mathbf{1}_2 & -\mathrm{i}\mathbf{1}_2\\ \mathbf{1}_2 & \mathbf{1}_2\end{pmatrix}.\end{array}\right\} \tag{4}$$

As for all polarized Abelian varieties, the complex structure of such Abelian fourfolds A is faithfully reflected in the polarized weight one Hodge structure on $H^1(A)$. The second cohomology carries an induced weight 2 Hodge structure. The extra complex structure on A (induced by multiplication with i) makes it possible to single out a rational sub Hodge structure $T(A)\subset H^2(A)$ of rank 6 with Hodge numbers $h^{2,0}=1$, $h^{1,1}=4$. Such Hodge structures are classified by points in the domain $\mathbf{D}_4$ we encountered before. Indeed, there is an isomorphism

$$\mathbf{H}_2 \xrightarrow{\sim} \mathbf{D}_4 \simeq \mathrm{O}(2,4)/\mathrm{O}(2)\times\mathrm{O}(4)$$

which is induced by an isomorphism between classical real Lie groups which on the level of Lie algebras gives the well known isomorphism $\mathfrak{su}(2,2)\simeq\mathfrak{so}(6)$. This isomorphism is implicit in [10]. One of the aims of this article is to review this using the explicit and classically known isomorphism between corresponding Lie groups. See § 1; the corresponding Hodge theoretic discussion is to be found in § 2. This gives an independent and coordinate free presentation of the corresponding results in [10, 9]. In particular, on the level of moduli spaces we find (see § 2.2):

$$\mathbf{M}:=\mathbf{H}_2/\mathrm{U}^*((2,2);\mathbb{Z}[\mathrm{i}]) \xrightarrow{\sim} \mathbf{D}_4/\mathrm{SO}^*(T).$$

The group $\mathrm{U}^*((2,2);\mathbb{Z}[\mathrm{i}])$ is an extension of the unitary group $\mathrm{U}((2,2);\mathbb{Z}[\mathrm{i}])$ (with coefficients in the Gaussian integers) by an involution as explained below (see eqn. (9)).

One of the new results in our paper is the study of the Néron-Severi lattice using the geometry of elliptic pencils. It allows us to determine the generic Néron-Severi lattice (in § 3.2) as well as the generic transcendental lattice. And indeed, we find $T(2)$ for the latter (here the brackets mean that we multiply the form by 2; this gives an even form as it should):

THEOREM (=Theorem 1). For generic X as above we have for the Néron-Severi lattice $\mathrm{NS}(X)=U\perp D_6^2\perp\langle-2\rangle^2$ and for the transcendental lattice $T(X)=T(2)$.[§]

Here D_k is the lattice for the corresponding Dynkin diagram. See the notation just after the introduction.

Of course, the divisors in X parametrizing special line configuration also have a moduli interpretation on the Abelian 4-fold side. This has been studied by Hermann in [5]. To compare the results from [10] and [5] turned out to be a non trivial exercise (at least for us). This explains why we needed several details from both papers. We

[§]See also [10, Prop. 2.3.1]

collected them in § 1 and § 2.1. We use this comparison in particular to relate (in § 3.1) Hermann's divisors D_Δ for small Δ to some of the divisors described in [10]. We need this in order to describe (in § 3.6) the geometry of the corresponding K3 surfaces in more detail. The technique here is the study of the degeneration of a carefully chosen elliptic pencil (when the surface moves to the special divisor) which reflects the corresponding lattice enhancements explained in § 3.5. This technique enables us to calculate the Néron-Severi and transcendental lattice of the generic K3 on the divisors D_Δ for $\Delta = 1, 2, 4, 6$ respectively (we use Hermann's notation).

For the full statement we refer to Theorem 3; here we want to single out the result for $\Delta = 1$:

THEOREM. The divisor $D_1 \subset \mathbf{M}$ corresponds to the divisor $XQ \subset \mathbf{X}$ and so (see (3)) is isomorphic to a Zariski-open subset in the moduli space of genus two curves. The generic point on D_1 corresponds to a K3 surface which is a double cover of the plane branched in 6 lines tangent to a common conic. Its Néron-Severi lattice is $\mathrm{NS}(X_1) = U \perp D_4 \perp D_8 \perp A_3$ and its transcendental lattice is $U(2)^2 \perp \langle -4 \rangle$.

This requires a little further explanation beyond what is stated in Theorem 3: we have seen in (3) that the image of XQ in $\mathbf{M} = \mathbf{X}/H$ we get a variety isomorphic to a Zariski-open subset of $\mathfrak{h}_2/\Gamma$, the moduli space of principally polarized Abelian varieties of dimension 2 (=the moduli space of genus 2 curves).

A further novelty of this paper is the role of the Kuga-Satake correspondence. For the general configuration of lines this has been done by one of us in [7], building on work of Paranjape [14]. One of these earlier results reviewed in § 4 (Theorem 9) states that the Kuga-Satake construction gives back the original Abelian 4–fold up to isogeny. In the present paper we explain what this construction specializes to for the K3 surface on the generic D_Δ, this time for *all* Δ. See Theorem 5.

Notation

The bilinear form on a lattice is usually denoted by $\langle -, - \rangle$. Several standard lattices as well as standard conventions are used:

- Orthogonal direct sums of lattices is denotes by $\perp$;
- For a lattice T the orthogonal group is denoted $\mathrm{O}(T)$ its subgroup of commutators is a subgroup of the special orthogonal group $\mathrm{SO}(T)$ and is denoted $\mathrm{SO}^+(T)$;
- Let $(T, \langle -, - \rangle)$ be a lattice. The dual of T is defined by

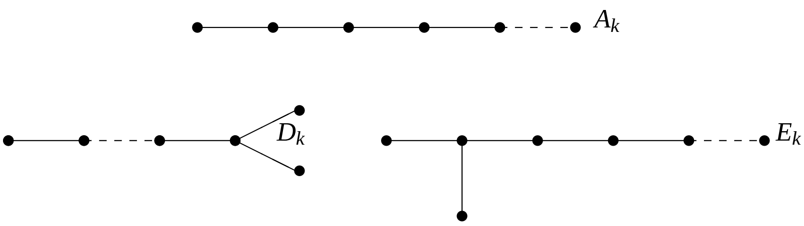
$$T^* := \{x \in T \otimes \mathbb{Q} \mid \langle x, y \rangle \in \mathbb{Z}, \quad \text{for all } y \in T\}.$$

  Note that $T \subset T^*$. The discriminant group $\delta(T)$ is the finite Abelian group T^*/T. If T is even, i.e. $\langle x, x \rangle \in 2\mathbb{Z}$, the form $\langle -, - \rangle$ induces a $\mathbb{Q}/\mathbb{Z}$-valued bilinear form b_T on $\delta(T)$ with associated $\mathbb{Q}/2\mathbb{Z}$-valued quadratic form q_T.

- The group of matrices with values in a subring $R \subset \mathbb{C}$ preserving the hermitian form with Gram matrix

$$\mathbf{J} := \begin{pmatrix} \mathbf{0}_2 & \mathbf{1}_2 \\ -\mathbf{1}_2 & \mathbf{0}_2 \end{pmatrix} \tag{5}$$

  will be denoted $\mathrm{U}((2,2);R)$ because it has signature $(2,2)$;
- For $a \in \mathbb{Z}$ the lattice $\langle a \rangle$ is the 1-dimensional lattice with basis e and $\langle e,e \rangle = a$;
- If L is a lattice, the lattice $L(a)$ is the same $\mathbb{Z}$–module, but the form gets multiplied by a;
- U is the standard hyperbolic lattice with basis $\{e,f\}$ and $\langle e,e \rangle = \langle f,f \rangle = 0$, $\langle e,f \rangle = 1$;
- A_k, D_k, E_k: the standard *negative definite* lattices associated to the Dynkin diagrams: if the diagrams have vertices $v_1,\ldots,v_k$, we put -2 on the diagonal, and in the entries ij and ji we put 1 or 0 if v_i and v_j are connected or not connected respectively:



Furthermore, for a projective surface X we let $\mathrm{NS}(X) \subset H^2(X;\mathbb{Z})$ and $T(X) = \mathrm{NS}(X)^{\perp}$ be the Néron-Severi lattice, respectively the transcendental lattice equipped with the lattice structure from $H^2(X;\mathbb{Z})$, i.e. the intersection product.

Finally we recall the convention to denote congruence subgroups. Suppose R is a subring of $\mathbb{C}$, and V_R a free R-module of finite rank and G a subgroup of the group $\mathrm{Aut}\, V_R$. For any principal ideal $(\omega) \subset R$ we set

$$G(\omega) := \{g \in G \mid g \equiv \mathrm{id} \mod (\omega)\}.$$

In what follows we restrict ourselves to

$$R = \mathbb{Z}[\mathrm{i}] \subset \mathbb{C}, \quad \omega = 1+\mathrm{i}, \quad G = \mathrm{SU}((2,2);\mathbb{Z}[\mathrm{i}]).$$

1. Two Classical Groups, Their Associated Domains and Lattices

1.1. The Groups

We summarize some classical results from [3, IV. § 8].

Let K be any field and V a 4-dimensional K–vector space. The decomposable elements in $W := \Lambda^2 V$ correspond to the 2-planes in V; the corresponding points in $\mathbb{P}(W)$ form the quadric G which is the image of the Grassmann variety of 2-planes in V under the Plücker-embedding. Concretely, choosing a basis $\{e_1, e_2, e_3, e_4\}$ and setting

$$x \wedge y = q(x,y)\, e_1 \wedge e_2 \wedge e_3 \wedge e_4, \quad x, y \in W, \tag{6}$$

the quadric G has equation $q(x,x) = 0$.

On G we have two types of planes: the first corresponds to planes contained in a fixed hyperplane of V, the second type corresponds to planes passing through a fixed line. Both types of planes therefore correspond to 3-dimensional subspaces of W on which the bilinear form q is isotropic. Its index, being the maximal dimension of q–isotropic subspace equals 3.

Any $A \in \mathrm{GL}(V)$ induces a linear map $B = \Lambda^2 A$ of W preserving the quadric G. Conversely such a linear map B is of this form, provided it preserves the two types of planes on the quadric G. This is precisely the case if $\det(B) > 0$ and so one obtains the classical isomorphism between *simple* groups

$$\begin{array}{rcl} \mathrm{SL}(4,K)/\mathrm{center} & \xrightarrow{\simeq} & \mathrm{SO}^+(6,q,K)/\mathrm{center}, \\ A & \mapsto & \Lambda^2 A \end{array}$$

where we recall that the superscript $+$ stands for the commutator subgroup.[¶]

From this isomorphism several others are deduced (loc. cit) through a process of field extensions. The idea is that if $K = k(\alpha)$, an imaginary quadratic extension of a real field k, the form q which over K has maximal index 3, over k can be made to have index 2. This is done as follows. One restricts to a subset of K-linear transformations of V which preserve a certain well-chosen anti-hermitian form f. The linear maps $\Lambda^2 A$ then preserve the *hermitian* form $g = \Lambda^2 f$ given by

$$g(x \wedge y, z \wedge t) = \det \begin{pmatrix} f(x,z) & f(y,z) \\ f(x,t) & f(y,t) \end{pmatrix} \tag{7}$$

Suppose that in some K–basis for W the Gram matrices for g and q coincide and both have entries in k. Then the matrix of $B = \Lambda^2 A$ being at the same time q–orthogonal and g–hermitian must be real. So this yields an isomorphism

$$\begin{array}{rcl} \mathrm{SU}(V,f,K)/\mathrm{center} & \xrightarrow{\simeq} & \mathrm{SO}^+(6,q,k)/\mathrm{center} \\ A & \mapsto & \Lambda^2 A \end{array}$$

In our situation $K = k(\mathrm{i})$ (with, as before, k a real field). We choose our basis $\{a_1, a_2, b_1, b_2\}$, for V in such a way that the anti-hermitian form f has Gram matrix $\mathrm{i}\mathbf{J}$ (see (5)).

The Gram matrix of $-q$ is the (integral) Gram matrix for $U \perp U \perp U$. The Gram matrix of the hermitian form $-g$ is found to be the Gram matrix for $U \perp U \perp \langle -1 \rangle \perp \langle -1 \rangle$ of signature $(2,4)$. In a different K-basis for W the Gram matrix of q is found to coincide with the Gram matrix for g:

[¶] The last subgroup can also be identified with the subgroup of elements whose spinor norm is 1, but we won't use this characterization.

LEMMA 1. *Let* $\omega = \frac{1}{2}(-1+\mathrm{i})$ *and set*

$$g_1 = f_1, \quad g_2 = f_2, \quad g_3 = f_3, \quad g_4 = f_4, \quad g_5 = \omega f_5 - \bar{\omega} f_6, \quad g_6 = (-\bar{\omega} f_5 + \omega f_6)$$
$$T := \mathbb{Z}\text{–lattice spanned by } \quad \{g_1, \dots, g_6\}.$$

Then the Gram matrices of $-q$ *(see* (6)*) and* $-g$ *(see* (7)*) on* T *are both equal to* $U \perp U \perp \langle -1 \rangle \perp \langle -1 \rangle$.

Indeed, in this basis we obtain the desired isomorphisms:

LEMMA 2 ([9, § 1.4]). *Set* $T_{\mathbb{Q}} := T \otimes \mathbb{Q}$. *We have an isomorphism of* $\mathbb{Q}$*–algebraic groups (see* (5)*)*

$$\begin{array}{ccc} \mathrm{SU}((2,2);\mathbb{Q}(\mathrm{i}))/\{\mathbf{1},-\mathbf{1}\} & \xrightarrow{\simeq} & \mathrm{SO}^+(T_{\mathbb{Q}};\mathbb{Q}) \\ A & \mapsto & \Lambda^2 A | T_{\mathbb{Q}}. \end{array}$$

On the level of integral points we have

$$\varphi : \mathrm{SU}((2,2);\mathbb{Z}[\mathrm{i}])/\{\mathbf{1},-\mathbf{1}\} \xrightarrow{\sim} \mathrm{SO}^+(T). \tag{8}$$

This isomorphism induces an isomorphism of real Lie groups

$$\mathrm{SU}((2,2);\mathbb{C}/)\{\mathbf{1},-\mathbf{1}\} \xrightarrow{\simeq} \mathrm{SO}^+((2,4);\mathbb{R}).$$

The target is the component of the identity of $\mathrm{SO}((2,4);\mathbb{R})$ *and is a simple group.*

Proof. As noted before, a matrix which is at the same time hermitian and orthogonal with respect to the same real matrix has to have real coefficients. So the map $A \mapsto \Lambda^2 A$ sends $\mathrm{SU}((2,2);\mathbb{C})$ injectively to a connected real subgroup of $\mathrm{SO}((6,q))$. A dimension count shows that we get the entire connected component of the latter group which is (isomorphic to) $\mathrm{SO}^+((2,4);\mathbb{R})$.

Assume now that A has coefficients in $\mathbb{Q}(\mathrm{i})$. It then also follows that $\Lambda^2 A | T_{\mathbb{Q}}$ must have rational coefficients, i.e. we have shown the first assertion of the lemma. The assertion about integral points follows since the change of basis matrix from the f-basis to the g-basis is unimodular and hence if A preserves a lattice, $\Lambda^2 A | T$ preserves the corresponding lattice. □

REMARK 1. This isomorphism can be extended to $\mathrm{U}((2,2))$ modulo its center $\mathrm{U}(1)$ provided one takes the semi-direct product of the latter group with an involution τ which acts on matrices $A \in \mathrm{U}((2,2))$ by $\tau(A) = \tau A \tau = \bar{A}$. We set

$$\mathrm{U}^*((2,2)) := \mathrm{U}((2,2)) \rtimes \langle \tau \rangle. \tag{9}$$

Now τ also induces complex conjugation on $\Lambda^2 V$ with respect to the real structure given by the real basis $\{f_1, \dots, f_6\}$. This involution preserves $\{g_1, \dots, g_4\}$ but interchanges g_5 and g_6. So on T the involution becomes identified with the involution

$$\tilde{\tau} : T \to T, \quad \tilde{\tau}(g_k) = g_k, k = 1, \dots, 4, \; \tilde{\tau}(g_5) = g_6 \tag{10}$$

and one can extend the homomorphism φ from (8) by sending τ to $\tilde{\tau}$.

Accordingly, we define a two component subgroup of $\mathrm{O}(2,4)$:

$$\mathrm{SO}^*(2,4) = \mathrm{SO}^+(2,4) \rtimes \langle\tilde{\tau}\rangle.$$

Note that $-\mathbf{1} \in \mathrm{SO}^*(2,4)$ so that

$$\begin{aligned}&\big(\mathrm{U}((2,2))/\mathrm{U}(1)\big) \rtimes \langle\tau\rangle \simeq \mathrm{SU}((2,2))/\{\pm\mathbf{1},\pm\mathrm{i}\mathbf{1}\} \rtimes \langle\tau\rangle \xrightarrow{\simeq} \mathrm{SO}^*(2,4)/\{\pm\mathbf{1}\}\\ &\mathrm{SU}((2,2))/\{\pm\mathbf{1},\pm\mathrm{i}\mathbf{1}\} \xrightarrow{\simeq} \mathrm{SO}^+(2,4)/\{\pm\mathbf{1}\}.\end{aligned}$$

REMARK 2. The (symplectic) basis $\{a_1,a_2,b_1,b_2\}$ of V can be used to define a linear isomorphism

$$\det : \Lambda^4 V_{\mathbb{C}} \xrightarrow{\sim} \mathbb{C}, \qquad a_1\wedge a_2\wedge b_1\wedge b_2 \mapsto 1. \tag{11}$$

We use this to obtain a $\mathbb{C}$-antilinear involution t of $\Lambda^2_{\mathbb{C}} V$ as follows:

$$t : \Lambda^2 V_{\mathbb{C}} \xrightarrow{\sim} \Lambda^2 V_{\mathbb{C}} \quad \text{such that} \quad g(u,v) = -\det(t(v)\wedge u).$$

By definition of t one has $t(f_i) = f_i$, $i = 1,\dots,4$ and $t(f_5) = f_6$, $t(f_6) = f_5$ and since t is $\mathbb{C}$ anti-linear we see that t preserves not only the first 4 basis vectors g_i of T but also the last two g_5, g_6. So the 12-dimensional real vector space $\Lambda^2 V_{\mathbb{C}}$ splits into two real 6-dimensional t-eigenspaces, namely $T_{\mathbb{R}} = T\oplus\mathbb{R}$ for eigenvalue 1 and $\mathrm{i}T_{\mathbb{R}}$ (for eigenvalue -1) respectively:

$$\Lambda^2 V_{\mathbb{C}} = T_{\mathbb{R}} \perp \mathrm{i}T_{\mathbb{R}}. \tag{12}$$

1.2. Congruence Subgroups

The quotient $\mathrm{SU}((2,2);R)/\mathrm{SU}((2,2);R)(\omega)$ acts naturally on $(R/(\omega R))^4 = \mathbb{F}_2^4$. Since the hermitian form $\mathrm{i}f$ in the basis $\{a_1,a_2,b_1,b_2\}$ descends to this $\mathbb{F}_2$–vector space to give a symplectic form, we then get an isomorphism

$$\mathrm{SU}((2,2);R)/\mathrm{SU}((2,2);R)(\omega) \xrightarrow{\sim} \mathrm{Sp}(4;\mathbb{F}_2).$$

By (12) we have $T\oplus \mathrm{i}T = \Lambda^2 V_R$. Lemma 2, states that the group $\mathrm{SU}((2,2);R)$ acts on T and so the subgroup $\mathrm{SU}((2,2);R)(\omega)$ acts on $(1+\mathrm{i})^2\Lambda^2 V_R = 2\Lambda^2 V_R$. It preserves the sublattice $2T\subset T$. It follows that under the isomorphism of Lemma 2 one gets an identification

$$\mathrm{Sp}(4;\mathbb{F}_2) \xleftarrow{\sim} \mathrm{SU}((2,2);R)/\mathrm{SU}((2,2);R)(\omega) \xrightarrow{\sim} \mathrm{SO}^+(T)/\mathrm{SO}^+(T)(2). \tag{13}$$

Since the involution $\tilde{\tau}\in \mathrm{O}^+(2,4)$ (see (10)) obviously does not belong to the congruence 2 subgroup, we may define extensions as follows:

$$\mathrm{SO}^*(T)(2) := \mathrm{SO}^+(T)(2) \ltimes \langle\tilde{\tau}\rangle, \quad \mathrm{U}^*((2,2);R)(\omega) := \mathrm{U}((2,2);R)(\omega) \ltimes \langle\tau\rangle.$$

In particular we have

$$\mathrm{Sp}(4;\mathbb{F}_2) \xleftarrow{\sim} \mathrm{U}^*((2,2);R)/\mathrm{U}^*((2,2);R)(\omega) \xrightarrow{\sim} \mathrm{SO}^*(T)/\mathrm{SO}^*(T)(2). \tag{14}$$

REMARK 1. 1. See also [9, § 1.5], where the result is shown by brute force. To compare, we need a dictionary. The group $\Gamma\mathcal{A}$ from loc. cit. is our $\mathrm{SO}^*(T)/\pm 1$. The group $\Gamma\mathcal{A}(2)$ is the congruence 2 subgroup which is equal to $\mathrm{O}^+(T)(2)/\pm 1$. It lacks the involution $\tilde{\tau}$ and hence has index 2 in the extended group $\mathrm{O}^+(T)(2) \ltimes \langle\tilde{\tau}\rangle$ modulo its center. This explains why $\Gamma\mathcal{A}/\Gamma\mathcal{A}(2) \simeq \mathrm{Sp}(4;\mathbb{F}_2)\times\mathbb{Z}/2\mathbb{Z}$.
2. As in [6, §3] it can be shown that the restriction homomorphism $\mathrm{O}(T(2)) \to \mathrm{O}(q_{T(2)})$ is surjective with kernel the congruence subgroup $\mathrm{O}(T(2))(2)$. The orbits of $\mathrm{O}(q_{T(2)})$ acting on $T(2)^*/T(2) \simeq \mathbb{F}_2^6$ have been described explicitly in loc. cit., using coordinates induced by the standard basis for $U(2) \perp U(2) \perp A_1 \perp A_1$. The form $q_{T(2)}$ is $\mathbb{Z}/2\mathbb{Z}$-valued on the sublattice $F = \{a = (a_1,\ldots,a_6) \in \mathbb{F}_2^6 \mid a_5 + a_6 = 0\}$ and $b_{T(2)}$ restricts to zero on $F^0 = \{0, \kappa = (1,1,1,1,0,0)\}$. Hence $b_{T(2)}$ induces a symplectic form on $F/F^0 \simeq \mathbb{F}_2^4$ (this explains anew that $\Gamma\mathcal{A}/\Gamma\mathcal{A}(2) \simeq \mathrm{Sp}(4;\mathbb{F}_2)\times\mathbb{Z}/2\mathbb{Z}$). The orbits are now as follows:

1. two orbits of length 1: 0 and κ;
2. $\{a \neq 0, q_{T(2)} = 0\}$, the orbit (of length 15) of $(1,0,0,0,0,0)$;
3. $\{a \neq \kappa, q_{T(2)} = 1\}$, the orbit (of length 15) of $(1,1,0,0,0,0)$;
4. $\{a, q_{T(2)} = \frac{1}{2}\}$, the orbit (of length 12) of $(1,1,0,0,1,0)$; it splits into two equal orbits under $\mathrm{SO}^*(T(2))/\mathrm{SO}^*(T(2))(2)$ (the involution exchanging the last two coordinates act as the identity in this quotient);
5. $\{a, q_{T(2)} = -\frac{1}{2}\}$, the orbit (of length 20) of $(0,0,0,0,1,0)$; it splits also into two equal orbits under $\mathrm{SO}^*(T(2))/\mathrm{SO}^*(T(2))(2)$.

1.3. The Corresponding Symmetric Domains

Recall [9, § 1.1] , that the symmetric domain associated to the group $\mathrm{U}(n,n)$, $n \geq 1$ is the n^2–dimensional domain[‖]

$$\mathbf{H}_n := \left\{ W \in M_{n\times n}(\mathbb{C}) \mid \frac{1}{2\mathrm{i}}(W - W^*) > 0 \right\} \simeq \mathrm{U}(n,n)/\mathrm{U}(n)\times\mathrm{U}(n).$$

Indeed, writing

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{U}(n,n), \quad A,B,C,D \in M_{n\times n}(\mathbb{C})$$

the action is given by $\gamma(W) = (AW+B)(CW+D)^{-1}$. The full automorphism group of $\mathbf{H}_n$ is the semi-direct product $[\mathrm{U}(n,n)/\mathrm{U}(1)] \rtimes \langle\tau\rangle$ where τ is the involution given by $\tau(W) = W^{-1}$. Since $\tau\circ\gamma\circ\tau = \bar{\gamma}$ this indeed corresponds to complex conjugation on $\mathrm{SU}(n,n)$ (see Remark 1).

[‖] Recall: for a matrix W, we abbreviate $W^* = {}^{\mathsf{T}}\bar{W}$.

The symmetric domain associated to a bilinear form b of signature $(2,n)$, $n \geq 2$ is the n-dimensional connected **bounded domain of type IV**

$$\mathbf{D}_n := \{z = [(z_1 : \ldots : z_{n+2})] \in \mathbb{P}^{n+1} \mid {}^{\mathsf{T}}zbz = 0, z^*bz > 0, \mathrm{im}(z_3/z_1) > 0\}.$$

Without the second defining inequality the resulting domain is no longer connected; the subgroup $\mathrm{O}^+(2,n)$ preserves each connected component.

This domain parametrizes polarized weight 2 Hodge structures (T,b) with Hodge numbers $(1,n,1)$. This can be seen as follows. The subspace $T^{2,0} \subset T \otimes \mathbb{C}$ is a line in $T \otimes \mathbb{C}$, i.e. a point $z \in \mathbb{P}(T)$. The polarizing form b is a form of signature $(2,n)$ and the two Riemann conditions translate into ${}^{\mathsf{T}}zbz = 0$ and $z^*bz > 0$. These two relations determine an open subset $D(T) \subset \mathbb{P}(T)$ and the moduli space of such Hodge structures is thus $D(T)/\mathrm{O}(T)$. Now $D(T)$ has two components, one of which is (isomorphic to) $\mathbf{D}_n$; both $\mathrm{SO}^+(T)$ and the involution $\tilde{\tau}$ preserve the components, $\mathrm{SO}^+(T)$ has index 4 in $\mathrm{O}(T)$ and hence

$$\mathrm{O}(T) = \underbrace{\mathrm{SO}^+(T) \cup \tilde{\tau}\,\mathrm{SO}^+(T)}_{\mathrm{SO}^*(T)} \cup \underbrace{\sigma\,\mathrm{SO}^+(T) \cup \sigma\tilde{\tau}\,\mathrm{SO}^+(T)}_{\sigma\,\mathrm{SO}^*(T)}$$

where σ permutes the two components of $D(T)$: our moduli space can be written as the orbit space

$$\mathbf{D}_n/\,\mathrm{SO}^*(T).$$

REMARK 3. Let us specify this to the case which interests us most, $n = 4$. Then, $\mathbf{D}_4 = \mathrm{SO}^+(2,4)/((\mathrm{O}(2) \times \mathrm{O}(4)) \cap \mathrm{SO}(6))$ and its automorphism group is $\mathrm{O}^+(2,4)/\langle -\mathbf{1}\rangle$ (acting as group of projectivities on the projective space $\mathbb{P}^5$ preserving the quadric in which $\mathbf{D}_4$ is naturally sitting).

The element $c := \mathrm{diag}(1,1,-1,-1,1,1) \in \mathrm{SO}(2,4)$ preserves the lattice T and exchanges the two components; every element of $\mathrm{SO}(2,4)$ can be written as $cg = g'c$ with $g, g' \in \mathrm{SO}^+(2,4)$. The element $a := \mathrm{diag}(1,1,1,1,U)$, $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has determinant -1 and every element in $\mathrm{O}(2,4)$ can be written as a product $acg = cag$ with $g \in \mathrm{SO}^+(2,4)$.

PROPOSITION 1 ([9, §1.1]). *There is a classical isomorphism between the two domains* $\mathbf{H}_2 \xrightarrow{\sim} \mathbf{D}_4$ *which is equivariant with respect to the isomorphism*

$$[\mathrm{U}((2,2))/\mathrm{U}(1)] \rtimes \langle \tau \rangle \xrightarrow{\simeq} \mathrm{SO}^*(2,4)/\{\pm \mathbf{1}\}$$

of Remark 1.

Sketch of proof: We describe the isomorphism briefly as follows. To the matrix $W \in \mathbf{H}_2$ one associates the 2–plane in $\mathbb{C}^4$ spanned by the rows of the matrix $(W|\mathbf{1}_2)$. This sends $\mathbf{H}_2$ isomorphically to an open subset in the Grasmannian of 2–planes in $\mathbb{C}^4$ which by the Plücker embedding gets identified with an open subset of the Plücker quadric in $\mathbb{P}^5$. This open subset is the type IV domain $\mathbf{D}_4$. □

REMARK 4. Recall we started § 1.1 with a pair (V, f), with $V = \mathbb{Q}(\mathrm{i})^4$ and f a non-degenerate skew-hermitian form on V. The above assignment $W \mapsto$ the plane $P = P_W \subset V$ spanned by the rows of the matrix $(W|\mathbf{1}_2)$ is such that the hermitian form $\mathrm{i}f|P$ is positive. In other words, the domain $\mathbf{H}_2$ parametrizes the complex 2–planes P in $V_{\mathbb{C}} = V \otimes \mathbb{C} = \mathbb{C}^4$ such $(\mathrm{i} \cdot f)|P > 0$.

1.4. Orbits in The Associated Lattice.

Recall that $T := U^2 \oplus \langle -1 \rangle^2$ is the lattice on which the group $\mathrm{O}^*(T)$ acts by isometries. To study the orbits of vectors in T we use the results from [20]. We summarize these for this example. Recall that a primitive vector x in a lattice is called **characteristic** if $\langle x, y \rangle \equiv \langle y, y \rangle \bmod 2$ for all vectors y in the lattice. Other vectors are called **ordinary**. In an even lattice all primitive vectors are characteristic. In the standard basis $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ for T we let $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6)$ be the coordinates. Then $\mathbf{x}$ is characteristic if and only if $x_1, \ldots, x_4$ are even and x_5, x_6 are odd. The **type** of a primitive lattice vector is said to be 0 for ordinary vectors and 1 for characteristic vectors. Wall's result formulated for T states that two primitive vectors of the same norm squared and of the same type are in the same $\mathrm{O}(T)$–orbit. So we have:

PROPOSITION 2. *$T = U^2 \oplus \langle -1 \rangle^2$ is a unimodular odd indefinite lattice of signature $(2,4)$, isometric to $\langle 1 \rangle^2 \perp \langle -1 \rangle^4$. Let $\mathbf{x} \in T$ be primitive with $\langle \mathbf{x}, \mathbf{x} \rangle = -(2k+1)$, respectively $-2k$, $k > 0$.*

- *In the first case $\mathbf{x}$ is always non-characteristic and $\mathrm{O}(T)$–equivalent to $(1, -k, 0, 0, 1, 0)$.*
- *In the second case, a vector $\mathbf{x}$ is $\mathrm{O}(T)$–equivalent to $(2, \frac{1}{2}(-k+1), 0, 0, 1, 1)$ if characteristic (and then $k \equiv 1 \bmod 4$) and to $(1, -k, 0, 0, 0, 0)$ if not.*

REMARK 5. 1) From the description of the subgroups $\mathrm{SO}(T)$ and $\mathrm{SO}^+(T)$ in Remark 3, we see that the "extra" isometries c and a do not change the two typical vectors $(1, -k, 0, 0, 0, 0)$, $(2, \frac{1}{2}(-k+1), 0, 0, 1, 1)$ while a replaces $(1, -k, 0, 0, 1, 0)$ by $(1, -k, 0, 0, 0, 1)$. This can be counteracted upon applying the map $\mathrm{diag}(-1, -1, -1, 1, U) \in \mathrm{SO}^+(T)$. In other words, the preceding Proposition remains true for orbits under the two subgroups $\mathrm{SO}(T)$ and $\mathrm{SO}^+(T)$.

2) Suppose that $-d = \langle \mathbf{x}, \mathbf{x} \rangle$ is a negative even number. It follows quite easily that in the non-characteristic case $\mathbf{x}^\perp$ is isometric to $\langle d \rangle \perp U \perp \langle -1 \rangle \perp \langle -1 \rangle$. In the characteristic case this is subtler. For instance, if $d = -2k$ and k is a sum of two squares, say $k = u^2 + v^2$, the vector $\mathbf{x}$ is in the orbit of $(0, 0, 0, 0, u+v, u-v)$ and so $\mathbf{x}^\perp \simeq U \perp U \perp \langle -d \rangle$. This is the case if $k = a^2 b$ with b square free and $b \equiv 1 \bmod 4$. However, in the general situation the answer is more complicated. The situation over the rational numbers is easier to explain. For later reference we introduce

$$\Delta(\mathbf{x}) := -\frac{1}{2} \langle \mathbf{x}, \mathbf{x} \rangle = \frac{1}{2}(x_5^2 + x_6^2 - 2(x_1 x_2 + x_3 x_4)) > 0.$$

Then, completing the square, one finds:

$$\mathbf{x}^{\perp} \sim_{\mathbb{Q}} U \perp \langle -2 \rangle^2 \perp \langle 2\Delta(\mathbf{x}) \rangle. \tag{15}$$

COROLLARY 1. *Consider the set $Y \subset T$ of vectors of the form*

$$\mathbf{y} = (2y_1, 2y_2, 2y_3, 2y_4, y_5 + y_6, y_5 - y_6) \in T, y_i \in \mathbb{Z}, \gcd(y_1, \dots, y_6) = 1.$$

We have

$$\Delta(\mathbf{y}) = y_5^2 + y_6^2 - 4(y_1 y_2 + y_3 y_4).$$

If $y_5 \not\equiv y_6 \bmod 2$, the vector is a characteristic primitive vector, $\Delta \equiv 1 \bmod 4$ and $\mathbf{y}$ is in the orbit of $(2, \frac{1}{2}(1-\Delta), 0, 0, 1, 1)$.

If $y_5 \equiv y_6 \bmod 2$ the vector $\frac{1}{2}\mathbf{y} \in T$ is primitive and non-characteristic and either $\Delta \equiv 0 \bmod 4$ and $\frac{1}{2}\mathbf{y}$ is in the orbit of $(1, -\frac{1}{4}\Delta, 0, 0, 0, 0)$, or $\Delta \equiv 2 \bmod 4$ and $\frac{1}{2}\mathbf{y}$ is in the orbit of $(1, \frac{1}{4}(2-\Delta), 0, 0, 1, 0)$.

Hence two vectors in Y with the same Δ–invariant are in the same $\mathrm{O}(T)$–orbit. Conversely, if the Δ–invariants are different the vectors are in different orbits.

To be able to make a comparison between [5] and [9] we have to realize that $T(2)$ is the transcendental lattice of the K3 surface in [9] while the lattice which plays a role in [5] is the lattice $T(-1)$. The special vectors $\mathbf{y} \in Y$ related to Hermann's paper are not necessarily primitive. A primitive vector $\mathbf{y}^*$ in the line $\mathbb{Z} \cdot \mathbf{y}$ is called a *primitive representative* for $\mathbf{y}$. This vector will be considered as a vector of $T(2)$. This makes the transition between the two papers possible.

EXAMPLE 1. As a **warning**, we should point out that it might happen that primitive vectors $\mathbf{y}^*$ with the same norm squared in $T(2)$ correspond to *different* $\Delta(\mathbf{y})$. For example $\mathbf{y} = (2, -2, 0, 0, 0, 0)$ and $(0, 0, 0, 0, 1, 1)$ correspond to $(1, -1, 0, 0, 0, 0)$, respectively $(0, 0, 0, 0, 1, 1)$. Both vectors in $T(2)$ have norm squared -4 while the first (with $y_1 = 1, y_2 = -1$) has $\Delta = 4$ and the second has $\Delta = 1$, since $y_5 = 1, y_6 = 0$. From the above it follows that the two are *not* in the same orbit under the orthogonal group of $T(2)$.

Observe now that divisors in our moduli space are cut out by hyperplanes in $\mathbb{P}(T \otimes \mathbb{C})$ orthogonal to elements $t \in T$ and any multiple of t determines the same divisor. We get therefore all possible divisors by restricting ourselves to the set $Y \subset T$. Accordingly we use the subgroup of $\mathrm{SO}^*(T)$ preserving the set Y of vectors of this form. Then it is natural to consider the basis $\{2g_1, 2g_2, 2g_3, 2g_4, g_5 + g_6, g_5 - g_6\}$ so that the new coordinates of $\mathbf{y}$ become $(y_1, y_2, y_3, y_4, y_5, y_6)$. We may identify this vector with $\mathbf{y}^*$.

REMARK 6. Suppose $\mathbf{y} \in Y$ as in Corr. 1. Set $\Delta(\mathbf{y}) = \Delta$. Let $n(\Delta)$ the number of different $\mathrm{SO}^*(T)(2)$–orbits in a given $\mathrm{SO}^*(T)$-orbit for $\mathbf{y} \in T$ when $\Delta \equiv 0 \bmod 4$, respectively $\frac{1}{2}\mathbf{y}$ else. Using Corr. 1 and Rem. 1.2 one sees the following:

- If $\Delta \equiv 0 \bmod 4$ then $n(\Delta) = 15$;

- If $\Delta \equiv 2 \bmod 8$ then $n(\Delta) = 10$;
- If $\Delta \equiv 6 \bmod 8$ then $n(\Delta) = 6$;
- If $\Delta \equiv 1 \bmod 4$ then $n(\Delta) = 1$.

Indeed, for instance a vector for which $\Delta \equiv 0 \bmod 4$ is in the $\mathrm{SO}^*(T)$-orbit of $(1, -\frac{\Delta}{4}, 0, 0, 0, 0)$ which corresponds to $(1,0,0,0,0,0)$ or $(1,1,0,0,0,0)$ in $\mathbb{F}_2^6$ according to whether $\Delta \equiv 0 \bmod 8$ or $\Delta \equiv 4 \bmod 8$. Both of these have orbitsize 15 under $\mathrm{O}(q_{T(2)})$. If $\Delta \equiv 2, 6 \bmod 8$, in applying Rem. 1 2., one has to take care of the extra involution explaining why the number of orbits $n(\Delta)$ is half the orbitsize under $\mathrm{O}(q_{T(2)})$.

In fact, this result is completely equivalent to [5, Prop. 2] which is stated below (Prop. 4).

2. A Moduli Interpretation: the Abelian Varieties Side

2.1. Special Abelian Varieties

We say that an even dimensional polarized Abelian variety (A, E) is of K-**Weil type** if $\mathrm{End}_{\mathbb{Q}}(A)$ contains an imaginary quadratic field $K = \mathbb{Q}(\alpha) \subset \mathbb{C}$ such that the action of α on the tangent space of A at 0 has half of its eigenvalues equal to α and half of its eigenvalues equal to $\bar{\alpha}$ (note that this does not depend on the embedding $K \hookrightarrow \mathbb{C}$). We say that α **has type** (k, k), where $2k = \dim A$. Moreover, we want that $E(\alpha x, \alpha y) = |\alpha|^2 E(x, y)$. If $\alpha = \mathrm{i}$ this means that A admits an automorphism M with $M^2 = -1$ which preserves the polarization. Equivalently, $R \subset \mathrm{End}(A, E)$.

As is well known (cf. [5, § 1.2], [19, §10]), the symmetric domain $\mathbf{H}_2$ parametrizes such Abelian 4–folds of $\mathbb{Q}(\mathrm{i})$–Weil type. We recall briefly how this can be seen. Consider the lattice $V_R = R^4$ equipped with the skew form $\mathbf{J}$ (see (5)). The complex vector space $V = \mathbb{C}^4$, considered as a real vector space, contains V_R as a lattice and $\mathbf{J}$ is a unimodular integral form on it. Weight -1 Hodge structures on V polarized by this form are given by complex structures J that preserve the form. They correspond to principally polarized Abelian 4-folds A, and if J commutes with multiplication by i the Abelian variety A admits an order 4 automorphism M of type $(2,2)$ which preserves the polarization. The converse is also true.

To get the link with the domain $\mathbf{H}_2$, recall from Remark 4 that points in $\mathbf{H}_2$ correspond precisely to complex 2-planes P in V for which $(\mathrm{i} \cdot f)|P > 0$. The direct sum splitting of $V = P \perp P^{\perp}$ can be used to define a complex structure J on the 8-dimensional real vector space V as desired by imposing $J|P = \mathrm{i}\mathbf{1}$, $J|P^{\perp} = -\mathrm{i}\mathbf{1}$. This complex structure commutes with multiplication by i on V and preserves if (since this is a hermitian form). The embedding (4) identifies the image with those $\tau \in \mathfrak{h}_4$ that form the fixed locus of the order 4 automorphism formed from $\mathbf{J}$ (see (5))

$$\begin{pmatrix} \mathbf{J} & \mathbf{0}_4 \\ \mathbf{0}_4 & \mathbf{J} \end{pmatrix} \in \mathrm{Sp}(4; \mathbb{Z}).$$

This automorphism corresponds to multiplication with i on the Abelian 4-fold.

The discrete group $\mathrm{U}^*((2,2);R)$ acts naturally on $\mathbf{H}_2$. It sends an Abelian variety of the given type to an isomorphic one. This is clear for $\mathrm{U}((2,2);R)$. Regarding τ, by [5, § 1.2] the embedding (4) is equivariant with respect to it; indeed, it acts as an integral symplectic matrix on $\mathfrak{h}_4$ and hence also τ permutes isomorphic Abelian varieties. Conversely, since $\mathrm{U}^*((2,2))$ modulo its center is the full group of isomorphisms of $\mathbf{H}_2$ it follows that two isomorphic Abelian varieties with multiplication by R are in the same $\mathrm{U}^*((2,2);R)$–orbit.** So the quotient

$$\mathbf{M} := \mathbf{H}_2/\mathrm{U}^*((2,2);R)$$

is the moduli space of principally polarized Abelian fourfolds with multiplication by R.

2.2. Relation With Special Weight 2 Hodge Structures

Consider the Hodge structures parametrized by the domain $\mathbf{D}_4$ introduced in § 1.3. The construction of § 2.1 relates such Hodge structures to polarized Abelian 4-folds A with multiplication by R. Indeed, $V = H_1(A;\mathbb{R})$ underlies an integral polarized Hodge structure of weight -1 and rank 8 admitting an extra automorphism M of order 4 induced by $\mathrm{i} \in \mathrm{End}(A)$. Giving a polarized integral Hodge structure on V of weight 1 is the same as giving a complex structure J preserving the polarization; moreover, M and J commute. According to [7, § 3] this can now be rephrased as follows. Since $V = H_1(A;\mathbb{R})$ underlies a rational polarized Hodge structure of weight -1 and rank 8, the second cohomology $H^2(A) = \Lambda^2 H_1(A)^*$, inherits a polarized Hodge structure of weight 2 and rank 28. We view (V,J) as a 4-dimensional complex vector space and hence we get a complex subspace $\Lambda^2_{\mathbb{C}} V^* \subset H^2(A;\mathbb{R})$ of complex dimension 6 and hence a real Hodge structure of dimension 12. In fact it can be seen to be rational. Recall from Remark 2 that there is a further $\mathbb{C}$–anti linear involution t on $\Lambda^2_{\mathbb{C}} V^*$. Its $(+1)$–eigenspace $T(A)$ has dimension 6 and gives a polarized Hodge substructure of $H^2(A)$ of weight 2 and Hodge numbers $(1,4,1)$ as desired. So, this construction explains the isomorphism

$$\mathbf{M} := \mathbf{H}_2/\mathrm{U}^*((2,2);R) \xrightarrow{\simeq} \mathbf{D}_4/\mathrm{SO}^*(T)$$

Hodge theoretically as the the one induced by $A \mapsto T(A)$.

If instead we divide out by the congruence subgroup $\mathrm{U}^*((2,2);R)(\omega)$ the quotient $\mathbf{M}^* := \mathbf{H}_2/\mathrm{U}^*((2,2);R)(\omega)$ under the natural morphism $\mathbf{M}^* \to \mathbf{M}$ is Galois over of $\mathbf{M}$ with group $\mathrm{Sp}(4;\mathbb{F}_2)$:

$$\begin{array}{ccc} \mathbf{M}^* & \xrightarrow{\simeq} & \mathbf{D}_4/\mathrm{SO}^*(T)(2) \\ \Big\downarrow p & & \Big\downarrow \\ \mathbf{M} & \xrightarrow{\simeq} & \mathbf{D}_4/\mathrm{SO}^*(T). \end{array}$$

**If we would consider such Abelian varieties up to *isogeny* we would classify the isomorphism classes of Abelian 4-folds of $\mathbb{Q}(\mathrm{i})$–Weil type.

2.3. Hypersurfaces in the Moduli Spaces

Any line $[a] \in \mathbb{P}(T \otimes \mathbb{Q})$ defines the divisor $D_{[a]} = \{x \in \mathbf{D}_4 \mid \langle a, x\rangle = 0\}$ inside the domain $\mathbf{D}_4$. As explained in § 1.4, we only consider representatives $a \in T$ which belong to the set $Y \subset T$ whose coordinates with respect to the basis $\{g_1, \ldots, g_6\}$ are of the form:

$$\mathbf{y} = (2y_1, 2y_2, 2y_3, 2y_4, y_5 + y_6, y_5 - y_6) \in T. \tag{16}$$

The corresponding divisor $D_{[\mathbf{y}]}$ inside $\mathbf{H}_2$ can be described by means of the skew symmetric matrix

$$M(\mathbf{y}) := \begin{pmatrix} 0 & -y_2 & \frac{1}{2}(y_5 - \mathrm{i}y_6) & -y_4 \\ y_2 & 0 & -y_3 & -\frac{1}{2}(y_5 + \mathrm{i}y_6) \\ -\frac{1}{2}(y_5 - \mathrm{i}y_6) & y_3 & 0 & -y_1 \\ y_4 & \frac{1}{2}(y_5 + \mathrm{i}y_6) & y_1 & 0 \end{pmatrix}.$$

Indeed, we have [5, p. 119]

$$D_{[\mathbf{y}]} := \left\{ W \in \mathbf{H}_2 \mid \begin{pmatrix} {}^{\mathsf{T}}W & \mathbf{1}_2 \end{pmatrix} M(\mathbf{y}) \begin{pmatrix} W \\ \mathbf{1}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0}_2 \end{pmatrix} \right\}.$$

Then $A \in \mathrm{SU}((2,2))$ acts by sending $M = M(\mathbf{y})$ to

$$A[M] := {}^{\mathsf{T}}AMA = M(\mathbf{z}), \qquad \mathbf{z} = \Lambda^2 A(\mathbf{y}).$$

For a skew symmetric matrix M with coefficients in any field K the determinant is always a square in the field and any root is called a pfaffian of M and denoted by $\mathrm{Pf}(M)$. If $K \subset \mathbb{R}$ we take the *positive root* and call it **the pfaffian**. By Corol. 1 and Remark 5, 1) we have

PROPOSITION 3. *Given a positive integer* Δ, *there is precisely one* $\mathrm{SO}^+(T)$ *orbit of primitive vectors* $\mathbf{y} \in Y$ *for which* $\Delta = -\frac{1}{2}\langle \mathbf{y}, \mathbf{y}\rangle$. *All such vectors* $\mathbf{y}$ *are* $\mathrm{SU}((2,2))$-*equivalent and the corresponding pfaffians* $\mathrm{Pf}(M_{\mathbf{y}})$ *are all equal.*

Moreover, such divisors $D_{[\mathbf{y}]}$ *define the same irreducible divisor* D_Δ *in the moduli space* $\mathbf{M}$.

REMARK 2. The image of such a divisor D_Δ in the moduli space $\mathbf{M}$ can be considered as a three dimensional modular variety $\mathfrak{h}_2/\Gamma$, where Γ is a discrete subgroup of a certain modular group $\mathrm{Sp}(2;\mathbb{R})$ (depending only on Δ). For this point of view see [5]. For the special case $\Delta = 1$ see also (3) in the Introduction.

Under the congruence subgroup $\mathrm{SU}((2,2);R)(\omega)$ there are more orbits corresponding to the fact that D_Δ may split under the cover $\mathbf{M}^* \to \mathbf{M}$. By Remark 6 we have:

PROPOSITION 4 ([5, Prop. 2]). *Under the* $\mathrm{Sp}(4,\mathbb{F}_2)$-*cover* $\pi : \mathbf{M}^* \to \mathbf{M}$ *the divisor* $\pi^{-1}D_\Delta$ *associated to a primitive class* $a \in T^*_{\mathbb{Z}}$ *with* $q(a) = \Delta$ *splits in* 15, 10, 6 *or* 1 *components if* $\Delta \equiv 0 \bmod 4$, $\equiv 2 \bmod 8$, $\equiv 6 \bmod 8$, *respectively* $\equiv 1 \bmod 4$.

3. Moduli Interpretation: Special K3 surfaces

3.1. Configurations of 6 Lines in the Plane

For this section see [10] and in particular Appendix A in it.

Let $\mathbf{P} = \mathbb{P}^2$ and $\mathbf{P}^*$ the dual projective space. For any integer $n \geq 4$ a configuration of n-tuples of points in $\mathbf{P}$ corresponds to a configuration for n-tuples of lines in $\mathbf{P}^*$. The notion of *good position* is easy to describe on the dual space as follows. An n-tuple of lines $(\ell_1, \cdots, \ell_n) \in (\mathbf{P}^*)^n$ is called *in good position* if the corresponding curve $\ell_1 \cup \cdots \cup \ell_n \subset \mathbb{P}^2$ has only ordinary double points. They form a Zariski open subset

$$U_n = \{(\ell_1, \cdots, \ell_n) \text{ in good position } \} \subset (\mathbf{P}^*)^n.$$

The linear group $\mathrm{GL}(3;\mathbb{C})$ acts on this space and we form the quotient

$$X_n := U_n / \mathrm{GL}(3;\mathbb{C}) : \textbf{configuration space of } n \textbf{ ordered lines in good position in } \mathbb{P}^2\textbf{.}$$

This is a Zariski open subset of $\mathbb{C}^{2(n-4)}$. The symmetric group $\mathfrak{S}_n$ acts on X_n and the quotient is the configuration space of n unordered lines in general position.

If $n = 6$ there is an extra involution on X_6 induced by the correlation

$$\delta : \mathbf{P} \to \mathbf{P}^*, \quad x \mapsto \text{ polar of } x \text{ with respect to a nonsingular conic } C$$

as follows: the 6 lines $\{\ell_1, \ldots, \ell_6\}$ form 2 triplets, say $\{\ell_1, \ell_2, \ell_3\}$ and $\{\ell_4, \ell_5, \ell_6\}$ each having precisely 3 intersection points. If we set $P_{ij} = \ell_i \cap \ell_j$ we thus get the triplets $\{P_{12}, P_{13}, P_{23}\}$ and $\{P_{45}, P_{46}, P_{56}\}$. A correlation δ is an involutive projective transformation: it sends the line through P and Q to the line through $\delta(P)$ and $\delta(Q)$. In particular, three distinct points which are the vertices of a triangle are sent to the three sides of some (in general different) triangle. This gives already a involution on the variety of three ordered non-aligned points which is easily seen to be holomorphic. However, it descends as the trivial involution on the space of unordered non-aligned triples since a projectivity maps any such triple to a given one.

The above procedure for six points gives a holomorphic involution on U_6

$$*_C(\ell_1, \ldots, \ell_6) = (\delta P_{12}, \delta P_{13}, \delta P_{23}, \delta P_{45}, \delta P_{46}, \delta P_{56}).$$

It descends to an involution on $\mathbf{X} = X_6$ which does not depend on the choice of C and commutes with the action of the symmetric group $\mathfrak{S}_6$. We set

$$\mathbf{Y} = \mathbf{X}/\{*\}.$$

The involution $*_C$ has as fixed point set on U_6 the 6-tuples of lines all tangent to the conic C. On X this gives a non-singular divisor $XQ \subset \mathbf{X}$, the configuration of 6-uples of lines tangent to some fixed conic. This shows in particular that $*_C$ is a non-trivial involution, in contrast to what happens for triplets of points.

We need an alternative description of this involution:

PROPOSITION 5. *The involution $*$ is, up to a projective transformation, induced by a standard Cremona transformation with fundamental points P_{14}, P_{25}, P_{36}.*

Proof. For the proof consult also Fig. 1. The points P_{14}, P_{25}, P_{36} form a triangle Δ_0. The lines ℓ_1, ℓ_2, ℓ_3 form a triangle Δ_1 and ℓ_4, ℓ_5, ℓ_6 another triangle Δ_2. In the dual plane the sides of the triangle Δ_0 correspond to three non-collinear points, say $p, q, r \in \mathbf{P}^*$. We denote the points in $\mathbf{P}^*$ corresponding to the lines ℓ_j by the same letter. The configuration of the three triangles Δ_0, Δ_1, Δ_2 is self-dual in the obvious sense. Note that the cubic curves $\ell_1\ell_2\ell_3 = 0$ and $\ell_4\ell_5\ell_6 = 0$ span a pencil of cubics passing through the vertices of the union of the triangles $\Delta_1 \cup \Delta_2$. The same holds for the dual configuration in $\mathbf{P}^*$.

By [4, p. 118–119] this implies that the standard Cremona transformation T^* with fundamental points p, q, r transforms these "dual" 6 vertices into a so-called associated 6-uple. To see this, let Y be the 3×6 matrix whose columns are the vectors of the six points $\{\ell_1, \ldots, \ell_6\}$ (in some homogeneous coordinate system). The corresponding matrix Y^* for the associated point set $\{T^*\ell_1, \ldots, T^*\ell_6\}$ by definition satisfies $Y\Lambda^{\mathsf{T}}Y^* = 0$ for some diagonal matrix Λ. In our case we can take coordinates in such a way that $Y = (I_3, A)$ with A invertible and after a projective transformation we may assume that $Y^* = (I_3, -\frac{1}{\det A}A^*)$, where A^* is the matrix of cofactors of A so that $A^{\mathsf{T}}A^* = \det(A)I_3$. This exactly means that the point set which gives Y is related to the point set given by Y^* by the involution $*_C$ where C is the conic $x^2 + y^2 + z^2 = 0$. See the calculations in [10, Appendix A2]. $\square$

The space $\mathbf{X}$ can be compactified to $\bar{\mathbf{X}}$ by adding certain degenerate configurations to which the involution $*$ extends and the resulting compactification $\bar{\mathbf{Y}} = \bar{\mathbf{X}}/\{*\}$ is naturally isomorphic to $\mathbb{P}^4$. The group $\mathfrak{S}_6$ acts on both sides giving a commutative diagram

$$\begin{array}{ccc} \bar{\mathbf{X}} & \xrightarrow{\sigma} & \bar{\mathbf{Y}} \simeq \mathbb{P}^4 \\ \downarrow & & \downarrow \pi \\ \bar{\mathbf{X}}/\mathfrak{S}_6 & \xrightarrow{\bar{\sigma}} & \bar{\mathbf{Y}}/\mathfrak{S}_6. \end{array}$$

Comparison with Hermann's work

We now compare the result with [5]. We need some more details of the above construction. To start with we choose coordinates in $\mathbf{Y}$ by representing first a point in $\mathbf{Y}$ by a 3×6 matrix (x_{ij}) (the 6 rows give the six lines) and let $d_{ijk}(x)$ be the minor obtained by taking columns i, j, k. Then for every permutation $\{ijk\ell mn\}$ of $\{1, \ldots, 6\}$, consider the 10 Plücker coordinates $Z_{ijk} := d_{ijk}d_{\ell mn}$ which one uses to embed $\mathbf{Y}$ in $\mathbb{P}^9$. The Plücker relations $Z_{ijk} - Z_{ij\ell} + Z_{ijm} - Z_{ijn} = 0$ then show that this embedding is a linear embedding into $\mathbb{P}^4 \subset \mathbb{P}^9$. Note that the permutation group $\mathfrak{S}_6$ interchanges the Plücker coordinates and the image 4-space is invariant under this action.

In [5, § 4] an embedding[††] of $\bar{\mathbf{Y}}$ into $\mathbb{P}^5$ with homogeneous coordinates

[††] What Hermann calls $\bar{X}(1+\mathrm{i})$ is in our notation $\bar{Y}$ and in Matsumoto's notation Y^*.

$(Y_0,\dots,Y_5)$ is constructed with image the hyperplane $(Y_0+\cdots+Y_5)=0$. We may assume that the Y_i, $0\le i\le 4$ coincide with some of our Plücker coordinates which we write accordingly as $\{Y_0,\dots,Y_9\}$. Indeed, Hermann's six coordinates are permuted as the standard permutation of $\mathfrak{S}_6$ on 6 letters and we can scale them in order that the hyperplane in which the image lies is given by the equation $(Y_0+\cdots+Y_5)=0$.

One of the divisors in $\bar{\mathbf{X}}$ added to $\mathbf{X}$ is needed below. It is called X_3 in [10] and has the 20 components X_3^{ijk} of configurations of 6 lines where precisely the three lines ℓ_i,ℓ_j,ℓ_k meet at one point which create one triple point. There are 12 further points where only 2 lines meet. We now show how to identify $\sigma(X_3)$ with the divisor $D_2'=\pi^{-1}D_2$ from Prop. 4. First invoke [10, Prop. 2.10.1] (see also Theorem 2 below) which makes the transition from $\mathbf{X}$ to $\mathbf{M}$ possible. Next, from [5, p. 122-123], we infer that the equation of D_2' reads

$$\prod_{abc}(Y_a+Y_b+Y_c)=0.$$

By the Plücker relations $Y_a+Y_b+Y_c=\pm Y_d$ for some $d\in\{0,\dots,9\}$. Hence $Y_a+Y_b+Y_c=\pm Y_d$, say $\pm Y_d=Z_{ijk}$ and the zero locus of that factor corresponds to $D_{ijk}\cdot D_{lmn}=0$. It follows that indeed $D_2'=X_3$. We observe that there are 20 irreducible components X_3^{ijk} but each Y_d gives two of them via the double cover σ, so we get indeed the 10 divisors of Hermann.

As a side remark, $\bar{\mathbf{X}}-\mathbf{X}$ contains further divisors, several of which parametrize K3 surfaces, namely whenever the double points in the configuration coalesce to triple points at worst.

We want to stress that the definition of *good* includes rather special configurations, one of which is needed below, namely the ones forming a divisor $\mathbf{X}_{\mathrm{coll}}\subset\bar{\mathbf{X}}$ corresponding to 6 lines $\{\ell_1,\dots,\ell_6\}$ where the intersection points $P_{12}=\ell_1\cap\ell_2, P_{34}=\ell_3\cap\ell_4, P_{56}=\ell_5\cap\ell_6$ of three pairs of lines are collinear. We can identify $\sigma\mathbf{X}_{\mathrm{coll}}$ with $\pi^{-1}D_4$ as follows. From [5, p. 123], we find that the equation of D_4 is

$$\prod_{ij}Y_i-Y_j=0.$$

The Plücker relations yield the equations

$$D_{abc}D_{efg}=D_{a'b'c'}D_{e'f'g'}$$

where in addition to $\{a,b,c,e,f,g\}=\{a',b',c',e',f',g'\}=\{1,\dots,6\}$ necessarily (up to commuting the factors of the products)

$$\#\{a,b,c\}\cap\{a',b',c'\}=\#\{e,f,g\}\cap\{e',f',g'\}=2.$$

Dually looking at 6 points in $\mathbb{P}^2$, we obtain the same result if three lines (spanned by different pairs of such points) meet in one and the same point. So indeed, we get $\sigma\mathbf{X}_{\mathrm{coll}}$. Observe that there are $\frac{1}{6}15\times 6=15$ ways to make the intersection points collinear giving 15 components as it should.

Relation to recent work of Kondō

In a recent preprint [6], Kondō also studies Heegner divisors on the moduli space of 6 lines in $\mathbb{P}^2$ using Borcherd's theory of automorphic forms on bounded symmetric domains of type IV. Specifically he singles out four divisors in [6, §3] which correspond to the cases $\Delta = 1,2,4,6$ studied extensively in this paper.

3.2. Double Cover Branched in 6 Lines in Good Position

We refer to [1, Ch VIII] for details of the following discussion on moduli of K3-surfaces. The second cohomology group of K3 surface X equipped with the cup product pairing is known to be isomorphic to the unimodular even lattice

$$\Lambda := U \perp U \perp U \perp E_8 \perp E_8.$$

The lattice underlies a weight 2 polarized Hodge structure with Hodge numbers $h^{2,0} = 1$, $h^{1,1} = 20$. The Néron-Severi lattice $\mathrm{NS}(X)$ gives a sub Hodge structure with Hodge numbers $h^{2,0} = 0$, $h^{1,1} = \rho$, the Picard number. Its orthogonal complement $T(X)$, the transcendental lattice, thus also is a polarized Hodge structure with Hodge numbers $h^{2,0} = 1$, $h^{1,1} = 20-\rho$. The Néron-Severi lattice is a Tate Hodge structure, but $T(X)$ has moduli. The two Riemann bilinear relations show that these kind of Hodge structures are parametrized by a type IV domain $\mathbf{D}_{20-\rho}$ (see § 1.3). Conversely, given a sublattice $T \subset \Lambda$ of signature $(2,n), n < 20$, the polarized Hodge structures on T with Hodge numbers $h^{2,0} = 1$, $h^{1,1} = n$ are parametrized by a domain $D(T)$ of type IV whose points correspond to K3 surfaces with the property that the transcendental lattice is contained in T. For generic such points the transcendental lattice will be exactly T, but upon specialization the surfaces may acquire extra algebraic cycles which show up in T. In other words, the transcendental lattice of the specialization becomes strictly smaller than T. Note also that $\dim D(T) = n$. These surfaces, commonly called **T-lattice polarized K3 surfaces**, thus have n moduli. It is not true that all points in $D(T)$ correspond to such K3 surfaces: one has to leave out the hyperplanes H_α that are perpendicular to the roots α in T; since T is an even lattice, these are elements $\alpha \in T$ with $\langle \alpha, \alpha \rangle = -2$. We quote the following result [1, VIII, §22] which makes this precise:

THEOREM. *Let*

$$D^0(T) := D(T) - \bigcup H_\alpha, \quad \alpha \text{ a root in } T.$$

The moduli space of T-lattice polarized K3 surfaces is the quotient of $D^0(T)$ by the group $\mathrm{SO}^*(T)$.

Examples of K3 surfaces with $\rho = 1$ (having 19 moduli) are the double covers of the plane branched in a generic smooth curve of degree 6 parametrized by a 19-dimensional type IV domain. If we let this curve acquire more and more singularities we get deeper and deeper into this domain. We are especially interested in those sextics that are the union of six lines in good position (see Fig. 1) and certain of their degenerations which were treated in § 3.1.

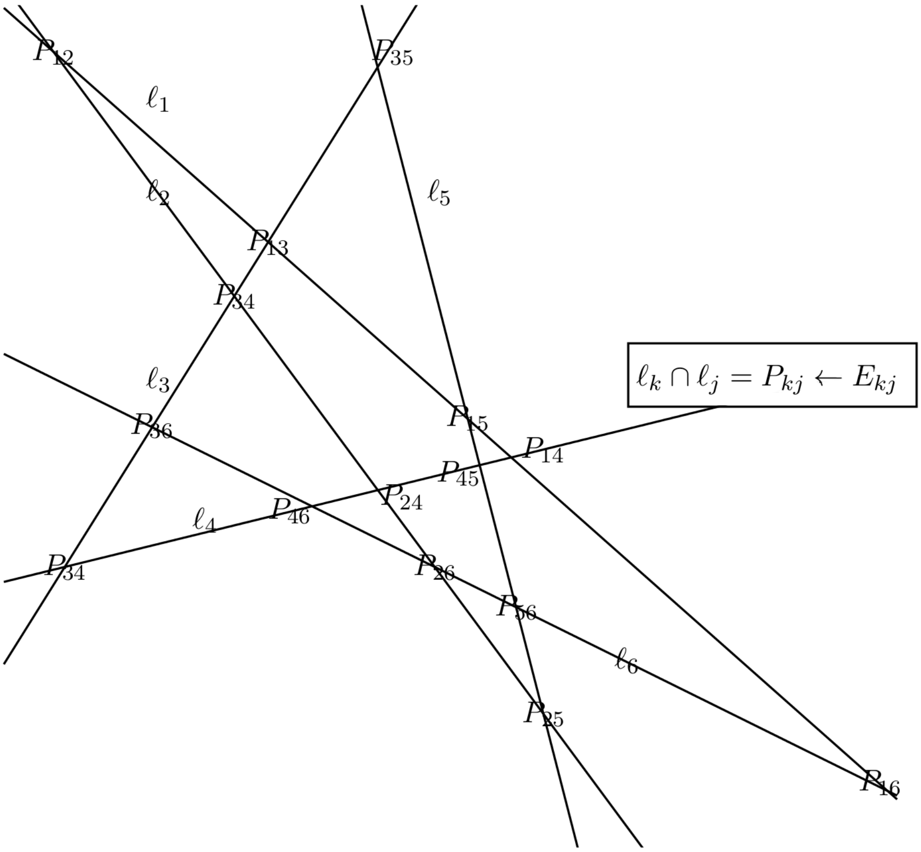


Figure 1: The 6 branch lines and the 15 exceptional curves E_{kj}

PROPOSITION 6. *Let X be the minimal resolution of the double of cover of the plane given by an equation:*

$$w^2 = \ell_1(x,y,z)\cdots\ell_6(x,y,z). \tag{17}$$

Assume that the 6 lines ℓ_i, $i = 1,\ldots,6$ are in general (and, in particular, in good) position. Then the Picard number $\rho(X)$ equals $\rho(X) = 16$.

Proof. The 15 ordinary double points $P_{ij}\,(1 \le i < j \le 6)$ in the configuration give 15 E_{ij} disjoint exceptional divisors on X; these are (-2)-curves, i.e. rational curves with self intersection (-2). The generic line gives one further divisor ℓ with $\ell^2 = 2$ and which is orthogonal to the E_{ij}. The divisors $\{\ell, E_{12}, \ldots, E_{56}\}$ thus form a sublattice N of rank 16 within the Néron-Severi lattice $\mathrm{NS}(X)$. So for the Picard number we have $\rho \geq 16$. As explained above, we have $20-\rho$ moduli where ρ is the Picard number of a generic member of the family. So $\rho = 16$ and hence for generic choices of lines ℓ_i the lattice $\mathrm{NS}(X)$ contains N as a sublattice of finite index. □

We need a simple consequence of the proof. To explain it we need a few notions from lattice theory. Recall that the dual of a lattice L is defined by

$$L^* := \{x \in L \otimes \mathbb{Q} \mid \langle x, y \rangle \in \mathbb{Z}, \quad \text{for all } y \in L\}$$

and that the discriminant group $\delta(L)$ is the finite Abelian group L^*/L. We say that L is p–**elementary**, if this is so for $\delta(L)$, i.e.

$$L^*/L \cong (\mathbb{Z}/p\mathbb{Z})^{\ell}, \quad \ell := \textbf{length of } L \leq \operatorname{rank}(L).$$

From the above proof we see that

$$\langle 2 \rangle \perp \langle -2 \rangle^{15} \subset \mathrm{NS}(X).$$

and so

COROLLARY 2. *The Néron-Severi lattice* $\mathrm{NS}(X)$ *is* 2*–elementary.*

In what follows we shall first of all determine both the Néron-Severi and the transcendental lattice for such a generic K3 surface X. We shall prove:

THEOREM 1. *For generic X as above we have* $\mathrm{NS}(X) = U \perp D_6^2 \perp A_1^2$ *and* $T(X) = U(2)^2 \perp A_1^2 = T(2)$.

This gives an interpretation of the previous results in terms of the moduli of K3 surfaces. Indeed we have $D(T(2)) = \mathbf{D}_4$ and we note that $T(X) = T(2)$ and T have the same orthogonal group. So the above result shows that our moduli space equals

$$D^0(T)/\mathrm{SO}^*(T) = \mathbf{D}_4^0/\mathrm{SO}^*(T).$$

Moreover, by the results of [10] we can now identify this moduli space with the configuration spaces from § 3.1.

THEOREM 2 ([10, Prop. 2.10.1]). *There is a commutative diagram*

$$\begin{tikzcd}
\mathbf{X}/\{*\} = \mathbf{Y} \arrow[r, "\tilde{p}", "\simeq"'] \arrow[d, "\pi"] & \mathbf{D}_4^0/\mathrm{SO}^*(T)(2) \subset \mathbf{M}^* \arrow[d] \\
\mathbf{X}/[\{*\} \times \mathfrak{S}_6] = \mathbf{Y}/\mathfrak{S}_6 \arrow[r, "\simeq", "p"'] & \mathbf{D}_4^0/\mathrm{SO}^*(T) \subset \mathbf{M}.
\end{tikzcd}$$

The holomorphic maps $\tilde{p}$ and p are biholomorphisms. They are the period maps.

In other words, the moduli space of T–lattice polarized K3's can be identified with the quotient of the configuration space of unordered 6-tuples of lines in $\mathbb{P}^2$ by the correlation involution $*$. Note also that the group $\mathfrak{S}_6$ on the left is indeed isomorphic to the quotient $\mathrm{SO}^*(T)/\,\mathrm{SO}^*(T)(2) = \mathrm{Sp}(4;\mathbb{F}_2)$ (see (13)).

REMARK 7. Recall that $*$ sends the corresponding double covering K3 surface X to a K3 surface $*X$. The involution $*$ sends the 6 branch lines defining X to the three branch lines defining $*X$. By Prop. 5 there is a standard Cremona transformation with fundamental points the three points P_{14}, P_{25}, P_{36} which induces this involution on the level of the plane and hence X and $*X$ are isomorphic K3-surfaces, the isomorphism being induced by the Cremona transformation. It follows that the quasi-polarization (given by the class of a line ℓ on the plane) is not preserved under this isomorphism: it is sent to $2\ell - e_{14} - e_{25} - e_{36}$ where e_{ij} is the class of the exceptional curve E_{ij}.

By [9, §1.4.] the involution $*$ corresponds to the involution j which on $T = U^2 \perp \langle -1 \rangle^2$ fixes the first 4 basis vectors and sends the fifth to minus the sixth. Since T as well as its orthogonal complement $S = T^{\perp}$ is 2-elementary, by [11, Theorem 3.6.2] the restriction $\mathrm{O}(S) \to \mathrm{O}(q_S)$ is surjective. Any lift of the image of j under the homomorphism $\mathrm{O}(T) \to \mathrm{O}(q_T)$ to S then can be glued together with j to obtain an isometry of the K3-lattice Λ. Such an isometry sends the period of X to the period of an isomorphic K3-surface which must be $*X$ by the Torelli theorem.

Next, we study what happens when the line configuraton degenerates.

THEOREM 3. *Put*

$$X_\Delta : \textit{the generic K3 surface on } D_\Delta \subset \mathbf{M}$$

We have

1. $\mathrm{NS}(X_2) = U \perp D_4^2 \perp E_7$, $T(X_2) = U(2)^2 \perp A_1$;
2. $\mathrm{NS}(X_4) = U \perp D_6^2 \perp A_3$, $T(X_4) = U(2) \perp \langle 4 \rangle \perp A_1^2$.
3. $\mathrm{NS}(X_1) = U \perp D_4 \perp D_8 \perp A_3$, $T(X_1) = U(2)^2 \perp \langle -4 \rangle$.
4. $\mathrm{NS}(X_6) = U \perp D_6^2 \perp A_1 \perp A_2$, $T(X_6) = U(2) \perp A_1^2 \perp \langle 6 \rangle$.

To prove this we will make substantial use of elliptic fibrations. We should point out that most, if not all computations can be carried out with explicit divisor classes on the K3 surfaces; elliptic fibrations have the advantage of easing the lattice computations as well as providing geometric insights, since the root lattices in the above decomposition of NS appear naturally as singular fibers of the fibration (conf. for instance [2], [13]).

After reviewing the basics on elliptic fibrations needed, we will first prove Theorem 1 in 3.4. Then using lattice enhancements the three cases of Theorem 3 will be covered in 3.6, 3.6 and 3.6.

3.3. Elliptic fibrations and the Mordell-Weil lattice

We start by reviewing basic facts on elliptic fibrations, and in particular on the Néron-Severi lattice for an elliptic fibration with section following Shioda as summarized in [17].

Let $S \to C$ be an elliptic fibration of a surface S, with a section s and general fiber f. These two span a rank 2 sublattice U of the Néron-Severi lattice $\mathrm{NS} = (\mathrm{NS}(S), \langle\, , \rangle)$, isomorphic to the hyperbolic plane. We call s the *zero section*; it meets every singular fiber in a point which figures as the neutral element in a group G_ν whose structure is given in the table below.

Fiber type	F_ν	e_ν	G_ν	$\mathrm{discr}(F_\nu)$	discr. gr.
I_n	A_{n-1}	n	$\mathbb{C}^* \times \mathbb{Z}/n\mathbb{Z}$	$(-1)^n(n+1)$	$\mathbb{Z}/n\mathbb{Z}$
II	–	1	$\mathbb{C}^*$	1	$\{1\}$
III	A_1	1	$\mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$	-2	$\mathbb{Z}/2\mathbb{Z}$
IV	A_2	2	$\mathbb{C} \times \mathbb{Z}/3\mathbb{Z}$	-3	$\mathbb{Z}/3\mathbb{Z}$
I_{2n}^*	D_{2n+4}	$2n+5$	$\mathbb{C} \times (\mathbb{Z}/2\mathbb{Z})^2$	4	$(\mathbb{Z}/2\mathbb{Z})^2$
I_{2n+1}^*	D_{2n+5}	$2n+6$	$\mathbb{C} \times \mathbb{Z}/4\mathbb{Z}$	-4	$\mathbb{Z}/4\mathbb{Z}$
II^*	E_8	9	$\mathbb{C}$	1	$\{1\}$
III^*	E_7	8	$\mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$	-2	$\mathbb{Z}/2\mathbb{Z}$
IV^*	E_6	7	$\mathbb{C} \times \mathbb{Z}/3\mathbb{Z}$	3	$\mathbb{Z}/3\mathbb{Z}$

In the table we enumerate Kodaira's list of singular fibers. The components of a singular fiber f_ν not met by the zero section define mutually orthogonal negative-definite sublattices F_ν of the Néron-Severi lattice, all orthogonal to the hyperbolic plane U. The Euler number of the fiber f_ν is abbreviated by e_ν in the table. The last two entries are the discriminant and the discriminant group of the lattice F_ν.

The lattice

$$T := U \perp \bigoplus_\nu F_\nu$$

is called the **trivial lattice** of the elliptic surface X. It is a sublattice of NS, but *not necessarily primitive*. Its orthogonal complement (inside $\mathrm{NS}(X)$)

$$L := T_{\mathrm{NS}}^\perp$$

is called the **essential lattice**. The group of sections forms the **Mordell Weil group** E. Its torsion part can be calculated as follows:

$$T' := \text{primitive closure of } T \text{ in NS}; \quad T'/T \simeq E_{\mathrm{tors}}.$$

It is one of the main results of the theory of elliptic surfaces that

$$E \cong \mathrm{NS}/T. \tag{18}$$

The most famous incarnation of this fact is often referred to as Shioda-Tate formula:

$$\mathrm{rank}(\mathrm{NS}) = 2 + \sum_\nu \mathrm{rank}(T_\nu) + \mathrm{rank}(E). \tag{19}$$

The main idea now is to endow E/E_{tors} with the structure of a positive definite lattice, the **Mordell-Weil lattice** $\text{MWL} = \text{MWL}(S)$. This can be achieved as follows. Since $L_{\mathbb{Q}} \perp T_{\mathbb{Q}} = \text{NS}_{\mathbb{Q}}$, the restriction to E

$$\pi_E : \text{NS}_{\mathbb{Q}}|_E \to L_{\mathbb{Q}}$$

of the orthogonal projection is well-defined with kernel E_{tors}. The **height pairing** on $\text{MWL} = E/E_{\text{tors}}$ by definition is induced from the pairing on the Néron-Severi group:

$$\langle P,Q\rangle := -\langle \pi_E(P), \pi_E(Q)\rangle, \quad P,Q \in E.$$

Note that by definition, this pairing need not be integral. Shioda has shown that the height pairing can be calculated directly from the way the sections P,Q meet each other, the zero section, and in particular the singular fibers f_ν. In the sequel we will need this only for the height $\langle P,P\rangle$ of an individual section $P \in E$. The only components of f_ν possibly met by a section are the multiplicity 1 components not met by the zero section. For I_n, this gives $n-1$ components that one enumerates successively, starting from the first component next to the one meeting the zero section (upto changing the orientation). For I_n^*, $n > 0$, there are 3 components: the *near* one (the first component next to the one meeting the zero section) and two far ones (for I_0^* fibers the three simple non-identity components are indistinguishable). In the end, the height formula reads

$$h(P) := \langle P,P\rangle = 2\chi(\mathcal{O}_S) + 2\langle P,s\rangle - \sum_\nu c_\nu, \tag{20}$$

where the local contribution c_ν for f_ν can be found in the following table and is determined by the component which the section P meets (numbered $i = 0,1,\dots,n-1$ as above for fibers of type I_n):

Fiber type	root lattice	c_ν
I_n $(n>1)$	A_{n-1}	$\frac{i(n-i)}{n}$
III	A_1	$\frac{1}{2}$
IV	A_2	$\frac{2}{3}$
I_n^* $(n \geq 0)$	D_{n+4}	1 (near), $1+\frac{n}{4}$ (far)
II^*	E_8	$-$
III^*	E_7	$\frac{3}{2}$
IV^*	E_6	$\frac{4}{3}$

The following formula for the discriminant of $\text{NS} = \text{NS}(S)$, the Néron-Severi lattice can be shown to follow from the above observations:

$$\text{discr}(N) = \frac{(-1)^{\text{rank}E}}{|E_{\text{tors}}|^2}\,\text{discr}(T)\cdot\text{discr}(\text{MWL}). \tag{21}$$

3.4. Generic $\mathrm{NS}(X)$ and the Proof of Theorem 1

For later use, we start by computing the Néron-Severi lattice $\mathrm{NS}(X)$ and the transcendental lattice $T(X)$ with the help of elliptic fibrations with a section, the so-called **jacobian elliptic fibrations**. It is a special feature of K3 surfaces that they may admit several jacobian elliptic fibrations. For instance, we can multiply any three linear forms from to the RHS to the LHS of (17) such as

$$X: \quad \ell_1 \cdots \ell_3 w^2 = \ell_4 \cdots \ell_5. \tag{22}$$

Here a fibration is simply given by projection onto $\mathbb{P}^1_w$; it is the quadratic base change $v = w^2$ of a cubic pencil with 9 base points $P_{14}, \dots, P_{36}$ as sections. However, this plentitude of sections (forming a Mordell-Weil lattice of rank 4) makes the lattice computations quite complicated, so we will rather work with two other elliptic fibrations on X.

Note that due to the freedom in arranging the lines in (22), the K3 surface X admits indeed several different fibrations of the above shape. This ambiguity will persist for all elliptic fibrations throughout this note.

Standard elliptic fibration

We shall now derive an elliptic fibration on X which will serve as our main object in the following. For this purpose we specify the elliptic parameter u giving the fibration by

$$u = \ell_1/\ell_2.$$

One easily computes the divisor of u as

$$(u) = 2\ell_1 + E_{13} + \ldots + E_{16} - 2\ell_2 - (E_{23} + \ldots + E_{26}).$$

Both zero and pole divisor encode divisors of Kodaira type I_0^*, hence the morphism

$$u : X \to \mathbb{P}^1$$

defines an elliptic fibration on X with sections $\ell_3, \dots, \ell_6$. Note that the exceptional divisors $E_{ij} (3 \leq i < j \leq 6)$ are orthogonal to both fibers; hence they comprise components of other fibers. We sketch some of these curves in the following figure:

There is an immediate sublattice N of $\mathrm{NS}(X)$ generated by the zero section and fiber components. Here this amounts to

$$N = U \perp D_4^2 \perp A_1^6.$$

Since the rank of N equals the Picard number $\rho = 16$ of X, the sublattice N has finite index in $\mathrm{NS}(X)$. We note two consequences. First, the (-2) curves $E_{ij} (3 \leq i < j \leq 6)$ generically sit on 6 fibers of type I_2 (because otherwise there would be an additional fiber component contributing to $\mathrm{NS}(X)$). For later reference, we denote the other component of the respective fiber by E'_{ij}; this gives another -2-curve on X. Secondly, we

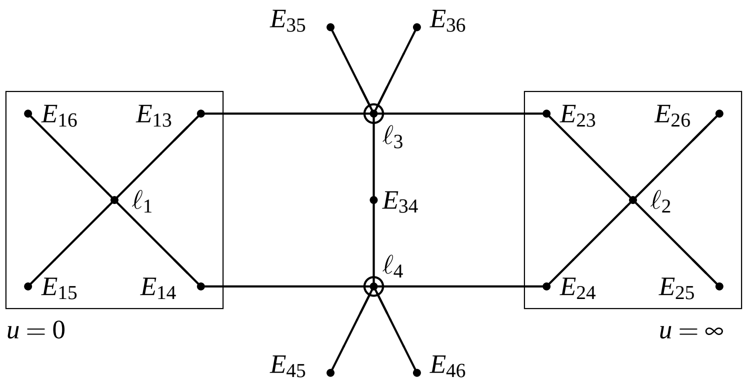


Figure 2: Some sections and fiber components of the standard fibration

deduce from (19) that generically the Mordell-Weil rank is zero. Since I_0^* fibers can only accommodate torsion section of order 2, the given four sections give the full 2-torsion. Alternatively, this can be computed with the height pairing as sketched in 3.3 or it can be derived from the actual equations which we give in 3.6. From (21) we deduce that $\mathrm{NS}(X)$ has discriminant

$$\operatorname{discr}\mathrm{NS}(X) = -2^{10}/2^4 = -2^6. \tag{23}$$

Since by Corollary 2 $\mathrm{NS}(X)$ is 2–elementary we find that (23) implies the length of the Néron–Severi lattice to equal 6.

Transcendental lattice

We want to compute the transcendental lattice $T(X)$. Again we need some general facts from lattice theory which we collect at this place for the reader's convenience.

For an even non-degenerate integral lattice $(L, \langle -,- \rangle)$ recall (see "Notation") the discriminant group $\delta(L) = L^*/L$ and the $\mathbb{Q}/2\mathbb{Z}$–valued **discriminant form** induced by $\langle -,- \rangle$ denoted q_L. The importance of this invariant stems from the following result of Nikulin [11]: two even lattices with the same signature and discriminant form are in the same genus, i.e. are isomorphic over the rationals.

Below we need a more precise result in a special situation:

PROPOSITION 7 ([12, Prop. 4.3.2]). *Any indefinite 2-elementary lattice is determined up to isometry by signature, length, and the property whether the discriminant form takes only integer values or not.*

We return to our double sextics X in the generic situation. Since $T(X)$ and $\mathrm{NS}(X)$ are orthogonal complements embedded primitively into the unimodular lattice

Λ, we find by construction an isomorphism of discriminant forms

$$q_{T(X)} \cong -q_{\mathrm{NS}(X)}. \tag{24}$$

In particular, $T(X)$ is again 2-elementary of length $l = 6$, and of signature $(2,4)$. So we may apply the above Prop. 7. To do so we need to be able to determine properties of the discriminant form. To decide this without going through explicit computations with divisor classes on X, we switch to another elliptic fibration on X.

Alternative elliptic fibration

In order to exhibit another elliptic fibration on X, we start by identifying two perpendicular divisors of Kodaira type I_2^*:

$$\begin{aligned} D_1 &= E_{15} + E_{16} + 2(\ell_1 + E_{13} + \ell_3) + E_{35} + E'_{46} \\ D_2 &= E_{25} + E_{26} + 2(\ell_2 + E_{24} + \ell_4) + E_{45} + E'_{36} \end{aligned}$$

Their linear systems induce an elliptic fibration with section induced by ℓ_6, since $\ell_6.D_i = 1$. In addition to the two fibers of type I_2^*, there are 2 further reducible fibers with identity component E'_{34} on the one hand and E_{56} on the other hand. Rank considerations imply that their type is generically I_2, so that the given fibers and the zero section generate the sublattice $U + D_6^2 + A_1^2$ of $\mathrm{NS}(X)$. In fact, since ranks and discriminants agree, we find the generic equality

$$\mathrm{NS}(X) = U \perp D_6^2 \perp A_1^2. \tag{25}$$

In particular, this singles out A_1^*/A_1 as an orthogonal summand of the discriminant group $\delta(\mathrm{NS}(X)) = \mathrm{NS}(X)^*/\mathrm{NS}(X)$. Its quadratic form thus takes non-integer values in $\frac{1}{2}\mathbb{Z}/2\mathbb{Z}$. As $T(X)$ has the same invariants as $U(2)^2 \perp A_1^2$, by Prop. 7 these must be isomorphic:

$$T(X) = U(2)^2 \perp A_1^2.$$

This concludes the proof of Theorem 1.

The representation of $\mathrm{NS}(X)$ in (25) is especially useful for the concept of lattice enhancements as it allows for writing an abstract isomorphism of discriminant forms as in (24). We will make this isomorphism explicit in 3.5 and exploit it on the level of elliptic fibrations.

3.5. Lattice Enhancements

General Theory

The theory of lattice polarised K3 surfaces as sketched in 3.2 predicts for a given even lattice L of signature $(1, r-1)$ that K3 surfaces admitting a primitive embedding

$$L \hookrightarrow \mathrm{NS}$$

come in $(20-r)$-dimensional families (if L admits a primitive embedding into the K3 lattice $\Lambda = U^3 \perp E_8^2$ at all). Equivalently, on the level of transcendental lattices, the primitive embedding has to be reversed for M the orthogonal complement of L in Λ:

$$T \hookrightarrow M.$$

Lattice enhancements provide an easy concept of specifying subfamilies of lattice polarised K3 surfaces. Namely one picks a vector $v \in M$ of negative square $v^2 < 0$ and enhances NS by postulating v to correspond to an algebraic class. Generically this leads to a codimension one subfamily of K3 surfaces with transcendental lattice

$$T = v^\perp \subset M.$$

The generic Néron-Severi lattice arises as primitive closure (or saturation)

$$\mathrm{NS} = (L + \mathbb{Z}v)' \subset \Lambda.$$

Explicitly NS can be computed with the discriminant form. Namely v induces a unique primitive element in the dual lattice M^*. The resulting equivalence class $\bar{v} \in M^*/M$ corresponds via the isomorphism of discriminant forms $q_M \cong -q_L$ as in (24) with an equivalence class $\bar{w} \in L^*/L$. Enhancing $L + \mathbb{Z}v$ by $\bar{v} + \bar{w}$ results in a well-defined even saturated lattice which exactly gives NS.

Note that presently $M = U(2)^2 \perp A_1^2$ has rank equalling its length, so any primitive vector $v \neq 0$ induces an order 2 element $\bar{v}$ in M^*/M. In other words, $(L + \mathbb{Z}v)$ has index 2 in its primitive closure. If v is assumed to be primitive in M, then we find the generic discriminant of the lattice enhancement

$$\mathrm{discr}\,\mathrm{NS} = (\mathrm{discr}\,L) \cdot v^2/4 = -16v^2. \tag{26}$$

In the following we want to study the K3 surfaces corresponding to the divisors D_Δ in the moduli space and relate them to Hermann's work [5]. To this end, we shall enhance NS by a primitive representative $\mathbf{y}^*$ as explained in Corollary 1 and the following paragraph.

Example: lattice enhancements by a -4 vector

We return to our double sextics branched along 6 lines. Following 3.5 we will enhance the Néron-Severi lattice by a -4-vector from $M = U(2)^2 \perp A_1^2$. We consider two ways to do so which we will soon see to be inequivalent and exhaustive. Following up on Example 1 we shall take either

$$v_1 = (0,0,0,0,1,1) \quad \text{or} \quad v_2 = (1,-1,0,0,0,0).$$

Computing their orthogonal complements in M, we find the generic transcendental lattices of the enhanced lattice polarised K3 surfaces:

$$T_1 = v_1^\perp \quad = \quad U(2)^2 \perp \langle -4 \rangle, \tag{27}$$
$$T_2 = v_2^\perp \quad = \quad U(2) \perp A_1^2 \perp \langle 4 \rangle. \tag{28}$$

Note that the first lattice (which corresponds to $\Delta = 1$ by Example 1, see also 3.6) is 2-divisible as an even lattice while the second lattice (corresponding to $\Delta = 4$, see also 3.6) certainly is not. This confirms that these two cases are indeed inequivalent. In what follows, we will interpret the enhancements in terms of elliptic fibrations. Along the way, we will verify that any other lattice enhancement by a -4-vector is equivalent to one of the above.

Interpretation in terms of elliptic fibrations

In view of the Picard number, a jacobian elliptic fibration can be enhanced in only 2 ways: either by a degeneration of singular fibers (changing the configuration of ADE-types) or by an additional section (which any multisection can be reduced to by (18)). For the second alternative, there are usually many possibilities, distinguished by the height of the section, but also by precise intersection numbers, for instance encoding the fiber components met. However, once we fix the discriminant which we are aiming at, this leaves only a finite number of possibilities.

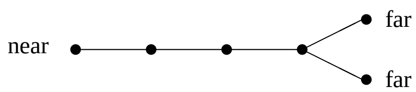
As an illustration, consider the fibration from 3.4 exhibiting the representation

$$\mathrm{NS}(X) = U \perp D_6^2 \perp A_1^2.$$

Enhancing NS as in 3.5, we reach a subfamily of lattice polarised K3 surfaces of Picard number $\rho = 17$ and discriminant 64 by (26). As the discriminant stays the same as before up to sign, there are only 3 possibilities of enhancement to start with:

- 2 fibers of type I_2 degenerate to I_4,
- I_1 and I_2^* degenerate to I_3^*,
- or a section P of height $h(P) = 1$.

Using the theory of Mordell-Weil lattices from 3.3, the third case can be broken down into another 3 subcases, depending on the precise fiber components met. Recall that the non-identity components of I_n^* fibers ($n > 0$) are divided into the near component (only one component away from the identity component) and the two far components as visible in the corresponding root diagram of Dynkin type D_{n+4}:

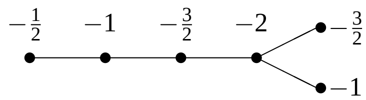


An easy enumeration of the possible configurations reveals the following possibilities for a section P of height $h(P) = 1$; all of them have P perpendicular to the zero section.

alternative	I_2^*'s	I_2's
(1)	far, far	id, id
(2)	far, near	id, non-id
(3)	near, near	non-id, non-id

Comparison of enhancements

We shall now compare our investigation of the above elliptic fibration with the concept of lattice enhancements by making the isomorphism (24) explicit. We start by calculating the discriminant form of D_6. The discriminant group is $(\mathbb{Z}/2\mathbb{Z})^2$ with generators $a^{\pm} \in D_6 \otimes \mathbb{Q}$ represented by elements meeting each one of the two far nodes $r^{\pm}$ precisely once and none of the remaining roots $r_1, \dots, r_4$ (enumerated from left to right).



The correct rational linear combination $a^{+} = -(\frac{1}{2}r + \frac{3}{2}r^{+} + r^{-})$, $r = r_1 + 2r_2 + 3r_3 + 4r_4$ a root, is shown in the figure. Of course the combination for a^{-} is similar and so we find $(a^{\pm})^2 = -\frac{3}{2}$ and $a^{+} \cdot a^{-} = -1$ so that

$$q_{D_6} = \begin{pmatrix} -\frac{3}{2} & -1 \\ -1 & -\frac{3}{2} \end{pmatrix}.$$

The discriminant group of $U(2)$ is also $(\mathbb{Z}/2\mathbb{Z})^2$ with basis e, f induced from the standard basis of $U(2)$. It follows that

$$q_{U(2)} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

Thus one easily verifies the isomorphism in the standard basis:

$$\begin{array}{ccc} q_{U(2)\perp A_1} & \xrightarrow{\cong} & -q_{D_6 \perp A_1} \\ (e, f, g) & \mapsto & (g+e, g+f, e+f+g) \end{array}$$

Duplicated this directly extends to the isomorphism (24). We continue by computing the impact of the enhancing vectors v_i from 3.5.

Starting out with v_1, this vector induces the element $(0,0,1)$ in either copy of $q_{U(2)\perp A_1} \hookrightarrow q_{T(X)}$. In each $q_{D_6 \perp A_1}$ this corresponds to the class $(1,1,1)$. Thus we obtain an algebraic class meeting each reducible fiber. A priori this would be a multisection, but using the group structure it induces a section, necessarily of height 1, of the third alternative in 3.5.

Next we turn to v_2. We have

$$\bar{v_2} = ((1,1,0),(0,0,0)) \in (q_{U(2)\perp A_1})^2 \cong q_{T(X)}.$$

Hence v_2 induces the same class in $(q_{D_6 \perp A_1})^2$. This corresponds to an algebraic class meeting only one I_2^* fiber non-trivially. By inspection of the alternatives in 3.5, this class cannot be a section of height 1 (which always meets both I_2^* fibers non-trivially), but it fits in with the degeneration of I_1 and I_2^* to I_3^*.

Connection with other enhancements

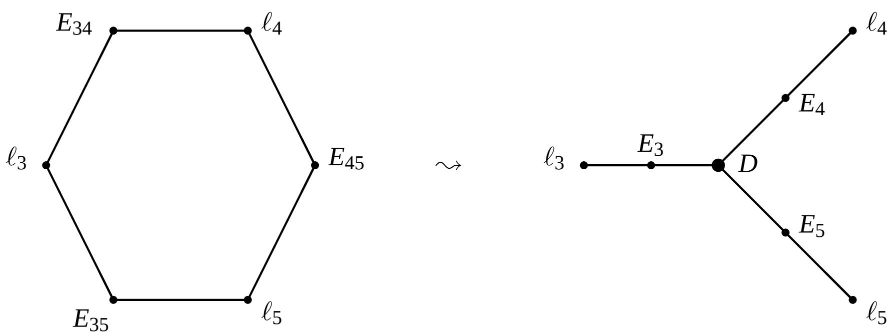
Before returning to the arrangement of the 6 lines, we comment on the other three possible enhancements of the elliptic fibration in 3.5. In fact, the freedom of choosing some lines out of the 6 carries over to these elliptic fibrations endowing X with several different ones of the same shape. We leave it to the reader to follow the degeneration of singular fibers on the given fibration through the other elliptic fibrations. Without too much effort, this enables us to identify all remaining degenerations with the one which was shown in 3.5 to correspond to the lattice enhancement by v_2.

3.6. Special Arrangements of the 6 Lines; Proof of Theorem 3

We are now in the position to investigate the subfamilies of our double sextics corresponding to the first few divisors on the moduli space of Abelian fourfolds of Weil type. In each case, we start from the special arrangement of lines to fill out the geometric and lattice theoretic details.

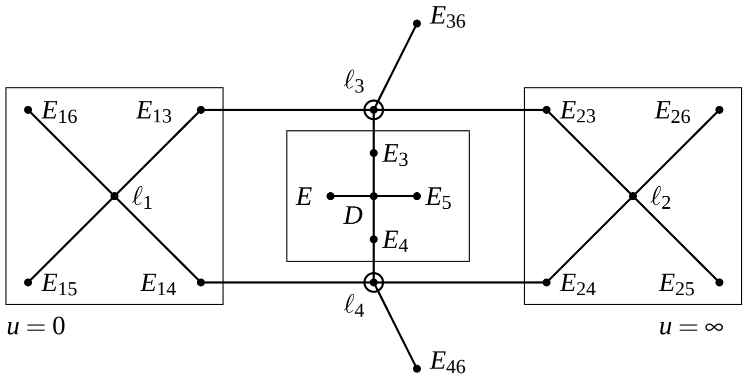
$\Delta = 2$

Recall from § 3.1 that D_2 corresponds to X_3. We now consider the component X_3^{345}, that is, when the lines ℓ_3, ℓ_4, ℓ_5 meet in a single point. On the double covering K3 surface, this results in a triple point whose resolution requires an additional blow-up. On the degenerate K3 surface, the original exceptional divisors can still be regarded as perpendicular (with notation adjusted, see the figure below); with the lines, however, they do not connect to a hexagon anymore, but to a star through the additional exceptional component D (Kodaira type IV^*):



On the standard elliptic fibration from 3.4, this degeneration causes three I_2 fibers to merge to a single additional fiber of type I_0^* (with a 'new' rational curve E as 4th simple component; compare Figure 2 where also some rational curves such as ℓ_5, ℓ_6 have been omitted) as indicated in the figure on the next page.

Thus NS has the index 4 sublattice $U \perp D_4^3 \perp A_1^3$ – which is again 2-elementary. The remaining generators of $\mathrm{NS}(X)$ can be given by the 2-torsion sections. To decide



on the discriminant form, we once more switch to the alternative elliptic fibration from 3.4. The fibration degenerates as follows: in the notation from 3.4 we have to replace E_{35}, E_{45} by E_3, E_4 as components of the I_2^* fibers, and E'_{34} by E_5 as component of one I_2 fiber. Then E_{56} still sits on a second I_2 fiber while E gives yet another one. Here D induces a 2-torsion section: visibly it meets both I_2^* fibers at far components. As for the I_2 fibers, it meets the one containing E_{56} at the other component (i.e. non-identity) and the one containing E_5 in this very component (non-identity again). Since the height of a section is non-negative, this already implies that the section D has height 0; then the fiber types predict that D can only be 2-torsion.

For completeness we study the fiber containing E in detail. Since ℓ_6 is also a section for the standard fibration, it meets some simple component of the degenerate I_0^* fiber. Obviously ℓ_6 does not meet any of E_3, E_4, E_5. Hence ℓ_6 has to meet E. In conclusion E is the identity component of the degenerate I_2 fiber of the alternative fibration. As D meets this fiber trivially, we find the orthogonal decomposition

$$\mathrm{NS}(X) = U \perp \langle D_6^2, A_1^2, D \rangle \perp A_1.$$

As before, we deduce from the orthogonal summand A_1 that the discriminant form takes non-integer values. Hence by Proposition 7 we deduce that

$$T(X) = U(2)^2 \perp A_1 \quad \text{corresponding to} \quad \Delta = 2. \tag{29}$$

In the language of lattice enhancements, the subfamily thus arises from a generator of either A_1 summand in the generic transcendental lattice. Recall that geometrically, this vector corresponds to the extra rational curve D involved in the resolution of the triple point where three lines come together. Conversely, we can derive from Proposition 7 again that the Néron-Severi lattice admits several representations purely in terms of U and root lattices such as

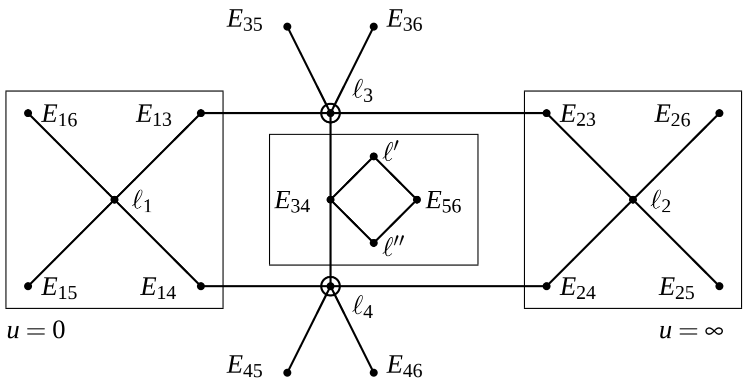
$$\mathrm{NS}(X) = U \perp D_4^2 \perp E_7.$$

This concludes the proof of Theorem 3 1. $\square$

$\Delta = 4$

Recall from § 3.1 that X_4 comes from the divisor $\mathbf{X}_{\text{coll}}$. Let ℓ be the line which contains the collinear points. Then ℓ splits on X as $\pi^*\ell = \ell' + \ell''$. Let $D = \ell' - \ell''$. Since D is anti-invariant for the covering involution, it defines an algebraic divisor on X which is orthogonal to the classes specialising from the generic member. By construction, $\ell'.\ell'' = 0$ so that $D^2 = -4$. In particular, D is primitive in $\mathrm{NS}(X)$, and X arises from a lattice enhancement by the -4-vector D as in 3.5. Presently we can even give a $\mathbb{Z}$-basis of $\mathrm{NS}(X)$ by complementing the generic basis by ℓ', say. To compute $\mathrm{NS}(X)$ and $T(X)$ without writing out intersection matrices etc, we make use of elliptic fibrations again.

For the standard fibration, it is convenient to choose the collinear points as P_{12}, P_{34}, P_{56}. From the obvious -2-curves, each ℓ' and ℓ'' then meets exactly the corresponding exceptional divisors E_{12}, E_{34}, E_{56} on X. On the standard fibration from 3.4, the two singular fibers of type I_2 at E_{34} and E_{56} are thus connected by ℓ', ℓ'', merging to a fiber of type I_4. Note that this indeed preserves the discriminant up to sign while raising the rank by one.



Switching to the alternative fibration from 3.4, the classes ℓ', E_{56}, ℓ'' which correspond to the root lattice A_3 remain orthogonal to the fibers D_1, D_2. Generically they are therefore contained in a fiber of type I_4, merging the I_2 fibers generically at E_{56} and at E'_{34}. That is, $\mathrm{NS}(X) = U \perp D_6^2 \perp A_3$. By 3.5, 3.5 we can thus verify that X arises from the lattice enhancement by v_2 with transcendental lattice $T(X) = U(2) \perp \langle 4 \rangle \perp A_1^2$ as stated in Theorem 3 2.

$\Delta = 1$

As the key part of this subsection, we now come to the case $\Delta = 1$ which will cover almost the rest of this section up to 3.6. Recall from §3.1 that we have the divisor $XQ \subset X$ of 6-uples of lines tangent to a fixed conic. This divisor can be identified with D_1 as follows from [9, Prop. 2.13.4]. Indeed, that hyperplane XQ is exactly the hyperplane orthogonal to our v_1 (see § 3.5).

This can also be read off directly from the fact that X is a Kummer surface. Indeed, by [10, § 0.19] the surface X arises from the jacobian of the genus 2 curve which is the double cover of the conic branched along the six intersection points with the lines. This gives $T(X) = U(2)^2 + \langle -4 \rangle$ in agreement with the lattice enhancement by v_1. We shall confirm this from our methods using elliptic fibrations.

The above argument, however, gives no information about the extra algebraic class needed to generate $\mathrm{NS}(X)$ over $\mathbb{Z}$. To overcome this lack of a generator, we shall work geometrically with elliptic fibrations, starting with the alternative fibration. Going backwards in our constructions, we first develop the corresponding section on the standard fibration and then interpret this in terms of a certain conic in $\mathbb{P}^2$ which splits on X. Finally we confirm our geometric arguments by providing explicit equations. Throughout we do not use any information about the Kummer surface structure.

First, denoting the conic by C, we have a splitting $C = C_1 + C_2$ on X. As in 3.6, the divisor $D = C_1 - C_2$ is anti-invariant for the covering involution and therefore orthogonal to the rank 16 sublattice of $\mathrm{NS}(X)$ generated by the classes of the lines and the exceptional divisors. The subtle difference, though, is that D is in fact 2-divisible in $\mathrm{NS}(X)$ since $C \sim 2H$:

$$\frac{1}{2}D = H - C_2 \in \mathrm{NS}(X).$$

Since $D^2 = -16$, we find that the latter class has square -4, hence we are indeed confronted with a lattice enhancement as in 3.5.

From alternative to standard fibration

In 3.5 we interpreted lattices enhancements in terms of the alternative elliptic fibration from 3.4. By 3.5, 3.5 it is alternative (3) which corresponds to the lattice enhancement by v_1. In detail, the alternative fibration admits a section P intersecting the following fiber components (see Figure 3 for the resulting diagram of -2-curves):

singular fiber	$D_1 = I_2^*$	$D_2 = I_2^*$	I_2	I_2
component met by ℓ'	E_{15}	E_{25}	opposite E'_{34}	opposite E_{56}

On the standard fibration, P defines a multisection whose degree is not immediate. Here we develop a backwards engineering argument to prove that the degree is actually 1, i.e. P is a section for both fibrations.

A priori the degree d of the multisection P need not be 1 on the standard fibration, but P always induces a section P' of height 1. The essential point of our argument is that we can read off from the alternative fibration which fiber components are not met by the multisection on the standard fibration. For each singular fiber this leaves only one fiber component with intersection multiplicity depending on the degree d. But then we can use the group structure to determine which fiber component will be met by the induced section P'. Thanks to the specific singular fibers, the argument only depends on the parity of d:

singular fiber	I_0^*	I_0^*	I_2	I_2	I_2	I_2	I_2	I_2
comp's met by P	E_{15}, $(d-1)E_{14}$	E_{25}, $(d-1)E_{23}$	dE_{34}	dE'_{35}	dE_{36}	dE'_{45}	dE_{46}	dE'_{56}
comp met by P'	E_{16}	E_{25}	E_{34}	E_{35}	E_{36}	E'_{45}	E'_{46}	E'_{56}
for even d	non-id	non-id	id	id	id	id	id	id
comp met by P'	E_{15}	E_{25}	E_{34}	E'_{35}	E_{36}	E'_{45}	E_{46}	E'_{56}
for odd d	non-id	non-id	id	non-id	id	id	non-id	id

Note that for even d the section P' would have even height by inspection of the fiber components met, contradicting $h(P') = h(P) = 1$. Hence d is odd, and the induced section P' meets the fiber components indicated in the last two rows of the table. With this section at hand, we can complete the circle: namely P' defines a section for both fibrations, of exactly the same shape as P, hence $P' = P$ (and $d = 1$).

For later reference, we point out the symmetry in the fiber components met by P on the standard fibration: on the I_0^* fibers, it is exactly those met by the section ℓ_5, while on the I_2 fibers it is exactly those not met by ℓ_5. This symmetry is essential for the section to be well-defined as it ensures that adding a 2-torsion section to P will always result in a section of height 1 (compare 3.6).

From standard fibration to double sextic

On the double sextic model, the section P arises from a curve Q in $\mathbb{P}^2$ which splits into 2 rational curves Q_1, Q_2 on X. Here we give an abstract description of Q and its components on X based on the geometry of the elliptic fibrations.

From the alternative fibration we know that P is perpendicular to the lines ℓ_i for $i \neq 5$. On the other hand, the standard fibration reveals by inspection of the above table which exceptional curves intersect P:

$$\text{exactly} \quad E_{15}, E_{25}, E_{34}, E_{36}, E_{46} \quad \text{plus possibly} \quad E_{12}.$$

The latter is the only exceptional curve which is not visible as section or fiber component on the standard fibration. The remaining two intersection numbers can be computed as follows: regarding ℓ_5 as a 2-torsion section of the standard fibration, the height pairing $\langle P, \ell_5 \rangle = 0$ implies by virtue of the fiber components met that $P.\ell_5 = 0$. As for E_{12}, consider the auxiliary standard fibration defined by $u' = \ell_1/\ell_5$. Then P defines a section for this fibration as well, as it meets the fiber

$$(u')^{-1}(\infty) = 2\ell_5 + E_{25} + E_{35} + E_{45} + E_{56}$$

exactly in E_{25} (transversally) by the above considerations. Looking at the fiber

$$(u')^{-1}(0) = 2\ell_1 + E_{12} + E_{13} + E_{14} + E_{16}$$

we deduce $P.E_{12} = 1$ from the fact that P does not intersect the other fiber components.

Turning to the double sextic model of X, P defines a curve meeting E_{12}, E_{15}, E_{25}, E_{34}, E_{36}, E_{46} transversally, but no other exceptional curves E_{ij} nor any of the ℓ_i. On the base $\mathbb{P}^2$, this curve necessarily corresponds to a conic Q through the six underlying nodes. On X, this conic splits into two disjoint rational curves Q_1, Q_2 where $Q_1 = P$, say, and Q_2 corresponds to the section $-P$ on the elliptic fibrations.

Explicit equations

We start out with the general equation of a jacobian elliptic K3 surface X with singular fibers of type I_0^* twice and 6 times I_2 over some field k of characteristic $\neq 2$. Necessarily this comes with full 2-torsion. Locating the I_0^* fibers at $t = 0, \infty$, we can write

(30) $$X: \quad tw^2 = x(x - p(t))(x - q(t))$$

where $p, q \in k[t]$ have degree 2. Here the I_2 fibers are located at $p = 0, q = 0$ $p = q$. Up to symmetries, there are only two ways to endow the above fibration with a section of height 1. Abstractly, restrictions are imposed by the compatibility with the 2-torsion sections. On the one hand, there is the symmetric arrangement encountered in 3.6. This will be investigated below. On the other hand, an asymmetric arrangement can be encoded, for instance, in terms of the standard fibration by a section with the following intersection behaviour:

singular fiber	I_0^*	I_0^*	I_2	I_2	I_2	I_2	I_2	I_2
component met	E_{15}	E_{26}	E_{34}	E_{35}	E_{36}	E_{45}	E_{46}	E'_{56}

To see that the arrangements do indeed generically describe different K3 surfaces, we switch to the alternative fibration from 3.4 for one final time. Here the section P with intersection pattern as in the above table induces a bisection meeting far and near component of D_1 and far and identity component of D_2. Using the group structure the induced section intersects both I_2^* fibers in a far component. In terms of 3.5 this corresponds to alternative (1) which was shown in 3.5, 3.5 to differ from alternative (3) which underlies the section arrangement in 3.6.

We shall now continue by deriving equations admitting a section of height 1 as encountered in 3.6. In agreement with this, we model the section P to intersect the same components of the I_0^* fibers as the 2-torsion section $(0,0)$. In terms of the RHS of (30), these correpond to the factor x. The section P therefore takes the shape

$$P = (at, \dots) \quad \text{for some constant} \quad a \in k.$$

Upon subsituting into (30), we now require the other two factors on the RHS to produce the same quadratic polynomial up to a constant. Concretely this polynomial can be given by

(31) $$g = at - p.$$

Then we consider the codimension 1 subfamily of all X such that there exists $b \in k, b \neq 1$ such that

$$q = at - bg. \tag{32}$$

By construction, these elliptic K3 surfaces admit the section

$$P = (at, \sqrt{ab}g).$$

This section has height 1: not only does it meet both I_0^* fibers non-trivially thanks to our set-up, but also the I_2 fibers at $p - q = (b-1)g = 0$ while being perpendicular to the zero section. As a whole, the family of K3 surfaces can be given by letting g, a, b vary and p, q depend on them as above. There are still normalisations in t and in (x, w) left which bring us down to the 3 moduli dimensions indeed.

Standard fibration reflecting the 6 lines

We shall now translate the above considerations to the standard fibration as it comes from the 6 lines in 3.4. With $u = \ell_1/\ell_2$, we naturally have an equation

$$uw^2 = \ell_3\ell_4\ell_5\ell_6. \tag{33}$$

Here we can use u to eliminate x, say, so that after clearing denominators the expressions ℓ_i are separately linear in both y and u. In particular, the family of elliptic curves over $\mathbb{P}^1_u$ becomes evident, with 2-torsion sections given by $\ell_i = 0$ $(i = 3, 4, 5, 6)$.

For ease of explicit computations, we normalise the lines to be

$$\ell_1 = x, \ell_2 = y, \ell_3 = x + y + z, \ell_4 = a_1x + a_2y + a_3z, \ell_5 = z, \ell_6 = b_1x + b_2y + b_3z.$$

Working affinely in the chart $z = 1$, equation (33) readily takes the shape of a twisted Weierstrass form

$$uw^2 = ((u+1)y + 1)((a_1u + a_2)y + a_3)((b_1u + b_2)y + b_3).$$

Standard variable transformations take this to:

$$uw^2 = (y + (a_1u + a_2)(b_1u + b_2))(y + a_3(u+1)(b_1u + b_2))(y + b_3(u+1)(a_1u + a_2)).$$

Translating to the shape of (30) and solving for (31), (32), we find

$$b_1 = ba_1b_3/(a_3 + (b-1)a_1), \quad b_2 = ba_2b_3/(a_3 + (b-1)a_2)$$

with $a = -ba_3b_3(a_1 - a_2)^2/[(a_3 + (b-1)a_1)(a_3 + (b-1)a_2)]$.

Conics on the double sextic

Finally we can trace back the section P to the double sextic model. Step by step, it leads to the following conic in the affine chart $z = 1$:

$$\begin{aligned} Q \quad = \quad & -a_1^2x^2a_2 + a_1^2x^2a_3 + a_1^2x^2ba_2 - a_1^2xa_2y + xa_1^2a_2b + a_1^2xa_2yb \\ & -a_1xa_2^2y + 2a_3xya_1a_2 - xa_1^2a_2 + xa_3a_1^2 + a_1xa_2^2yb + a_1a_2^2yb \\ & -a_1a_2^2y + a_3ya_2^2 - a_2^2y^2a_1 + a_2^2y^2a_3 + a_2^2y^2ba_1. \end{aligned}$$

One directly verifies that Q indeed passes through the nodes P_{12}, P_{15}, P_{25}, P_{34}, P_{36}, P_{46} in $\mathbb{P}^2$ as in 3.6. Hence Q splits into two components on the double sextic X one of which is P.

We conclude this paragraph by verifying that the subfamily of double sextics constructed in 3.6 does in fact admit a conic which is tangent to each of the 6 lines. For this purpose, set

$$\alpha = a_1(a_2 - a_3), \beta = a_2(a_3 - a_1), \gamma = a_3(a_1 - a_2).$$

Then the conic in $\mathbb{P}^2$ given by

$$\alpha^2x^2 + \beta^2y^2 + \gamma^2z^2 - 2(\alpha\beta xy + \alpha\gamma xz + \beta\gamma yz) = 0$$

meets each of the 6 lines ℓ_i tangentially.

Conclusion

By comparison of moduli dimensions, it follows conversely that the K3 surfaces X_1 with a conic tangent to each of the 6 lines of the branch locus also admits a conic through a selection of 6 nodes as above. From 3.6 we therefore deduce that X_1 generically arises from X via the lattice enhancement by the vector v_1; that is, by 3.5

$$T(X_1) = U(2)^2 \perp \langle -4 \rangle.$$

The Néron-Severi lattice $\mathrm{NS}(X_1)$ is thus generically generated by the sublattice $U \perp D_6^2 \perp A_1^2$ coming from X enhanced by the section P from 3.6. The simple representation of $\mathrm{NS}(X_1) = U \perp D_4 \perp D_8 \perp A_3$ in Theorem 3 is derived from the above fibration by switching to yet another jacobian elliptic fibration as depicted below.

$\Delta = 6$

Our aim for the final bit of this section is to understand the geometry of the K3 surfaces for D_6, i.e. the case $\Delta = 6$ (the final Heegner divisor singled out in [6]). By Corollary 1 and the discussion succeeding it, this corresponds to a lattice enhancement by a vector $v \in T(2)$ of square $v^2 = -6$. Here we choose the primitive representative

$$v = (1, -1, 1) \text{ in one copy of } U(2) \perp A_1 \subset T(2),$$

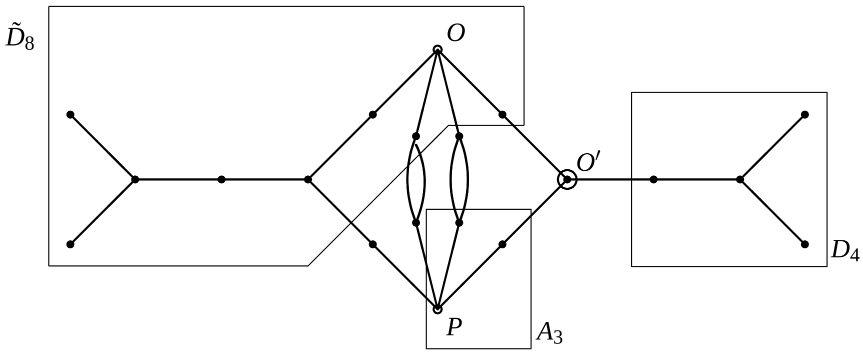


Figure 3: From alternative fibration to $U \perp D_4 \perp D_8 \perp A_3$ on X_1

augmented by zeroes in $T(2)$. Then $v/2$ defines a class in the discriminant group $T(2)^*/T(2)$ which via the isomorphism in 3.5 maps to the class

$$(0,0,1) \in (D_6 \perp A_1)^*/(D_6 \perp A_1) \hookrightarrow \mathrm{NS}^*/\mathrm{NS},$$

augmented by zeroes in $\mathrm{NS}^*/\mathrm{NS}$. The Néron-Severi lattice NS is thus enhanced by a divisor which only meets one reducible fiber of the alternative fibration in a non-identity component (corresponding to A_1). If this divisor were a section, then it would have height $h \geq 4 - 1/2 = 7/2$ by 3.3, but certainly not $3/2$. Hence the lattice enhancement can only result in a fiber degeneration

$$A_1 \rightsquigarrow A_2$$

on the alternative fibration. Thus we find the enhanced Néron-Severi lattice

$$\mathrm{NS}' = U \perp 2D_6 \perp A_1 \perp A_2 \tag{34}$$

in agreement with the generic transcendental lattice of the enhanced subfamily,

$$T(X) = U(2) \perp A_1^2 \perp \langle 6 \rangle \quad \text{corresponding to} \quad \Delta = 6.$$

In order to determine the corresponding special curve in $\mathbb{P}^2$ explicitly, we assume without loss of generality that the I_2 fiber with E_{56} as non-identity component degenerates to Kodaira type I_3. That is, there are two other smooth rational curves D_1, D_2 as fiber components. Both give sections of the standard fibration, meeting exactly the following fiber components:

singular fiber	I_0^*	I_0^*	I_2	I_2	I_2	I_2	I_2	I_2
component met	E_{14}	E_{23}	E_{34}	E'_{35}	E_{36}	E'_{45}	E_{46}	E'_{56}
	non-id	id	id	non-id	id	id	non-id	non-id

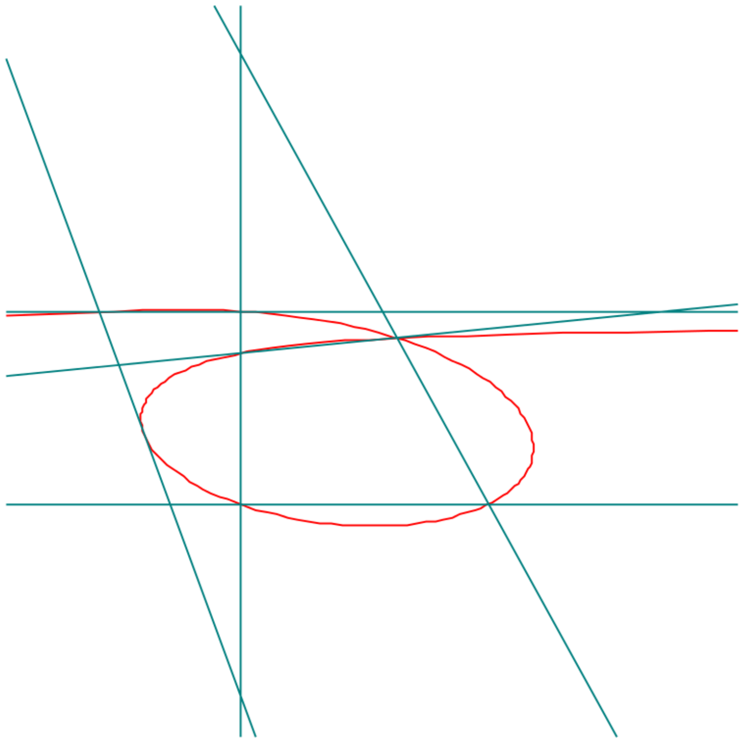


Figure 4: The 6 lines and the singular cubic

One directly checks that the height pairing from 3.3 gives $h(D_1) = h(D_2) = 3/2$; in fact the sections are inverse to each other, since they meet the same fiber components and $\langle D_1, D_2 \rangle = -3/2$. The intersection numbers with all other rational curves from the line arrangement are zero except for E_{12} and ℓ_5 which are neither visible on the alternative fibration nor fiber components of the standard fibration. Here, since ℓ_5 defines a 2-torsion section of the standard fibration, the height pairing

$$0 = \langle D_1, \ell_5 \rangle = 2 - D_1.\ell_5 - \underbrace{1/2}_{I_0^* \text{ at } E_{14}, E_{15}} - \underbrace{1/2}_{I_2 \text{ at } E_{46}} \quad \text{gives} \quad D_1.\ell_5 = 1,$$

and likewise for D_2. As for E_{12}, arguing with an auxiliary standard fibration such as the one induced by ℓ_1/ℓ_4, we find

$$D_1.E_{12} = D_2.E_{12} = 2.$$

It follows that D_1 and D_2 correspond to a cubic curve $C \subset \mathbb{P}^2$ of the following shape:

- with a singularity at the node underlying E_{12},

- through the nodes underlying $E_{14}, E_{23}, E_{34}, E_{36}, E_{46}, E_{56}$,
- meeting ℓ_5 tangentially in a smooth point.

4. The Kuga-Satake Construction

4.1. Clifford Algebras

Let V be a finite dimensional k-vector space equipped with a non-degenerate bilinear form q. Its tensor algebra is $\mathsf{T}V = \oplus_{p\geq 0} V^{\otimes p}$, where the convention is that $V^{(0)} = k$. Recall that the **Clifford algebra** is the following quotient algebra of this algebra:

$$\mathrm{Cl}(V) = \mathrm{Cl}(V,q) := \mathsf{T}V/\text{ideal generated by } \{x\otimes x - q(x,x)\cdot 1 \mid x \in V\}.$$

Then we have $xy + yx = 2q(x,y)$ in the algebra $\mathrm{Cl}(V)$; in particular x and y anti-commute whenever they are orthogonal.

The Clifford algebra has dimension 2^n where $n = \dim_k V$. Let us make this explicit for $k = \mathbb{Q}$. Then Q can be diagonalised in some basis, say $\{e_1, \ldots, e_n\}$. Consider $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{F}_2^n$. Taking all 2^n possibilities, we find a basis for $\mathrm{Cl}(V)$:

$$e^{\mathbf{a}} := e_1^{a_1} \cdots e_n^{a_n}.$$

The even Clifford algebra $\mathrm{Cl}^+(V)$ is generated by those $e^{\mathbf{a}}$ for which $\sum a_j$ is even.

To describe Clifford algebras certain quaternion algebras play a role. Let F be a field and $a, b \in F^\times$. The quaternion algebra $(a,b)_F$ over a field F has an F–basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ such that $\mathbf{i}^2 = a, \mathbf{j}^2 = b, \mathbf{ij} = -\mathbf{ji} = \mathbf{k}$. The Clifford algebra $\mathrm{Cl}(\langle a\rangle \perp \langle b\rangle)$, $a, b \in \mathbb{Q}^\times$ is isomorphic to $(a,b)_{\mathbb{Q}}$ while $\mathrm{Cl}^+(\langle a\rangle \perp \langle b\rangle) = \mathbb{Q}(\sqrt{-ab})$. One can see (cf. [16]) that for rank 3, 4 and 5 the results are:

LEMMA 3. *Suppose $Q = \mathrm{diag}(a_1, \ldots, a_m)$. Put $d = (-)^m a_1 \cdots a_m$. Then*

1. *For $m = 3$ we have $\mathrm{Cl}^+(Q) = (-a_1a_2, -a_2a_3)_{\mathbb{Q}}$;*
2. *For $m = 4$ we have $\mathrm{Cl}^+(Q) = (-a_1a_2, -a_2a_4)_{\mathbb{Q}} \otimes F$ with $F = \mathbb{Q}\sqrt{d}$;*
3. *For $m = 5$ we have $\mathrm{Cl}^+(Q) = (-a_1a_2, -a_2a_3)_{\mathbb{Q}} \otimes_{\mathbb{Q}} (a_1a_2a_3a_4, -a_4a_5)_{\mathbb{Q}}$.*

4.2. From Certain Weight 2 Hodge Structures to Abelian Varieties

Next suppose that (V,q) carries a weight 2 Hodge structure polarized by q with $h^{2,0} = 1$. Then $V^{2,0} \oplus V^{0,2}$ is the complexification of a real plane $W \subset V$ carrying a Hodge substructure and q polarizes it. Then $b(x,y) := -q(x,y)$ is a metric on this plane since $C = -1$ is the Weil-operator ‡‡ of this Hodge structure. A choice of orientation for W then defines a unique almost complex structure which is the rotation over $\pi/2$

‡‡Recall that C is defined by $C|H^{p,q} = \mathrm{i}^{p-q}$.

in the positive direction. Equivalently, this almost complex structure is determined by any positively oriented orthonormal basis $\{f_1, f_2\}$ for W. Such a choice also defines an almost complex structure $J = f_1 f_2$ on $\mathrm{Cl}^+(V)$ since $f_1 f_2 f_1 f_2 = -f_1^2 f_2^2 = -1$. Then J defines a weight 1 Hodge structure: the eigenspaces of J for the eigenvalues $\pm\mathrm{i}$ are the Hodge summands $H^{1.0}$, respectively $H^{0,1}$.

It turns out that the Hodge structure is polarized by a very natural skew form

$$E : \mathrm{Cl}^+(V) \times \mathrm{Cl}^+(V) \to \mathbb{Q}, \quad (x,y) \mapsto \mathrm{tr}(\varepsilon\iota(x)y)$$

built out of the canonical involution

$$\iota : \mathrm{Cl}^+(V) \to \mathrm{Cl}^+(V), \quad e_1^{a_1} \cdots e_n^{a_n} \mapsto e_1^{a_n} \cdots e_n^{a_1},$$

the trace map

$$\mathrm{tr} : \mathrm{Cl}^+(V) \to \mathbb{Q}, \quad c \mapsto \mathrm{tr}(R_c), R_c : x \mapsto cx, \text{(left multiplication by } c)$$

and $\varepsilon \in \mathrm{Cl}^+(V)$ any element with $\iota \circ \varepsilon = -\varepsilon$, for instance $\varepsilon = e_1 e_2$. If we set

$$U := \text{ vector space dual to } \mathrm{Cl}^+(V),$$

any choice of a free $\mathbb{Z}$-module $U_{\mathbb{Z}}$ makes $U/U_{\mathbb{Z}}$ into a complex torus which is polarized by E. This Abelian variety by definition is the **Kuga Satake variety** and is denoted by $A(V,q)$. It is an Abelian variety of dimension 2^{n-2}, half of the real dimension of $\mathrm{Cl}^+(V)$.

Recall (e.g. [15]):

THEOREM 4. *One has* $\mathrm{Cl}^+(V) \subset \mathrm{End}_{\mathbb{Q}}(A(V,q))$. *If*

$$\mathrm{Cl}^+(V) = M_{n_1}(D_1) \times \cdots \times M_{n_d}(D_d), D_j \textit{ division algebra, } j = 1, \ldots, d,$$

then we have a decomposition into simple polarized Abelian varieties (here $\sim$ denotes isogeny):

$$A(V,q) \sim A_1^{n_1} \times \cdots \times A_d^{n_d}, A_j \textit{ with } D_j \subset \mathrm{End}_{\mathbb{Q}}(A_j), j = 1, \ldots, d. \tag{35}$$

4.3. Abelian Varieties of Weil Type

In this note we are mainly interested in the following two examples which arise as certain Kuga-Satake varieties. The first class of examples is related to Abelian varieties in the full moduli space $\tilde{\mathbf{M}}$, and they are described by the following proposition.

PROPOSITION 8 ([7, Theorem 6.2]). *Let V be a rational vector space of dimension 6 and let $q = \langle 2\rangle \oplus \langle 2\rangle \oplus \langle -2\rangle \oplus \langle -2\rangle \oplus \langle -a\rangle \oplus \langle -b\rangle$, $a,b \in \mathbb{Q}^+$. We put $\alpha = \mathrm{i}\sqrt{ab}$.*

Suppose that (V,q) is a polarized weight 2 Hodge structure with $h^{2,0} = 1$. Then the Kuga-Satake variety $A(V,q)$ is a 16-dimensional Abelian variety of $\mathbb{Q}(\alpha)$-Weil-type. For a generic such Hodge structure $\mathrm{End}_{\mathbb{Q}}(A(V,q)) = M_4(\mathbb{Q}(\alpha))$.

In the situation of Theorem 4 we have $d = 1$ and $n_1 = 4$, i.e. :

$$A(V,q) \sim B^4,\ B \text{ simple 4-dim. Abelian variety with } \mathbb{Q}(\alpha) \subset \mathrm{End}_{\mathbb{Q}}(B).$$

On the other hand, the procedure of § 2.2 describes how one may associate to any Abelian variety A of dimension 4 of $\mathbb{Q}(\mathrm{i})$–Weil-type a polarized Hodge substructure $T(A)$ of $H^2(A)$ with $h^{2,0} = 1$. We have:

PROPOSITION 9 ([7, Theorem 6.5]). *For an Abelian variety A of dimension 4 which is of* $\mathbb{Q}(\mathrm{i})$*–Weil type we have*

$$A(T(A)) \sim A^4,$$

in other words, the Kuga-Satake procedure applied to $T(A)$ *gives back the original Abelian variety A up to isogeny.*

The second class relates to the Abelian varieties in the hypersurfaces D_Δ of the moduli space **M**. We recall equation (15) where we found the generic transcendental subspace $(T_\Delta)_{\mathbb{Q}} \subset (T(A_\Delta))_{\mathbb{Q}}$ of such an Abelian variety A_Δ.

We have:

THEOREM 5. *Let* $T_\Delta = \langle 2\Delta\rangle \perp U \perp \langle -2\rangle^2$ *be a polarized Hodge structure of type* $(1,3,1)$ *and let* $A(T_\Delta)$ *be its associated Kuga-Satake variety. If* A_Δ *is any Abelian variety with moduli point in the hypersurface* D_Δ *of the moduli space* **M** *such that* $T(A_\Delta) = T_\Delta$ *as polarized* $\mathbb{Q}$*–Hodge structures, then we have an isogeny*

$$A(T_\Delta) \sim A_\Delta^2, \quad (-1,\Delta)_{\mathbb{Q}} \subset \mathrm{End}_{\mathbb{Q}}(A_\Delta).$$

Proof. We consider the Clifford algebra associated to the Hodge structure T_Δ, since $U \cong_{\mathbb{Q}} \langle 2\rangle \oplus \langle -2\rangle$ the quadratic form is Q=diag$(2,-2,-2,-2,2\Delta)$. For the corresponding Clifford algebra we deduce from Lemma 3 that

$$\begin{aligned} \mathrm{Cl}^+(Q) &\cong (4,-4)_{\mathbb{Q}} \otimes (-16,4\Delta)_{\mathbb{Q}} \\ &\cong (1,-1)_{\mathbb{Q}} \otimes (-1,\Delta)_{\mathbb{Q}} \cong M_2((-1,\Delta)_{\mathbb{Q}}). \end{aligned}$$

From Thm. 4 the Kuga-Satake variety can be decomposed as $A(T_\Delta) \sim B^2$ where B is an Abelian fourfold with $(-1,\Delta)_{\mathbb{Q}}$ contained in $\mathrm{End}_{\mathbb{Q}}(B)$. Since T_Δ is a sub Hodge structure of $T(A_\Delta)$), the corresponding Kuga-Satake variety is a factor of $A(T(A_\Delta)) \sim A_\Delta^4$, i.e. $B \sim A_\Delta$ so that $A(T_\Delta) \sim A_\Delta^2$. □

REMARK 3. The quaternion algebra $(-1,\Delta)_{\mathbb{Q}}$ has zero divisors (i.e. $(-1,\Delta)_{\mathbb{Q}} \simeq M_2(\mathbb{Q})$) precisely when Δ is a sum of two squares in $\mathbb{Z}$ which is the case if and only if all primes $p \equiv 3$ mod 4 divide Δ with even power. In these cases the Abelian fourfold is isogeneous to a product B^2 with B an Abelian surface. It is an interesting open question if and how the Kummer surface of B and the K3 double plane are related. This occurs for instance if $\Delta = 1,2,4$ and for $\Delta = 1$ we have a candidate for B.

Acknowledgements

We thank the anonymous referee for her/his detailed comments. CP thanks the University of Turin as well as the Riemann Center for Geometry and Physics of Leibniz Universität Hannover for its hospitality. MS gratefully acknowledges support from ERC through StG 279723 (SURFARI).

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AMS Subject Classification: primary: 14J28

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Lavoro pervenuto in redazione il 02.07.2013.

Rend. Sem. Mat. Univ. Politec. Torino
Vol. 71, 3–4 (2013), 385 – 388

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NON-RATIONALITY OF THE $\mathfrak{S}_6$-SYMMETRIC QUARTIC THREEFOLDS

Abstract. We prove that the quartic hypersurfaces defined by $\sum x_i = t\sum x_i^4 - (\sum x_i^2)^2 = 0$ in $\mathbb{P}^5$ are not rational for $t \neq 0, 2, 4, 6, \frac{10}{7}$ ·

Pour Alberto, à l'occasion de son 70^e *anniversaire*

1. Introduction

Let V be the standard representation of $\mathfrak{S}_6$ (that is, V is the hyperplane $\sum x_i = 0$ in $\mathbb{C}^6$, with $\mathfrak{S}_6$ acting by permutation of the basis vectors). The quartic hypersurfaces in $\mathbb{P}(V)$ ($\cong \mathbb{P}^4$) invariant under $\mathfrak{S}_6$ form the pencil

$$X_t : \ t\sum x_i^4 - (\sum x_i^2)^2 = 0 \ , \quad t \in \mathbb{P}^1 \ .$$

This pencil contains two classical quartic hypersurfaces, the Burkhardt quartic X_2 and the Igusa quartic X_4 (see for instance [6]); they are both rational.

For $t \neq 0, 2, 4, 6$ and $\frac{10}{17}$, the quartic X_t has exactly 30 nodes; the set of nodes $\mathcal{N}$ is the orbit under $\mathfrak{S}_6$ of $(1, 1, \rho, \rho, \rho^2, \rho^2)$, with $\rho = e^{\frac{2\pi i}{3}}$ ([7], §4). We will prove:

THEOREM. *For $t \neq 0, 2, 4, 6, \frac{10}{7}$, X_t is not rational.*

The method is that of [1] : we show that the intermediate Jacobian of a desingularization of X_t is 5-dimensional and that the action of $\mathfrak{S}_6$ on its tangent space at 0 is irreducible. From this one sees easily that this intermediate Jacobian cannot be a Jacobian or a product of Jacobians, hence X_t is not rational by the Clemens-Griffiths criterion. We do not know whether X_t is unirational.

I am indebted to A. Bondal and Y. Prokhorov for suggesting the problem, to A. Dimca for explaining to me how to compute explicitly the defect of a nodal hypersurface, and to I. Cheltsov for pointing out the rationality of $X_{\frac{10}{7}}$.

2. The action of $\mathfrak{S}_6$ on $T_0(JX)$

We fix $t \neq 0, 2, 4, 6, \frac{10}{7}$, and denote by X the desingularization of X_t obtained by blowing up the nodes. The main ingredient of the proof is the fact that the action of $\mathfrak{S}_6$ on JX is non-trivial. To prove this we consider the action of $\mathfrak{S}_6$ on the tangent space $T_0(JX)$, which is by definition $H^2(X, \Omega^1_X)$.

LEMMA 1. *Let $\mathcal{C}$ be the space of cubic forms on $\mathbb{P}(V)$ vanishing along $\mathcal{N}$. We have an isomorphism of $\mathfrak{S}_6$-modules $\mathcal{C} \cong V \oplus H^2(X, \Omega^1_X)$.*

Proof : The proof is essentially contained in [2]; we explain how to adapt the arguments there to our situation. Let $b : P \to \mathbb{P}(V)$ be the blowing-up of $\mathbb{P}(V)$ along $\mathcal{N}$. The threefold X is the strict transform of X_t in P. The exact sequence

$$0 \to N^*_{X/P} \longrightarrow \Omega^1_{P|X} \longrightarrow \Omega^1_X \to 0$$

gives rise to an exact sequence

$$0 \to H^2(X, \Omega^1_X) \longrightarrow H^3(X, N^*_{X/P}) \longrightarrow H^3(X, \Omega^1_{P|X}) \to 0$$

([2], proof of theorem 1), which is $\mathfrak{S}_6$-equivariant. We will compute the two last terms.

The exact sequence

$$0 \to \Omega^1_P(-X) \longrightarrow \Omega^1_P \longrightarrow \Omega^1_{P|X} \to 0$$

provides an isomorphism $H^3(X, \Omega^1_{P|X}) \xrightarrow{\sim} H^4(P, \Omega^1_P(-X))$, and the latter space is isomorphic to $H^4(\mathbb{P}(V), \Omega^1_{\mathbb{P}(V)}(-4))$ ([2], proof of Lemma 3). By Serre duality $H^4(\mathbb{P}(V), \Omega^1_{\mathbb{P}(V)}(-4))$ is dual to $H^0(\mathbb{P}(V), T_{\mathbb{P}(V)}(-1)) \cong V$. Thus the $\mathfrak{S}_6$-module $H^3(X, \Omega^1_{P|X})$ is isomorphic to V^*, hence also to V.

Similarly the exact sequence $0 \to \mathcal{O}_P(-2X) \longrightarrow \mathcal{O}_P(-X) \longrightarrow N^*_{X/P} \to 0$ and the vanishing of $H^i(P, \mathcal{O}_P(-X))$ ([2], Corollary 2) provide an isomorphism of $H^3(X, N^*_{X/P})$ onto $H^4(P, \mathcal{O}_P(-2X))$, which is naturally isomorphic to the dual of $\mathcal{C}$ ([2], proof of Proposition 2). The lemma follows. $\square$

LEMMA 2. *The dimension of $\mathcal{C}$ is* 10.

Proof : Recall that the *defect* of X_t is the difference between the dimension of $\mathcal{C}$ and its expected dimension, namely :

$$\operatorname{def}(X_t) := \dim \mathcal{C} - (\dim H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(3)) - \#\mathcal{N}) .$$

Thus our assertion is equivalent to $\operatorname{def}(X_t) = 5$.

To compute this defect we use the formula of [5], Theorem 1.5. Let $F = 0$ be an equation of X_t in $\mathbb{P}^4$; let $R := \mathbb{C}[X_0, \ldots, X_4]/(F'_{X_0}, \ldots, F'_{X_4})$ be the Jacobian ring of F, and let R^{sm} be the Jacobian ring of a *smooth* quartic hypersurface in $\mathbb{P}^4$. The formula is

$$\operatorname{def}(X_t) = \dim R_7 - \dim R^{sm}_7 .$$

In our case we have $\dim R^{sm}_7 = \dim R^{sm}_3 = 35 - 5 = 30$; a simple computation with Singular (for instance) gives $\dim R_7 = 35$. This implies the lemma. $\square$

PROPOSITION 1. *The $\mathfrak{S}_6$-module $H^2(X, \Omega^1_X)$ is isomorphic to V.*

Proof : Consider the homomorphisms a and b of $\mathbb{C}^6$ into $H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(3))$ given by $a(e_i) = x_i^3$, $b(e_i) = x_i \sum x_j^2$. They are both $\mathfrak{S}_6$-equivariant and map V into $\mathcal{C}$; the subspaces $a(V)$ and $b(V)$ of $\mathcal{C}$ do not coincide, so we have $a(V) \cap b(V) = 0$. By Lemma 2 this implies $\mathcal{C} = a(V) \oplus b(V)$, so $H^2(X, \Omega_X^1)$ is isomorphic to V by Lemma 1. $\square$

REMARK 1. Suppose $t = 2, 6$ or $\frac{10}{7}$. Then the singular locus of X_t is $\mathcal{N} \cup \mathcal{N}'$, where $\mathcal{N}'$ is the $\mathfrak{S}_6$-orbit of the point $(1, -1, 0, 0, 0, 0)$ for $t = 2$, $(1, -1, 1, -1, 1, -1)$ for $t = 6$, $(-5, 1, 1, 1, 1, 1)$ for $t = \frac{10}{7}$ [7]. Since $x_1^3 - x_0^3$ does not vanish on $\mathcal{N}'$, the space of cubics vanishing along $\mathcal{N} \cup \mathcal{N}'$ is strictly contained in $\mathcal{C}$. By Lemma 1 it contains a copy of V, hence it is isomorphic to V; therefore $H^2(X, \Omega_X^1)$ and JX are zero in these cases. We have already mentioned that X_2 and X_4 are rational. The quartic $X_{\frac{10}{7}}$ is rational: it is the image of the anticanonical map of $\mathbb{P}^3$ blown up along 6 lines which are permuted by $\mathfrak{S}_6$ (see [4], proof of Lemma 4.5, and the references given there). We do not know whether this is the case for X_6.

3. Proof of the theorem

To prove that X is not rational, we apply the Clemens-Griffiths criterion ([3], Cor. 3.26): it suffices to prove that JX is not a Jacobian or a product of Jacobians.

Suppose $JX \cong JC$ for some curve C of genus 5. By the Proposition $\mathfrak{S}_6$ embeds into the group of automorphisms of JC preserving the principal polarization; by the Torelli theorem this group is isomorphic to $\mathrm{Aut}(C)$ if C is hyperelliptic and $\mathrm{Aut}(C) \times \mathbb{Z}/2$ otherwise. Thus we find $\#\mathrm{Aut}(C) \geq \frac{1}{2} 6! = 360$. But this contradicts the Hurwitz bound $\#\mathrm{Aut}(C) \leq 84(5-1) = 336$.

Now suppose that JX is isomorphic to a product of Jacobians $J_1 \times \ldots \times J_p$, with $p \geq 2$. Recall that such a decomposition is *unique* up to the order of the factors: it corresponds to the decomposition of the Theta divisor into irreducible components ([3], Cor. 3.23). Thus the group $\mathfrak{S}_6$ permutes the factors J_i, and therefore acts on $[1, p]$; by the Proposition this action must be transitive. But we have $p \leq \dim JX = 5$, so this is impossible. $\square$

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AMS Subject Classification: primary: 14M20, secondary; 14E08, 14K30

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Lavoro pervenuto in redazione il 02.07.2013.

Rend. Sem. Mat. Univ. Politec. Torino
Vol. 71, 3–4 (2013), 389 – 400

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ON CREMONA CONTRACTIBILITY

Dedicated to Professor Alberto Conte on the occasion of his 70th birthday

Abstract. In this note we give a constructive proof of a classical theorem which determines irreducible plane curves that are contractible to a point by a Cremona transformation. The problem of characterizing Cremona contractible (not necessarily irreducible) hypersurfaces in a projective space is in general widely open: we report on the only known result about reducible plane curves consisting of two components, due to Iitaka, and we discuss a couple of examples concerning plane curves with more components. Finally, we prove that all varieties of codimension at least two in a projective space are Cremona contractible to a point.

Introduction

Let $\mathbb{P}^2$ be the projective plane over $\mathbb{C}$. Plane Cremona transformations are birational maps $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ and they form the Cremona group $\mathrm{Cr}(\mathbb{P}^2)$. Cremona studied properties of geometric objects which are invariants under the action of $\mathrm{Cr}(\mathbb{P}^2)$. In this setting, a general problem is the classification of plane curves and linear systems up to Cremona transformations. See [3] for a classification, in particular of Cremona *minimal degree* models of plane curves. Such minimal degree is called the *Cremona degree* of the plane curve.

In this paper we deal with reduced, not necessarily irreducible, plane curves with Cremona degree 0, i.e. curves which are contracted to a set of points by a Cremona transformation. We say that these curves are *Cremona contractible*, shortly *Cr-contractible*.

Classical tools to study a plane curve C are the *m-adjoint linear systems* to C:

$$\mathrm{ad}_m(C) = f_*(|\tilde{C} + mK_S|),$$

where $f\colon S \to \mathbb{P}^2$ is a birational morphism which resolves the singularities of the curve C and $\tilde{C}$ denotes the strict transform of C on S. Relevant invariants of C can be read off from adjoint linear systems: for example, $\dim \mathrm{ad}_1(C) + 1 = g(C)$, where $g(C) = p_a(\tilde{C})$ is the *geometric genus* of C (e.g., if C is a union of rational curves, then $\mathrm{ad}_1(C) = \emptyset$).

In [4] and in [6, III, §21, p. 188], one finds the following:

THEOREM 1. *Let C be an irreducible plane curve. There exists a plane Cremona transformation which maps C to a line (which in turn is Cr-contractible) if and only if* $\mathrm{ad}_m(C) = \emptyset$ *for each* $m \geq 1$.

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The proof in [4] is incomplete, because of a careless use of infinitely near points of order two which are *satellite*, see Section 1 for notation and definitions, while the proof in [6] lacks some details. In Section 2 we give a complete and quick proof of this theorem, which is also constructive, in the sense that it describes the steps one has to make in order to find the Cremona transformation contracting the curve.

In 1982, Kumar and Murthy proved a stronger version of Theorem 1 with different methods which however are not constructive (in modern terms, they run a log minimal model program). It is interesting to notice that Kumar and Murthy, apparently unaware of the gap in Coolidge's proof, reported it *verbatim* in [10]. Their result is as follows:

THEOREM 2 (Kumar-Murthy). *Let C be an irreducible plane curve. There exists a plane Cremona transformation which maps C to a line if and only if* $\mathrm{ad}_1(C) = \mathrm{ad}_2(C) = \emptyset$.

In general, one can consider *pairs* (D,S), where D is a smooth curve on a smooth rational surface S. This viewpoint has been introduced by Suzuki in [17] and extensively carried out by Iitaka in several papers, cf. e.g. [7]. One defines the (log) *plurigenera* of the pair (D,S) as $P_m(D,S) = h^0(S,\mathcal{O}_S(m(D+K_S)))$, for each $m \geq 1$. One shows that the plurigenera are birational invariants of the pair (D,S). Moreover, one says that the pair (D,S) has (log) *Kodaira dimension* $-\infty$, shortly $\kappa(D,S) = -\infty$, if $P_m(D,S) = 0$ for each $m \geq 1$. The study of such pairs may be considered as the basic step of the classification problem in the birational setting.

If C is a reduced plane curve, we consider the pair $(\tilde{C},S)$ as above. By abusing notation, we also denote such a pair by $(C,\mathbb{P}^2)$.

In this setting, Kumar-Murthy's Theorem 2 says that the pair $(C,\mathbb{P}^2)$, with C irreducible, has Kodaira dimension $-\infty$ if and only if $P_2(C,\mathbb{P}^2) = 0$, which is a log-analogue of Castelnuovo's rationality criterion for regular surfaces.

Since C is effective, if $(C,\mathbb{P}^2)$ has Kodaira dimension $-\infty$, then the adjoint linear systems $\mathrm{ad}_m(C)$ are empty, for each $m \geq 1$. Hence Theorem 2 says that being all the adjoint systems to C are empty is equivalent to $\kappa(C,\mathbb{P}^2) = -\infty$.

An interesting problem is the extension of Kumar-Murthy's Theorem 2 to the case of reducible curves. The only result in this direction is due to Iitaka, see [8, 9]:

THEOREM 3 (Iitaka). *Let $C = C_1 + C_2$ be a reduced plane curve with two irreducible components. There exists a plane Cremona transformation which maps C to two lines (which are Cr-contractible) if and only if* $\mathrm{ad}_1(C) = \mathrm{ad}_2(C) = \emptyset$*, or equivalently* $P_2(C,\mathbb{P}^2) = 0$.

Again Cr-contractibility is equivalent to $\kappa(C,\mathbb{P}^2) = -\infty$ in this case.

After Iitaka's extension of Kumar-Murthy's result, an optimistic and naif conjecture would be the equivalence between Cr-contractibility and $P_2(C,\mathbb{P}^2) = 0$, or equivalently $\mathrm{ad}_1(C) = \mathrm{ad}_2(C) = \emptyset$.

Interestingly enough, things are more complicated. An old example of Pompilj in [16] shows that Iitaka's Theorem 3 cannot be extended to the case of three components and moreover that the implication

$$\mathrm{ad}_1(C) = \mathrm{ad}_2(C) = \emptyset \Longrightarrow \mathrm{ad}_m(C) = \emptyset, \text{ for each } m \geq 1,$$

is not true if C has more than two components, see Example 1 in Section 3.

Note, however, the following result:

THEOREM 4 (Kojima-Takahashi). *Let C be a reduced plane curve with at most four irreducible components. Then, $\kappa(C,\mathbb{P}^2) = -\infty$ if and only if $P_6(C,\mathbb{P}^2) = 0$.*

Kojima and Takahashi do not consider the Cr-contractibility of C.

A further remark is that, in general, for a reducible plane curve, the property that $\kappa(C,\mathbb{P}^2) = -\infty$ is not equivalent to $\mathrm{ad}_m(C) = \emptyset$, for each $m \geq 1$, as it happens in the irreducible case, see Example 2 in Section 3. Example 2 is also a counterexample to the assertion in [6, III, §21, p. 190] that the irreducibility assumption in Theorem 1 can be removed.

All this given, one can consider the following:

PROBLEM 1. Is it true that a reduced plane curve C is Cr-contractible if and only if $\kappa(C,\mathbb{P}^2) = -\infty$?

In a work in progress, we deal with this problem for a reduced union of lines.

Concerning higher dimensional analogues, in Section 4 we show that any Zariski closed subset of $\mathbb{P}^r$ of codimension at least two is contractible to a point by a Cremona transformation $\mathbb{P}^r \dashrightarrow \mathbb{P}^r$. Thus the analogue to Problem 1 is meaningful only for hypersurfaces in $\mathbb{P}^r$: for example, which reduced and irreducible hypersurfaces in $\mathbb{P}^r$ can be contracted to a point by a Cremona transformation? When $r = 3$, Mella and Polastri in [13] gave a criterion, which is difficult to use, because one needs information on infinitely many birational models of the pair given by the surface and $\mathbb{P}^3$. The case of cones has been solved by Mella in this volume, see [12].

1. Notation and preliminaries

We will use standard notation in surface theory, e.g. $K = K_S$ will denote a canonical divisor, the linear equivalence of divisors will be denoted by $\equiv$, etc.

1.1. Infinitely near, proximate and satellite points (cf., e.g., [1, 3, 6])

Let S be a rational smooth irreducible projective surface. Any birational morphism $\sigma\colon S \to \mathbb{P}^2$ is the composition of blowing-ups $\sigma_i\colon S_i \to S_{i-1}$ at a point $p_i \in S_{i-1}$, $i = 1,\ldots,n$:

$$\sigma\colon S = S_n \xrightarrow{\sigma_n} S_{n-1} \xrightarrow{\sigma_{n-1}} \cdots \xrightarrow{\sigma_2} S_1 \xrightarrow{\sigma_1} S_0 = \mathbb{P}^2. \tag{1}$$

Let $p \in \mathbb{P}^2$ be a point. One says that q is an *infinitely near point to p of order n*, and we write $q >^n p$, if there exists a birational morphism $\sigma\colon S \to \mathbb{P}^2$ as in (1), such that $p_1 = p$, $\sigma_i(p_{i+1}) = p_i$, $i = 1, \dots, n-1$, and $q \in Z_n = \sigma_n^{-1}(p_{n-1})$. For each $i = 1, \dots, n$, let $E_i = \sigma_i^{-1}(p_i) \subset S_i$ be the exceptional curve of σ_i. If $i > j$, let $\sigma_{i,j}$ be the morphism $S_i \to S_j$. For each $i = 1, \dots, n-1$, set $Z_i = \sigma^*_{n,i+1}(E_i)$ and let E'_i be the strict transform of E_i on S. Recall that $Z_1, \dots, Z_{n-1}, Z_n = E_n$ generate the Picard group $\mathrm{Pic}(S)$ of S over $\mathrm{Pic}(\mathbb{P}^2)$.

One says that q is *proximate to p*, and we write $q \to p$, if either $q >^1 p$ or $q >^n p$ with $n > 1$ and q lies on the strict transform E'_1 of E_1 on S. In the latter case, one says that q is *satellite to p*, and we write $q \odot p$. This may happen only if p_i lies on the strict transform of E_1 on S_{i-1}, for each $i = 2, \dots, n$.

We will refer to *points on* the plane $\mathbb{P}^2$ including infinitely near ones. We will say that a point p is *proper*, and we will write $p \in \mathbb{P}^2$, if p is not infinitely near to any point of $\mathbb{P}^2$.

Let now C' be a curve on S and $C = \sigma_*(C')$. Then $C' = \sigma^*(C) - \sum_{i=1}^n m_i Z_i$, where $m_1, \dots, m_n$ are integers. If C is a curve, i.e. if C' is not contracted by σ, one says that m_i, $i = 1, \dots, n$, is the *(virtual) multiplicity* of C at the point p_i. If no component of C' is contracted by σ, then, for each $i = 1, \dots, n$, one has $C' \cdot E'_i \geqslant 0$, equivalently

$$m_i \geqslant \sum_{j\colon\, p_j \to p_i} m_j. \tag{2}$$

which is the *proximity inequality at p_i*.

A complete linear system $\mathcal{L}$ on S has the form $\mathcal{L} = |dH - \sum_{i=1}^n m_i Z_i|$ where d and the m_i are integers and $H = \sigma^*(L)$, with L a general line in $\mathbb{P}^2$. Abusing notation, we also write $\mathcal{L}$ as $\mathcal{L} = |dL - \sum_{i=1}^n m_i p_i|$.

1.2. Simplicity of a plane curve

Let C be a reduced plane curve. Let $d = \deg(C)$ and let $m_1 \geq m_2 \geq \cdots \geq m_n$ be the respective multiplicities of the singular points $p_1, p_2, \dots, p_n$ of C. If C is smooth, we assume $m_1 = 1$ and p_1 is a general point of C. By the proximity inequalities, we may and will assume that $p_i > p_j$ implies $i > j$. Therefore $p_1 \in \mathbb{P}^2$ and either $p_2 \in \mathbb{P}^2$ or $p_2 >^1 p_1$.

Set $h = (d - m_1)/2$, let t be the number of points p_i of multiplicity $m_i > h$ and let s be the number of *satellite* points among them.

The triplet (h, t, s) is called the *simplicity* of the curve C. A curve C' is said to be *simpler* than C if the simplicity of C' is lexicographically lower than the simplicity of C.

1.3. Cremona transformations

A linear system $\mathcal{L}$ of plane curves, with no divisorial fixed component, is called a *net* if $\dim(\mathcal{L}) = 2$. If, in addition, $\mathcal{L}$ defines a birational map $\gamma\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, i.e. a

Cremona transformation, the net is called *homaloidal*. In that case, the general curve C of $\mathcal{L}$ is irreducible and rational. If d is the degree of the members of $\mathcal{L}$, one says that γ has *degree* d. Cremona transformations of degree 1 are *projective* or *linear* transformations. When $d = 2$, one says that the Cremona transformation γ is *quadratic*. In that case, the homaloidal net defining γ has three simple base points p_1, p_2, p_3 and one says that γ is *centered* at p_1, p_2, p_3 and that p_1, p_2, p_3 are the fundamental points of γ. When γ has degree $d \geq 2$ and the homaloidal net $\mathcal{L}$ has a base point p_1 of multiplicity $d-1$, then $\mathcal{L}$ also has $2d-2$ simple base points $p_2, \dots, p_{2d-1}$ and one says that γ is a *de Jonquières* transformation centered at $p_1, p_2, \dots, p_{2d-1}$.

The famous Noether-Castelnuovo Theorem states that each Cremona transformation is the composition of finitely many linear and quadratic transformations (see [3] for a short proof that uses the simplicity).

The next lemmas show that quadratic transformations can make a curve simpler.

LEMMA 1. *Let (h,t,s) be the simplicity of an irreducible plane curve C. Assume there exists the quadratic Cremona transformation γ centered at points p_1, p_i, p_j where C has respective multiplicities m_1, m_i, m_j with $m_1 = d - 2h \geq m_i \geq m_j > h$. Then $\gamma_*(C)$ is simpler than C.*

Proof. Let $\tilde{p}_k$, $k = 1, i, j$, be the fundamental points of γ^{-1}. Setting $\delta = m_1 + m_i + m_j - d > m_1 + 2h - d = 0$, the curve $\gamma_*(C)$ has degree $\tilde{d} = d - \delta$ and multiplicity $\tilde{m}_k = m_k - \delta$ at $\tilde{p}_k$, $k = 1, i, j$, while all the other multiplicities of $\gamma_*(C)$ are the same as the corresponding ones of C. Denote by $(\tilde{h}, \tilde{t}, \tilde{s})$ the simplicity of $\gamma_*(C)$. Setting $\tilde{m}$ the maximal multiplicity of points of $\gamma_*(C)$, one has that $2\tilde{h} = \tilde{d} - \tilde{m} \leq \tilde{d} - \tilde{m}_1 = d - m_1 = 2h$. If $\tilde{h} < h$, then $\gamma_*(C)$ is simpler than C. Otherwise $\tilde{h} = h$ but $\tilde{m}_i = m_i - \delta = 2h - m_j < h$ and $\tilde{m}_j = m_j - \delta = 2h - m_i < h$, hence $\tilde{t} = t - 2$ and $\gamma_*(C)$ is simpler than C. $\square$

LEMMA 2. *Let (h,t,s) be the simplicity of an irreducible plane curve C. Suppose that C has multiplicity m_1, m_i, m_j at the points p_1, p_i, p_j with $m_1 = d - 2h \geq m_i \geq m_j > h$, $p_j >^1 p_i >^1 p_1$ and p_j is satellite to p_1. Then, a quadratic Cremona transformation γ centered at p_1, p_i, p, where p is a general point in $\mathbb{P}^2$, is such that $\gamma_*(C)$ is simpler than C.*

Proof. Let $\tilde{p}_1, \tilde{p}_i, \tilde{p}$ be the fundamental points of γ^{-1}. Setting $\lambda = d - m_1 - m_i$, one has $0 \leq \lambda = 2h - m_i < h$. The curve $\gamma_*(C)$ has degree $\tilde{d} = d + \lambda \geq d$ and multiplicity $\tilde{m}_1 = m_1 + \lambda \geq m_1$ at $\tilde{p}_1$, multiplicity $\tilde{m}_i = m_i + \lambda$ at $\tilde{p}_i$ and multiplicity λ at $\tilde{p}$, while the remaining multiplicities are not affected by γ. Denote by $(\tilde{h}, \tilde{t}, \tilde{s})$ the simplicity of $\gamma_*(C)$. Then, $2\tilde{h} = \tilde{d} - \tilde{m}_1 = d - m_1 = 2h$ and furthermore $\tilde{t} = t$, but $\tilde{s} = s - 1$ because the point $\tilde{p}_j$ corresponding to p_j via γ is no longer satellite, in fact $\tilde{p}_j >^1 \tilde{p}_1$. $\square$

2. A proof of Theorem 1

The proof is by induction on the simplicity.

One has that $h \geq 0$ and that $h = 0$ if and only if C is a line, that is the assertion.

If $h = 1/2$, then C is a curve of degree $d \geq 2$ with a point p_1 of multiplicity $m_1 = d-1$ and no other singular point, otherwise the line through it and p_1 would be a component of C, contradicting the irreducibility assumption. The curve C is mapped to a line by a de Jonquières transformation of degree d centered at p_1, with multiplicity $d-1$, and $2d-2$ general points of C.

Let now $h \geq 1$. The proof of Theorem 1 will be concluded by the following

PROPOSITION 1. *Let C be an irreducible plane curve such that its maximal multiplicity is $m_1 \leq d-2$, where d is the degree of C. If*

$$\mathrm{ad}_m(C) = \emptyset, \quad \text{for each } m \geq 1, \tag{3}$$

then there exists a quadratic transformation which maps C to a simpler curve.

First we need some lemmas.

LEMMA 3. *In the above setting, one has $m_1 > h$, or equivalently $m_1 > d/3$.*

Proof. The equivalence between $m_1 > h$ and $m_1 > d/3$ is clear. The assertion is trivial if $d < 3$, so we assume $d \geq 3$.
Suppose by contradiction that $m_1 < d/3$. Then the $[d/3]$-adjoint to C, where $[x] = \max\{m \in \mathbb{N} \mid m \leq x\}$, would be non-empty:

$$|\tilde{C} + [d/3]K| = \left| \left(d - 3\left[\frac{d}{3}\right]\right) L - \sum_{i=1}^{n} \left(m_i - \left[\frac{d}{3}\right]\right) p_i \right| \supseteq |(d \bmod 3)L| \neq \emptyset,$$

contradicting (3). □

Set $\overline{h} = [h] \in \mathbb{Z}$ and $\varepsilon = 2(h - \overline{h}) \in \{0, 1\}$. Lemma 3 reads $m_1 \geq \overline{h} + 1$.

LEMMA 4. *In the above setting, one has $m_2 \geq m_3 \geq \overline{h} + 1$.*

Proof. Suppose by contradiction that $m_3 \leq \overline{h}$. Then

$$\begin{aligned} |\tilde{C} + \overline{h}K| &\supseteq \left|\varepsilon L + (m_1 - \overline{h})(L - p_1) - (m_2 - \overline{h})p_2\right| = \\ &= (m_2 - \overline{h})L_{12} + |\varepsilon L + (m_1 - m_2)(L - p_1)| \neq \emptyset, \end{aligned}$$

where L_{12} is (the strict transform of) the line passing through p_1 and p_2, against (3). □

Either $p_2 \in \mathbb{P}^2$ or $p_2 >^1 p_1$. Moreover, either $p_3 \in \mathbb{P}^2$, or $p_3 >^1 p_1$, or $p_3 >^1 p_2$. In any case, there is no line passing through p_1, p_2, p_3, because $m_1 + m_2 + m_3 > m_1 + 2h = d$ and C is irreducible of degree $d \geq 3$. By Lemma 1, the curve C is mapped to a simpler curve by a quadratic transformation γ centered at p_1, p_2, p_3, unless either

1. $p_3 >^1 p_2 >^1 p_1$ and p_3 is satellite to p_1, or
2. $p_2 >^1 p_1$ and $p_3 >^1 p_1$,

that are the cases when there is no quadratic transformation centered at p_1, p_2, p_3.

In case (1), the curve C is mapped to a simpler curve by the quadratic transformation centered at p_1, p_2 and at a general point in $\mathbb{P}^2$ by Lemma 2.

We are left with case (2). Let j be the largest integer such that $p_i >^1 p_1$ for each $2 \leq i \leq j$. By construction, one has $j \geq 3$.

LEMMA 5. *In the above setting, one has* $j+1 \leq n$ *and* $m_{j+1} \geq \overline{h}+1$.

Proof. The proximity inequality at p_1 implies that $m_1 \geq m_2 + m_3 + \cdots + m_j$, therefore

$$m_1 - \overline{h} \geq m_2 + m_3 + \cdots + m_j - (j-1)\overline{h} = (m_2 - \overline{h}) + (m_3 - \overline{h}) + \cdots + (m_j - \overline{h}).$$

Suppose by contradiction that either $j = n$ or $m_{j+1} \leq \overline{h}$. Then

$$\begin{aligned} |\tilde{C} + \overline{h}K| &\supseteq \left| \varepsilon L + (m_1 - \overline{h})(L - p_1) - \sum_{i=2}^{j} \left(m_i - \overline{h}\right) p_i \right| \supseteq \\ &\supseteq \sum_{i=2}^{j} \left(m_i - \overline{h}\right) L_{1i} + \left| \varepsilon L + \left(m_1 - \overline{h} - \sum_{i=2}^{j} (m_i - \overline{h}) \right) (L - p_1) \right| \neq \emptyset, \end{aligned}$$

where L_{1i} is (the strict transform of) the line passing through p_1 and p_i, contradicting (3). $\square$

Then, either $p_{j+1} \in \mathbb{P}^2$ or $p_{j+1} >^1 p_i >^1 p_1$, with $1 < i \leq j$. In the latter case, either p_{j+1} is satellite or it is not. If p_{j+1} is satellite, then we get a simpler curve by using Lemma 2. Otherwise, there is no line passing through p_1, p_i, p_{j+1}, because of the irreducibility of C and we get a simpler curve by using Lemma 1.

This ends the proof of Proposition 1 and hence of Theorem 1.

3. Examples and a remark

We recall an interesting example due to Pompilj in [16].

EXAMPLE 1 (Pompilj). Let C_1, C_2 be two irreducible rational plane quartic curves and let C_3 be a line such that $C = C_1 + C_2 + C_3$ is a reduced curve of degree 9 with 10 triple points, where the multiplicities of the C_i, $i = 1, 2, 3$, are as follows:

	deg	p_0	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
C_1	4	2	2	2	1	1	1	1	1	1	1
C_2	4	1	1	1	2	2	2	1	1	1	1
C_3	1	0	0	0	0	0	0	1	1	1	1
C	9	3	3	3	3	3	3	3	3	3	3

One checks that $\mathrm{ad}_1(C) = \mathrm{ad}_2(C) = \emptyset$, but $\mathrm{ad}_3(C) \neq \emptyset$, therefore C is not Cr-contractible. The existence of C can be proved by giving equations: for example we take $C_1 : x^2y^2 + 2x^2 + 3y^2 + 6xy(x+y+1) = 0$, $C_3 : x+y+1 = 0$, and C_2 the orthogonal symmetric of C_1 with respect to C_3.

REMARK 1. Let C be a reduced union of lines in the plane. By definition, $(C,\mathbb{P}^2)$ has Kodaira dimension $-\infty$ if and only if $|m(C+K)|=\emptyset$, for each $m\geqslant 1$. On the other hand, since $C+K$ intersects negatively all components of C, then $|mC+mK|\neq\emptyset$ if and only if $|(m-1)C+mK|\neq\emptyset$. When $m=2$, this means that

$$P_2(C,\mathbb{P}^2)=\dim(\mathrm{ad}_2(C))+1.$$

When $m\geq 3$, as the next example shows, it is possible that

$$P_m(C,\mathbb{P}^2)>\dim(\mathrm{ad}_m(C))+1.$$

EXAMPLE 2. Let C be the union of $d\geq 9$ distinct lines $L_1,\dots,L_d$ with the first $d-3$ passing through a point p_0, while the remaining three in general position. Thus C has an ordinary point p_0 of multiplicity $d-3$ and $3(d-2)$ nodes $p_1,\dots,p_{3(d-2)}$. One checks that $\mathrm{ad}_m(C)=\emptyset$ for each $m\geq 1$. Since

$$\tilde{C}=L_1+L_2+\cdots+L_d\equiv dL-(d-3)p_0-2(p_1+p_2+\cdots+p_{3(d-2)}),$$

where we denote by L_i, $i=1,\dots,d$, also the strict transform of L_i in S, one has

$$\begin{aligned}|2\tilde{C}+3K|&=|(2d-9)L-(2d-9)p_0-1(p_1+\cdots+p_{3(d-2)})|=\\&=\{L_1+\cdots+L_{d-3}+L_1'+L_2'+L_3'\}\neq\emptyset,\end{aligned}$$

where L_i', $i=1,2,3$, is the line passing through p_0 and a vertex of the triangle whose sides are the three general lines L_{d-2},L_{d-1},L_d.

4. Cremona contractibility in codimension at least 2

4.1. Monoids.

A *monoid* in $\mathbb{P}^r$ is a hypersurface of degree d with a point of multiplicity $d-1$, called the *vertex* of the monoid. A monoid will be called a *true* monoid if it is irreducible and the vertex has multiplicity exactly one less than the degree. Such a monoid is rational, because its *stereographic projection* from the vertex to a hyperplane not containing the vertex is birational.

REMARK 2. The linear system of monoids of degree d and fixed vertex in $\mathbb{P}^r$ has dimension

$$\binom{d+r}{r}-1-\binom{d+r-2}{r}=\frac{2}{(r-1)!}\,d^{r-1}+\frac{r-1}{(r-2)!}\,d^{r-2}+O(d^{r-3}).$$

LEMMA 6. *Consider a Zariski closed subset Z of $\mathbb{P}^r$, no component of which is a hypersurface. For a general point $p\in\mathbb{P}^r$, Z is contained in a true monoid with vertex p.*

Proof. Let $n \leq r-2$ be the dimension of Z, i.e. the maximal dimension of the components of Z.

Let $\pi\colon Z \dashrightarrow Z' \subset \Pi$ be the projection of Z to a hyperplane Π not containing p. Let us take affine coordinates so that p is the point at infinity of the x_r-axis and $\Pi : x_r = 0$.

Assume first that $n < r-2$. Then we may consider two polynomials $f, g \in k[x_1, \dots, x_{r-1}]$ of degree d and $d-1$, respectively, with no common factor, vanishing on Z'. They clearly exist for $d \gg 0$. Then the monoid with equation $f + x_r g = 0$ is a true monoid with vertex p containing Z.

Suppose now $n = r-2$ and let δ be the degree of the codimension 2 part Z_1 of Z. The linear system $\mathcal{L}$ of monoids with vertex p containing Z contains the linear system $\mathcal{L}'$ of cones with vertex p over Z_1 plus a monoid of degree $d-\delta$ with vertex p containing $Z_2 = \overline{Z \setminus Z_1}$.

By Remark 2 and by the Riemann-Roch Theorem, we have

$$\dim \mathcal{L} \geq \frac{2}{(r-1)!}\, d^{r-1} + \frac{r-1-\delta}{(r-2)!}\, d^{r-2} + O(d^{r-3}).$$

On the other hand, by the same argument,

$$\dim \mathcal{L}' = \frac{2}{(r-1)!}\, d^{r-1} + \frac{r-1-2\delta}{(r-2)!}\, d^{r-2} + O(d^{r-3}),$$

therefore $\dim \mathcal{L}' < \dim \mathcal{L}$, so that $\mathcal{L}'$ is strictly contained in $\mathcal{L}$.

If Z_1 is irreducible, the assertion follows by Bertini's Theorem. If Z_1 is reducible, one has to iterate this argument using Bertini's Theorem by stepwise eliminating all cones over the components of Z_1. □

REMARK 3. In the hypothesis of the previous lemma, the generality assumption on p can be relaxed: it suffices to assume that the projection from p preserves the degree of the components of Z of codimension 2, if it preserves the dimension of Z.

4.2. Monoidal transformations.

We call a Cremona transformation $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ *monoidal* if it is defined by a homaloidal linear system $\mathcal{L}$ whose general member is a monoid. The classification of monoidal linear systems is an interesting problem which, to the best of our knowledge, has never been considered in its generality: the case of quadrics is classical but a complete classification in $\mathbb{P}^3$ has been obtained only recently [14]; contributions to special quadratic ones is due to [2] and [15]. See also [5] as a general reference.

A *special* case is the one in which the vertex p of the monoids of the system is fixed.

EXAMPLE 3 (De Jonquières transformations). Let X be a true monoid of degree d with vertex p. Let us consider a true monoid Y of degree $d-1$ with vertex p and let C be the intersection scheme of X and Y. Let $\mathcal{L}$ be the linear system of monoids of degree d with vertex p containing C. Then $\mathcal{L}$ is a special homaloidal linear system.

Indeed, the restriction sequence of $\mathcal{L}$ to X (or, for all that matters, to the general monoid of $\mathcal{L}$) shows that $\dim \mathcal{L} = r$ and that the rational map defined by $\mathcal{L}$ restricts to X to the stereographic projection from the vertex.

The examples implies:

PROPOSITION 2. *Every true monoid is Cr-contractible, i.e. it is contained in a homaloidal linear system.*

4.3. Cr-contractibility in codimension at least 2.

THEOREM 5. *Let Z be a Zariski closed subset of $\mathbb{P}^r$, no component of which is a hypersurface. For a general point $p \in \mathbb{P}^r$, there exists a Cremona transformation $\gamma\colon \mathbb{P}^r \dashrightarrow \mathbb{P}^r$ such that $\gamma(Z)$ is the projection of Z from p to a hyperplane not containing p.*

Proof. It follows by Lemma 6 and Example 3. □

REMARK 4. The generality of the point p is not necessary in the statement of the previous theorem. It suffices to take the point off the 0-dimensional components of Z.

Let $n \leq r-2$ be the dimension of Z. By iterating Theorem 5, there exists a Cremona transformation $\gamma\colon \mathbb{P}^r \to \mathbb{P}^r$ such that $\gamma_{|Z}\colon Z \to \mathbb{P}^n$ is a general projection, hence it is a finite morphism.

LEMMA 7. *Fix a linear subspace $\mathbb{P}^n$ in $\mathbb{P}^r$, $r > n$. Then there exists a Cremona transformation $\mathbb{P}^r \dashrightarrow \mathbb{P}^r$ which contracts $\mathbb{P}^n$ to a point.*

Proof. Follows from Theorem 5 and Remark 4. □

COROLLARY 1. *Let Z be a Zariski closed subset of $\mathbb{P}^r$, no component of which is a hypersurface. Then there exists a Cremona transformation $\gamma\colon \mathbb{P}^r \dashrightarrow \mathbb{P}^r$, which is defined at the general point of each component of Z, such that $\gamma(Z)$ is a point.*

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AMS Subject Classification: 14E07, 14H50, 14E08

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Lavoro pervenuto in redazione il 02.07.2013.

Rend. Sem. Mat. Univ. Politec. Torino
Vol. 71, 3–4 (2013), 401 – 420

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JETS OF SINGULAR FOLIATIONS*

Abstract. Given a singular foliation satisfying locally everywhere the Frobenius condition, even at the singularities, we show how to construct its global sheaves of jets. Our construction is purely formal, and thus applicable in a variety of contexts.

1. Introduction

Let M be a complex manifold of complex dimension m. A holomorphic foliation $\mathcal{L}$ of dimension n of M is a decomposition of M in complex submanifolds, called leaves, of dimension n. Also, locally the leaves must pile up nicely, like the fibers of a holomorphic map. In other words, for each point p of M there must exist an open neighborhood U and a holomorphic submersion $\varphi\colon U \to V$ to an open subset $V \subseteq \mathbf{C}^{m-n}$ such that the fibers of φ are the intersection of the leaves with U. We say that φ defines $\mathcal{L}$ on U.

A holomorphic foliation $\mathcal{L}$ of M induces a vector subbundle of the tangent bundle T_M of M: for each $p \in M$, the vector subspace of $T_{M,p}$ is the tangent space at p of the unique leaf passing through p. Thinking in dual terms, $\mathcal{L}$ induces a surjection $w\colon T_M^* \to E$ from the bundle of holomorphic 1-forms to a holomorphic rank-n vector bundle E. The bundle E, also denoted by $T_{\mathcal{L}}^*$, is regarded as the bundle of 1-forms of $\mathcal{L}$.

Not all surjections $w\colon T_M^* \to E$ to a holomorphic vector bundle E arise from foliations. The necessary and sufficient condition for this is given by the Frobenius Theorem: locally at each point p of M, choose a trivialization of E, and consider the vector fields $X_1, \dots, X_n$ induced by w; if their Lie brackets $[X_i, X_j]$ can be expressed as sums $\sum_\ell g_\ell X_\ell$, where the g_ℓ are holomorphic functions on a neighborhood of p, then w arises from a foliation.

The surjection w can be seen, locally on an open subset $U \subseteq M$ for which there is a submersion $\varphi\colon U \to V \subseteq \mathbf{C}^{m-n}$ defining $\mathcal{L}$, as the natural map $T_U^* \to T_{U/V}^*$ from the bundle of 1-forms on U to the bundle of relative 1-forms on U over V. Also, on such a U, we may consider the natural surjection $J_U^q \to J_{U/V}^q$ from the bundle J_U^q of q-jets (or principal parts of order q) on U to the bundle $J_{U/V}^q$ of relative jets of φ, for each integer $q \geq 0$. These patch to form surjections $w^q\colon J_M^q \to J_{\mathcal{L}}^q$ to bundles $J_{\mathcal{L}}^q$ that can be regarded as the bundles of q-jets of the foliation.

But what happens if all the data are algebraic? More precisely, assume M is algebraic, E and w are algebraic, and there is an algebraic trivialization of E at each $p \in M$ such that the resulting vector fields $X_1, \dots, X_n$ are involutive, i.e. satisfy $[X_i, X_j] =$

*Supported by CNPq, Proc. 478625/03-0 and 301117/04-7, FAPERJ, Proc. E-26/170.418/00, and CNPq/FAPERJ, Proc. E-26/171.174/03.

$\sum_\ell g_\ell^{i,j} X_\ell$ for $g_\ell^{i,j}$ algebraic. Are the bundles $J^m_{\mathcal{L}}$ algebraic? In principle, they are just holomorphic, since the local submersions φ from which they arise are just holomorphic, constructed by means of the implicit function theorem.

Moreover, it is rare for a projective manifold to admit interesting foliations. For this reason, one has started to study *singular foliations*, in a variety of ways. For instance, a singular foliation of M of dimension n may be defined to be a map $w\colon T^*_M \to E$ to a rank-n holomorphic vector bundle E which, on a dense open subset $M^0 \subseteq M$, arises from a foliation $\mathcal{L}$. We still regard E as the bundle of 1-forms of the foliation. But the bundles of jets $J^m_{\mathcal{L}}$ are, in principle, only defined on M^0. Under which conditions do they extend to M?

In the present paper, we will show that if all the data are algebraic, then the bundles $J^m_{\mathcal{L}}$ are algebraic. Furthermore, if the same Frobenius' conditions, appropriately formulated, are verified at each point of $M - M^0$, then the bundles $J^m_{\mathcal{L}}$ extend to the whole M. For the proofs, we will completely bypass Frobenius Theorem, giving an entirely formal construction of the bundles of jets that applies in many categories, for instance that of differentiable manifolds, or of schemes over any base. Furthermore, not only will we consider maps to bundles E, but also to sheaves of modules, locally free or not, obtaining thus sheaves of jets.

Our construction of the sheaves of jets is by "iteration", so will only apply in characteristic zero. For an approach in positive characteristic, albeit limited in scope, see [1] or [10]. From now on, all rings are assumed to be $\mathbf{Q}$-algebras.

Here is what we do. Let X be a topological space, $\mathcal{O}_B$ a sheaf of $\mathbf{Q}$-algebras and $\mathcal{O}_X$ a sheaf of $\mathcal{O}_B$-algebras. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules and $D\colon \mathcal{O}_X \to \mathcal{F}$ an $\mathcal{O}_B$-derivation. We may think of $\mathcal{O}_X$ as the sheaf of "functions" on X and of $\mathcal{O}_B$ as the sheaf of "constant functions." For each integer $i \geq 0$, let $\mathcal{T}^i(\mathcal{F})$ be the tensor product over $\mathcal{O}_X$ of i copies of $\mathcal{F}$, and denote by $\mathcal{S}^i(\mathcal{F})$ and $\mathcal{A}^i(\mathcal{F})$ its symmetric and exterior quotients. Let $\mathcal{S}(\mathcal{F})$ be the direct product of all the $\mathcal{S}_i(\mathcal{F})$ for all integers $i \geq 0$, with its natural graded $\mathcal{O}_X$-algebra structure. We may think of $\mathcal{S}(\mathcal{F})$ as the sheaf of "formal power series" on the sections of $\mathcal{F}$.

We define a D-connection to be a map of $\mathcal{O}_B$-modules $\gamma\colon \mathcal{F} \to \mathcal{T}^2(\mathcal{F})$ satisfying

$$\gamma(am) = D(a) \otimes m + a\gamma(m)$$

for all local sections a of $\mathcal{O}_X$ and m of $\mathcal{F}$. We will see in Construction 2 how γ can be used to iterate D to obtain a sequence of maps $T_i : \mathcal{F} \to \mathcal{S}^i(\mathcal{F}) \otimes \mathcal{F}$, for $i = 0, 1, \dots$, where $T_0 := \mathrm{id}_{\mathcal{F}}$, $T_1 = \gamma$, and the T_i satisfy properties similar to that of a connection, i.e. Equations 3. We call any sequence of maps $T = (T_0, T_1, \dots)$ with these properties, for any D-connection γ, an extended D-connection.

From an extended D-connection T we get a map $h\colon \mathcal{O}_X \to \mathcal{S}(\mathcal{F})$ of $\mathcal{O}_B$-algebras, with $h_0 := \mathrm{id}_{\mathcal{O}_X}$ and $h_1 := D$, by letting $h_i(a)$ be the class in $\mathcal{S}^i(\mathcal{F})$ of $(1/i)T_{i-1}D(a)$ for each local section a of $\mathcal{O}_X$; see Construction 3. We may think of h as a way of computing "Taylor series" of functions on X. In this way, $\mathcal{S}(\mathcal{F})$ may be regarded as a sheaf of jets. We call such an h an iterated Hasse derivation.

However, D-connections are usually local gadgets. So, to be able to patch the local maps h, we need D-connections to be canonical. But there is nothing canonical about γ: for every O_X-linear map $\nu\colon \mathcal{F} \to \mathcal{T}^2(\mathcal{F})$, the sum $\gamma+\nu$ is also a D-connection! So we study special D-connections.

A D-connection γ is called flat if the image of $\gamma D(O_X)$ in $\mathcal{A}^2(\mathcal{F})$ is zero. When such γ exists, we say that D is integrable. Our main result, Theorem 5, shows that, given a flat D-connection γ, there is an extended D-connection T with $T_1=\gamma$, which is also flat, meaning that $T_iD(O_X)$ lies in the subsheaf of $\mathcal{S}^i(\mathcal{F})\otimes\mathcal{F}$ generated locally by

$$\sum_{\ell=1}^{i} m_1m_2\cdots m_{\ell-1}m_{\ell+1}m_{\ell+2}\cdots m_i\otimes m_\ell$$

for all local sections $m_1,\dots,m_i$ of $\mathcal{F}$, for each $i\geq 1$.

Now, our Proposition 1 says that a flat, extended D-connection is "comparable" to any other extended D-connection. So, Theorem 5 and Proposition 1 can be coupled to yield that all iterated Hasse derivations are equivalent; see Corollary 1. More precisely, if h and h' are iterated Hasse derivations, there is an O_X-algebra automorphism $\phi\colon \mathcal{S}(\mathcal{F})\to\mathcal{S}(\mathcal{F})$ such that $h'=\phi h$. Furthermore, the degree-i part of this ϕ is zero if $i<0$ and the identity if $i=0$.

Now, for the patching we also need the automorphisms ϕ to be canonical, so that cocycle conditions are satisfied. For this, we make a technical assumption on D, that bounds its singularities, and holds in all applications we know of; see Corollary 1 and the remark thereafter.

Finally, assume that D is locally integrable, and has bounded singularities, in the sense alluded to above. The patching of the local iterated Hasse derivations and the O_X-algebra automorphisms comparing them is straightforward. We obtain an O_X-algebra $\mathcal{J}$ and a map $h\colon O_X\to\mathcal{J}$ of O_B-algebras. Also, since the local O_X-algebra automorphisms do not decrease degrees, and their degree-0 parts are the identity, $\mathcal{J}$ comes naturally with a filtration by O_X-algebra quotients $\mathcal{J}^q$, for $q=0,1,\dots$, and natural exact sequences

$$0\to\mathcal{S}^q(\mathcal{F})\to\mathcal{J}^q\to\mathcal{J}^{q-1}\to 0$$

for each $q>0$; see Construction 6. We say that $\mathcal{J}^q$ is the sheaf of q-jets of D.

How does this formal construction fit with the geometric setting? If M is a holomorphic manifold, let (X,O_X) be the ringed space where X is the underlying topological space of M and O_X is its sheaf of holomorphic functions. Let O_B be the sheaf of constant complex functions. A map of vector bundles $w\colon T_M^*\to E$ corresponds to a derivation $D\colon O_X\to\mathcal{F}$, where $\mathcal{F}$ is the sheaf of holomorphic sections of E. To say that w arises from a foliation on an open subset $U\subseteq M$ is equivalent to say that $\mathcal{F}|_U=O_UD(O_U)$ and $D|_U$ is locally integrable on U; see Example 2. Now, assume that D is locally integrable on the whole X, and that $D(O_X)$ generates $\mathcal{F}$ as an O_X-module on a dense open subset $M^0\subseteq M$. Then there exists a sheaf of q-jets $\mathcal{J}^q$ on X, as explained above. Also, w defines a foliation $\mathcal{L}$ on M^0, and $\mathcal{J}^q|_{M^0}$ is the sheaf of sections of the bundle of q-jets $J^q_{\mathcal{L}}$; see Example 4.

Bundles of jets associated to foliations or derivations were considered by a number of people. In algebraic geometry, to my knowledge, the first was Letterio Gatto, who in his thesis [6] constructed jets from a family of stable curves $f\colon \mathcal{X}\to\mathcal{S}$ and its canonical derivation $\mathcal{O}_X \to \omega_{X/S}$, where $\omega_{X/S}$ is the relative dualizing sheaf. Afterwards, in [1], jets were constructed for more general families, of local complete intersection curves, over any base and in any characteristic.

Also, Dan Laksov and Anders Thorup constructed bundles of jets in a series of articles in different setups; see [8], [9] and [10]. In characteristic zero, their more general work is [9]. Actually, in [9], Laksov and Thorup construct larger sheaves of "jets", that are naturally noncommutative. The true generalization of the sheaf of jets is what they call "symmetric" jets. They show that the sheaf of (symmetric) jets is uniquely defined when $\mathcal{F}$ is free and has an $\mathcal{O}_X$-basis under which D can be expressed using commuting derivations of $\mathcal{O}_X$. As we have observed above, the uniqueness of the definition is important for patching. However, the commutativity is stronger than Frobenius' conditions, at least in the algebraic category — in the analytic category, at nonsingular points, the local existence of commuting derivations follows from the existence of the foliation. The present work arose from the feeling that the Frobenius' conditions should be enough to construct sheaves of jets.

There have already been applications of the sheaves of jets associated to a foliation or a derivation. They were used by Gatto [7], and Gatto and myself [3] in enumerative aplications. They were used by myself in understanding limits of ramification points [2], and of Weierstrass points, with Nivaldo Medeiros, [4] and [5]. They were also used by Jorge Vitório Pereira in the study of foliations of the projective space [12].

Finally, we will see an example where the integrability condition holds and $\mathcal{F}$ is not locally free; see Example 3. That will be the example of the canonical derivation on a special non-Gorenstein curve, arguably the simplest non-Gorenstein unibranch curve there is, whose complete local ring at the singular point is of the form $\mathbf{C}[[t^3,t^4,t^5]]$, as a subring of $\mathbf{C}[[t]]$. It could be that the integrability condition holds for the canonical derivation on any curve, Gorenstein or not. If so, the sheaf of jets might be used to define Weierstrass points on non-Gorenstein integral curves, and show that these points are limits of Weierstrass points on nearby curves, in the way done by Robert Lax and Carl Widland for Gorenstein curves; see [11] and [13]. However, this problem will not be pursued here.

Here is a layout of the article. In Section 2, we define connections, extended connections and iterated Hasse derivations, and explain a few preliminary constructions. In Section 3, we define integrable derivations and flat (extended) connections, and show that a flat, extended connection is comparable with any other extended connection. Finally, in Section 4, we show that, if a derivation D is integrable, then flat, extended connections exist, and all iterated Hasse derivations are equivalent; then we construct the sheaves of jets for locally integrable derivations.

Throughout the paper, X will stand for a topological space, $\mathcal{O}_B$ for a sheaf of $\mathbf{Q}$-algebras, $\mathcal{O}_X$ for a sheaf of $\mathcal{O}_B$-algebras, $\mathcal{F}$ for a sheaf of $\mathcal{O}_X$-modules and $D\colon \mathcal{O}_X \to \mathcal{F}$ for an $\mathcal{O}_B$-derivation.

This work started as a joint work with Letterio Gatto. However, he felt he did not contribute to it as much as he wished. Though a few discussions with him were vital to how this work came to be, and though I feel that this could be classified as a

joint work, I had to respect his decision to not coauthor it. Anyway, being the only thing left for me to do, I thank him for his great contributions.

2. Derivations and connections

Recall the notation for X, $\mathcal{O}_B$, $\mathcal{O}_X$, $\mathcal{F}$ and D.

CONSTRUCTION 1. (*Tensor operations*) We denote by

$$\mathcal{T}(\mathcal{F}) := \prod_{n=0}^{\infty} \mathcal{T}^n(\mathcal{F}), \quad \mathcal{S}(\mathcal{F}) := \prod_{n=0}^{\infty} \mathcal{S}^n(\mathcal{F}), \quad \mathcal{A}(\mathcal{F}) := \prod_{n=0}^{\infty} \mathcal{A}^n(\mathcal{F})$$

the *formal tensor*, *symmetric* and *exterior* graded sheaf of $\mathcal{O}_X$-algebras of $\mathcal{F}$, respectively. (Note that we take the direct product and not the direct sum.)

Set $\mathcal{R}^0(\mathcal{F}) := \mathcal{O}_X$. Also, set $\mathcal{R}^n(\mathcal{F}) := \mathcal{S}^{n-1}(\mathcal{F}) \otimes \mathcal{F}$ for each integer $n \geq 1$, and

$$\mathcal{R}(\mathcal{F}) := \prod_{n=0}^{\infty} \mathcal{R}^n(\mathcal{F}).$$

Then $\mathcal{R}(\mathcal{F})$ is a graded $\mathcal{O}_X$-algebra quotient of $\mathcal{T}(\mathcal{F})$, in a natural way.

As usual, we let $\mathcal{T}_+(\mathcal{F})$, $\mathcal{S}_+(\mathcal{F})$, $\mathcal{A}_+(\mathcal{F})$ and $\mathcal{R}_+(\mathcal{F})$ denote the ideals generated by formal sums with zero constant terms in each of the indicated $\mathcal{O}_X$-algebras.

We view $\mathcal{T}(\mathcal{F})$ as a sheaf of algebras, with the (noncommutative) product induced by tensor product. So, given local sections $m_1, \dots, m_n$ of $\mathcal{F}$, we let $m_1 \cdots m_n$ denote their product in $\mathcal{T}^n(\mathcal{F})$. Also, we view $\mathcal{S}(\mathcal{F})$, $\mathcal{A}(\mathcal{F})$ and $\mathcal{R}(\mathcal{F})$ as sheaves of $\mathcal{T}(\mathcal{F})$-algebras, and $\mathcal{S}(\mathcal{F})$ as a sheaf of $\mathcal{R}(\mathcal{F})$-algebras, under the natural quotient maps. So, given a local section ω of $\mathcal{T}^n(\mathcal{F})$ (resp. $\mathcal{R}^n(\mathcal{F})$), we will use the same symbol ω to denote its image in $\mathcal{S}^n(\mathcal{F})$, $\mathcal{A}^n(\mathcal{F})$ or $\mathcal{R}^n(\mathcal{F})$ (resp. $\mathcal{S}^n(\mathcal{F})$). These simplifications should not lead to confusion, and will clean the notation enormously.

Define the *switch operator* $\sigma \colon \mathcal{R}_+(\mathcal{F}) \to \mathcal{R}_+(\mathcal{F})$ as the homogeneous $\mathcal{O}_X$-linear map of degree 0 given by $\sigma|_{\mathcal{F}} := 0$, and on each $\mathcal{R}^n(\mathcal{F})$, for $n \geq 2$, by the formula:

$$\sigma(m_1 \cdots m_n) = \sum_{i=1}^{n-1} m_n m_{n-1} \cdots m_{n-i+2} m_{n-i+1} m_1 m_2 \cdots m_{n-i-1} m_{n-i}$$

for all local sections $m_1, \dots, m_n$ of $\mathcal{F}$. The reader may check that σ is actually well-defined on $\mathcal{R}_+(\mathcal{F})$, and not only on $\mathcal{T}_+(\mathcal{F})$.

Let $\sigma^\star := 1 + \sigma$. Notice that $\sigma^\star$ factors through $\mathcal{S}_+(\mathcal{F})$. For each integer $n \geq 1$, let $\mathcal{K}^n(\mathcal{F}) := \sigma^\star(\mathcal{R}^n(\mathcal{F}))$. Then $\mathcal{K}^n(\mathcal{F})$ is also the kernel of $n - \sigma^\star$. In particular, $\mathcal{K}^2(\mathcal{F})$ is the kernel of the surjection $\mathcal{T}^2(\mathcal{F}) \to \mathcal{A}^2(\mathcal{F})$. Indeed, that $\mathcal{K}^n(\mathcal{F})$ is in the kernel of $n - \sigma^\star$ follows from the equality

$$\sigma^\star \sigma^\star|_{\mathcal{R}^n(\mathcal{F})} = n\sigma^\star|_{\mathcal{R}^n(\mathcal{F})},$$

a fact checked locally. And if ω is a local section of $\mathcal{R}^n(\mathcal{F})$ such that $(n - \sigma^\star)(\omega) = 0$, then $\omega = \sigma^\star((1/n)\omega)$, and thus ω is a local section of $\mathcal{K}^n(\mathcal{F})$.

Put

$$\mathcal{K}_+(\mathcal{F}) := \prod_{n=1}^{\infty} \mathcal{K}^n(\mathcal{F}).$$

DEFINITION 1. *A* Hasse derivation of $\mathcal{F}$ *is a map of $\mathcal{O}_B$-algebras*

$$h = (h_0, h_1, \dots) \colon \mathcal{O}_X \longrightarrow \prod_{i=0}^{\infty} \mathcal{S}^i(\mathcal{F})$$

with $h_0 = id_{\mathcal{O}_X}$. We say that h extends D *if $h_1 = D$.*

If $h = (h_0, h_1, \dots)$ is a Hasse derivation of $\mathcal{F}$, then $h_1 \colon \mathcal{O}_X \to \mathcal{F}$ is an $\mathcal{O}_B$-derivation of $\mathcal{F}$. Conversely, given D, we may ask when there is a Hasse derivation $h = (h_0, h_1, \dots)$ of $\mathcal{F}$ extending D. We will see in Construction 3 that such h exists when there is a D-connection.

DEFINITION 2. *A D-*connection *is a map of $\mathcal{O}_B$-modules*

$$\gamma \colon \mathcal{F} \to \mathcal{T}^2(\mathcal{F})$$

satisfying

$$\gamma(am) = D(a)m + a\gamma(m) \tag{1}$$

for each local sections a of $\mathcal{O}_X$ and m of $\mathcal{F}$.

EXAMPLE 1. There may not exist a D-connection. For instance, assume X is the union of two transversal lines in the plane, or $X = \mathrm{Spec}(\mathbf{C}[x,y]/(xy))$. Assume $\mathcal{F} = \Omega^1_X$, the sheaf of differentials, and $D \colon \mathcal{O}_X \to \Omega^1_X$ is the universal $\mathbf{C}$-derivation. The sheaf Ω^1_X is generated by $D(x)$ and $D(y)$, and the sheaf of relations is generated by the single relation $yD(x) + xD(y) = 0$. In particular, $D(x)$ and $D(y)$ are $\mathbf{C}$-linearly independent at the node. Suppose there were a D-connection $\gamma \colon \Omega^1_X \to \mathcal{T}^2(\Omega^1_X)$. Then

$$0 = \gamma(yD(x) + xD(y)) = D(y)D(x) + D(x)D(y) + y\gamma(D(x)) + x\gamma(D(y)).$$

However, $D(x)D(x)$, $D(x)D(y)$, $D(y)D(x)$ and $D(y)D(y)$ are linearly independent sections of $\mathcal{T}^2(\Omega^1_X)$ at the node. Hence the above relation is not possible.

When a D-connection γ exists, it is not unique, since for every $\mathcal{O}_X$-linear map $\lambda \colon \mathcal{F} \to \mathcal{T}^2(\mathcal{F})$, the sum $\gamma + \lambda$ is a D-connection. However, these are all the D-connections.

A D-connection allows us to iterate D to a Hasse derivation, as we will explain in Construction 3. First, we will see how to extend a connection.

CONSTRUCTION 2. (*Extending connections*) Let $\gamma \colon \mathcal{F} \to \mathcal{T}^2(\mathcal{F})$ be a D-connection. Define a homogeneous map of degree 1 of $\mathcal{O}_B$-modules,

$$\nabla \colon \mathcal{R}_+(\mathcal{F}) \to \mathcal{R}_+(\mathcal{F}), \tag{1}$$

given on each graded part $\mathcal{R}^n(\mathcal{F})$ by

$$\nabla(m_1\cdots m_n) := \sum_{i=1}^{n} m_1\cdots m_{i-1}\gamma(m_i)m_{i+1}\cdots m_n \tag{2}$$

for each local sections $m_1,\ldots,m_n$ of $\mathcal{F}$.

At first, it seems ∇ would be a well-defined map from $\mathcal{T}_+(\mathcal{F})$ to $\mathcal{T}_+(\mathcal{F})$. This would indeed be true, were γ a map of $\mathcal{O}_X$-modules. But γ is not! To check that ∇, as given above, is well-defined, we need to check the following three properties for all local sections $m_1,\ldots,m_i,m_i',\ldots,m_n'$ of $\mathcal{F}$ and a of $\mathcal{O}_X$, and each permutation τ of $\{1,\ldots,n-1\}$:

1. For each $i = 1,\ldots,n$,
$$\nabla(m_1\cdots(m_i+m_i')\cdots m_n) = \nabla(m_1\cdots m_i\cdots m_n)+\nabla(m_1\cdots m_i'\cdots m_n).$$
2. For each $i,j = 1,\ldots,n$,
$$\nabla(m_1\cdots(am_i)\cdots m_n) = \nabla(m_1\cdots(am_j)\cdots m_n).$$
3. $\nabla(m_1\cdots m_{n-1}m_n) = \nabla(m_{\tau(1)}\cdots m_{\tau(n-1)}m_n)$.

The first (multilinearity) and third (symmetry) properties are left for the reader to check. The second property is the key to why ∇ must take values in $\mathcal{R}_+(\mathcal{F})$, so let us check it: from the definition of ∇, and using $\gamma(am_i) = D(a)m_i + a\gamma(m_i)$, we get

$$\nabla(m_1\cdots(am_i)\cdots m_n) = m_1\cdots m_{i-1}D(a)m_i\cdots m_n + a\sum_{j=1}^{n} m_1\cdots\gamma(m_j)\cdots m_n.$$

So $\nabla(m_1\cdots(am_i)\cdots m_n)$ would depend on i, were ∇ to take values in $\mathcal{T}_+(\mathcal{F})$. But instead, ∇ takes values in $\mathcal{R}_+(\mathcal{F})$, and hence

$$m_1\cdots m_{i-1}D(a)m_i\cdots m_n = D(a)m_1\cdots m_{i-1}m_i\cdots m_n.$$

So the second property (scalar multiplication) is checked, and the three properties imply that ∇ is well-defined. Also, we proved the formula

$$\nabla(a\omega) = D(a)\omega + a\nabla(\omega)$$

for all local sections ω of $\mathcal{R}_+(\mathcal{F})$ and a of $\mathcal{O}_X$.

Now, for each integer $n \geq 1$, put

$$T_n := \frac{1}{n!}\nabla^n|_{\mathcal{F}} : \mathcal{F} \to \mathcal{R}^{n+1}(\mathcal{F}).$$

Also, set $T_0 := \mathrm{id}_{\mathcal{F}}$. Then $T_n = (1/n)\nabla T_{n-1}$ for each $n \geq 1$. Furthermore, for each integer $i \geq 1$, and each local sections a of $\mathcal{O}_X$ and m of $\mathcal{F}$,

$$T_i(am) = aT_i(m) + \sum_{j=1}^{i}\frac{1}{j}T_{j-1}D(a)T_{i-j}(m). \tag{3}$$

Indeed, Formula (3) holds for $i=1$, because $T_1=\gamma$ and γ is a D-connection. And if, by induction, Formula (3) holds for a certain $i\geq 1$, then

$$\begin{aligned}
T_{i+1}(am) =&\frac{1}{(i+1)}\nabla T_i(am)\\
=&\frac{1}{(i+1)}\nabla\Big(aT_i(m)+\sum_{j=1}^{i}\frac{1}{j}T_{j-1}D(a)T_{i-j}(m)\Big)\\
=&\frac{1}{(i+1)}\Big(a\nabla T_i(m)+D(a)T_i(m)\Big)\\
&+\frac{1}{(i+1)}\sum_{j=1}^{i}\frac{1}{j}\Big(\nabla T_{j-1}D(a)T_{i-j}(m)+T_{j-1}D(a)\nabla T_{i-j}(m)\Big)\\
=&aT_{i+1}(m)+\frac{1}{(i+1)}D(a)T_i(m)\\
&+\frac{1}{(i+1)}\sum_{j=1}^{i}\Big(T_jD(a)T_{i-j}(m)+\frac{(i+1-j)}{j}T_{j-1}D(a)T_{i+1-j}(m)\Big)\\
=&aT_{i+1}(m)+\frac{1}{(i+1)}D(a)T_i(m)\\
&+\sum_{j=2}^{i}\Big(\frac{1}{j}T_{j-1}D(a)T_{i+1-j}(m)\Big)+\frac{1}{(i+1)}T_iD(a)m+\frac{i}{(i+1)}D(a)T_i(m)\\
=&aT_{i+1}(m)+\sum_{j=1}^{i+1}\frac{1}{j}T_{j-1}D(a)T_{i+1-j}(m).
\end{aligned}$$

The maps T_n form an extended D-connection, according to the definition below.

DEFINITION 3. *Let n be a positive integer or $n:=\infty$. A map of O_B-modules*

$$T=(T_0,T_1,T_2,\dots)\colon \mathcal{F}\longrightarrow\prod_{i=0}^{n}\mathcal{R}^{i+1}(\mathcal{F})$$

is called an extended D-connection *if $T_0=id_{\mathcal{F}}$ and Formula* (3) *holds for each $i\geq 1$ and all local sections a of O_X and m of $\mathcal{F}$.*

If nothing is noted otherwise, extended D-connections are assumed *full*, that is, with $n:=\infty$.

CONSTRUCTION 3. (*Hasse derivations extending D*) Let $T\colon \mathcal{F}\to\mathcal{R}_+(\mathcal{F})$ be an extended D-connection. Notice that the map $T_1\colon\mathcal{F}\to\mathcal{F}\otimes\mathcal{F}$ is a D-connection. However, T need not arise from T_1 as in Construction 2.

Put $h_0:=\mathrm{id}_{O_X}$, and for each integer $i\geq 1$ let $h_i\colon O_X\to\mathcal{S}^i(\mathcal{F})$ be the O_B-linear map given by $h_i(a):=(1/i)T_{i-1}D(a)$ for each local section a of O_X. Then

$$h:=(h_0,h_1,h_2,\dots)\colon O_X\longrightarrow\prod_{i=0}^{\infty}\mathcal{S}^i(\mathcal{F})$$

is a Hasse derivation of $\mathcal{F}$ extending D. Indeed, clearly $h_1 = D$. Now, if a and b are local sections of $\mathcal{O}_X$, and $i \geq 1$, then

$$\begin{aligned}
h_i(ab) =& \frac{1}{i} T_{i-1}\Big(aD(b) + bD(a)\Big) \\
=& \frac{1}{i}\Big(aT_{i-1}D(b) + \sum_{j=1}^{i-1} \frac{1}{j} T_{j-1}D(a)T_{i-1-j}D(b)\Big) \\
+& \frac{1}{i}\Big(bT_{i-1}D(a) + \sum_{j=1}^{i-1} \frac{1}{j} T_{j-1}D(b)T_{i-1-j}D(a)\Big) \\
=& ah_i(b) + bh_i(a) \\
+& \frac{1}{i}\sum_{j=1}^{i-1}\Big(\frac{1}{j}T_{j-1}D(a)T_{i-1-j}D(b) + \frac{1}{i-j}T_{i-1-j}D(b)T_{j-1}D(a)\Big) \\
=& ah_i(b) + bh_i(a) + \sum_{j=1}^{i-1}\Big(\frac{1}{j(i-j)}T_{j-1}D(a)T_{i-1-j}D(b)\Big) \\
=& \sum_{j=0}^{i} h_j(a)h_{i-j}(b),
\end{aligned}$$

where in the fourth equality we used that the computation is done in $\mathcal{S}^i(\mathcal{F})$.

DEFINITION 4. *Let h be a Hasse derivation extending D. We say that h is* iterated *if there is an extended D-connection T such that $h_i = (1/i)T_{i-1}D$ for each* $i \geq 1$.

3. Flat connections and integrable derivations

Recall the notation for X, $\mathcal{O}_B$, $\mathcal{O}_X$, $\mathcal{F}$ and D.

DEFINITION 5. *A D-connection $\gamma\colon \mathcal{F} \to \mathcal{T}^2(\mathcal{F})$ is called* flat *if $\gamma D(\mathcal{O}_X) \subseteq \mathcal{K}^2(\mathcal{F})$. We say that D is* integrable *if there exists a flat D-connection.*

EXAMPLE 2. Assume that $\mathcal{F}$ is the free sheaf of $\mathcal{O}_X$-modules with basis $e_1, \ldots, e_n$. Then $D = D_1e_1 + \cdots + D_ne_n$, where the D_i are $\mathcal{O}_B$-derivations of $\mathcal{O}_X$. Conversely, a n-tuple $(D_1, \ldots, D_n)$ of $\mathcal{O}_B$-derivations of $\mathcal{O}_X$ defines an $\mathcal{O}_B$-derivation of $\mathcal{F}$.

There is a natural D-connection $\gamma\colon \mathcal{F} \to \mathcal{T}^2(\mathcal{F})$, satisfying

$$\gamma(\sum_{i=1}^{n} a_ie_i) = \sum_{i=1}^{n}\sum_{j=1}^{n} D_j(a_i)e_je_i$$

for all local sections $a_1, \ldots, a_n$ of $\mathcal{O}_X$. Any other D-connection is of the form $\gamma + \nu$, where $\nu\colon \mathcal{F} \to \mathcal{T}^2(\mathcal{F})$ is a map of $\mathcal{O}_X$-modules. The map ν is defined by global sections $c_\ell^{j,i}$ of $\mathcal{O}_X$, for $1 \leq i, j, \ell \leq n$, satisfying

$$\nu(e_\ell) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_\ell^{j,i} e_j e_i.$$

To say that $(\gamma + \nu)D(a) = 0$ in $\mathcal{A}^2(\mathcal{F})$ for a local section a of $\mathcal{O}_X$ is to say that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} D_j D_i(a) e_j e_i + \sum_{\ell=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} c_\ell^{j,i} D_\ell(a) e_j e_i = 0$$

in $\mathcal{A}^2(\mathcal{F})$ or, equivalently,

$$D_j D_i(a) - D_i D_j(a) = \sum_{\ell=1}^{n} (c_\ell^{i,j} - c_\ell^{j,i}) D_\ell(a)$$

for all distinct i and j. In other words, D is integrable if and only if the collection $\{D_1, \ldots, D_n\}$ is *involutive*, i.e., if and only if there are sections $b_\ell^{j,i}$ of $\mathcal{O}_X$ such that

$$[D_j, D_i] = \sum_{\ell=1}^{n} b_\ell^{j,i} D_\ell$$

for all distinct i and j.

Let T be the extended D-connection of Construction 2, derived from γ. From the definition of γ we have $T_q(e_i) = 0$ for each integer $q > 0$ and each $i = 1, \ldots, n$. Suppose $\gamma D = 0$ in $\mathcal{A}^2(\mathcal{F})$, or in other words $[D_j, D_i] = 0$ for all i and j. Then the Hasse derivation h associated to T satisfies

(1) $$h_q(a) = \sum_{j_1=1}^{n} \cdots \sum_{j_q=1}^{n} \frac{D_{j_1} \cdots D_{j_q}(a)}{q!} e_{j_1} \cdots e_{j_q}$$

for each integer $q > 0$ and each local section a of $\mathcal{F}$.

EXAMPLE 3. It is not necessary that $\mathcal{F}$ be locally free for a flat D-connection to exist. For instance, assume $X = \mathrm{Spec}(\mathbf{C}[t^3, t^4, t^5])$. Viewed as a sheaf of regular meromorphic differentials, Rosenlicht-style, the dualizing sheaf ω_X is generated by dt/t^2 and dt/t^3. Assume $\mathcal{F} = \omega_X$. Let $\eta_1 := dt/t^2$ and $\eta_2 := dt/t^3$. The following relations generate all relations η_1 and η_2 satisfy with coefficients in $\mathcal{O}_X$:

(1) $$t^3\eta_1 = t^4\eta_2, \quad t^4\eta_1 = t^5\eta_2, \quad t^5\eta_1 = t^6\eta_2.$$

Assume $D\colon \mathcal{O}_X \to \omega_X$ is the composition of the universal derivation with the canonical map $\Omega_X^1 \to \omega_X$. Then D satisfies

$$D(t^3) = 3t^4\eta_1 = 3t^5\eta_2, \quad D(t^4) = 4t^5\eta_1 = 4t^6\eta_2, \quad D(t^5) = 5t^6\eta_1 = 5t^7\eta_2.$$

So $D(a)\eta_i = 0$ in $\mathcal{A}^2(\omega_X)$ for each local section a of $\mathcal{O}_X$ and each $i = 1, 2$.

Define $\gamma\colon \omega_X \to \mathcal{T}^2(\omega_X)$ by letting

$$\gamma(a\eta_1 + b\eta_2) = D(a)\eta_1 + D(b)\eta_2 + 4t^3 a\eta_2\eta_2 + 3b\eta_1\eta_1$$

for all local sections a and b of $\mathcal{O}_X$. To check that γ is well defined we need only check that the values of γ on both sides of the three relations (1) agree. This is the case; for instance,

$$\gamma(t^3\eta_1) = 3t^4\eta_1\eta_1 + 4t^6\eta_2\eta_2 = 7t^4\eta_1\eta_1 = 4t^5\eta_1\eta_2 + 3t^4\eta_1\eta_1 = \gamma(t^4\eta_2).$$

Now, since $D(a)\eta_1 = D(b)\eta_2 = 0$ in $\mathcal{A}^2(\omega_X)$, we have that $\gamma = 0$ in $\mathcal{A}^2(\omega_X)$. So γ is a flat D-connection, and hence D is integrable.

DEFINITION 6. *An extended D-connection* $T\colon \mathcal{F} \to \mathcal{R}_+(\mathcal{F})$ *is said to be* flat *if* $TD(\mathcal{O}_X) \subseteq \mathcal{K}_+(\mathcal{F})$.

We will see in Theorem 5 that flat extended D-connections exist, when D is integrable. Also, by Proposition 1 below, any two of them are “comparable”. First, a piece of notation.

CONSTRUCTION 4. (*Generating maps*) Let n be a positive integer or $n := \infty$. Let

$$\lambda = (\lambda_0, \lambda_1, \dots)\colon \mathcal{F} \longrightarrow \prod_{i=0}^{n} \mathcal{R}^{i+1}(\mathcal{F})$$

be a map of $\mathcal{O}_X$-modules. For each integer $p > 0$ and each sequence $i_1, \dots, i_p$ of non-negative indices at most equal to n, let $q := i_1 + \cdots + i_p$ and define

$$(\lambda_{i_1} \cdots \lambda_{i_p})\colon \mathcal{T}^p(\mathcal{F}) \to \mathcal{R}^{p+q}(\mathcal{F})$$

to be the map of $\mathcal{O}_X$-modules satisfying

$$(\lambda_{i_1} \cdots \lambda_{i_p})(m_1 \cdots m_p) := \lambda_{i_1}(m_1) \cdots \lambda_{i_p}(m_p)$$

for all local sections $m_1, \dots, m_p$ of $\mathcal{F}$.

The maps $(\lambda_{i_1} \cdots \lambda_{i_p})$ are not defined on $\mathcal{R}^p(\mathcal{F})$, but the sum

$$s_q(\lambda) := \sum_{i_1+\cdots+i_p=q} (i_p + 1)\big(\lambda_{i_1} \cdots \lambda_{i_p}\big)\colon \mathcal{R}^p(\mathcal{F}) \longrightarrow \mathcal{R}^{p+q}(\mathcal{F})$$

is, for all integers $p > 0$ and each integer q with $0 \le q \le n$. Analogously, the sum

$$\widetilde{s}_q(\lambda) := \sum_{i_1+\cdots+i_p=q} \big(\lambda_{i_1} \cdots \lambda_{i_p}\big)\colon \mathcal{S}^p(\mathcal{F}) \longrightarrow \mathcal{S}^{p+q}(\mathcal{F})$$

is well-defined, for all integers $p > 0$ and each integer q with $0 \le q \le n$.

Notice that, for each local section ω of $\mathcal{K}^p(\mathcal{F})$,

$$s_q(\lambda)(\omega) = \frac{p+q}{p}\widetilde{s}_q(\lambda)(\omega) \text{ in } \mathcal{S}^{p+q}(\mathcal{F}). \tag{1}$$

Indeed, locally,

$$\omega = \sum_{i=1}^{p} m_1 \cdots \widehat{m_i} \cdots m_p m_i$$

for local sections $m_1, \dots, m_p$ of $\mathcal{F}$. Thus, in $\mathcal{S}^{p+q}(\mathcal{F})$,

$$\begin{aligned}
s_q(\lambda)(\omega) &= \sum_{i=1}^{p} \sum_{j_1+\cdots+j_p=q} (j_p+1) \prod_{s=1}^{i-1} \lambda_{j_s}(m_s) \prod_{s=i}^{p-1} \lambda_{j_s}(m_{s+1}) \lambda_{j_p}(m_i) \\
&= \sum_{j_1+\cdots+j_p=q} \sum_{i=1}^{p} (j_i+1) \prod_{s=1}^{p} \lambda_{j_s}(m_s) \\
&= (p+q) \sum_{j_1+\cdots+j_p=q} \prod_{s=1}^{p} \lambda_{j_s}(m_s) \\
&= \frac{p+q}{p} \sum_{j_1+\cdots+j_p=q} \sum_{i=1}^{p} \prod_{s=1}^{i-1} \lambda_{j_s}(m_s) \prod_{s=i}^{p-1} \lambda_{j_s}(m_{s+1}) \lambda_{j_p}(m_i) \\
&= \frac{p+q}{p} \sum_{j_1+\cdots+j_p=q} (\lambda_{j_1} \cdots \lambda_{j_p})(\omega) \\
&= \frac{p+q}{p} \widetilde{s}_q(\lambda)(\omega).
\end{aligned}$$

PROPOSITION 1. *Let n be a positive integer. Let*

$$T = (T_0, T_1, \dots, T_n) \colon \mathcal{F} \longrightarrow \prod_{i=0}^{n} \mathcal{R}^{i+1}(\mathcal{F}),$$

$$S = (S_0, S_1, \dots, S_n) \colon \mathcal{F} \longrightarrow \prod_{i=0}^{n} \mathcal{R}^{i+1}(\mathcal{F})$$

be two O_B-linear maps. Assume that $T_i D(O_X) \subseteq \mathcal{K}^{i+1}(\mathcal{F})$ for each $i = 0, \dots, n-1$. Then any two of the following three statements imply the third:

1. *The map S is an extended D-connection.*
2. *The map T is an extended D-connection.*
3. *There is a (unique) map of O_X-modules*

$$\lambda = (\lambda_0, \dots, \lambda_n) \colon \mathcal{F} \longrightarrow \prod_{i=0}^{n} \mathcal{R}^{i+1}(\mathcal{F})$$

such that $\lambda_0 = \mathrm{id}_{\mathcal{F}}$ and

$$S_i = \sum_{\ell=0}^{i} s_{i-\ell}(\lambda) T_\ell \tag{1}$$

for each $i = 0, 1, \dots, n$.

Proof. We will argue by induction. For $n = 1$, the proposition simply says that, given a D-connection T_1, a map of $\mathcal{O}_B$-modules $S_1\colon \mathcal{F} \to \mathcal{R}^2(\mathcal{F})$ is a D-connection if and only if $S_1 - T_1$ is $\mathcal{O}_X$-linear, a fact already observed.

Suppose now that $n \geq 2$, and that the statement of the proposition is known for $n-1$ in place of n. So we may assume that $(S_0,\dots,S_{n-1})$ and $(T_0,\dots,T_{n-1})$ are extended D-connections, and that there exists a map of $\mathcal{O}_X$-modules

$$\lambda = (\lambda_0,\dots,\lambda_{n-1})\colon \mathcal{F} \longrightarrow \prod_{i=0}^{n-1} \mathcal{R}^{i+1}(\mathcal{F})$$

such that $\lambda_0 = \mathrm{id}_{\mathcal{F}}$ and Equations (1) hold for each $i < n$.

Define

$$R_n := \sum_{\ell=1}^{n-1} s_{n-\ell}(\lambda) T_\ell.$$

We need only show that, for each local sections a of $\mathcal{O}_X$ and m of $\mathcal{F}$,

$$R_n(am) - aR_n(m) + \sum_{z=0}^{n-1} \frac{T_{n-z-1}D(a)}{n-z} T_z(m) = \sum_{z=0}^{n-1} \frac{S_{n-z-1}D(a)}{n-z} S_z(m). \tag{2}$$

Indeed, if S and T are D-connections, Formula (2) implies that $S_n - R_n - T_n$ is $\mathcal{O}_X$-linear. So, setting $\lambda_n := (1/(n+1))(S_n - R_n - T_n)$, Equation (1) holds for $i = n$ as well. Conversely, if there is an $\mathcal{O}_X$-linear map $\lambda_n\colon \mathcal{F} \to \mathcal{R}^{n+1}(\mathcal{F})$ such that Equation (1) holds for $i = n$, then $(n+1)\lambda_n + R_n = S_n - T_n$. So, from Formula (2) we see that S is a D-connection if and only if T is.

Now, on the one hand, since $(T_0,\dots,T_{n-1})$ is an extended D-connection,

$$\begin{aligned} R_n(am) - aR_n(m) &= \sum_{\ell=1}^{n-1} \sum_{j_0+\cdots+j_\ell = n-\ell} (j_\ell+1)\big(\lambda_{j_0}\cdots\lambda_{j_\ell}\big)\Big(T_\ell(am) - aT_\ell(m)\Big) \\ &= \sum_{\ell=1}^{n-1} \sum_{j_0+\cdots+j_\ell = n-\ell} \sum_{p=0}^{\ell-1} \frac{j_\ell+1}{\ell-p}\big(\lambda_{j_0}\cdots\lambda_{j_\ell}\big)\Big(T_{\ell-1-p}D(a)T_p(m)\Big). \end{aligned}$$

Thus the left-hand side of Formula (2) is equal to

$$\sum_{\ell=1}^{n} \sum_{j_0+\cdots+j_\ell = n-\ell} \sum_{p=0}^{\ell-1} \frac{j_\ell+1}{\ell-p}\big(\lambda_{j_0}\cdots\lambda_{j_\ell}\big)\Big(T_{\ell-1-p}D(a)T_p(m)\Big). \tag{3}$$

On the other hand, using Equations (1) for $i < n$, the right-hand side of (2) becomes

$$\sum_{z=0}^{n-1} \frac{1}{n-z}\left(\sum_{k=0}^{n-z-1} \sum_{j_0+\cdots+j_k = n-z-1-k} (j_k+1)\big(\lambda_{j_0}\cdots\lambda_{j_k}\big)T_k D(a)\right)\left(\sum_{p=0}^{z} \omega_{z-p}\right),$$

where

$$\omega_\ell := \sum_{j'_0+\cdots+j'_p = \ell} (j'_p+1)\big(\lambda_{j'_0}\cdots\lambda_{j'_p}\big)T_p(m)$$

for each $\ell = 0,\dots,n-1$. Now, for each $z = 0,\dots,n-1$ and $k = 0,\dots,n-z-1$, using that $T_kD(a)$ is a local section of $\mathcal{K}^{k+1}(\mathcal{F})$, Formula (1) yields the following equation in $\mathcal{S}^{n-z}(\mathcal{F})$:

$$\sum (j_k+1)(\lambda_{j_0}\cdots\lambda_{j_k})T_kD(a) = \sum \frac{n-z}{k+1}(\lambda_{j_0}\cdots\lambda_{j_k})T_kD(a),$$

where the sum on both sides runs over the $(k+1)$-tuples $(j_0,\dots,j_k)$ such that $j_0+\cdots+j_k = n-z-1-k$. Thus, introducing $\ell := k+p+1$, the right-hand side of (2) becomes

$$\sum_{\ell=1}^{n}\sum_{p=0}^{\ell-1}\frac{1}{\ell-p}\sum_{z=p}^{n-\ell+p}\sum_{j_0+\cdots+j_{\ell-p-1}=n-z-\ell+p}(\lambda_{j_0}\cdots\lambda_{j_{\ell-p-1}})T_{\ell-p-1}D(a)\omega_{z-p},$$

whence, introducing $u := z-p$, equal to

$$\sum_{\ell=1}^{n}\sum_{p=0}^{\ell-1}\frac{1}{\ell-p}\sum_{u=0}^{n-\ell}\sum_{j_0+\cdots+j_{\ell-p-1}=n-\ell-u}(\lambda_{j_0}\cdots\lambda_{j_{\ell-p-1}})T_{\ell-p-1}D(a)\omega_u,$$

which is equal to (3). □

4. Jets

Recall the notation for X, $\mathcal{O}_B$, $\mathcal{O}_X$, $\mathcal{F}$ and D.

THEOREM 5. *If D is integrable, then there exists a flat, extended D-connection.*

Proof. Since D is integrable, there is a flat D-connection $T_1\colon \mathcal{F} \to \mathcal{T}^2(\mathcal{F})$. Set $T_0 := \mathrm{id}_{\mathcal{F}}$. Suppose, by induction, that for an integer $n \geq 2$ we have constructed an extended D-connection

$$T = (T_0, T_1, \dots, T_{n-1})\colon \mathcal{F} \longrightarrow \prod_{i=0}^{n-1}\mathcal{R}^{i+1}(\mathcal{F}).$$

We will also suppose the maps T_i satisfy one additional property, Equations (2), after we make a definition.

For each $j = 0,\dots,n-1$, define a map of $\mathcal{O}_B$-modules $T'_j\colon \mathcal{R}^2(\mathcal{F}) \to \mathcal{R}^{j+2}(\mathcal{F})$ by letting

$$T'_j(m_1m_2) := \sum_{i=0}^{j} T_i(m_1)T_{j-i}(m_2)$$

for all local sections m_1 and m_2 of $\mathcal{F}$. To check that T'_j is well defined, we need only check that $\sum_i T_i(am_1)T_{j-i}(m_2) = \sum_i T_i(m_1)T_{j-i}(am_2)$ for each local section a of $\mathcal{O}_X$. In fact, using (3), we see that both sides are equal to

$$\sum_{i=0}^{j} aT_i(m_1)T_{j-i}(m_2) + \sum_{\ell=1}^{j}\sum_{i=0}^{j-\ell}\frac{1}{\ell}T_{\ell-1}D(a)T_i(m_1)T_{j-\ell-i}(m_2).$$

Furthermore, we see from this computation that

$$T'_j(a\omega) = aT'_j(\omega) + \sum_{i=1}^{j} \frac{1}{j+1-i} T_{j-i} D(a) T'_{i-1}(\omega)$$

for all local sections ω of $\mathcal{R}^2(\mathcal{F})$ and a of $\mathcal{O}_X$.

Also, notice that $\sigma^\star T'_{i-1}(1-\sigma) = 0$. Indeed, for local sections m_1 and m_2 of $\mathcal{F}$,

$$\begin{aligned}
\sigma^\star T'_{i-1}(1-\sigma)(m_1m_2) =& \sigma^\star T'_{i-1}(m_1m_2 - m_2m_1) \\
=& \sigma^\star \left(\sum_{j=0}^{i-1} \Big(T_j(m_1) T_{i-1-j}(m_2) - T_{i-1-j}(m_2) T_j(m_1) \Big) \right) \\
=& \sum_{j=0}^{i-1} \Big(\sigma^\star\big(T_j(m_1)T_{i-1-j}(m_2)\big) - \sigma^\star\big(T_{i-1-j}(m_2)T_j(m_1)\big) \Big),
\end{aligned}$$

which is zero because $\sigma^\star(\omega_1\omega_2) = \sigma^\star(\omega_2\omega_1)$ for all local sections ω_1 and ω_2 of $\mathcal{R}_+(\mathcal{F})$. Then

$$T'_{i-1}(1-\sigma) = \frac{i-\sigma}{i+1} T'_{i-1}(1-\sigma). \tag{1}$$

Now, suppose that

$$(i-\sigma)\big((i+1)T_i - T'_{i-1}(1-\sigma)T_1\big) = 0 \tag{2}$$

for each $i = 1, \ldots, n-1$. (Notice that Equation (2) holds automatically for $i = 1$, because $(1-\sigma)\sigma^\star(\mathcal{R}^2(\mathcal{F})) = 0$.) Also, from Equation (2), and the flatness of T_1, we get $T_i D(\mathcal{O}_X) \subseteq \mathcal{K}^{i+1}(\mathcal{F})$ for each $i = 1, \ldots, n-1$.

Let

$$T := \frac{1}{n+1} T'_{n-1}(1-\sigma) T_1.$$

Then

$$(n-\sigma)\Big(T(am) - aT(m) - \sum_{i=1}^{n} \frac{1}{i} T_{i-1} D(a) T_{n-i}(m)\Big) = 0 \tag{3}$$

for all local sections m of $\mathcal{F}$ and a of $\mathcal{O}_X$. Indeed,

$$
\begin{aligned}
(n-\sigma)T(am) =& \frac{n-\sigma}{n+1} T'_{n-1}(1-\sigma)T_1(am) \\
=& \frac{n-\sigma}{n+1}\Big(T'_{n-1}\Big(D(a)m - mD(a) + a(1-\sigma)T_1(m)\Big)\Big) \\
=& \frac{n-\sigma}{n+1}\Big(T_{n-1}D(a)m + \sum_{i=1}^{n-1} T_{n-1-i}D(a)T_i(m)\Big) \\
& - \frac{n-\sigma}{n+1}\Big(mT_{n-1}D(a) + \sum_{i=1}^{n-1} T_i(m)T_{n-1-i}D(a)\Big) \\
& + \frac{n-\sigma}{n+1}\Big(aT'_{n-1}(1-\sigma)T_1(m)\Big) \\
& + \frac{n-\sigma}{n+1}\Big(\sum_{i=1}^{n-1} T_{n-i-1}D(a)T'_{i-1}\frac{(1-\sigma)}{n-i}T_1(m)\Big).
\end{aligned}
$$

Now, first observe that, for each $i = 0, \dots, n-1$, we have $(n-i-\sigma^\star)T_{n-1-i}D(a) = 0$, since $T_{n-1-i}D(a)$ is a local section of $\mathcal{K}^{n-i}(\mathcal{F})$. Then

$$
\begin{aligned}
(n-\sigma)\Big(T_i(m)T_{n-1-i}D(a)\Big) =& (n-\sigma)\Big(T_i(m)\frac{\sigma^\star}{n-i}T_{n-1-i}D(a)\Big) \\
=& -\frac{n-\sigma}{n-i}\Big(T_{n-1-i}D(a)\sigma^\star T_i(m)\Big).
\end{aligned}
$$

Also, from (1) and (2),

$$
\begin{aligned}
T_{n-i-1}D(a)T'_{i-1}(1-\sigma)T_1(m) =& T_{n-i-1}D(a)\frac{i-\sigma}{i+1}T'_{i-1}(1-\sigma)T_1(m) \\
=& T_{n-i-1}D(a)(i-\sigma)T_i(m)
\end{aligned}
$$

for each $i = 1, \dots, n-1$. So

$$
\frac{n-\sigma}{n+1}\Big(T_{n-1}D(a)m - mT_{n-1}D(a)\Big) = \frac{n-\sigma}{n}\Big(T_{n-1}D(a)m\Big),
$$

and, for each $i = 1, \dots, n-1$,

$$
\frac{n-\sigma}{n+1}\Big(T_{n-1-i}D(a)T_i(m) - T_i(m)T_{n-1-i}D(a) + T_{n-i-1}D(a)T'_{i-1}\frac{1-\sigma}{n-i}T_1(m)\Big)
$$

is equal to

$$
\frac{n-\sigma}{n+1}\Big(T_{n-1-i}D(a)\big(1 + \frac{1+\sigma}{n-i} + \frac{i-\sigma}{n-i}\big)T_i(m)\Big),
$$

whence equal to

$$
\frac{n-\sigma}{n-i}\Big(T_{n-1-i}D(a)T_i(m)\Big).
$$

Applying these equalities in the above expression for $(n-\sigma)T(am)$ we get Equation (3).

We want to show that there exists a map of $\mathcal{O}_B$-modules $T_n\colon \mathcal{F}\to\mathcal{R}^{n+1}(\mathcal{F})$ such that (2) holds for $i=n$, and such that $(T_0,\dots,T_n)$ is an extended D-connection. First we claim that there exists a map of $\mathcal{O}_B$-modules $T_n\colon \mathcal{F}\to\mathcal{R}^{n+1}(\mathcal{F})$ such that $(T_0,\dots,T_n)$ is an extended D-connection. Indeed, from T_1 construct an extended D-connection $(S_0,\dots,S_n)$ by iteration, as described in Construction 2. By Proposition 1, there is a map of $\mathcal{O}_X$-modules

$$\lambda=(\lambda_0,\lambda_1,\dots,\lambda_{n-1})\colon \mathcal{F}\longrightarrow\prod_{i=0}^{n-1}\mathcal{R}^{i+1}(\mathcal{F})$$

such that $\lambda_0=\mathrm{id}_{\mathcal{F}}$ and such that Equations (1) hold for $i=0,\dots,n-1$. Now, just set $\lambda_n:=0$ in Equation (1) for $i=n$, and let it define T_n. Then, by Proposition 1, the map $(T_0,\dots,T_n)$ is an extended D-connection.

The above map T_n does not necessarily make (2) hold for $i=n$. So, rename it by U. At any rate, since $(T_0,\dots,T_{n-1},U)$ is an extended D-connection, it follows from Equation (3) that $(n-\sigma)(U-T)$ is $\mathcal{O}_X$-linear. Set

$$T_n:=U-\frac{(n-\sigma)}{n+1}(U-T).$$

Then T_n differs from U by an $\mathcal{O}_X$-linear map, and thus $(T_0,\dots,T_n)$ is an extended D-connection. Now,

$$(n-\sigma)T_n=(n-\sigma)U-\frac{(n-\sigma)^2}{n+1}(U-T).$$

Since

$$(n-\sigma)^2|_{\mathcal{R}^{n+1}(\mathcal{F})}=(n+1)(n-\sigma)|_{\mathcal{R}^{n+1}(\mathcal{F})},$$

we get $(n-\sigma)T_n=(n-\sigma)T$. So (2) holds for $i=n$.

The induction argument is complete, showing that there is an infinite extended D-connection

$$T=(T_0,T_1,\dots)\colon \mathcal{F}\longrightarrow\prod_{i=0}^{\infty}\mathcal{R}^{i+1}(\mathcal{F})$$

such that (2) holds for each $i\geq 1$, and thus $T_iD(\mathcal{O}_X)\subseteq\mathcal{K}^{i+1}(\mathcal{F})$ for each $i\geq 0$. $\square$

DEFINITION 7. *Two Hasse derivations h and h' of $\mathcal{F}$ are said to be* equivalent *if there is an $\mathcal{O}_X$-algebra automorphism ϕ of $\mathcal{S}(\mathcal{F})$ such that $\phi_0|_{\mathcal{F}}=id_{\mathcal{F}}$ and $h'=\phi h$. We say that h and h' are* canonically equivalent *when there is only one such automorphism.*

COROLLARY 1. *Let h and h' be iterated Hasse derivations of $\mathcal{F}$ extending D. If D is integrable, then h and h' are equivalent. Furthermore, if $\nu D(\mathcal{O}_X)\neq 0$ for every nonzero $\mathcal{O}_X$-linear map $\nu\colon \mathcal{F}\to\mathcal{S}(\mathcal{F})$, then h and h' are canonically equivalent.*

Proof. First, we prove the existence of an equivalence. By Theorem 5, there is a flat, extended D-connection $T = (T_0, T_1, \dots)$. We may suppose h arises from T. Let $S = (S_0, S_1, \dots)$ be an extended D-connection from which h' arises.

By Proposition 1, there is a map of $\mathcal{O}_X$-modules

$$\lambda = (\lambda_0, \lambda_1, \dots) \colon \mathcal{F} \longrightarrow \prod_{i=0}^{\infty} \mathcal{R}^{i+1}(\mathcal{F})$$

such that $\lambda_0 = \mathrm{id}_{\mathcal{F}}$ and

$$S_i = \sum_{\ell=0}^{i} s_{i-\ell}(\lambda) T_\ell \quad \text{for each } i \geq 0. \tag{1}$$

Let $\phi \colon \mathcal{S}(\mathcal{F}) \to \mathcal{S}(\mathcal{F})$ be the map of $\mathcal{O}_X$-algebras whose graded part ϕ_q of degree q satisfies $\phi_q|_{\mathcal{S}^p(\mathcal{F})} = \widetilde{s}_q(\lambda)|_{\mathcal{S}^p(\mathcal{F})}$ for all integers $p > 0$ if $q \geq 0$, and $\phi_q = 0$ if $q < 0$. Since $\lambda_0 = \mathrm{id}_{\mathcal{F}}$, the homogeneous degree-0 part ϕ_0 is the identity, and thus ϕ is an automorphism. We claim that $h' = \phi h$.

Indeed, clearly $h'_0 = (\phi h)_0$. Now, since T is flat, $T_\ell D(\mathcal{O}_X) \subseteq \mathcal{K}^{\ell+1}(\mathcal{F})$ for each $\ell \geq 0$. Thus, for each $i \geq 0$ and each local section a of $\mathcal{O}_X$, using Equations (1) and (1), the following equalities hold on $\mathcal{S}^{i+1}(\mathcal{F})$:

$$h'_{i+1}(a) = \frac{S_i D(a)}{i+1} = \sum_{\ell=0}^{i} s_{i-\ell}(\lambda) \frac{T_\ell D(a)}{i+1} = \sum_{\ell=0}^{i} \widetilde{s}_{i-\ell}(\lambda) \frac{T_\ell D(a)}{\ell+1} = \sum_{\ell=0}^{i} \phi_{i-\ell} h_{\ell+1}(a).$$

Since $\phi_{i+1}|_{\mathcal{O}_X} = 0$, we have $h'_{i+1} = (\phi h)_{i+1}$. So $h' = \phi h$.

Now, assume that $\nu D(\mathcal{O}_X) \neq 0$ for every nonzero $\mathcal{O}_X$-linear map $\nu \colon \mathcal{F} \to \mathcal{S}(\mathcal{F})$. Let ϕ be an $\mathcal{O}_X$-algebra automorphism of $\mathcal{S}(\mathcal{F})$ such that $\phi_0|_{\mathcal{F}} = \mathrm{id}_{\mathcal{F}}$ and $h' = \phi h$. To see that ϕ is unique, we just need to show that $\phi_q|_{\mathcal{F}}$ is uniquely defined for each $q \geq 0$. We do it by induction. Since $h' = \phi h$, for each $q \geq 0$ the following equality holds:

$$h'_{q+1} = \phi_q h_1 + \phi_{q-1} h_2 + \cdots + \phi_1 h_q + h_{q+1}.$$

Then $\phi_q D$ is determined by $\phi_1, \dots, \phi_{q-1}$. Since ϕ_q is $\mathcal{O}_X$-linear and takes values in $S^{q+1}(\mathcal{F})$, it follows from our extra assumption that ϕ_q is determined. $\square$

If $(X, \mathcal{O}_X)$ is a Noetherian scheme over a Noetherian $\mathbf{Q}$-scheme $(B, \mathcal{O}_B)$, if $\mathcal{F}$ is a locally free sheaf on X, and if $D \colon \mathcal{O}_X \to \mathcal{F}$ is an $\mathcal{O}_B$-derivation such that $\mathcal{F}$ is generated by $D(\mathcal{O}_X)$ at the associated points of X, then $\nu D(\mathcal{O}_X) \neq 0$ for all nonzero $\mathcal{O}_X$-linear maps $\nu \colon \mathcal{F} \to \mathcal{S}(\mathcal{F})$.

CONSTRUCTION 6. (*Jets*) Assume that $D \colon \mathcal{O}_X \to \mathcal{F}$ is locally integrable, and that the sheaf of $\mathcal{O}_X$-linear maps from $\mathcal{F}$ to $\mathcal{S}(\mathcal{F})$ sending $D(\mathcal{O}_X)$ to zero has only trivial local sections. Let $\mathcal{U}$ be the collection of open subspaces $U \subseteq X$ such that $D|_U$ is integrable. For each $U \in \mathcal{U}$, there exist iterated Hasse derivations extending $D|_U$. Let $\mathcal{C}_U$ be the collection of these Hasse derivations. By Corollary 1, for any two $h, h' \in \mathcal{C}_U$ there is a unique $\mathcal{O}_U$-algebra automorphism $\phi_{h,h'}$ of $\mathcal{S}(\mathcal{F})|_U$ such that $h' = \phi_{h,h'} h$. Now, consider the collection of all the $h \in \mathcal{C}_U$ for all $U \in \mathcal{U}$. Consider also the collection of all the $\phi_{h,h'}$ for all $U \in \mathcal{U}$ and all $h, h' \in \mathcal{C}_U$. If $h, h', h'' \in \mathcal{C}_U$, then $h'' = \phi_{h',h''}\phi_{h,h'}h$. From the uniqueness of $\phi_{h,h''}$, we get $\phi_{h',h''}\phi_{h,h'} = \phi_{h,h''}$. The cocycle condition being satisfied, the $\phi_{h,h'}$ patch the $\mathcal{S}(\mathcal{F}|_U)$ to an $\mathcal{O}_X$-algebra $\mathcal{J}$, and the h patch to a map of $\mathcal{O}_B$-algebras $\tau\colon \mathcal{O}_X \to \mathcal{J}$. Since the $\phi_{h,h'}$ do not decrease degrees, for each integer $n \geq 0$ the truncated sheaves $\prod_{i=0}^{n} \mathcal{S}^i(\mathcal{F}|_U)$ patch to an $\mathcal{O}_X$-algebra quotient $\mathcal{J}^n$ of $\mathcal{J}$. Also, since the $(\phi_{h,h'})_0$ are the identity maps, there is a natural map of $\mathcal{O}_X$-modules $\mathcal{S}^n(\mathcal{F}) \to \mathcal{J}^n$. This map is an isomorphism for $n = 0$. Also, for each integer $n > 0$, the $\mathcal{O}_X$-algebra $\mathcal{J}^{n-1}$ is a subquotient of $\mathcal{J}^n$, and there is a natural exact sequence,

$$0 \to \mathcal{S}^n(\mathcal{F}) \to \mathcal{J}^n \to \mathcal{J}^{n-1} \to 0.$$

We say that $\mathcal{J}$ is the *sheaf of jets of D*, and that $\tau\colon \mathcal{O}_X \to \mathcal{J}$ is its *Hasse derivation*. For each integer $n \geq 0$, we say that $\mathcal{J}^n$ is the *sheaf of n-jets of D*, and that the induced $\tau_n\colon \mathcal{O}_X \to \mathcal{J}^n$ is the *n-th order truncated Hasse derivation*.

EXAMPLE 4. Let M be a complex manifold of complex dimension m, and $\mathcal{L}$ a foliation of dimension n of M. Let $w\colon T_M^* \to E$ be the surjection associated to $\mathcal{L}$. Then w induces a derivation $D\colon \mathcal{O}_M \to \mathcal{F}$, where $\mathcal{F}$ is the sheaf of holomorphic sections of E. The Frobenius conditions imply that D is integrable. Also, since w is surjective, $\mathcal{F}$ is generated by $D(\mathcal{O}_M)$. Thus, applying Construction 6, we have an associated sheaf of q-jets $\mathcal{J}^q$ on M for each integer $q \geq 0$. Also we may consider the bundle of q-jets $J_{\mathcal{L}}^q$ of $\mathcal{L}$. Then $\mathcal{J}^q$ is the sheaf of holomorphic sections of $J_{\mathcal{L}}^q$.

Indeed, for each point p of M, there exist a neighborhood X of p in M, and an open embedding of X in $\mathbf{C}^n \times \mathbf{C}^{m-n}$ whose composition $\varphi_X\colon X \to \mathbf{C}^{m-n}$ with the second projection defines $\mathcal{L}$ on X. The sheaf $\mathcal{F}|_X$ is the pullback of the sheaf of 1-forms on $\mathbf{C}^n$, and the canonical vector fields on $\mathbf{C}^n$ yield a basis $e_1, \dots, e_n$ of $\mathcal{F}|_X$ such that $D = D_1e_1 + \cdots + D_ne_n$, where the D_i are the pullbacks of these vector fields. Since they commute, so do the D_i.

Using the D-connection γ given in Example 2, and the associated extended D-connection T given in Construction 2, we obtain an iterated Hasse derivation h_X on X extending $D|_X$, and given by Formula (1) for each integer $q > 0$ and each local section a of $\mathcal{F}$. Then the truncation in order q of h_X has exactly the same form of the canonical Hasse derivation of the sheaf of sections of the bundle of relative q-jets $J_{\varphi_X}^q$. So, patching the h_X is compatible with patching the $J_{\varphi_X}^q$. The patching of the latter yields the bundle of jets $J_{\mathcal{L}}^q$, and hence we get that $\mathcal{J}^q$ is the sheaf of sections of $J_{\mathcal{L}}^q$.

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AMS Mathematics Subject Classification: 14F10, 13N15, 37F75.

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Lavoro pervenuto in redazione il 02.07.2013.

Rend. Sem. Mat. Univ. Politec. Torino
Vol. 71, 3–4 (2013), 421 – 461

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THE ITALIAN SCHOOL OF ALGEBRAIC GEOMETRY AND THE TEACHING OF MATHEMATICS IN SECONDARY SCHOOLS: MOTIVATIONS, ASSUMPTIONS AND STRATEGIES

Abstract. In this paper I intend to illustrate the reasons which led some members of the Italian School of algebraic geometry – in particular, Corrado Segre (1863-1924), Guido Castelnuovo (1865-1952), Federigo Enriques (1871-1946) and Francesco Severi (1879-1961) – to become so concerned with problems pertaining to mathematics teaching; describe the epistemological vision which inspired them; discuss the various ways in which this commitment manifested itself (school legislation, teacher training, textbooks, publishing initiatives, university lectures, etc.); make evident the influence of the reform movements abroad, particularly that of Klein; finally, show how, in this respect as well, Italian geometers projected an unquestionable image of a "School".

1. Introduction

I am very pleased to take part in this conference dedicated to Alberto Conte in celebration of his seventieth birthday. The reason I was invited to a such a strictly gathering of mathematicians is due to the fact that Alberto Conte, along with his research work in the field of algebraic geometry, has always cultivated an interest in history of mathematics, both that of his specific area of study and that more generally related to Turin's rich scientific tradition. Further, with great sensitivity he has encouraged and supported initiatives aimed at bringing that tradition to broader attention.

I limit myself to recalling that in 1987, on the occasion of the 13th Congress of the Italian Mathematical Union, he encouraged Silvia Roero and myself to explore the patrimony of books and manuscripts housed in the libraries of Turin in order to highlight, by means of an exhibition and a catalogue, the wealth of mathematical studies in Piedmont and connections with international scientific research.[1] Noteworthy among his various works in the history of mathematics is the critical edition of the letters of Federigo Enriques to Guido Castelnuovo, entitled *Riposte armonie*, edited together with Umberto Bottazzini and Paola Gario[2], an extraordinary document testifying to the scientific and human fellowship of the two mathematicians. But I would also like mention one of the research projects we carried out together, regarding the

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This research was carried out as a part of the project PRIN 2009, *Mathematical Schools in North Italy in* 18^{th} *and* 20^{th} *Centuries and Relationships with the International Community*, University of Turin.

[1]The Catalogue of the exhibition *Bibliotheca Mathematica. Documenti per la storia della matematica nelle biblioteche torinesi*, L. Giacardi and C. S. Roero, Eds. (Torino, Allemandi, 1987) was given as a gift to participants at the congress and has become a point of departure for many successive studies on mathematics in Piedmont.

[2]Bottazzini, Conte, Gario 1996.

physical-mathematical sciences in Turin during the period of French domination at the beginning of the nineteenth century, and to cite the contribution made by Alberto Conte to the reconstruction, based on in-depth examination of archives, of the large number of young students from Piedmont, including Giovanni Plana, who attended the École Politechnique, and of the kind of studies that they carried out.[3]
The topic I have chosen for this conference conjoins both of Alberto Conte's interests: algebraic geometry and the history of mathematics.

As it is well known, the Italian School of algebraic geometry was born in Turin at the end of the nineteenth century, under the guidance of Corrado Segre. It soon brought forth such significant results that it became a leading light (*führende Stellung*) on an international level, as F. Meyer and H. Mohrmann affirm in the *Encyclopädie der mathematischen Wissenschaften*.[4] Segre inspired an atmosphere of work characterised by highly prolific, enthusiastic, and frenetic activity, which Guido Castelnuovo, remembering his years in Turin, would refer to as "Turin's geometric orgies". The mathematicians involved were gifted students preparing their degree theses with Segre, such as Gino Fano (1892), Beppo Levi (1896), Alberto Tanturri (1899), Francesco Severi (1900), Giovanni Zeno Giambelli (1901), Alessandro Terracini (1911), and Eugenio Togliatti (1912). A number of newly graduated students from Italy and abroad were also drawn to Turin by Segre's fame. Amongst these, the most famous were Castelnuovo (1887-1891), Federico Amodeo (1890-91), Federigo Enriques (November 1992, November 1893-January 1894), Gaetano Scorza (1899-1900), the English couple William H. Young and Grace Chisholm Young (1898-99), and, from the United States, Julian Coolidge (1903-04), and C. H. Sisam (1908-09).

The great significance of the scientific results obtained by the School has led historians of mathematics to overlook, or at best to attach only secondary importance to the issues related to mathematics teaching that would occupy many of its members, including Segre himself, throughout their lives. It is only recently that various studies – which I will cite presently – have begun to explore this aspect as well of the activities of Italian geometers.
Broadly speaking, they shared a vision of mathematics teaching which derived, on the one hand, from their contacts with Felix Klein and his important movement of reforming the teaching of mathematics in secondary and higher education, on the other, from the way in which the authors themselves conceived of advanced scientific research. If, however, we look more closely at their contributions, from these common roots there emerge diverse motivations underlying their involvement with and different approaches to the problem.

The period of reference here goes from the final years of the nineteenth century through the first half of the twentieth century, and the institutional context which frames their commitment to education is still characterised, in the first two decades, by the Casati Law (1859), with attempts at reform either unsuccessful, or carried out

[3] A. Conte, L. Giacardi, *La matematica a Torino, in Ville de Turin 1798-1814*, Città di Torino, Torino, pp. 281-329.

[4] W. Fr. Meyer and H. Mohrmann, *Vorrede zum dritten Banden*, Encyklopädie der mathematischen Wissenschaften, III.I.1, 1907-1910, p. VI.

only in part. Such was the fate of the important reform project proposed by the Royal Commission (1909) to which Giovanni Vailati contributed. The project was never approved, although some of Vailati's proposals, as we shall see, were adopted by Castelnuovo in designing the programs for the *liceo moderno* (1913). The rise of Fascism and the Gentile Reform (1923) nullified any attempt at renovation in the area of science, notwithstanding the battle (carried out by some mathematicians such as Castelnuovo and Enriques) to restore dignity to mathematics, which had been strongly devalued by that reform.[5]

2. The spread in Italy of Felix Klein's ideas on education

It is well known that Felix Klein (1849-1925) always combined advanced-level research with serious attention to organizational and didactic problems pertaining to mathematics teaching at both the secondary and the advanced level.[6] Klein's interest in such problems can be dated back to 1872 when he obtained the professorship at the Erlangen University: in his *Antrittsrede* he described his own conception of mathematics, outlining a vision of teaching that would mature over the next thirty years. In the mid-1890s he became concerned with teacher training, organising a series of seminars for secondary teachers and publishing the *Vorträge über Ausgewählte Fragen der Elementargeometrie* (1895), with the aim of opposing the trend, then prevailing, towards approaches to mathematics that were too formal and abstract, and to resist overspecialization. Further, from the first he believed that the whole sector of mathematics teaching, from its very beginnings at elementary school right up to the most advanced research level, should be regarded as an organised whole:

> "It grew ever clearer to me that, without this general perspective, even the purest scientific research would suffer, inasmuch as, by alienating itself from the various and lively cultural developments going on, it would be condemned to the dryness which afflicts a plant shut up in a cellar without sunlight".[7]

Klein's document, *Gutachten*,[8] from May 1900 illustrates the evolution of his thoughts regarding education, and presents an overview of the cardinal points of his famous program for reforming mathematics teaching, which would be expressed publicly for the first time five years later at a congress held in Merano.[9]
The key points on which Klein based his reform movement are the following: he desired to bridge the gap between secondary and higher education and so he proposed

[5] See Giacardi 2006 and the texts of the programs on the website `http://www.mathesistorino.it/?page_id=564`.

[6] Regarding Klein and his movement to reform mathematics teaching, cf. Rowe, 1985; Schubring, 1989, 2008; Nastasi, 2000; Gario, 2006.

[7] F. Klein, *Göttinger Professoren. Lebensbilder von eigener Hand. 4. Felix Klein*, Mitteilungen des Universitätsbundes Göttingen, 5, 1923, p. 24.

[8] This document is transcribed in Schubring 1989, Appendix II.

[9] *Reformvorschläge für den mathematischen und naturwissenschaftlichen Unterricht*, Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht, 36, 1905, pp. 533-580.

transferring the teaching of differential and integral calculus, to the middle school level; he favoured a "genetic" teaching method, because developing a theory according to the way in which it is formed represents a good guide to scientific research; he believed that teachers should capture the interest and attention of their pupils by presenting the subject in an intuitive manner; he suggested highlighting the applications of mathematics to all the natural sciences; he believed in looking at the subject from a historical perspective; he argued that more space should be dedicated to the *Approximationsmathematik* that is, "the exact mathematics of approximate relations"; finally he firmly believed that the elementary mathematics from an advanced standpoint should play a key role in teacher training.

These tenets about the teaching of mathematics are intimately connected with Klein's vision of mathematics and mathematical research. In particular he believed that pure research had to be very closely tied to experimental research and he refused the axiomatic point of view, "the death of all sciences". He was convinced that the progress in science originates from the combined use of intuition and logic:

> "The science of mathematics may be compared to a tree thrusting its roots deeper and deeper into the earth and freely spreading out its shady branches to the air. Are we to consider the roots or the branches as its essential part? Botanists tell us that the question is badly framed, and that the life of the organism depends on the mutual action of its different parts".[10]

Moreover, Klein distinguished between *naïve intuition* and *refined intuition* and highlighted the fact that naïve intuition is important in the discovery phase of a theory (as an example he cites the genesis of differential and integral calculus), while refined intuition intervenes in the elaboration of data furnished by naïve intuition, and in the rigorous logical development of the theory itself (for example, in Euclid's *Elements*).[11]

Klein's epistemological vision of mathematics and his initiatives to improve mathematics teaching at the secondary and university levels in Germany arrived in Italy through various channels.[12]

First of all, towards the end of the nineteenth century a large number of young Italian mathematicians frequently attended German universities – in particular Leipzig and Göttingen, where Klein himself taught – in the course of postgraduate programs or on study trips: among these Giuseppe Veronese (1880–1881), Ernesto Pascal (1888–1889), Segre (summer 1891), Fano (1893–1894), and Enriques (1903) were the most noteworthy. They returned to Italy not only with new ideas about methods and areas of research, but also with new perspectives on teaching mathematics. Upon his return to Turin after several months of advanced study at Göttingen, Fano gave an enthusiastic account of Klein's work, referring also to Klein's teacher training seminars, observing

[10] F. Klein,*The Arithmetizing of Mathematics*, Isabel Maddison, trans. Bulletin of the American Mathematical Society 2, 8, 1896, pp. 241-249, at pp. 248-249.

[11] F. Klein, *On the mathematical character of space-intuition and the relation of pure mathematics to the applied sciences* (1893), pp. 41-50 in *Lectures on mathematics delivered from Aug. 28 to Sept.9, 1893 . . . at Northwestern University Evanston, Ill. by F. Klein, reported by A. Ziwet*, New York, Macmillan and C. 1894, at p. 42.

[12] See Giacardi 2010, pp. 2-4.

that, "We have a great deal to learn from Germany as far as the relationship between secondary and higher education is concerned".[13]
Klein himself visited Italy first in 1874 to meet the Italian mathematicians, to listen and to learn,[14] in 1878[15] and again in 1899, stopping over in Turin, Florence, Bologna, Rome, and Padua and meeting, among others, Segre, Enriques, Castelnuovo, Cremona, Veronese and Fano.[16]
Further, at the end of the 1800s several of Klein's writings touching upon mathematics education were translated into Italian, even before he became the president of the International Commission on Mathematical Instruction (Rome, 1908) and his ideas spread worldwide. Besides the *Erlanger Programm*, which Fano translated at Segre's request, the *Vorträge über Ausgewählte Fragen der Elementargeometrie*, was translated by Francesco Giudice at the request of the geometer Gino Loria, and the lecture "Über Arithmetisierung der Mathematik", appeared in the *Rendiconti del Circolo Matematico di Palermo*, translated by Salvatore Pincherle.[17]
Finally the rich correspondence between Klein and the Italian geometers, Segre, Fano, Loria, Enriques, and Castelnuovo,[18] bears witness to Klein's twofold influence in Italy, in scientific research and in mathematics teaching.
As Enriques would write twenty years later in his review of Klein's *Gesammelte mathematische Abhandlungen*, it was precisely the "tendency to consider the objects to be studied in the light of visual intuition" [19] that brought Klein and the Italian geometers so close together intellectually.

3. Corrado Segre and Teacher Training

Segre had been in epistolary correspondence with Klein since 1883, and considered him a *Maestro* at a distance. Klein's influence can be clearly perceived in his scientific work and in his methodological approach to mathematical research, but Klein

[13] G. Fano, *Sull'insegnamento della matematica nelle Universitá tedesche e in particolare nell'Universitá di Gottinga*, Rivista di matematica, 4, 1894, pp. 170-187, at p. 181.

[14] See for example, Klein to Cremona, 25 July 1874, 23 August 1874, 21 November 1874 in Menghini 1994, pp. 59-61.

[15] See, for example, Klein to Brioschi, 30 March 1878, Casorati to Brioschi, 4 November 1878, in *Francesco Brioschi e il suo tempo* (1824–1897) (2000), Milan, Franco Angeli, II Inventari, pp. 160 and 316.

[16] See Cremona to Fano, 21 March 1899 and Veronese to Fano, 21 March 1899, in *Fondo Fano*, Biblioteca matematica "G. Peano", Turin; Enriques to Castelnuovo, 17 March 1899 and 28 March 1899, in Bottazzini, Conte, & Gario, 1996, pp. 402 and 404; see also William Young, *Christian Felix Klein* -1949-1925, Proceedings of the London Royal Society, Series A 121(1928), pp. i-xix, at p. xiii.

[17] F. Klein, *Considerazioni comparative intorno a ricerche geometriche recenti*. Annali di matematica pura ed applicata, (2), 17, 1889, pp. 307–343; F. Klein, *Conferenze sopra alcune questioni di geometria elementare*. Torino, Rosenberg & Sellier 1896; F. Klein, *Sullo spirito aritmetico nella matematica*. Rendiconti del Circolo matematico di Palermo, 10, 1896, pp. 107–117.

[18] The letters are conserved in the dossier "*F. Klein*" of the Niedersächsische Staats-und Universitätsbibliothek of Göttingen and in the "Fondo Guido Castenuovo" of the Accademia Nazionale dei Lincei, Roma. See Luciano, Roero 2012 and P. Gario (ed.) *Lettere e Quaderni dell'Archivio di Guido Castelnuovo*: `http://archivi-matematici.lincei.it/Castelnuovo/Lezioni_E_Quaderni/menu.htm`.

[19] F. Enriques, (Review of) "F. Klein: *Gesammelte mathematische Abhandlungen, zweiter Band*." Periodico di matematiche, (4), 3, 1923, p. 55.

also left a strong imprint on Segre's conceptions of mathematics teaching.
This influence is evident in the lectures he gave at the *Scuola di Magistero* (Higher Teacher Training School) of Turin University; in fact, along with his courses of advanced geometry Segre also taught a course of mathematics for prospective teachers in this School for eighteen years, from 1887-88 to 1890-91 and from 1907-08 to 1920, the year when the Scuole di Magistero were suppressed. From that time until his death he taught the course in "complementary mathematics" that had just been established for the combined degree in mathematics and physics with the aim of preparing new graduates for teaching.
Leading Segre, a mathematician committed above all to preparing young people for research, to take an interest in the training of future teachers, was surely the influence of Klein, but other factors could have influenced him as well. First of all, the very close connection that Segre saw between research and teaching, which emerges quite clearly from his notebooks, led him to assign great importance to teaching methods. Further, the Turinese environment was very stimulating thanks to the presence of the Mathesis Association, a society of mathematics teachers, and the Peano School, which was particularly sensitive to problems related to teaching.[20] The example set by his mentor Enrico D'Ovidio, author of successful textbooks for secondary schools, and his friendship with Gino Loria, who quite soon became the referent abroad for questions regarding Italian education, may have influenced Segre as well.
Segre's teaching skills are attested to by forty handwritten notebooks (1888-1924) in which, every summer, he made a careful record of his lectures for the courses he was to teach the following autumn, dealing with different subjects each year. In fact, these not only constitute an important historical documentation of his research activity – of which, as Alessandro Terracini remarked, they are sometimes a preliminary stage, sometimes a reflection – but also provide extraordinary evidence of his gifts as a teacher.

The tenets which inspired Segre in advanced teaching are illustrated in his 1891 article *Su alcuni indirizzi nelle investigazioni geometriche. Osservazioni dirette ai miei studenti*.[21] He maintained that a good Maestro should invite his pupils only to deal with "relevant" problems and teach them to distinguish the significant questions from the sterile and useless ones; advise them to study, along with theories, their applications; urge them not to be "slave to one method", and not to restrict their scientific research within a too limited field, so as to be able to look at things "from a higher vantage point"; take due account of the didactic needs; suggest that they read the works of the great masters. These same issues are taken again into consideration in relation to secondary school teaching, in Segre's lessons at the Scuola di Magistero.

[20] See Luciano, Roero 2010.

[21] C. Segre, *Su alcuni indirizzi nelle investigazioni geometriche. Osservazioni dirette ai miei studenti*, Rivista di Matematica, 1, 1891, pp. 42-66 (Opere, 4, pp. 387-412). This article also reached American mathematicians thanks to the translation done by John Wesley Young and revised by Segre himself, who made several additions and modifications to it.

3.1. "Teach to discover"

Among Segre's handwritten notebooks, three in particular, *Lezioni di Geometria non euclidea* (1902-03), *Vedute superiori sulla geometria elementare* (1916-17) and [*Appunti relativi alle lezioni tenute per la Scuola di Magistero*], together with archival documents, show Segre's vision of mathematics and its teaching.[22]
Segre's lessons at the Scuola di Magistero were characterized by a threefold approach: theory, methodology, and practice. In fact, he took up anew the themes of elementary mathematics, making evident each time the connections to higher mathematics;[23] he also examined questions of methodology and didactics. Then, in the laboratories-classes, students were taught to impart real lessons, documented and stimulating.
In the notebook [*Appunti relativi alle lezioni tenute per la Scuola di Magistero*], Segre begins with some considerations on the nature of mathematics, the objectives of teaching, and the importance of intuition and rigor, then provides future teachers with some methodological instructions which are closely related to his particular way of approaching research, and are the fruit of his own teaching experience and of an attentive examination of legislative measures in various European countries and of educational issues debated at the time.

Between the two ways of addressing mathematics – considering it in relation to applications, or seeing it from an exclusively logical point of view – Segre's preferences tended towards the first. This first approach, typical of Klein, is characterised by three phases: gathering information derived from experience, putting the data obtained into mathematical form and proceeding to a purely mathematical treatment of the problem, and finally, translating the mathematical results into the form most suitable for the applications. With regards to the second approach, Segre cites Peano and Hilbert and observes:

> "Let us say immediately that this second line is of great importance, philosophically as well. It has made it quite clear what pure mathematics is; and has contributed to making various parts of mathematics more rigorous. But, by detaching itself from reality, there is a risk of ending up with constructions, which, even while logical, are too unnatural, and cannot be of lasting scientific importance" (pp. 13-14).[24]

For Segre, the aim of mathematics is to teach how "to reason well; not to be satisfied with empty words; to draw conclusions from the hypothesis, to reflect and discover on one's own; . . . to speak precisely" (p. 42). In secondary teaching mathematics should not be considered an end in itself, "it must arise from the external world and then

[22] For further information and an overview of the international context see Giacardi 2003 and 2010.

[23] In the notebook *Vedute superiori sulla geometria elementare* (1916-17) Segre deals with the following subjects: non-Euclidean geometry, foundations of mathematics, elementary geometry and projective geometry, Analysis situs, geometrical constructions, linkages, problems that can be solved with straightedge and compass, algebraic equations that can be solved by extracting square roots, the problem of the division of the circumference, the problem of squaring the circle.

[24] This and successive page numbers refer to the notebook [*Appunti relativi alle lezioni tenute per la Scuola di Magistero*].

be applied to it" (p. 15); therefore the first approach to it must be experimental and intuitive, so that the student learns "not only to demonstrate truths already known, but to make discoveries as well, to solve the *problems* on his own" (p. 16), while "perfect rigor in certain things can be reached at a later time" (pp. 25-26).

Consequently, the objective of mathematics teaching is to develop not only the powers of reasoning but equally those of intuition; it is no coincidence that, as regards the method to be used, Segre prefers the *heuristic* method for presenting the subject, the *analytic* for the proofs, and the *genetic* for the development of theories. The first, the Socratic method, permits the student to discover mathematical truths on his own; the second allows him to enter into the mathematics "workshop" and understand the "why" of each step in a proof; the third represents a good guide to scientific research. Segre, however, never tired of underlining the importance of varying the methods, and above all, of choosing them according to "the subject, the pupils, and the time available" (p. 44).

The teacher should therefore seek the right balance between rigour and intuition. There are in Segre no specific philosophical reflections on the concept of intuition; for him, intuition is that used in scientific research: freedom of creative imagination, freedom of choice of methods, is "perceiving a truth spontaneously, without reasoning and without experiences, but it is the fruit of unconscious reasoning or experiences" (p. 15), based on previous knowledge that is unconsciously chosen and combined in new ways or that suggest analogies. How he understands the relationship between intuition and rigour in the practice of teaching emerges from the following considerations. According to Segre, the postulates on which the development of a theory is based must be intuitive and not necessarily independent. He also observes that in modern works are listed some "very obvious" postulates (for example, "the successor to a number is a number"), remarking that "a young person cannot understand the purpose of a series of such statements!" (p. 20). Further, he invites the teacher to use the postulates as need arises in a reasoning, without having to state them all at the beginning of the treatment of a theory. With regard to proofs, he observes that it is not necessary to prove propositions that are intuitively evident and that it can be useful to provide sketches of proofs rather than proofs that are rigorous but long and heavy:

> "A . . . sketch, or non-rigorous proof may show how discoveries are made, how intuition works; or serve to provide an idea that is more synthetic, easier to remember, than the rigorous proof that will be explained later . . . it is only necessary to warn the students that the proof presented is incomplete" (p. 25).

Concerning the definitions, Segre affirms:

> "to define for the young person the things that he already knows with a long discourse, is to bore him"(p. 46).
> "If we consider the exclusively logical point of view, the word line or curve, should be eliminated from elementary teaching because there are no means to define it. But that is absurd!" (pp. 18-19).

> "Don't give rigorous definitions, but clarifications, when the definition would be too difficult" (p. 46).

To highlight the difference between the intuitive approach and the logical approach in his lessons at the Scuola di Magistero, Segre takes as an example the textbook by Sebastiano Catania, *Trattato di Aritmetica ed Algebra* (1910) and compares the way the commutative property of the product of two numbers is presented there with the presentation used by Émile Borel in his textbook *Arithmétique* (1907) (p. 41). Catania, a follower of the Peano School, demonstrates it by induction,[25] while Borel illustrates it with an example that exploits the definition of the product as a sum.[26] On the other hand, Segre stresses the fact that it is important to underline the insufficiency of intuition for conceiving certain notions, as for example a curve that has no tangents (p. 43). As far as geometry is concerned, Segre agrees with Vailati's point of view, according to which teaching must be experimental and operative and must avail itself of teaching aids such as squared paper, drawing, and geometric models, that make it possible to "see certain properties that with deductive reasoning alone cannot be obtained".[27]

Moreover, like Klein, he believes that it is important to bridge the gap between secondary and university teaching by introducing, beginning in secondary schools, the concepts of function and transformation, and also to present some applications of mathematics to other sciences (physics, astronomy, political economy, geography, ...), in order to make the subject more interesting and stimulating.

In addition to considerations of a methodological nature, Segre does not hesitate to offer future teachers various bits of practical advice now and then, showing how aware he was of students' errors, bad habits, weak points, and idiosyncrasies:

> "Avoid being boring!" (p. 24);
> "Try to stimulate the activity of the student's mind" (pp. 26-27);
> "Sometimes satisfy the request for a proof which wouldn't be given, but which a more intelligent youngster can understand" (p. 27);
> "Vary the notations and figures. It shouldn't happen that a youngster does not know how to solve an equation only because the unknown is not called x. Or a geometric proof because the position of the figure has changed" (p. 28);
> "The calculations should not be too long, because there is no reason to try the patience of children" (p. 32);
> "Prepare the lesson perfectly ... Don't dictate: use a textbook ... Be patient with the students; repeat if they have not understood; don't be aghast at errors; try to persuade the students ... that they needn't have a gift for mathematics" (p. 42).

Segre's notebook related to the courses for the Scuola di Magistero, includes an annotated bibliography divided into sections, with references to texts then very recent

[25] S. Catania, *Trattato di Aritmetica ed Algebra ad uso degli Istituti Tecnici*, Catania, N. Giannotta, 1910, p. 32.

[26] E. Borel, *Arithmétique*, Paris, A. Colin, 1907, pp. 28-29.

[27] C. Segre, *Su alcuni indirizzi nelle investigazioni geometriche*, cit. p. 54.

both Italian and foreign. In addition to specific books on various topics of elementary mathematics, there are texts on didactics, as Segre himself calls them, textbooks for secondary schools and workbooks of exercises, books on the history of mathematics and on recreational mathematics. In addition to Klein, Segre's principal points of reference were C. A. Laisant, E. Borel, J. Hadamard and H. Poincaré in France, and P. Treutlein and M. Simon, in Germany, mathematicians all critical of a mathematics teaching too marked by logical rigour.

Segre's contribution to the field of mathematics education remained limited to the lessons at the Scuola di Magistero, nevertheless for eighteen years he trained the mathematics teachers who came out of the University of Turin, contributed to the spread of Klein's vision of mathematics teaching among his students, and by his very example transmitted a certain way of teaching mathematics, both at the university and the secondary level, exploiting intuition, encouraging creativity, using more than one method, and establishing connections between different sectors in a unitary vision of mathematics. Moreover, through individual discussions with his students as well as his university lectures, he had communicated his vision of mathematics teaching to his School, although each of its members interpreted it in their own way, according to their different individual experiences.

4. Guido Castelnuovo: the involvement in education as a social duty

Thanks to the intervention of Segre, Castelnuovo was called to Turin as assistant to D'Ovidio in 1887, and remained until 1891, when he obtained a professorship in Rome. As is well known, the scientific collaboration between the two young mathematicians – at the time, Segre was 28, Castelnuovo 26 – led to the creation of the Italian line of research on the geometry of algebraic curves and laid the bases for all of Italian algebraic geometry.[28] After leaving Turin, Castelnuovo kept up the correspondence with his friend and mentor that had begun in 1885, and in Rome inaugurated the extraordinary scientific fellowship with Enriques.

Already at this time in his correspondence, along with questions regarding research there appear topics concerning education,[29] but it is in 1907 that Castelnuovo began to turn his thoughts and efforts to the improvement of mathematics teaching in secondary schools, becoming involved also at an institutional level. That year, in fact, appeared the article entitled "Il valore didattico della matematica e della fisica", which can be considered a manifesto of Castelnuovo's thinking on education. That same year Castelnuovo was involved in the organisation of the International Congress of Mathematicians, which was held in Rome from April 6 to 11, 1908, and paid a great deal of attention also to the section devoted to education, with the help of Vailati, who at the time was working with the Royal Commission on the project to reform secondary

[28] Brigaglia, Ciliberto 1995, §1.5.

[29] The 255 letters from Segre to Castelnuovo dating from 1885 to 1905 can be read in P. Gario (ed.) *Lettere e Quaderni dell'Archivio di Guido Castelnuovo*: `http://operedigitali.lincei.it/Castelnuovo/Lettere_E_Quaderni/menu.htm` (see also Bottazzini, Conte, Gario 1996, pp. 669-678). For the letters from Enriques to Castelnuovo, see Bottazzini, Conte, Gario 1996.

schools.[30] During the congress an international committee dedicated to issues pertaining to mathematics teaching was created, the International Commission on Mathematical Instruction (later ICMI):[31] its first president was Klein, and Castelnuovo, Enriques and Vailati were the Italian delegates. It is thus natural that the contacts with Klein, who he had already met in 1899, became more intense.
In 1909, during the Congress of the Mathesis Association in Padua Castelnuovo explicitly proposed following Klein's example with regard to teacher training:

> "At Klein's suggestion, during the spring holidays a number of German universities hold short courses for Middle school teachers. Couldn't we too set up similar courses in our universities?".[32]

In the years immediately following he himself began to include in his courses in higher geometry in Rome a number of topics designed specifically for the cultural training of future mathematics teachers following the examples of Klein and Segre. Of particular interest from this point of view are the following notebooks: *Geometria non-euclidea* (1910–11), *Matematica di precisione e matematica di approssimazione* (1913–14), *Indirizzi geometrici* (1915–16), *Equazioni algebriche* (1918–19) and *Geometria non-euclidea* (1919–20).[33] In the introduction to the 1913–1914 course on the relationship between precise and approximate mathematics, Castelnuovo explicitly discusses the various ways in which future teachers can be trained and quotes Klein:

> "The educational value of mathematics would be much enriched if, in addition to the logical procedures needed to deduce theorems from postulates, teachers included brief digressions on how these postulates derive from experimental observations and indicated the coefficients with which theoretical results are verified in real experience . . . The relationship between problems pertaining to pure mathematics and those pertaining to applied mathematics is very interesting and instructive. Klein, who dedicated a series of lectures to the subject (1901), describes the first of these as problems of 'precise mathematics' and the second as problems of 'approximate mathematics'. In this course we will . . . more or less follow the general outline of Klein's course.[34] Klein also had another reason for pursuing this line of enquiry, that is, his desire to bridge the gap between mathematicians engaged in pure research and those who have to solve problems relating to applied mathematics."[35]

[30] See G. Castelnuovo to G. Vailati, s. l., February 16, 1907, and D. E. Smith to G. Loria, New York, January 12, 1906, *Fondo Vailati*, Library of Philosophy, University of Milan.

[31] See Giacardi, 2008: `http://www.icmihistory.unito.it/19081910.php`.

[32] G. Castelnuovo, *Sui lavori della Commissione Internazionale pel Congresso di Cambridge. Relazione del prof. G. Castelnuovo della R. Università di Roma*, in *Atti del II Congresso della Mathesis - Società italiana di matematica*, Padova, 20-23 Settembre 1909, Padova, Premiata Società Cooperativa Tip. 1909, Allegato F, pp. 1–4, at p. 4.

[33] The notebooks can be read in P. Gario (ed.) *Lettere e Quaderni dell'Archivio di Guido Castelnuovo*, cit. See also Gario 2006.

[34] Castelnuovo is referring to F. Klein, *Anwendung der Differential-und Integralrechnung auf Geometrie*, Leipzig Teubner 1902.

[35] G. Castelnuovo, *Matematica di precisione e matematica di approssimazione*, 1913–14, pp. 2–3.

Thus, there is no doubt that Klein's influence was important, but Castelnuovo's interest in educational issues also arose from social concerns, as he himself affirmed in the lecture he gave in Paris in the stead of ICMI President Klein and at his express request:

> "Nous nous demandons parfois si le temps que nous consacrons aux questions d'enseignement n'aurait pas été mieux employé dans la recherche scientifique. Eh bien, nous répondons que s'est un devoir social qui nous force à traiter ces problèmes."[36]

4.1. "Break down the wall separating schools from the real world"

Castelnuovo's approach to education grew out of a lucid critique of the Italian school system: in his opinion the teaching of mathematics was too abstract and theoretical, all reference to practical application was neglected and an excessive specialization of different areas led to a distorted cultural perspective.
In order to define exactly what secondary schools should be offering to young people, Castelnuovo asked himself the following three questions: At whom is middle school education aimed?; What should the ultimate goal of schooling be?; What skills should teaching develop? He believed that schools should cater above all to young people aiming to go into one of the so-called "free" professions "both because they constitute the majority of school pupils and because the progressive development of the country will rest mainly on their shoulders".[37] The primary aim of middle schools should be to form the future member of civil society, because "education cannot be truly effective if it is not aimed at average levels of intelligence and if it is unable to create that refined democracy which forms the basis of every modern nation".[38] The qualities which teachers must foster and cultivate in their pupils are the creative imagination, the spirit of observation and the faculties of logic. Excessive rigour is to be avoided:

> "Middle schools should not furnish [their pupils] with knowledge so much as with a desire and a need for knowledge; they should not seek to provide an encyclopaedic knowledge, but must only offer a clear, although necessarily very limited, idea of the principal questions of the various branches of knowledge, and of some of the methods which have been employed in tackling them. [...] Of course, this kind of teaching will not be sufficient to provide middle school students with preparation specific to one or another of the faculties of the university. However, this is not the aim of middle schools. They serve simply to provide students with the aptitude to move on to more advanced studies."[39]

[36] G. Castelnuovo, *Discours de M.G. Castelnuovo*, L'Enseignement Mathématiques, 16, 3, 1914, pp. 188–191, at p. 191.

[37] G. Castelnuovo, *La scuola nei suoi rapporti colla vita e colla Scienza moderna*, in *Atti del III Congresso della Mathesis - Società italiana di matematica, Genova, 21-24 ottobre 1912*, Roma, Tip. Manuzio, 1913, pp. 15–21, at pp. 18–19.

[38] Castelnuovo, *Sui lavori della Commissione Internazionale pel Congresso di Cambridge*, cit. p. 4.

[39] G. Castelnuovo (1910), *La scuola media e le attitudini che essa deve svegliare nei giovani*, in *Guido Castelnuovo, Opere Matematiche. Memorie e Note*, vol. III, Roma, Accademia dei Lincei, 2004, pp. 21–30.

In the article "Il valore didattico della matematica e della fisica", mentioned above, the placing of mathematics and physics side by side is by no means coincidental.[40] Here, in fact, Castelnuovo emphasises the importance of observations and experiment, the usefulness of constantly confronting abstraction with reality because "it in fact makes it possible to clarify the two different meanings given to the adjective exact in the theory and in the practice", and the importance of practical application as a means of "shedding light on the value of science". Furthermore, he claims that heuristic procedures should be favoured for two reasons: "the first, and the most important reason, is that this type of reasoning is the best way to attain to truth, not just in experimental sciences, but also in mathematics itself"; the second is that it is "the only kind of logical procedure that is applicable in everyday life and in all the knowledge involved with it."[41]
He concludes his article by recommending that teachers draw on the history of science so that young people understand the relative and provisional nature of every theory.

To illustrate more incisively his conception of mathematics teaching, Castelnuovo often introduced veritable slogans in his speeches and in his articles: "Rehabilitate the senses"; "Break down the wall separating schools from the real world"; "Teaching should proceed hand in hand with nature and with life".

4.2. Various directions of Castelnuovo's commitment to education

Castelnuovo's commitment to education manifested itself in various forms: in his activities in the ICMI and the Mathesis Association, of which he was president from 1911 to 1914, in the courses he taught at university, in the various articles he dedicated to issues relating to mathematics teaching, and in the syllabi of *liceo moderno* he designed for secondary education.

As a delegate and, later, first as a member of the Central Committee of the ICMI and then as vice-president, Guido Castelnuovo built up an international network and promoted the exchange of information about the new movements for reform in Europe, in particular that proposed by Klein, whose methodological approach he wholeheartedly endorsed.[42] Further, he guided the work of the Italian subcommission of the ICMI, encouraging its members not to "occupy themselves only with statistical data" but to "turn the investigation to more elevated fields and to treat pedagogical and psychological questions." [43]

As president of the Mathesis Association, he inserted into the *Bollettino* of the Mathesis summaries of the activities of the subcommission, translations of lectures, inquiries into problems concerning mathematics teaching in the various orders of schools. He also encouraged debates regarding method. For example he wrote to Giovanni Vacca:

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[40] See Brigaglia 2006.

[41] This and the above quotations are from G. Castelnuovo, *Il valore didattico della matematica e della fisica*, Rivista di Scienza, 1, 1907, pp. 329–337, respectively at p. 336 and at p. 333.

[42] See Giacardi 2008.

[43] Castelnuovo, *Sui lavori della Commissione Internazionale pel Congresso di Cambridge*, cit. p. 2.

> "Almost unexpectedly and against my will, I have been elected president of the Mathesis. I accept the nomination only because I think that it might be helpful for the affairs of the Italian Commission for mathematics teaching, for which the Bollettino of the Math[esis] will become the publishing organ. I would like keep the level of the Bollettino high, reducing to a minimum the Byzantine discussions in which secondary teachers too often delight. I am therefore very much counting on your cooperation."[44]

When in 1911 Minister of Education Luigi Credaro set up a *liceo moderno*, in which Greek was replaced with a modern language (German or English) and more attention was dedicated to the scientific subjects, Castelnuovo drew up the mathematics syllabus and related instructions.[45] He put a number of Klein's proposals into practice by introducing the notion of function and the concepts of derivative and integral, attaching a greater importance to numerical approximations, and coordinating mathematics and physics teaching. He wrote:

> "But if we truly wish the middle school student to feel an inspiring breeze in this modern mathematics, and perceive something of the grandeur of its whole structure, it is necessary to speak to him of the concept of function and show him, even summarily, the two operations that constitute the foundation of infinitesimal calculus. In this way, if he will have a scientific spirit, he will acquire a more correct and balanced idea of the exact sciences nowadays ... If the pupil's mind is more disposed towards other subjects, he will at least find mathematics to be, instead of a logical drudge, a set of methods and results which can be easily applied to concrete problems." [46]

On that occasion Castelnuovo wrote to Klein:

> "A propos de l'enseignement, certain que vous agréerez la nouvelle, je vais vous communiquer que les programmes (modernes) de l'enseignement mathématique que j'ai fait adopter dans les lycées modernes, ont été si bien accueillis que le Ministére de L'Instr. P. pense maintenant de les introduire même dans les lycées classiques et dans les instituts techniques, en développant davantage, dans ces dernières écoles, le programme de calcul infinitésimal." [47]

He also presented the programs to the international community during the ICMI meeting held in Paris in 1914.[48] Unfortunately, the *liceo moderno* was short-lived. The reorganisation of secondary schools was introduced in 1923 by the Gentile Reform in

[44] See the letter of G. Castelnuovo to G. Vacca, Roma, January 27,1911, in Nastasi, Scimone 1995, p. 46.

[45] See *Ginnasio - Liceo Moderno. Orario - Istruzioni - Programmi*, 1913, in `http://www.mathesistorino.it/wordpress/wp-content/uploads/2012/09/liceomod.pdf`.

[46] G. Castelnuovo, *La riforma dell'insegnamento matematico secondario nei riguardi dell'Italia*, Bollettino della Mathesis, XI, 1919, pp. 1-5, at p. 5.

[47] See the letter of G. Castelnuovo to F. Klein, Rome, March 10, 1915, in Luciano, Roero 2012, p. 212.

[48] G. Castelnuovo, *Italie*, L'Enseignement Mathématique, 16, 1914, p. 295. The programs for the liceo

completely different terms: the liberal democratic culture was defeated by new political trends – firmly opposed by Castelnuovo – and by the triumph of Neo-Idealism. Castelnuovo would reformulate some of his cherished ideas in *Progetto di riforma dell'insegnamento secondario* that he presented in 1947 for the political party called *Partito d'Azione*. Here he maintained the importance of education's being developed in a liberal climate, advancing the proposal of a single, common middle school without Latin, a proposal dictated, this time as well by social needs: "The social needs are the most pressing; we must first of all satisfy these, as far as possible, and formulate the cultural problem accordingly."[49]

In Castelnuovo, as in Segre, we find no lofty philosophical or epistemological reflections on the foundations or on problems of rigour and intuition; these appear not to interest him.[50] What he did take to heart was a mathematics teaching, and in particular a geometry teaching, that was strictly tied to reality, in which the deductive aspect was only one of three stages to be gone through: a first stage consists in "the passage from reality to the symbolic scheme", with resort to experience and intuition; in the second stage work must be done on the symbols by means of logical procedures to deduce new truths; and in the third the abstract propositions are translated into results that can be applied to reality.[51] For this reason, in his opinion, a good teacher had to develop above all "the creative imagination", which grows out of the union of intuition and the spirit of observation. In the same way, a good textbook had to find the balance between intuition and rigour, as he forcefully maintained, for example, in the debate with Catania, already cited by Segre, regarding algebra textbooks for secondary schools:

> "If I had to teach in middle schools, I would avoid adopting a textbook in which, along with procedures that are perfectly rigorous, ample space was not given to intuition and experience. Not because I refuse to recognise the immense value of deductive logic . . . but I would be blind if I did not see that with logic alone science would never have been born!". [52]

Again, in the comparative investigation that he undertook for the ICMI in 1911 regarding rigour in the teaching of geometry in classical schools, he was clearly inclined towards a deductive development of geometry teaching which, however, began from empirical bases.[53]

There are two other significant aspects in Castelnuovo's work that must be made

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moderno and the instructions regarding methodology were also translated into French: see G. Loria, *Les Gymnases-lycées "modernes" en Italie*, Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht aller Schulgattungen, 1914, pp. 188–193.

[49] G. Castelnuovo, *Progetto di riforma dell'insegnamento secondario*, in Castelnuovo (1944), *Opere Matematiche* 2004, cit. p. 410.

[50] See, for example, the letters from F. Enriques to G. Castelnuovo of May 1896, particularly that of 4 May 1896, in Bottazzini, Conte, Gario 1996, pp. 260–261.

[51] Castelnuovo, *Il valore didattico della matematica e della fisica*, cit., pp. 331–332.

[52] G. Castelnuovo, *Risposta ad un'osservazione del Prof. Catania*, Bollettino della Mathesis, 1913, pp. 119–120.

[53] G. Castelnuovo, *Commissione internazionale dell'insegnamento matematico – Congresso di Milano*, Bollettino della Mathesis, 1911, pp. 172–184, and L'Enseignement Mathématique 13, 1911, pp. 461–468.

evident: the importance attached to the formation of a "cultured democracy", where the sciences were considered as important as letters; and the need for an adequate scientific and didactic training for secondary school teachers,[54] for whom he, among other things, made room in the activities of the Italian subcommission of the ICMI.

A final point that must be underlined is his firm opposition to the Gentile Reform, which led to his refusal to collaborate with the minister on drafting of the programs for secondary schools,[55] and which was clearly explained in his report "Sopra i problemi dell'insegnamento superiore e medio a proposito delle attuali riforme",[56] compiled for a commission of the Accademia dei Lincei, of which Vito Volterra was president.

5. Federigo Enriques: Teacher training and scientific humanitas[57]

As is well known, after earning his degree Enriques' aspiration was to specialise with Segre in Turin, but instead he was appointed to Rome where he began his extraordinary fellowship with Castelnuovo. Their collaboration led to the publication of important works on algebraic surfaces[58] and developed into a lasting friendship.

However, Enriques did come to Turin, first in November 1892 for several weeks and then from November 1893 to January 1894.[59] The stay in Turin was intense from a scientific point of view, but also inspired his first reflections on the foundations of geometry, which he would later develop during the time spent in Bologna, where he was called to teach projective and descriptive geometry. In January 1922 he transferred to Rome to teach complementary mathematics and then higher geometry, thanks to Castelnuovo having renounced that professorship.

In addition to strictly scientific interests, the two friends also shared a profound attentiveness to education, although they were motivated by different reasons. While Castelnuovo's engagement in educational issues sprang from social concerns, Enriques' involvement was rooted in his interests in philosophical, historical, and interdisciplinary issues and his studies on the foundations of geometry. In 1896, the teaching of projective geometry at the University of Bologna stimulated him to study the genesis of the postulates of geometry, taking the psychological and physiological

[54] Regarding this point, see, for example, *Atti del II Congresso della Mathesis - Società italiana di matematica. Padova 20.23 Settembre 1909*, Padova, Premiata Società Cooperativa Tipografica, 1909, pp. 42–43, and Castelnuovo's *Relazione*, appended to the minutes of the meeting of the Facoltà di Scienze Matematiche, Fisiche e Naturali of the University di Rome of 14 March 1922, in Gario 2004, pp. 117–119.

[55] Assisting the minister in the preparation of the mathematics program was Gaetano Scorza, an active member of the Mathesis Association and one of the Italian representatives in the International Commission on Mathematical Instruction; see *Relazione del Congresso di Livorno, 25-27 Settembre 1923*, Periodico di matematiche, s. IV, 3, 1923, p. 465.

[56] See G. Castelnuovo, *Sopra i problemi dell'insegnamento superiore e medio a proposito della attuali riforme*, in Castelnuovo, *Opere Matematiche* 2004, cit. pp. 358–367.

[57] For further details about this, see Giacardi 2012.

[58] Brigaglia, Ciliberto 1995, Chap. 1, 4; Bottazzini, Conte, Gario 1996.

[59] See, for example, the letters of F. Enriques to G. Castelnuovo, November, 6, 9, 1892 and November, 23, 1893, in Bottazzini, Conte, Gario 1996, pp. 3-4 and p. 44.

studies of H. Helmholtz, E. Hering, E. Mach, W. Wundt, as his starting point.[60]

From the correspondence, writings and documents conserved in the archives of the University of Bologna it is possible to see how already in these early years Enriques had turned his attention to the needs of education, stressing in particular the refusal to resort to artifices in the proofs; the importance of using intuition; the connections between elementary and higher mathematics; the use of the history of mathematics as a tool for understanding the genesis of the concepts presented, and a unified vision of science and culture.[61]

In the evolution of Enriques' cultural project and his vision of mathematics teaching, along with his experience in teaching, an important role was played by the influence of Klein.[62] During Klein's second trip to Italy in 1899 the principal theme of conversation was the psychological genesis of postulates[63] and when Klein invited him to write a chapter on the foundations of geometry for the *Encyklopädie der mathematischen Wissenschaften*, Enriques reached him in Göttingen in 1903 to discuss this subject:

> "In addition to talking about the foundations of geometry, we discussed didactic issues at length, and in just a few hours I learned a great deal from him about a lot of things I knew nothing about – specifically about the way in which mathematics teaching is developing in England and Germany".[64]

It was thanks to Klein's that a German translation of Enriques' *Lezioni di geometria proiettiva* was published in 1903. In his introduction to this book, Klein expresses particular appreciation for Enriques' treatment of the subject, which "is always intuitive, but thoroughly rigorous", and underlines the impact of this kind of research on didactics, writing

> "Italian researchers are also well ahead of us from a practical point of view. They have by no means disdained exploring the didactic consequences of their investigations. The high quality textbooks for secondary schools which came out from this exploration could be made available to a broader audience through good translations. And it would seem particularly desirable in Germany when we consider that our own textbooks are completely out of touch with active research".[65]

[60] See F. Enriques, *Lezioni di Geometria proiettiva*. Bologna, Zanichelli 1898; *Sulla spiegazione psicologica dei postulati della geometria*, Rivista di Filosofia 4, 1901, pp. 171–195 (rpt. Federigo Enriques, *Memorie scelte di Geometria*, Roma, Accademia Nazionale dei Lincei, vol. II, 1959, pp. 145–161); *Problemi della Scienza*, Bologna, Zanichelli 1906; and F. Enriques to G. Castelnuovo, 4 May 1896, in Bottazzini, Conte, Gario 1996, pp. 260–261.

[61] See, for example, Bottazzini, Conte, Gario 1996, p. 224; Chisini 1947, p. 119; *Registri e Relazioni*, in Archivio Storico dell'Università di Bologna.

[62] See Nurzia 1979; Israel 1984; Giacardi 2012, pp. 223-229 and Appendix 2, "Letters to Klein".

[63] F. Enriques to G. Castelnuovo, 28 March 1899, in Bottazzini, Conte, Gario 1996, p. 404.

[64] F. Enriques to G. Castelnuovo, 24 October 1903, in Bottazzini, Conte, Gario 1996, p. 536.

[65] F. Klein, *Zur Einführung*, in F. Enriques, *Vorlesungen über projektive Geometrie*, Introduction (*Zur Einfürung*) by F. Klein. H. Fleisher, trans. Leipzig, Teubner, 1903 (2nd. ed. 1915), p. III.

5.1. Enriques' cultural programme and epistemological assumptions at the basis of his idea of Mathematics Teaching

From the first years of the twentieth century Enriques had in mind a very precise cultural program, one in which active research in the field of algebraic geometry and philosophical, psychological and historical reflections are all closely intertwined. Enriques' aim was to communicate to his intended audience – mathematicians, scientists, philosophers, and educators – his vision of a scientific *humanitas* in which the boundaries between disciplines were overcome and the abyss between science and philosophy was bridged. The history of science constituted the way for achieving this end, or at least it was the tool used by Enriques in his university courses from the very first years, and over time gradually became a very important one in the various initiatives aimed at teacher training.

It is not easy to outline in a few strokes the epistemological vision on which all of Enriques' scientific work was founded, so I will confine myself to indicating the most important factors which inspired his idea of mathematics education.

– Enriques held a dynamic and genetic view of the scientific process, describing it as a "process of continuous development, which establishes a generative relationship between theories and perceives in their succession only an approximation to truth."[66] In such a vision of science, errors become valuable as well, because in the dynamic process of science truth and error are constantly mixed: "every error always contains a partial truth that must be kept, just as every truth contains a partial error to be corrected".[67]

  As a consequence of this idea, Enriques criticises the tendency to present a mathematical theory in strictly deductive manner, because in this way though it appears closed and perfect, it leaves no room for further discovery. Instead, teachers should approach problems with a number of different methods, paying attention to the errors which have allowed science to move forward and indicating those questions which remain open as well as new fields of discovery.[68]

– These views on science are connected to Enriques' conception of the nature of mathematical research – typical of the Italian school of algebraic geometry – as something aiming above all at discovery and particularly emphasizing the inductive aspects and intuition: 'As a rule, the main thing is to discover ... A posteriori it will always be possible to give a proof' ..., [which] 'translating the intuition of the discoverer into logical terms, will provide everyone with the means to recognise and verify the truth'.[69]
  Much has been written on the working method of the Italian geometers, and

[66] F. Enriques, *Scienza e razionalismo*, Bologna, Zanichelli, 1912, p. 132.

[67] F. Enriques 1911, *Esiste un sistema filosofico di Benedetto Croce?*, Rassegna contemporanea, 4.6, 1911, pp. 405–418, at p. 417.

[68] See F. Enriques [Adriano Giovannini], *L'errore nelle matematiche*, Periodico di Matematiche 4, 22, 1942, pp. 57–65.

[69] F. Enriques, O. Chisini, *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche*, 4 vols. Bologna, Zanichelli, 1915–1934, II 1918, pp. 307, 318.

about Enriques in particular, so here I will limit myself to underlining by means of a quotation the importance that he attached to intuition in scientific research:

> "The faculty which comes into play in the construction of science and which thus expresses the actual power of the mathematical spirit is intuition. ... There are in any case different forms of intuition. The first is the intuition or imagination of what can be seen. ... But there is another form of intuition that is more abstract, that – for example – which makes it possible for the geometer to see into higher dimensional space with the eyes of the mind. And there is also a sense of formal analogies which, in the work of many analysts, takes the place of the visual representation of things. ... [I]ntuition protracts and surpasses itself in the unifying power of reason, which is not something exclusive to the mathematician, but – in every field of science and application – marks the greatest reaches of the spirit".[70]

This belief is naturally reflected in the style of teaching, which should, according to Enriques, take into account the inductive as well as the rational aspect of theories. Logic and intuition represent two inextricable aspects of the same process, therefore teachers should find the right balance between the two: the important thing is to distinguish clearly between empirical observation and intuition on the one hand, and logic on the other. Teaching should above all take into account "large scale logic" (which considers the organic connections in science), not so much the "small scale logic" (refined, almost microscopic analysis of exact thought), and prepare young people gradually to develop a more refined and rigorous analysis of thought:

> "It is of no use to develop with impeccable deduction the series of theorems of Euclidean geometry, if the teacher does not go back to contemplate the edifice constructed, inviting the students to distinguish the truly significant geometric properties from those which are valuable only as links in the chain".[71]

With regard to the fact that many Italian teachers resisted the introduction of methods that were more intuitive and empirical, lamenting that a certain incompleteness and a non-rigorous way of reasoning is inherent in these, Enriques observed with a touch of humour:

> "Resisting the ideas that ... relate to the eye, the ear, the sense of touch, and seeing in sensations, not the doors to knowledge, but only occasions for sinful errors, this strange chastity of mathematical logicians brings to mind Plotinus and those Christian ascetics of the Middle Ages who were ashamed of having a body".[72]

[70] F. Enriques, *Le matematiche nella storia e nella cultura*. Bologna, Zanichelli, 1938, pp. 173–174.
[71] F. Enriques, *Insegnamento dinamico*, Periodico di Matematiche 4, 1, 1921, pp. 6–16, at p. 10.
[72] Enriques, *Le matematiche nella storia e nella cultura*, cit, p. 145.

– Moreover, for Enriques science is a "conquest and activity of the spirit, which . . . merges in the unity of the spirit with the ideas, feelings and aspirations which find expression across all the different aspects of culture"[73], so it is important to establish links between scientific knowledge (mathematics, physics, biology, etc.) and other intellectual activities (psychology, physiology, philosophy, history, etc.). In this Enriques ran counter to Croce and Gentile, the leading exponents of Italian Neo-Idealism, who tended to devalue science, recognising in it only a practical function and a role that was completely instrumental, and separating it from the world of philosophy and culture.[74] For Enriques, the fact that science does not have purely utilitarian goals does not imply a separation between pure and applied science, but means that scientific research is valuable in itself, and does not necessarily have to aim at applications. Like Klein, he believed it was useful and necessary to maintain close ties between abstract science and applied sciences because pure sciences offer instruments that are needed for the purposes of applied science, and in their turn, applied sciences perform functions that are essential for stimulating the development of theoretical sciences, as history makes amply clear.[75]

  As a consequence of this conviction, according to Enriques, the duty of the teacher consists in inducing the students to acquire knowledge for themselves, because only in this way can they arrive at the true comprehension of mathematics. So he suggests using the Socratic method, which consists in conversing with them, acting 'a little ignorant' and, through dialogue and a guided search, leading them to a personal discovery of mathematical truth. Furthermore teachers should transmit a unified vision of knowledge to their pupils, because only by overcoming narrow specialisation can mathematics achieve its true humanistic and formative value.

– Another central aspect of Enriques' epistemological vision is his belief that scientific developments can only be fully understood in their historical connection:

> "A dynamic vision of science leads us naturally into the territory of history. The rigid distinction that is usually made between science and history of science is founded on the concept of this [history] as pure literary erudition; . . . But the historical comprehension of scientific knowledge that aims at . . . clarifying the progress of an idea has a very different meaning. . . . Such a history becomes an integral part of science."[76]

  Furthermore, history also offers the cultural legitimisation of the function of mathematics, and thus for Enriques has a central educational role in both teacher

[73] F. Enriques, *L'importanza della storia del pensiero scientifico nella cultura nazionale*, Scientia 63, 1938, pp. 125–134, at p. 130.

[74] See for example, Israel 1984; Pompeo Faracovi 2006.

[75] F. Enriques, *Il significato umanistico della scienza nella cultura nazionale*, Periodico di Matematiche 4, 4, 1924, pp. 1-6, at p. 4.

[76] Enriques, Chisini, *Lezioni sulla teoria geometrica delle equazioni*, cit., I, p. XI.

training as well as in teaching proper: future teachers should study the origins of each theory, together with its developments, not some static formulation and young people too should be "educated in the masterpieces of the masters".[77] The history of science can also constitute an important auxiliary tool for education in making it possible to better understand certain concepts or properties.

5.2. The Battle for a Scientific Humanitas in the first decades of the twentieth century

Enriques used various strategies for carrying out his cultural project and imposing his vision of a scientific *humanitas*. These followed different lines depending on context (world of culture, institutions, editorial projects, university courses) and the intended audience (university professors, secondary school teachers, philosophers and scientists in general). I will limit myself to mentioning the initiatives with the strongest links to teaching.[78]

- In 1900 Enriques published the *Questioni riguardanti la geometria elementare*, specifically designed for teacher training purposes and inspired by Klein's *Vorträge über Ausgewählte Fragen der Elementargeometrie* (1895). The topics treated were congruence, equivalence, the parallel theory and non Euclidean geometry, problems that could or could not be solved with straightedge and compass, and the constructibility of regular polygons. In fact, although for Enriques Euclidean geometry remained "the most effective tool for educating the mind, the most consistent with geometric reality", he, like Klein, nevertheless believed that the teaching of geometry could "take advantage of the progress made, in the field of the elements as well, by a more mature criticism and recent developments in higher mathematics", and that "the teacher entrusted with secondary school education must possess a much broader knowledge of such progress so that his work is inspired by much larger perspective".[79] This work also shows that Enriques did not at all disdain "the microscope" (small scale logic), and that he was aware of its usefulness in science, but while he held that questions about the foundations were important for teacher training, he was convinced that it was necessary to maintain a knowing balance between rigour and intuition in secondary teaching.[80]

- In 1903 Enriques published, with Ugo Amaldi, the *Elementi di geometria*, the first in a long and successful series of textbooks. The treatment followed a "rational inductive" method: beginning with a series of observations, the authors enunciate certain postulates from which the theorems that depend on them are developed by logical reasoning; from these theorems, they then continually return to observations or intuitive explanations. In this case as well Enriques ac-

[77] Enriques, *Insegnamento dinamico*, cit. p. 16.
[78] For further details see Giacardi 2012.
[79] F. Enriques, *Questioni riguardanti la geometria elementare*. Bologna, Zanichelli, 1900, pp. I–II.
[80] Enriques, *Insegnamento dinamico*, cit., p. 11.

knowledged Klein's influence,[81] and Klein mentions this textbook in his essay on geometry teaching in Italy, *Der Unterricht in Italien*, praising the authors for having taken didactic requirements into consideration, thus reconciling logical rigour and intuition.[82]

– In 1906 Enriques founded the Italian Philosophical Society and was its president until 1913.

– In 1906 on the occasion of the congress of the Federazione Nazionale Insegnanti Scuola Media held in Bologna, he explained his opinion regarding teacher training. He then suggested the establishment of a *pedagogical degree* in addition to the *scientific degree*: the first two years of study would be dedicated to acquiring basic knowledge of the discipline, and by the end of that time, a distinction would be made between those who intended to dedicate themselves to research and those who wanted to teach. For the future teachers, the next two years would be aimed at providing professional training by means of "1) courses on those parts of science that aim at a more profound understanding of the elements, 2) lectures on concrete questions of pedagogy that interest the various areas of teaching, particularly in relation to the analysis of the textbooks, 3) exercises comprising practice teaching, partly in the university and partly in secondary schools, drawing, and experimental technique".[83]

– In 1906, during the first International Congress of Philosophy in Milan, Enriques proposed a project to reform the Italian universities, and in accordance with his strategy, that same year explained his point of view to the middle school teachers in Bologna, to the mathematicians and scientists in his 1908 article in the *Rivista di Scienza,* and finally to university professors in 1911. His project had grown out of the ascertainment of the defects of the Italian university system which had, in his opinion, serious repercussions for research, teaching and the work world: the lack of interaction between the various faculties; the excessive fragmentation of disciplines with programs too heavy; the tendency of each professor to defend his own discipline favouring the pre-eminence of already consolidated areas of research over those which were interdisciplinary or unexplored. The solution he proposed was that of conjoining in a single faculty of philosophy all of the theoretical disciplines: mathematics, physics, physiology, history, law, economy, etc., so as to "correspond to the synthesis required by renewed philosophical consciousness and practical life, as opposed to the scientific-educational particularism of the previous era".[84] He also proposed the institution of "special schools

[81] See F. Enriques to F. Klein, 10 January 1905, in Giacardi 2012, p. 263, and now also in Luciano, Roero 2012.

[82] F. Klein, *Elementarmathematik vom höheren Standpunkte aus, I Arithmetik, Algebra, Analysis, II Geometrie, III PräzisionsundApproximationsmathematik*. Berlin, Springer, 1925–1933 (1st ed. 1908–1909), II, pp. 245–250.

[83] F. Enriques 1907, *Sulla preparazione degli insegnanti di Scienze*, pp. 69–78 in *Quinto Congresso nazionale degli insegnanti delle scuole medie. Bologna, 25-26-27-28 settembre 1906*, Atti. Pistoia, Tip. Sinibuldiana, p. 78.

[84] Quoted in Simili 2000, p. 114.

of Application" which were to group together professional teaching aimed at a specific career, the polytechnic schools for engineers and the polyclinical schools for physicians, and the Scuole di Magistero for the training of teachers.

– In 1907 Enriques founded, with Eugenio Rignano, the *Rivista di Scienza* (from 1910 *Scientia*), "an international organ of scientific synthesis", aimed at fighting excessive specialisation in the field of science and putting an end to the hegemony of literary and historic studies;[85] his imprint is particularly evident in the early years.

– In 1911 he organised the fourth International Congress of Philosophy in Bologna.

– From 1908 to 1920 he was Italian delegate to the ICMI, together with Castelnuovo and Vailati, who was substituted, after his death, by Gaetano Scorza.

– From 1912 to 1915 he was President of the National Association of University Professors.

5.3. Institutional and editorial initiatives for teacher training

Enriques' efforts and commitment to the training of teachers, and thus more generally to the improvement of mathematics education in secondary schools, are truly remarkable at both the institutional and the editorial levels.

– From 1919 to 1932 Enriques was president of the Mathesis Association. His desire to open up to other sciences is evident in the new charter for the association, which, on 7 May 1922, welcomed teachers of physics into its ranks, and led the society to assume a new name: *Società italiana di scienze fisiche e matematiche "Mathesis"*. Under the leadership of Enriques, the number of members grew from 775 in 1920 to more than 1,200 in 1924. Then, since in 1921 the *Periodico di Matematiche* had gone back to being the association's publishing venue, he assumed its direction (1921–1937, 1946)[86], giving it a strong imprint.

– From 1903 to his death he wrote many secondary school textbooks (for geometry, algebra, trigonometry, and infinitesimal calculus), in which he translates his vision of mathematics teaching into practice, thanks also to the valuable collaboration of Amaldi.

– In 1923 he founded the National Institute for the History of Sciences and the following year the University School for the History of the Sciences connected to it with the threefold aim of giving an impulse to historical research, of achieving his ideal of a scientific *humanitas*, and of training mathematics teachers.

[85] See Linguerri 2005.

[86] From 1921 to 1934 he was co-director with Giulio Lazzeri. From 1938 to 1943 Enriques' name does not appear on the title page as a consequence of the racial laws passed in 1938. In 1946 he was co-director with Oscar Chisini.

– Enriques also fostered important and successful publishing projects. First of all he broadened the 1900 work on elementary geometry, publishing his *Questioni riguardanti le matematiche elementari* (2nd ed. 1912–1914, 3rd ed. 1924–1927), a collective work whose declared aim was to contribute to the scientific and didactic training of mathematics teachers, and offer a broader vision of the theories and historical progress of mathematical science, but also to "illuminate the most elevated research as well as open the fruitful field of historical investigation to a larger number of scholars".[87] In carrying out this undertaking, he involved his friends and colleagues and various secondary school teachers.
  Less well known is the book series begun in 1925 by Enriques, *Per la storia e la filosofia delle matematiche*, for which he also drew on the collaboration of secondary school teachers. The idea behind the series came, as he himself wrote, "from teaching experience at the Scuola di Magistero";[88] the series' target audience was not only teachers, but also secondary school students and the educated public in general. In the choice of the subjects Enriques particularly favoured translations with commentaries, often accompanied by historical notes, of works by important authors of the past (Euclid, Archimedes, Bombelli, Newton, Dedekind, etc.) which might be of relevance to mathematics teaching. In fact he believed that:

> "The training of mathematics teachers who are capable of carrying out their educational responsibilities requires, generally speaking, that they understand science not only in its static aspect, but also in its developing state; and thus that they learn from history to reflect on the genesis of the ideas, and on the other hand, take an active interest in research."[89]

Each of these initiatives deserves to be presented in detail, but I will limit myself to mentioning the change in style that Enriques impressed on the *Periodico di matematiche* because it so aptly exemplifies his cultural project.[90]
The imprint of the fourth series, which began with the 1921 volume, is exquisitely Enriques', as evident starting with the title – *Periodico di Matematiche. Storia - Didattica - Filosofia* – and continuing with the introductory sentence that appears on the inside of the front cover of each issue of the journal:

> "The *Periodico* publishes above all articles regarding elementary mathematics in a broad sense, and others that tend towards a wider comprehension of the spirit of mathematics. It also contains reports on movements in mathematics abroad, notes on bibliographies and treatises, miscellany (problems, games, paradoxes, etc.) as well as news of a professional

[87] F. Enriques, *Prefazione*, in *Questioni riguardanti le matematiche elementari. Raccolte e coordinate da Federigo Enriques*. Bologna, Zanichelli 1924–1927 (Anastatic rpt. Bologna, Zanichelli 1983).

[88] F. Enriques, *Gli Elementi d'Euclide e la critica antica e moderna (Libri I–IV)*, Rome. Alberto Stock, 1925, p. 7.

[89] Enriques, *Le matematiche nella storia e nella cultura*, cit. p. 190.

[90] For the other initiatives, see Giacardi 2012.

> nature, and finally, the Proceedings of the Italian Mathematical Society Mathesis".

According to Enriques' project, the *Periodico* was intended to disseminate the idea of mathematics as an integral part of philosophical culture, an idea he had always supported, as well as to fill the gap that existed in scientific education at that time in Italy. For this reason he gave ample space to questions of methodology and philosophy, to elementary mathematics from an advanced standpoint, to physics and to history of mathematics and science, availing himself of the collaboration of mathematicians, physicists and historians of science (Ugo Cassina, Giulio Vivanti, Enrico Persico, Enrico Fermi, Ettore Bortolotti, Gino Loria, Amedeo Agostini, etc.). Enriques also encouraged an active collaboration on the part of mathematicians and young people "provided with a solid scientific preparation", but also on the part of "teachers who wished to offer the contribution of their experience".[91]
In the letter to the readers that opened the 1921 issue, he presented a working program for the journal, which was at the same time a working program for teachers. The cardinal points are: teachers should study the science that they are teaching in depth from various points of view, so as to master it from new and higher points of view, and thus make evident the connections between elementary mathematics and higher mathematics; use the history of science in the attempt to attain, not so much erudite knowledge as a dynamic consideration of concepts and theories, through which students can recognise the unity of thought; bring out the relationships between mathematics and the other sciences, and physics in particular, in order to offer a broader vision of science and of the aims and significance of the many different kinds of research.[92]
This open letter was followed by Enriques' famous article, "Insegnamento dinamico", which is almost a manifesto of his working program and of his particular vision of mathematics education: active teaching, Socratic method, learning as discovery, the right balance between intuition and logic, the importance of error, the historic view of problems, the connections between mathematics and physics, elementary mathematics from an advanced standpoint, and the educational value of mathematics.[93]
In his 1931 preface to the index of the first ten years of the second series of the *Periodico di matematiche*, Enriques underlined with pride the role played by the journal in teacher training:

> "No other journal of this sort, in no other country in the world, has been able to realise a program that is as lofty and attuned to the exigencies of education and culture of teachers of middle schools."[94]

Almost all of these last initiatives of Enriques, in the final analysis aimed at creating a scientific humanism, took place after the Gentile Reform, when the Fas-

[91] Enriques, *Ai lettori*, Periodico di matematiche, 4, 1, 1921, pp. 1–5.

[92] Enriques, *Ai lettori*, cit., pp. 3–4.

[93] Enriques, *Insegnamento dinamico*, cit.; see also the new edition accompanied by essays by F. Ghione and M. Moretti published by the Centro Studi Enriques: La Spezia, Agorà 2003.

[94] See "Indice generale Serie IV - Volumi I a X – Anni MCMXX-MCMXXX", *Periodico di matematiche*, (4) 11, 1931, pp. 3–21.

cist period had well begun. In 1923 Giovanni Gentile, minister for Education in Italy's Fascist government, carried out a full and organic reform of the school system in accordance with the pedagogical and philosophical theories he had been developing since the beginning of the 1900s. As is well known, according this reform, the secondary school system was dominated by the classical-humanistic branch, which was designed for the ruling classes and considered absolutely superior to the technical-scientific branch. The principles of Fascism and the neo-idealist ideology were opposed to the widespread diffusion of scientific culture and, above all, to its interaction with other sectors of culture. Humanistic disciplines were to form the main cultural axis of national life and, in particular, of education. This point of view was, of course, opposed to the scientific *humanitas* to which Enriques aspired. As president of the Mathesis Association, he engaged in intense negotiation with Gentile, both before and after the law on secondary education was enacted, in the hope of avoiding the devaluation of science teaching. However, the pleas of the Mathesis fell on deaf ears.

Unlike Volterra and Castelnuovo, who were in absolute opposition to the Gentile Reform, Enriques assumed and maintained a conciliatory position. In fact, his ideal was to achieve a fusion between "scientific knowledge" and "humanistic idealism" in a "superior awareness of the universality of thought".[95] However, he was opposed to all forms of nationalistic isolation, as is shown, for example, by the organisation of the international meeting entitled "Settimana della Scuola di Storia delle Scienze" (Rome, 15-22 April 1935). Among the participants, in addition to lecturers at his School for the History of the Sciences (Roberto Almagià, Silvestro Baglioni, Giuseppe Montalenti, Giovanni Vacca), there were Castelnuovo, Enrico Bompiani and Giuseppe Armellini and twenty-six members from London's Unity History School as well as scholars from other European countries, including the Belgian Paul Libois, who would draw various aspects of his own vision of mathematics teaching from Enriques, and the French historian Hélène Metzger, who shared Enriques' unitary concept of science. The topics addressed ranged from philosophy to the history of physics, astronomy, biology and technology, and the debate was lively, as can be seen from the detailed summary of the week's activity written by Metzger herself.[96]

During the same period Enriques also participated in the meetings (Paris, Vienna, Berlin) and congresses (Heidelberg, 1927; Barcelona, 1929; Paris, 1933; Budapest, 1934; Zurich, 1938) of the *Fédération internationale des Unions intellectuelles*, in addition to various other international congresses of philosophy, history of philosophy and philosophy of science: it was no coincidence that Enriques remained in contact with the *Fédération* whose aim was to promote international cultural exchange. He also directed two sections – *Philosophie et histoire de la pensée scientifique and Histoire de la pensée scientifique* – of the book series *Actualités scientifiques et industrielles* published by Hermann in Paris. Between 1934 and 1939 eight volumes were published in the first series, with the collaboration of Metzger, Ferdinand Gonseth and Castelnuovo, and six in the second series, written in collaboration with Giorgio de Santillana.

[95] Enriques, *Il significato umanistico della scienza nella cultura nazionale*, cit. p. 4. See also Israel 1984, Guerraggio, Nastasi 1993, Pompeo Faracovi 2006

[96] See Metzger 1935.

Up the end in 1946 Enriques fought his battle for a scientific humanitas and was involved in teacher training, which he believed to be the crucial element for the formation of good schools and one of the channels for achieving his cultural project.

6. Francesco Severi: politics and education.

Graduating with Segre in 1900, Severi was, as is well known, a top level mathematician who made very significant contributions in the field of algebraic geometry as well as various other areas in mathematics, which earned him numerous prizes, recognitions and prestigious positions over the course of his career.

Two factors are of prime importance for fully understanding Severi's stance with regard to education: his singular political path, and his relationship, first of collaboration and then of conflict, with Enriques.

Severi was a socialist during the period he was in Padua; as rector in Rome, he resigned after the murder of Giacomo Matteotti; he was a signer of Benedetto Croce's Manifesto of the Anti-Fascist Intellectuals; and a supporter of those who opposed the fascistization of the University of Rome. Nevertheless, following his nomination to the Accademia d'Italia in the spring of 1929, he supported Fascism without reserve and later had no compunction about using the racial laws to assume absolute control over Italian mathematics.

His collaboration with Enriques began right after his degree, intensified during the period in which Severi was Enriques' assistant in Bologna, and reached its peak in their work on hyperelliptic surfaces, which was awarded the Prix Bordin of the Académie des Sciences in Paris (1907). In the years that followed their relationship was increasingly marked by divergences on scientific, academic, editorial and cultural levels.[97]

6.1. Enriques' influence and successive rivalries

To be sure, the influence of Enriques is one of the principal factors underlying Severi's interests in mathematical epistemology and teaching. To see this we need only look at the writings and events of the period from 1902 to 1920. In 1906 Severi published his *Complementi di geometria proiettiva* (1906) as an integration to Enriques' *Lezioni di geometria proiettiva* (1903). The two textbooks were born in symbiosis, and Severi accepted the epistemological and didactic vision of his mentor. Between 1906 and 1920 he wrote various articles and reviews[98] which also demonstrate an acceptance

[97] Cf. Brigaglia, Ciliberto 1995, pp. 24–32 and 36–41; the essays of Brigaglia, Ciliberto and Vesentini, in Pompeo Faracovi 2004.

[98] 1903, *Estensione e limiti dell'insegnamento della matematica, in ciascuno dei due gradi, inferiore e superiore, delle Scuole Medie*, Il Bollettino di Matematica, 2, pp. 50–56 (with F. Enriques and A. Conti); 1906, Review of *F. Enriques, "Problemi della scienza"*, Rivista di filosofia e scienze affini, 8, 2, pp. 527–541; 1908, Review of G. Loria, *"Il passato e il presente delle principali teorie geometriche"*, Rivista di Scienza, 4, pp. 376–378; 1910, *Ipotesi e realtà nelle scienze geometriche*, in *Atti della Società Italiana per il Progresso delle Scienze*, 3, pp. 191–217 (also in Scientia, pp. 1-29 and the French translation in Suppl. pp. 3-32); 1911, *La nostra scuola*, Padova; 1914, *Razionalismo e spiritualismo*, Conferenze e prolusioni, 10, pp. 181–189; 1919, *La matematica*, Energie Nuove, 9, pp. 196–199; 1920, *L'istruzione professionale*, in *Atti del Congresso degli Agricoltori e Bonificatori*, Padova.

of many of Enriques' methodological assumptions: knowledge proceeds by successive approximations; geometry is seen as a part of physics; the historical and psychological genesis of mathematical concepts; the importance of analogies and induction in discovery; the use of an experimental, intuitive approach in mathematics teaching.
In particular, in the 1914 article entitled "Razionalismo e spiritualismo" Severi sided with Enriques against the idealism of Croce, proclaiming the cognitive and aesthetic value of science and illustrating the harmful consequences of the "movement against science" on the levels of society and education.[99]

Severi's burning ambition to occupy top level positions within the mathematical and academic communities inevitably led to his first clashes with Enriques on scientific, academic and personal planes. He knew how driven he was; he himself said, "My will is tenacious to the point of obstinacy".[100]
When, in 1909, he became president of the Mathesis Association, Severi attempted to insert himself into the work of the Italian subcommission of the ICMI, whose three delegates at the time, nominated directly by ICMI Central Committee, were, as we have said, Castelnuovo, Enriques and Vailati. In fact, the Mathesis Association was not officially part of the delegation. To reach his objective, Severi sought the support of Volterra, and even suggested that Vailati should be encouraged to resign: "… poor Vailati, afflicted as he is by his long illness, might do well to step down … and then much could be put to rights by having a replacement elected by the Mathesis".[101] His attempts to impose himself were not successful because Enriques and Castelnuovo believed that it was important that the subcommission, while collaborating with the Mathesis, maintain its "freedom to act" and not be obliged to conform to the directives of the Association.[102]
This first setback was followed by another. During his term as president, Severi sent repeated requests (January 1909, February and April 1910) to the different Ministers for Education at the time asking them to consider the proposals of the Mathesis regarding the reform of the Scuole di Magistero for teacher training, the abolition of the choice between Greek and mathematics beginning in the second year of liceo that had been introduced by the Orlando Decree of 1904, and the reinstatement of the written exam in mathematics for all categories of schools. Severi was able to obtain from the Minister only a few general promises, and in all likelihood these setbacks drove him to look for different ways to achieve his ends and impose his power on the mathematical and academic communities. Thus on 6 November 1901 he announced his resignation and that of the Mathesis executive committee:

> "And we intend to communicate our decision to the largest daily newspapers, so that public opinion will pause, at least for a moment, to consider whether the slight regard in which cultural Societies, such as ours, are held by executive power, constitutes the most suitable means for stimulating that disinterested attachment to Education, which, despite everything,

[99] F. Severi, *Razionalismo e spiritualismo*. Conferenze e prolusioni, 10, 1914, pp. 181–189, at p. 187.
[100] F. Severi, *Confidenze*, La scienza per i giovani, 1952, II, pp. 65–69, at p. 69.
[101] F. Severi to V. Volterra, Padova, 20 April 1909 in Nastasi 2004, p. 180.
[102] See the letters of Severi to Volterra in Nastasi 2004, pp. 176–181.

teachers still show themselves to hold."[103]

In any case, Severi deserves the credit for having put his finger, during his brief term as president, on the two main weaknesses of the Mathesis, calling for, on one hand, the reform of the *Bollettino della Mathesis*, which was supposed to be transformed from a simple administrative tool into a journal with articles about science and education, and on the other hand, a strengthening of the Association's congresses, which were to offer rich programs and, above all, fighting absenteeism.[104] His wishes would be carried out by the presidents who succeeded him, first Castelnuovo and then Enriques.

After Croce's sharp attack on his article "Razionalismo e spiritualismo",[105] Severi began to distance himself from Enriques, which led to scientific and cultural battles to dominate Italian mathematics. These, as has been said, led to a genuine "pursuit" on scientific, academic, educational, editorial and cultural planes.[106] In 1921 Severi brought to light an error in an article of Enriques, leading to a polemic that would last over twenty years.[107] That same year, supported by Tullio Levi Civita, Severi had the better of Enriques for the transfer to Rome to the chair of algebraic analysis left vacant by Alberto Tonelli. Enriques would assume the chair in higher geometry in 1923, thanks only to Castelnuovo's renunciation of it.[108]

Quick to understand the mechanisms of political power and exploit them to his own advantage, with his nomination to the Accademia d'Italia in 1929,[109] Severi went, as mentioned above, from being an anti-Fascist to being a fervent Fascist. In 1929-1931 he had no qualms about collaborating on the draft of a new form of oath of loyalty to the Fascist party.[110] He then began to cooperate in the process of the fascistization of culture, contributing to widen that breach between Italian mathematicians and the international mathematics community which was one of the reasons for the weakening of mathematics research in Italy that ensued. When he later became conscious of this fact, he attempted to halt the process of decline by creating in 1939 of the Istituto di Alta matematica (Institute for Higher Mathematics).[111]
On this aspect of Severi's personality, Francesco Tricomi wrote:

> "Severi . . . wanted to be (and to a certain extent, was) the 'godfather' of Italian mathematics during the Fascist period. We in any case have the consolation of knowing that — while, as a rule, totalitarian regimes put

[103] See *Dimissioni del CD*, Bollettino della "Mathesis", 2, 1910, p. 90.

[104] See *Sezione veneta, Adunanza del 20 maggio*, Bollettino della Mathesis, 1, 1909, pp. 31–32, and *Programma del prossimo Congresso sociale*, Ibid., pp. 51–52.

[105] B. Croce, *Se parlassero di matematica?*, La Critica. Rivista di letteratura, storia e filosofia, XII, 1914, pp. 79–80.

[106] See Faracovi 2004, especially the essays of Vesentini, Ciliberto, Brigaglia, Bolondi, Faracovi and Linguerri.

[107] Brigaglia 2004, pp. 66–77, Ciliberto 2004, pp. 44–49.

[108] See T. Nastasi 2010, Appendice 2, *Il trasferimento di Enriques a Roma.*

[109] Enriques' name was included on the early list of candidates of scientific disciplines but was stricken at the last moment; see Goodstein 1984, p. 294.

[110] F. Severi to G. Gentile, Barcelona, 15 February 1929, in Guerraggio, Nastasi 1993, pp. 211–213.

[111] On the effects of his activities on Italian research in mathematics, see for example, Israel 1984 §5–6; Guerraggio, Nastasi, 1993; Israel 2010, Chap. 6.

> the worst elements in positions of control, only because they are violent or subservient or both — in the case of Severi, the man was, from a scientific point of view, irreproachable".[112]

The "Severi case" has been amply studied by historians, so here I will only mention Severi's open opposition to Enriques. He refused to collaborate with the *Enciclopedia italiana* on the mathematics section, of which Enriques was director, writing: "with a man such as Enriques, ... I can no longer have anything in common, much less a relationship akin to subordination."[113] He opposed the request that university chairs be established for history of science, presented by Enriques to the Accademia dei Lincei in 1938.[114] That same year Italy's shameful racial laws were put into effect, and Severi unhesitatingly exploited them in order to rise to a position of absolute predominance in Italian mathematics. In fact, when Enriques was dismissed from the University because of the racial laws against Jews, he immediately transferred to the chair of higher geometry held by Enriques, and in February 1939 he assumed direction of the University School for the History of the Sciences created by Enriques, leading finally to its closure. As president of the Vallecchi publishing house in Florence, he took advantage of the circular issued by Minister of Education Giuseppe Bottai in August 1938, which ordered school principals to eliminate from use all textbooks written by Jewish authors, to replace the geometry textbooks for secondary schools by Enriques and Amaldi with his own textbooks, published by Vallecchi.[115]

Severi's opinions regarding the Gentile Reform were in many respects similar to those of Enriques: he was convinced of the superiority of the *ginnasio-liceo*,[116] he was in favour of combining mathematics and physics but held that too few hours were dedicated to mathematics, and that the number of hours assigned to teachers (22) was too heavy.[117] There were however points where their opinions differed: Severi tended to share the nationalistic and autarchic vision of scientific research,[118] while Enriques instead observed:

> "In scientific discovery there is ... a universal value that transcends the person of the discoverer, and also the forma mentis that he may have received from his people. ... greatness and decline of culture alternate to the

[112] Tricomi 1967, p. 55.

[113] F. Severi to G. Gentile, Arezzo 24 May 1928, in Guerraggio Nastasi 1993, pp. 209-210; see also G. Bolondi, *Enriques, Severi, l'Enciclopedia Italiana e le istituzioni culturali*, in Pompeo Faracovi 2004, pp. 79–106.

[114] Enriques, *L'importanza della storia del pensiero scientifico nella cultura nazionale*, cit.

[115] S. Linguerri, *Federigo Enriques e Francesco Severi: una concorrenza editoriale*, in Pompeo Faracovi 2004, pp. 151-154.

[116] *Riunione straordinaria promossa dal consiglio direttivo, Roma 11 febbraio 1923*, Periodico di Matematiche, 1923, pp. 156–157.

[117] See F. Severi, *L'insegnamento della geometria nei suoi rapporti colla riforma*, Annali dell'istruzione media, 3, 1927, pp. 108–116, at p. 116.

[118] See, for example, F. Severi, *Scienza pura e applicazioni della scienza*, in *Atti del I Congresso dell'Unione Matematica Italiana*, Zanichelli, Bologna, 1937, pp. 13–25; also in Scienza e tecnica, 1, 1937, n. 4, pp. 8-1-89, at p. 89; F. Severi, *Interventi al Convegno di Padova per l'istruzione media, classica, scientifica e magistrale*, Scuola e cultura,1939, pp. 62–65, at p. 65.

> extent that the exchange of ideas with other peoples, near or far, in space and time, either grow and intensify, or oppositely, weaken."[119]

How Severi adapted himself to Fascist directives can also be seen in his *Curriculum vitae*, where he states that he "had contributed also with his textbooks concerning the most elementary fields of mathematics, to renovate teaching methods" in middle schools, "adapting them to new lines of knowledge and new pedagogical needs determined by Fascism".[120] On the other hand, Gentile, in the preface to one of Severi's geometry textbooks wrote:

> "I am pleased to see that books such as these by Prof. Severi are beginning to be published for the study of mathematics in middle schools.
> The new Italian school must be an active school, one which sets, at all levels and in all forms of teaching, the student's spiritual strengths into motion, allowing him to feel the fatigue and joy of understanding for himself, of discovering for himself and acquiring his own truth . . . And to me these books seem to correspond wonderfully to our desire that these subjects as well . . . be presented in the most suitable form for beginners: the heuristic form of the concept arrived at by means of intuitions that are concrete, evident and attractive."[121]

Furthermore, when in 1939 the Grand Council of Fascism approved the twenty-nine declarations contained in the *Carta della Scuola* (School Charter) presented by Minister of Education Bottai with the aim of further fascistizing Italian schools, Severi declared that he agreed "to every single part of it".[122]

6.2. Mathematics teaching: methodological assumptions and their effects on textbooks

This said, the cornerstones of Severi's methodological and pedagogical vision were nevertheless very close indeed to those of Enriques, although the epistemological considerations upon which they were founded were not as broad and detailed:

– secondary school must have an essential formative aim and a "frank humanistic basis"; to these ends mathematics plays an important role because it trains the faculties of intuition and abstraction and develops an aptitude for "observing, abstracting, and deducing";[123]

[119] F. Enriques, *L'Italia nella collaborazione universale della cultura*, Nuova Antologia, s. 7, 247, 1926, pp. 129–134, at p. 132, 133.

[120] F. Severi, *Curriculum vitae* (1938-XVI), in `http://dm.unife.it/matematicainsieme/riforma_gentile/pdf/Gentile09.pdf`.

[121] F. Severi, *Elementi di geometria pel ginnasio e pel corso inferiore dell'istituto tecnico, Volume I*, Firenze, Vallecchi, 1926, p. V.

[122] Severi, *Interventi al Convegno di Padova*, cit. p. 63.

[123] F. Severi, *Relazione al Convegno di Firenze per l'istruzione classica scientifica e magistrale*, Scuola e cultura, 1940, pp. 70–73, at p. 72–73.

– humanism must not be disjoined from scientific thought, in fact true humanism is integral by nature[124] – thus it is necessary to transmit to the student a unitary vision of culture, and scientific, historic, literary and philosophic teaching must be "maintained in the same plane";[125]

– mathematics teaching must have an intuitive character in lower middle schools and a rational character in upper middle schools, proceeding by "successive approximations" from the concrete to the abstract, and allowing time for the ideas to "filter slowly through the brains, if it is desired that they leave traces that are useful and lasting".[126] In any case, in teaching precedence must be given to intuition because it develops in a way that is natural and direct, as a "synthesis of sensations, observations and experiences", almost without any wilful effort at attention on the student's part,[127] and because only intuition provides the raw material to the logical machine:

> "[students are] taught to reason . . . by reasoning well; not by dissecting the reasoning";[128]
> "It is necessary to take middle school teaching of mathematics back to its practical and intuitive origins; and this not only for practical reasons (which in middle school could have no prevailing weight), but above all precisely for the educational goals of secondary studies;[129]

– it is important to use "the utmost parsimony in formulating programs, reducing them for each discipline to things which are truly essential and which have unquestionable educational value".[130] In particular, Severi suggests abandoning the cyclical method by which subjects already treated in an intuitive way in middle schools are repeated in a rationally developed way in secondary schools, and to "bring teaching closer to the current state of science";[131]

– it is useful for teachers to link mathematics teaching to that of physics in order to "give new impetus to their own teaching by means of continuous and fruitful contact with the real world";[132]

– the teacher must play a central role in guiding the student in learning:

> "Having discovered the main path [to learning], it is necessary to travel it anew, and to clear away the difficulties that are too serious

[124] F. Severi, *Relazione al Convegno di Messina per l'istruzione media, classica, scientifica e magistrale*, Scuola e cultura, 1940, pp. 136–138, at p. 137.
[125] Severi, *Relazione al Convegno di Firenze*, cit., p. 70.
[126] F. Severi, *La matematica*, Energie nove, II serie, 9, 1919, pp. 196–199, at p. 197; see also F. Severi, *Didattica della matematica*, Enciclopedia delle Enciclopedie: Pedagogia, Roma, Formiggini, 1931, pp. 362–370, at p. 365.
[127] Ibid., p. 198.
[128] Severi, *Didattica della matematica*, cit., p. 368.
[129] Ibid., p. 368.
[130] Severi, *Relazione al Convegno di Messina*, cit. p. 138.
[131] Severi, *Relazione al Convegno di Firenze*, cit. p. 72, 73.
[132] Severi, *Didattica della matematica*, cit., p. 365.

> for non-experts, so that the student can travel them along with us, following us, without excessive effort, in the process of constructing knowledge";[133]

– it is necessary to stimulate "the youthful desire for conquest", to involve the students in the process of constructing knowledge and exhort them to acquire mathematical truths for themselves: "allowing them to find everything nice and ready, does them no good";[134]

– the history of science can play a significant educational role: Severi, like Enriques, believed that in order to facilitate students' comprehension of certain mathematical concepts it is useful to take their historical origins as a point of departure,[135] and he himself used history in his lessons at university[136] as well as in the courses of specialisation:

> "don't forget the masters, because an ingenious idea is worth more in creative power than all of its consequences. And in order to follow the thought of the masters it is necessary to not distance ourselves from historical development of the ideas and from that troublesome but indispensable instrument, the bibliography."[137]

A brief overview of the way in which Severi conceived mathematics teaching appears in the entry "Didattica della matematica" that he wrote for the *Enciclopedia delle Enciclopedie* (1931), which includes an historical excursus about the teaching of this discipline in Italy that goes from the use of the textbooks by Legendre and Bertrand at the beginning of the nineteenth century up to the Gentile Reform.

How this vision of teaching is translated into practical terms emerges above all from the textbooks for lower and upper secondary schools, which constitute Severi's most important and lasting legacy regarding secondary teaching. Beginning in 1926, he directed the book series entitled *Collezione di testi di matematica per le scuole medie* for the Vallecchi publishing house in Florence. The series included textbooks for geometry, arithmetic, algebra (with trigonometry, financial mathematics and infinitesimal analysis), which were often written in collaboration with two teachers, his niece Maria Mascalchi[138] and Umberto Bini.[139] The distinguishing features of the books in this

[133] F. Severi, *Elementi di geometria pei licei e pel corso superiore dell'istituto tecnico, Volume II*, Firenze, Vallecchi, 1927, p. V.

[134] Ibid.

[135] See for example Severi, *Didattica della matematica*, cit., pp. 362-370.

[136] See R. Migliari, *L'insegnamento della Geometria Descrittiva e delle sue applicazioni*, in *La Facoltà di Architettura di Roma "La Sapienza" dalle origini al duemila. Discipline, docenti, studenti*, a cura di Vittorio Franchetti Pardo, Roma, Edizioni Gangemi, 2001, pp. 279–282.

[137] F. Severi, *Del teorema di Riemann-Roch per curve, superficie e varietà. Le origini storiche e lo stato attuale*. Varenna, CIME 1955, Roma, Istituto matematico dell'Università, 1955, p. 38.

[138] Maria Mascalchi (1902-1976), with a degree in mathematics at the University of Turin in 1923, and in physics in 1931, obtained in 1928 the chair in mathematics and physics at the Liceo classico d'Azeglio in Turin. See Archivio Storico of the University of Turin and the Archivio storico of the Liceo classico d'Azeglio in Turin.

[139] Umberto Bini taught at the R. Liceo scientifico in Rome.

series are the use of an intuitive approach, but with due attention to rational aspects, suitably arranged according to school level and type of school, brevity of treatment, mentions of history of mathematics, questions to facilitate learning, good exercises, clarity, precision and conciseness. In particular, the textbook entitled *Elementi di geometria*, adapted for the various types of schools, is distinguished by its particular approach to the principal topics of geometry (congruence, equivalence, the parallel theory, theory of proportions), as well as for the methodological framework dictated by the concern that the student not overlook the intuitive underpinnings of each notion introduced.

Rather than going into detail regarding the individual textbooks, I will mention only the numerous reflections of Severi scattered throughout his writings regarding the criteria to be respected in order to produce a good mathematical textbook.[140]
First of all, with regard to the use of textbooks, Severi observes that while in primary school teaching of geometry and arithmetic must be essentially oral, and "the importance of the textbooks is minimal",[141] "in secondary schools the advice to follow the texts, without the deplorable system ... of taking notes, must be strictly respected".[142] In general, "the exposition of the subject must ... allow it to be assimilated by mediocre minds and yet encompass a more hidden meaning, which induces better minds to more profitable meditation";[143] the treatment must be rich in intuitive observations, and must make "clearly distinguishable, for those who have the capacity, that which is taken from intuition and that which must be deduced".[144]

With regard to the kind of language to be adopted, it is necessary that "particular care be taken, both from the point of view of correctness, sobriety, correspondence between word and idea, and that of simplicity".[145]

Concerning proof, according to Severi this must be presented at the first stage as equivalent "to a reduction to the evidence", and the postulates must appear to the students as "explicit expressions, which could have remained unstated, of intuitive facts":[146] it is not necessary that the purely logical function of these be understood. Rigour can be arrived at gradually, taking care to give greater importance to "rigour in substance" – in line with which the framework of the treatment must appear to be "impeccable from a rational point of view" – rather than to "formal rigour".[147]

[140] See Giacardi, L., Tealdi, A., *Francesco Severi and mathematics teaching in secondary schools. Science, politics and schools in the first half of the twentieth century*, in "Dig where you stand" 3. Proceeedings of the Third International Conference on the History of Mathematics Education, K. Bjarnadóttir, F. Furinghetti, J. Prytz, G. Schubring (Eds.), Uppsala: Uppsala University, 2015, pp. 187-202.

[141] Severi, *Didattica della matematica*, cit., pp. 368–369.

[142] Severi, *Relazione al Convegno di Firenze*, cit. p. 71.

[143] F. Severi, *Elementi di geometria I pel ginnasio e pel corso inferiore dell'istituto tecnico*, edizione completa, 4º ristampa, Firenze, Vallecchi, 1933, p. IX.

[144] Severi, *Didattica della matematica*, cit., p. 368.

[145] F. Severi, M. Mascalchi, *Nozioni di Aritmetica pratica, con cenni storici per il* 1º *e* 2º *anno della Scuola media*, (con M. Mascalchi), Firenze, Vallecchi, 1941, p. 1.

[146] F. Severi *Elementi di geometria pel ginnasio e pel corso inferiore dell'istituto tecnico*, Firenze, Vallecchi 1926, pp. VIII, IX.

[147] Ibid., p. VIII.

Another point which Severi particularly emphasises is the importance of having opportune explanations precede the definitions of the mathematical objects:

> "I have taken the most scrupulous care to avoid definitions *ex abrupto*. These are most irksome. I do not give definitions without an appropriate prelude ... And I have no fear of being verbose. ... The time given to a good understanding of the meaning of a definition is ... excellently spent, always ... He [the student] must construct the definitions himself, beginning with the common sense notions that he possesses".[148]

Finally, Severi was a fervent supporter of the need for brevity of treatment, stripping it of anything that is not essential to the comprehension of the structure of a mathematical theory, with the aim of both "reducing the burden of the students, without damaging the educational function of the teaching of mathematics, and geometry in particular",[149] and of making room for more modern topics. Nevertheless, he emphasises that "simplicity and clarity deriving from implied statements, that are neither educationally nor scientifically honest, are not what I aspire to, and nothing could induce me to introduce such an approach into my work".[150]

Conclusions

From our examination of the commitment to questions regarding mathematics teaching of these four eminent exponents of the Italian School of algebraic geometry emerges a core set of shared assumptions whose roots lie in their common way of conceiving mathematical research, and which constitute an additional indicator of the appropriateness of the term "School" in speaking of the Italian geometers. Using the word "School", we are referring both to a group of researchers trained by the same *maestri*, from whom they draw topics of investigation, methodologies, approaches to research and a particular scientific style, and a place where talents are developed and contacts made, as well as an environment, as we have tried to show, in which a common vision of the transmission of knowledge matures, while still considering the opportune *distinguo*.

Thus, if we attempt to draw a conclusion from their multifaceted activities and provide a comprehensive overview, we first of all observe that the common vision of scientific research, and the influence of Klein, so clearly documented above all in Segre, Enriques and Castelnuovo – who were able to meet him personally and work for some time alongside him – led these mathematicians to share the following pedagogical assumptions:

– the attribution of an educational value to mathematics, in the hope of creating a scientific *humanitas* (an integral humanism);

[148] Severi, *Elementi di geometria I pel ginnasio e pel corso inferiore dell'istituto tecnico*, 1933, cit. p. X.
[149] F. Severi, *Geometria, Volume I*, Firenze, Vallecchi, 1934, p. V.
[150] Ibid., p. VII.

– using to the best of their advantage the faculty of intuition and the heuristic procedures in teaching;

– aiming at rigour in substance (large-scale logic), rather than formal rigour (small-scale logic, microscope, etc.);

– the establishment of connections between mathematics and other sciences, and between mathematics and applications;

– the attribution of importance to the history of mathematics in teaching and in research.

These were the assumptions that directed their activities in education, in spite of the fact that their motivations and even the strategies they employed sometimes followed different channels.

For Segre it was above all the intimate connection that he saw between teaching and research that led him to become personally involved in teacher training, and at the same time present to his university students topics that were useful for teaching. Instead, Castelnuovo's motivation was mainly social, because he believed it was important to train people to be capable of understanding the reality in which they live and work in order to improve it. His idea for the *liceo moderno* grew out of his belief in a cultured democracy, one capable of providing the basis of a modern nation. The channels he used to make his ideas concrete were essentially three: involvement with national and international institutions – ICMI, the Mathesis Association, the Ministry of Education; teaching university courses; and directly involving teachers.

What led Enriques to become involved in problems of education and in secondary teaching were his strong philosophical, historical and interdisciplinary interests, especially the studies on the foundations of geometry. He adopted a range of strategies and, as we have seen, worked on different fronts: institutional, editorial (periodicals, book series, textbooks), and cultural. Further, he addressed his activities to different categories – secondary school teachers, researchers, philosophers, scientists, people of culture – asking for their cooperation. His direction of the *Periodico di Matematiche* is significant in this respect.

Severi's intellectual itinerary was of yet a different nature: his interest in problems concerning the secondary teaching of mathematics was inspired both by his relationship, first of collaboration and then of rivalry for leadership, with Enriques, and by political reasons. After his unsuccessful bid to insert himself into the ICMI, and the sparse results of his presidency of the Mathesis Association, Severi supported the school policies of the Fascist regime, while holding firmly to the pedagogical assumptions of the Italian School of algebraic geometry. His channel of choice for improving mathematics teaching was the publishing of schoolbooks, and as might be expected of such a great mathematician, he produced textbooks that were paragons of clarity, precision and conciseness.

To a much greater extent than Segre, Castelnuovo and Severi, and above all Enriques, took special advantage of journals to spread their point of view. Castelnuovo, during his presidency of the Mathesis Association, used the *Bollettino della Mathesis* to further his agenda, inserting reports on the work of the Italian subcommission of the ICMI. Without relying on any particular periodical, Severi published his reflections on teaching in different journals aimed at diverse readerships: before 1923, *Rivista di Scienza, Energie Nuove, Rivista di filosofia e scienze affini*, and after 1923, *Annali dell'istruzione media, L'illustrazione italiana*, and above all *Scuola e cultura*, a magazine that was particularly supportive of the Fascist Regime. It was, however, especially Enriques who exploited periodicals to his advantage in spreading his idea of scientific *humanitas*: for the *Rivista di Scienza* (later named *Scientia*) he wrote no fewer than 22 articles, 63 reviews and 25 digests of magazines, and to the *Periodico di matematiche* he contributed 26 articles, 33 reviews and numerous other interventions in his role as president, covering topics in mathematics, philosophy, history of science, and education.

Finally, these mathematicians differed in their reactions to the Gentile Reform and to the problems of the devaluation of science and the autarchy touted by Fascism. We have seen that Castelnuovo always maintained an attitude of determined opposition, while Enriques, even while continuing to uphold the educational and cultural value of mathematics, maintained a conciliatory position. In fact, he agreed with Gentile on many points: he was convinced that among the various kinds of secondary schools, those which best performed the function of education were the *ginnasi-licei* (schools with an emphasis on the humanities); he conceived of knowledge as a personal conquest; he was in agreement with the need to fight encyclopaedism and he considered education to be the free and unfettered development of inner energy. Moreover, as we mentioned above, he did not want to renounce his idea of the fusion of scientific knowledge and humanistic idealism which was the basis of the cultural program he had dedicated his whole life. With regard to the ideological tendency of the Fascism to give pre-eminence to applied sciences as a means of solving problems arising from autarchy, Enriques never abandoned his idea of linking pure mathematics to its applications, even though this originated primarily from philosophical and didactic considerations. Moreover, he did not share the principle of autarchy, but was open and ready to engage in a dialogue with the international scientific community.
Instead, Severi more than once expressed his acceptance of the cultural directives of the Fascist Government, and only later became aware of the harm that scientific isolation could lead to. In spite of this, he always maintained the pre-eminence of pure science, which he believed to be a source of beauty and art, over applied science, without any concession made to its social function:

> "more often than not, ideas, which are the true engines of human society, descend from abstract regions to enliven the applications and provide their most useful and powerful orientations. Where abstract science is neglected, practice soon becomes arid and civilisation begins a rapid decline".[151]

What the legacy of the wide-ranging and diversified activities of these Italian

[151] Severi, *Scienza pura e applicazioni della scienza*, cit. p. 83, 84; see also Israel, Nurzia 1989, at pp. 139–143.

geometers has been for the development of mathematics teaching in Italy is a subject that has been examined up to now only sporadically, and is not the aim of this present paper. However, to me their intellectual bequest seems evident, and I would like to highlight three aspects in particular: the belief that it is important that mathematicians who are active in research be involved in problems related to teaching; the need to invest substantial resources in teacher training and to bring about a greater interaction between universities and secondary schools; and finally, the belief that it is important to develop an integrated humanism in schools, a goal to which the history of mathematics can make a valid and valuable contribution.

I am very grateful to Gabriella Viola for her invaluable help in finding some of Severi's textbooks and other documents concerning him, and to Antonio Salmeri for his generous aid in bibliographic research. Thank you also to the personnel of the Biblioteca "Giuseppe Peano" of the Department of Mathematics of the University of Torino, and to Kim Williams for linguistic advice.

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AMS Subject Classification: 01A60, 01A72

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Lavoro pervenuto in redazione il 15.12.2012.

Rend. Sem. Mat. Univ. Politec. Torino
Vol. 71, 3–4 (2013), 463 – 472

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EQUIVALENT BIRATIONAL EMBEDDINGS III: CONES

Dedicated to Alberto Conte on the occasion of his 70th birthday.

Abstract. Two divisors in $\mathbb{P}^n$ are said to be Cremona equivalent if there is a Cremona modification sending one to the other. In this paper I study irreducible cones in $\mathbb{P}^n$ and prove that two cones are Cremona equivalent if their general hyperplane sections are birational. In particular I produce examples of cones in $\mathbb{P}^3$ Cremona equivalent to a plane whose plane section is not Cremona equivalent to a line in $\mathbb{P}^2$.

Introduction

Let $X \subset \mathbb{P}^n$ be an irreducible and reduced projective variety over an algebraically closed field. A classical question is to study the birational embedding of X in $\mathbb{P}^n$ up to the Cremona group of $\mathbb{P}^n$. In other words let X_1 and X_2 be two birationally equivalent projective varieties in $\mathbb{P}^n$. One wants to understand if there exists a Cremona transformation of $\mathbb{P}^n$ that maps X_1 to X_2, in this case we say that X_1 and X_2 are Cremona equivalent. This projective statement can also be interpreted in terms of log Sarkisov theory, [3], and is somewhat related to the Abhyankar–Moh problem, [1] and [10]. In the latter paper it is proved, using techniques derived form A–M problem, that over the complex field the birational embedding is unique as long as $\dim X < \frac{n}{2}$. The problem is then completely solved in [14] where it is proved that this is the case over any algebraically closed field as long as the codimension of X_i is at least 2. Examples of inequivalent embeddings of divisors are well known, see also [14], in all dimensions. The problem of Cremona equivalence is therefore reduced to study the equivalence classes of divisors. This can also be interpreted as the action of the Cremona group on the set of divisors of $\mathbb{P}^n$.

The special case of plane curves received a lot of attention both in the old times, [5], [17], [11], and in more recent times, [16], [8], [12], [4], and [15], see also [7] for a nice survey. In [4] and [15] a complete description of plane curves up to Cremona equivalence is given and in [4] a detailed study of the Cremona equivalence for linear systems is furnished. In particular it is interesting to note that the Cremona equivalence of a plane curve is dictated by its singularities and cannot be divined without a partial resolution of those, [15, Example 3.18]. Due to this it is quite hard even in the plane curve case to determine the Cremona equivalence class of a fixed curve simply by its equation.

The next case is that of surfaces in $\mathbb{P}^3$. In this set up using the $\sharp$-Minimal Model Program, [13] or minimal model program with scaling [2], a criterion for detecting surfaces Cremona equivalent to a plane is given. The criterion, inspired by the previous work of Coolidge on curves Cremona equivalent to lines [5], allows to determine all rational surfaces that are Cremona equivalent to a plane, [15, Theorem 4.15]. Unfortunately, worse than in the plane curve case, the criterion requires not only the resolution of singularities but also a control on different log varieties attached to the pair $(\mathbb{P}^3, S)$. As a matter of fact it is impossible to guess simply by the equation if a rational surface in $\mathbb{P}^3$ is Cremona equivalent to a plane and it is very difficult in general to determine such equivalences. The main difficulty comes from the condition that the sup-threshold, see Definition 4, is positive. This is quite awkward and should be very interesting to understand if this numerical constrain is really necessary. Via $\sharp$-MMP it is easy, see remark 4, to reduce this problem to the study of pairs (T,S) such that T is a terminal $\mathbb{Q}$-factorial 3-fold with a Mori fiber structure $\pi : T \to W$ onto either a rational curve or a rational surface, and S is a smooth Cartier divisor with $S = \pi^* D$ for some divisor $D \subset W$. The first "projective incarnation" of such pairs are cones in $\mathbb{P}^3$.

In this work I develop a strategy to study cones in arbitrary projective space. If two cones in $\mathbb{P}^n$ are built on varieties Cremona equivalent in $\mathbb{P}^{n-1}$ then also the cones are Cremona equivalent, see Proposition 1. The expectation for arbitrary cones built on birational but not Cremona equivalent varieties was not clear but somewhat more on the negative side. The result I prove is therefore quite unexpected and shows once more the amazing power of the Cremona group of $\mathbb{P}^n$.

THEOREM 1. Let S_1 and S_2 be two cones in $\mathbb{P}^n$. Let X_1 and X_2 be corresponding general hyperplane sections. If X_1 and X_2 are birational then S_1 is Cremona equivalent to S_2.

The main ingredient in the proof is the reduction to subvarieties of codimension 2 to be able to apply the main result in [14]. To do this I produce a special log resolution of the pair $(\mathbb{P}^n, S)$ that allows me to blow down the strict transform of S to a codimension 2 subvariety. Despite the fact that this step cannot produce a Cremona equivalence for S it allows me to work out Cremona equivalence on the lower dimensional subvariety and then lift the Cremona equivalence.

In the special case of cones in $\mathbb{P}^3$ the statement can be improved to characterize the Cremona equivalence of cones with the geometric genus of the plane section, see Corollary 1. In particular this shows that any rational cone in $\mathbb{P}^3$ has positive sup-threshold, see Definition 4. From the $\sharp$-MMP point of view we may easily translate it as follows.

THEOREM 2. Let $S \subset \mathbb{P}^3$ be a rational surface. Assume that there is a $\sharp$-minimal model of the pair (T, S_T) such that T has a scroll structure $\pi : T \to W$ onto a rational surface W and $S_T = \pi^* C$, for some rational curve $C \subset W$. Then $\overline{\rho}(T, S_T) = \overline{\rho}(\mathbb{P}^3, S) > 0$.

The next candidate for the sup-threshold problem are pairs whose $\sharp$-minimal model is a conic bundle and the surface is trivial with respect to the conic bundle structure. With the technique developed in this paper I am only able to treat a special class of these, see Corollary 2.

1. Notations and preliminaries

I work over an algebraically closed field of characteristic zero. I am interested in birational transformations of log pairs. For this I introduce the following definition.

DEFINITION 1. *Let $D \subset X$ be an irreducible and reduced divisor on a normal variety X. We say that (X,D) is birational to (X',D'), if there exists a birational map $\varphi : X \overset{X}{\longrightarrow}{}'$ with $\varphi_*(D) = D'$. Let $D,D' \subset \mathbb{P}^n$ be irreducible reduced divisors then we say that D is Cremona equivalent to D' if $(\mathbb{P}^n,D)$ is birational to $(\mathbb{P}^n,D')$.*

Let us proceed recalling a well known class of singularities.

DEFINITION 2. *Let X be a normal variety and $D = \sum d_iD_i$ a $\mathbb{Q}$-Weil divisor, with $d_i \leq 1$. Assume that $(K_X + D)$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a log resolution of the pair (X,D) with*

$$K_Y = f^*(K_X + D) + \sum a(E_i,X,D)E_i$$

We call

$$disc(X,D) := \min_{E_i}\{a(E_i,X,D) \mid E_i \text{ is an } f\text{-exceptional divisor for some log resolution}\}$$

Then we say that (X,D) is

$$\left.\begin{array}{l} terminal \\ canonical \end{array}\right\} \quad \text{if } disc(X,D) \quad \left\{\begin{array}{l} > 0 \\ \geq 0 \end{array}\right.$$

REMARK 1. Terminal surfaces are smooth, this is essentially the celebrated Castelnuovo theorem. Any log resolution of a smooth surface can be obtained via blow up of smooth points. Hence a pair (S,D), with S a smooth surface has canonical singularities if and only if $\mathrm{mult}_p D \leq 1$ for any point $p \in S$.

Note further that one direction is true in any dimension. Assume that X is smooth and $\mathrm{mult}_p D \leq 1$ for any point $p \in X$. Let $f : Y \to X$ be a smooth blow up, with exceptional divisor E. Then $K_Y = f^*(K_X) + aE$ for some positive integer a and $a(E,X,D) \geq a - 1 \geq 0$. This proves that (X,D) has canonical singularities if X is smooth and $\mathrm{mult}_p D \leq 1$ for any $p \in X$. This simple observation allows to produce many inequivalent embeddings of divisors, see [14, §3]

For future reference we recall a technical result on pseudoeffective divisors, i.e. the closure of effective divisors.

LEMMA 1 ([15, Lemma 1.5]). *Let (X,D_X) and (Y,D_Y) be birational pairs with canonical singularities. Then $K_X + D_X$ is pseudoeffective if and only if $K_Y + D_Y$ is pseudoeffective.*

The main difficulty to study Cremona equivalence in $\mathbb{P}^r$ with $r \geq 3$ is the poor knowledge of the Cremona group. The case of surfaces in $\mathbb{P}^3$ is already quite mysterious. It is easy to show that Quadrics and rational cubics are Cremona equivalent to a plane. Rational quartics with either 3-ple or 4-uple points are again easily seen to be Cremona equivalent to planes, the latter are cones over rational curves Cremona equivalent to lines. It has been expected that Noether quartic should be the first example of a rational surface not Cremona equivalent to a plane, but this is not the case as proved in [15, Example 4.3]. Having in mind these examples and the $\sharp$-MMP developed in [13] for linear systems on uniruled 3-folds, I recall the definition of (effective) threshold.

DEFINITION 3. *Let (T,H) be a terminal $\mathbb{Q}$-factorial uniruled variety and H an irreducible and reduced Weil divisor on T. Let*

$$\rho(T,H) =: \ \sup\,\{m \in \mathbb{Q} | H + mK_T \text{ is an effective } \mathbb{Q}\text{-divisor }\} \geq 0,$$

be the (effective) threshold of the pair (T,H).

REMARK 2. The threshold is not a birational invariant of the pair and it is not preserved by blowing up. Consider a Quadric cone $Q \subset \mathbb{P}^n$ and let $Y \to \mathbb{P}^n$ be the blow up of the vertex then $\rho(Y,Q_Y) = 0$, while $\rho(\mathbb{P}^n,Q) > 0$.

To study Cremona equivalence, unfortunately, we have to take into account almost all possible thresholds.

DEFINITION 4. *Let (Y,S_Y) be a pair birational to a pair (T,S). We say that (Y,S_Y) is a good birational model if Y has terminal $\mathbb{Q}$-factorial singularities and S_Y is a Cartier divisor with terminal singularities. The sup-threshold of the pair (T,S) is*

$$\overline{\rho}(T,S) := \sup\{\rho(Y,S_Y)\},$$

where the sup is taken on good birational models.

REMARK 3. It is clear that any pair $(\mathbb{P}^n,S)$ Cremona equivalent to a hyperplane satisfies $\overline{\rho}(\mathbb{P}^n,S) > 0$. The pair $(\mathbb{P}^n,H)$, where H is a hyperplane, is a good model with positive threshold.

Considering birationally super-rigid MfS's one can produce examples of pairs, say (T,S), with $\overline{\rho}(T,S) = 0$. It is not clear to me if such examples can exist also on varieties with bigger pliability, see [6] for the relevant definition.

We are ready to state the characterization of surfaces Cremona equivalent to a plane.

THEOREM 1 ([15, Theorem 4.15]). *Let $S \subset \mathbb{P}^3$ be an irreducible and reduced surface. The following are equivalent:*

a) S is Cremona equivalent to a plane,

b) $\overline{\rho}(T,S) > 0$ and there is a good model (T,S_T) with $K_T + S_T$ not pseudoeffective.

REMARK 4. As remarked in the introduction the main drawback of the above criterion is the bound on the sup-threshold. It is very difficult to compute it. While the requirement that $K_T + S_T$ is not pseudoeffective is natural and justified also by Lemma 1, it is not clear if pairs with vanishing sup threshold may exist on a rational 3-fold. This naturally leads to study good models with vanishing threshold.

Let (X,S) be a good pair with X rational and $\rho(X,S) = 0$. Then the $\sharp$-MMP applied to this pair may lead to a Mori fiber space $\pi : T \to W$ such that S_T is trivial with respect to π and it is a smooth surface, see [13, Theorem 3.2] and the proof of [15, Theorem 4.9]. In particular $S_T = \pi^* D$ for some irreducible divisor $D \subset W$. Then if W is a curve S_T is a smooth fiber of π, that is a del Pezzo surface. If W is a surface then S_T is a (not necessarily minimally) ruled surface and π is a conic bundle structure. In the latter case if π has a section it is easy, see for instance the proof of Corollary 2, to prove that (T, S_T) is birational to a cone in $\mathbb{P}^3$.

2. Cremona equivalence for cones

Here I am interested in cones in $\mathbb{P}^n$. Let $S \subset \mathbb{P}^n$ be an irreducible reduced divisor of degree d with a point p of multiplicity d. Let H be a hyperplane in $\mathbb{P}^n \setminus \{p\}$ and $C = S \cap H$. Then S can be viewed as the cone over the variety C. It is easy to see that if $C_1, C_2 \subset \mathbb{P}^{n-1}$ are Cremona equivalent divisors then the cones over them are Cremona equivalent.

PROPOSITION 1. *Let C_1 and C_2 be Cremona equivalent divisors in $\mathbb{P}^{n-1}$ and S_1, S_2 cones over them in $\mathbb{P}^n$. Then S_1 is Cremona equivalent to S_2.*

Proof. Without loss of generality I may assume that S_1 and S_2 have the same vertex in the point $[0,\dots,0,1]$ and $C_1 \cup C_2 \subset (x_n = 0)$. Let $\mathcal{H} \subset |\mathcal{O}_{\mathbb{P}^{n-1}}(h)|$ be a linear system realizing the Cremona equivalence between C_1 and C_2. Hence I have a Cremona map $\psi : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$ given by a n-tuple $\{f_0, \dots, f_{n-1}\}$, with $f_i \in k[x_0, \dots, x_{n-1}]_h$. This allows me to produce a map $\Psi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ considering the linear system

$$\mathcal{H}' := \{f_0, \dots, f_{n-1}, x_n x_0^{h-1}\}.$$

Note that the general element in $\mathcal{H}'$ has multiplicity $h-1$ at the point p and, in the chosen base, there is only one element of multiplicity exactly $h-1$. This shows that lines through p are sent to lines through a fixed point. Moreover the restriction $\Psi_{|(x_n=0)}$ is the original map ψ. Hence the map Ψ is birational and realizes the required Cremona equivalence. $\square$

Next I want to understand what happens if $C_1, C_2 \subset \mathbb{P}^{n-1}$ are simply birational as abstract varieties. To do this I need to produce "nice" good models of the pairs $(\mathbb{P}^n, S_1)$ and $(\mathbb{P}^n, S_2)$.

Let $C \subset H \subset \mathbb{P}^n$ be a codimension two subvariety and S be a cone with vertex $p \in \mathbb{P}^n \setminus H$ over C. I produce a good model of the pair $(\mathbb{P}^n, S)$ as follows. First I blow up p producing a morphism $\varepsilon : Y \to \mathbb{P}^n$ with exceptional divisor E. Note that Y has a scroll structure $\pi : Y \to \mathbb{P}^{n-1}$ given by lines through p, and S_Y, the strict transform, is just $\pi^*(C)$. Let $\nu : W \to \mathbb{P}^{n-1}$ be a resolution of the singularities of C and take the fiber product

$$\begin{tikzcd} Z \arrow[r, "\nu_Y"] \arrow[d, "\pi_W"'] & Y \arrow[d, "\pi"] \\ W \arrow[r, "\nu"] & \mathbb{P}^{n-1}. \end{tikzcd}$$

Then the strict transform S_Z is a smooth divisor and (Z, S_Z) is a good model of $(\mathbb{P}^n, S)$. Note that the threshold $\rho(Z, S_Z)$ vanishes. According to the $\sharp$-MMP philosophy this forces us to produce different good models.

The following Lemma is probably well known, but I prefer to state it, and prove it, to help the reader.

LEMMA 2. *Let $C \subset X$ be an irreducible and reduced subvariety. Assume that there exists a birational map $\chi : X \dashrightarrow Y$ such that χ is an isomorphism on the generic point of C. Let $D := \chi(C)$ be the image and X_C, respectively Y_D the blow up of C and D with morphism f_C, f_D, and exceptional divisors E_C, E_D respectively. Then there is a birational map $\chi_C : X_C \dashrightarrow Y_D$ mapping E_C onto E_D. In other words (X_C, E_C) is birational to (Y_D, E_D).*

Proof. Let $U \subset X$ be an open and dense subset intersecting C such that $\chi_{|U}$ is an isomorphism. Then considering the fiber product

$$\begin{tikzcd} X_C \supset U_C \arrow[r, "\chi_C"] \arrow[d, "f_U"'] & Y_D \arrow[d, "f_D"] \\ X \supset U \arrow[r, "\chi_{|U}"] & Y, \end{tikzcd}$$

I conclude, by the Universal Property of Blowing Up, the existence of the morphism χ_C with the required properties. □

REMARK 5. Let me stress that the above result is in general not true with the weaker assumption that χ is a morphism on the general point of W. On the other hand if Y is $\mathbb{P}^n$ the statement can be rephrased also in this weaker form.

Let us go back to the pair (Z, S_Z). Let Γ_Z be the strict transform of a general hyperplane section of S. Then I may consider the following elementary transformation of the scroll structure π_W

$$\begin{tikzcd} & Bl_{\Gamma_Z}Z \arrow[dl, "\gamma"'] \arrow[dr, "\eta"] & \\ Z \arrow[rr, dotted, "\Phi"] \arrow[d, "\pi_W"] & & V \arrow[d] \\ W \arrow[rr, "\cong"] & & W, \end{tikzcd}$$

where γ is the blow up of Γ_Z and η is the blow down of the strict transform of S_Z to a codimension 2 subvariety, say Γ_S. Then there is a birational map $\varphi : V \dashrightarrow \mathbb{P}^n$ sending Γ_S to a codimension 2 subvariety, say Γ', and such that φ is an isomorphism on the generic point of Γ_S. The map φ can be easily constructed again via an elementary transformation of the scroll structure followed by blow downs of exceptional divisors. We may summarize the above construction in the following proposition and diagram.

PROPOSITION 2. *Let $S \subset \mathbb{P}^n$ be a cone and C a general hyperplane section. Then there are birational maps $\varepsilon : \mathbb{P}^n \dashrightarrow Y$ and $\eta : Y \dashrightarrow V$ such that $\varepsilon_* S =: E$ and η is the blow down of E to a codimension 2 subvariety Γ. In particular S is the valuation associated to the ideal I_Γ. Moreover there is a third birational map $\varphi : V \dashrightarrow \mathbb{P}^n$ sending Γ to a codimension 2 subvariety C' and such that again S is the valuation associated to $I_{C'}$*

$$\begin{tikzcd} & Y \supset E \arrow[dr, "\eta"] & \\ & & V \supset \Gamma \arrow[d, dotted, "\varphi"] \\ \mathbb{P}^n \supset S \arrow[uur, dotted, "\varepsilon"] \arrow[rr, dotted, "\Phi"] & & \mathbb{P}^n \supset C' \end{tikzcd}$$

The composition $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is a birational map sending S to the codimension 2 subvariety C' and such that S is the valuation associated to the ideal $I_{C'}$.

We are now ready to prove the main result on cones in $\mathbb{P}^n$.

THEOREM 2. *Let $S_1, S_2 \subset \mathbb{P}^n$ be cones and C_1, C_2 general hyperplane sections. If C_1 is birational to C_2 then S_1 is Cremona equivalent to S_2. In particular all divisorial cones over a rational variety are Cremona equivalent to a hyperplane.*

REMARK 6. I doubt the other direction is true. Let $X \subset \mathbb{P}^n$ be a non rational but stably rational variety. Assume that $X \times \mathbb{P}^a$ is rational but $X \times \mathbb{P}^{a-1}$ is not rational for some $a \geq 1$. Then $X \times \mathbb{P}^a$ can be birationally embedded as a cone, say S, with hyperplane section birational to $X \times \mathbb{P}^{a-1}$. In principle S could be Cremona equivalent to a hyperplane but its hyperplane section cannot be rational. This cannot occur for surfaces, see Corollary 1.

Proof. Let S_1 and S_2 be two cones and C_1, respectively, C_2 general hyperplane sections. Then by Proposition 2 there are birational maps $\Phi_i : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ sending S_i to a codimension 2 subvariety C'_i and such that:

- S_i is the valuation associated to $I_{C'_i}$,
- C'_i is birational to C_i.

I am assuming that C_1 is birational to C_2. Then the main result and its proof [14, p. 92] states that there is a Cremona map $\chi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ sending C'_1 to C'_2 and such that χ is an isomorphism on the generic point of C'_1. Then by Lemma 2 I may extend χ to the blow up of the C'_i to produce the required Cremona equivalence Ψ

$$\begin{tikzcd}
 & Y_1 \supset E_1 \arrow[rrr, dotted, "\Psi"] \arrow[dr, "\eta_1"] & & & Y_2 \supset E_2 \arrow[dl, "\eta_2"'] & \\
 & & V_1 \supset \Gamma_1 \arrow[d, dotted, "\varphi_1"] & V_2 \supset \Gamma_2 \arrow[d, dotted, "\varphi_2"'] & & \\
\mathbb{P}^n \supset S_1 \arrow[uur, dotted, "\varepsilon_1"] \arrow[rr, dotted, "\Phi_1"] & & \mathbb{P}^n \supset C'_1 \arrow[r, dotted, "\chi"] & \mathbb{P}^n \supset C'_2 & & \mathbb{P}^n \supset S_2 \arrow[uul, dotted, "\varepsilon_2"'] \arrow[ll, dotted, "\Phi_2"']
\end{tikzcd}$$

□

As observed before the result can be strengthened in lower dimension.

COROLLARY 1. *Let $S_1, S_2 \subset \mathbb{P}^3$ be cones and C_1, C_2 general hyperplane sections. Then C_1 is birational to C_2 if and only if S_1 is Cremona equivalent to S_2. In particular rational cones are Cremona equivalent to a plane.*

Proof. I have to prove that if S_1 and S_2 are Cremona equivalent then also C_1 and C_2 are birational. Note that the irregularity of a resolution of S_i is a birational invariant and it is the geometric genus of the curve C_i. This yields $g(C_1) = g(C_2)$ and concludes the proof. □

REMARK 7. As observed, in the special case of rational surfaces in $\mathbb{P}^3$ this gives the Cremona equivalence to a plane for any rational cone. In particular any rational cone has positive sup-threshold.

It remains to translate the statement in $\sharp$-MMP dictionary for conic bundles.

COROLLARY 2. *Let $S \subset \mathbb{P}^3$ be a rational surface. Assume that there is:*

a) *a $\sharp$-minimal model of the pair (T, S_T) such that T has a conic bundle structure $\pi : T \to W$ onto a rational surface W, $S_T = \pi^* C$, for a curve $C \subset W$,*

b) *a birational map $\chi : T \dashrightarrow \mathbb{P}^3$ that contracts S_T to a curve, say Γ, such that S_T is the valuation associated to I_Γ.*

Then $\overline{\rho}(T,S_T)=\overline{\rho}(\mathbb{P}^3,S)>0$*. Assumption b) is always satisfied if* π *has a section, i.e. if* π *is a scroll structure.*

Proof. Let (T,S_T) be a good model as in assumption a). By hypothesis there is a birational map $\chi: T \dashrightarrow \mathbb{P}^3$ that contracts S_T onto a curve Γ and such that S_T is the valuation associated to I_Γ. The curve Γ is dominated by a rational surface, and it is therefore rational. The extension trick used in Theorem 2 yields that (T,S_T) is birationally equivalent to a plane in $\mathbb{P}^3$. This is enough to prove that $\overline{\rho}(\mathbb{P}^3,S)>0$.

If π has a section then all fibers are irreducible. Let $\phi: T \dashrightarrow Y$ be an elementary transformation that blows down S_T to a curve Γ. Then Y has a scroll structure onto W and I may run a $\sharp$-MMP on the base surface W, as described in [13, p. 700], that is an isomorphism in a neighborhood of Γ. This yields a new 3-fold model Z with a Mori fiber space structure onto either $\mathbb{P}^2$ or a ruled surface and then via elementary transformation of the scroll structure I produce the required map χ. $\square$

REMARK 8. Unfortunately the birational geometry of rational conic bundles without sections is very poorly understood and it is difficult to understand whether condition b) is always satisfied or not, even assuming the standard conjectures [9].

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AMS Subject Classification: Primary 14E25; Secondary 14E05, 14N05, 14E07

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Lavoro pervenuto in redazione il 02.07.2013.

Rend. Sem. Mat. Univ. Politec. Torino
Vol. 71, 3–4 (2013), 473 – 492

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ON THE JOINT WORK OF ALBERTO CONTE, MARINA MARCHISIO AND JACOB MURRE

To my friend, Alberto Conte, for his 70^{th} *birthday.*

This is - a slightly expanded - written version of the lecture delivered by the author on the occasion of the seventieth birthday of Alberto Conte. It is a *survey* of joint work done by Alberto and the author starting from around 1975; later in the end of the nineties we were joined by Marina Marchisio.

The topics of our joint work are:

1. The non-rationality of the quartic threefold with a double line [9]

2. The Hodge conjecture for special fourfolds [10] and [11]

3. On Fano's theorem on threefolds whose hyperplane sections are Enriques surfaces [13]

4. On the definition and on the nature of the singularities of Fano threefolds [13]

5. On Morin's work on unirationality of hypersurfaces [14], [15], [16], [17].

The majority of these subjects have their roots in classical Italian algebraic geometry (topics 1, 3 and 5).

In the lecture I have concentrated on topics 1 and 5 and we do the same here. The purpose was to give the audience some idea of the beautiful geometry involved in these subjects. The author wants to stress however that this is *only a survey lecture*, for the details one has to go to the original papers.

The basic concepts in topic 1 and 5 are the notions of *rationality* and *unirationality*. Therefore we want to recall these notions in the precise form they will be used here.

Basic notions

Let X_n be a projective irreducible variety of dimension n defined over a field k. We assume throughout that k is of *characteristic zero* but not necessarily algebraically closed.

X is *rational over* k if there exists a *birational map* $f : \mathbb{P}_n \overset{\sim}{\dashrightarrow} X$ which is *itself also defined over* k.

X is *rational* if this is true over $\bar{k}$, where $\bar{k}$ is the algebraic closure of k (i.e. f is defined over a finite extension k_1 of k).

X is *unirational over* k if there exists a *dominant rational map* $f : \mathbb{P}_n - - \twoheadrightarrow X$ *defined over* k which is generically finite to one (i.e. there exists a Zariski open set $U \subset X$ over which f is finite).

X is *unirational* if this is true over $\overline{k}$ (again f is then defined over a finite extension k_1 of k).

Acknowledgement

The author thanks the organizers for the invitation to lecture in this special conference. Very special thanks to Marina Marchisio for TeXing the manuscript.

1. Quartic threefolds with a double line

1.1. The theorem which we proved is the following.

THEOREM 1. ([9])
Let $V = V_3(4)$ be a threefold af degree 4 in projective space $\mathbb{P}_4$ defined over a field k. Assume that V has a double line l_0 but is "otherwise general". Then V is unirational, but not *rational.*

Remarks

1. We shall explain the term "otherwise general" below; we also use sometimes the expression "sufficiently general".

2. The inspiration for the above theorem came from the famous paper [6] of Clemens and Griffiths where they proved the non-rationality of a smooth cubic threefold (a fact claimed classically by Fano). The proof of Clemens-Griffiths is in characteristic zero and their basic tool is the intermediate jacobian. In the papers [37] and [38] the author studied over any field k with $char(k) \neq 2$ the group of 1-dimensional algebraic cycles on a cubic threefold and related that group to a so-called Prym variety. Based upon Mumford's theory of Prym varieties ([36]) he obtained in this way another proof of the non-rationality of the cubic threefold. The proof of the above theorem follows this more algebraic proof.

1.2. Equation for V

Let k_1 be a field. Assume that the double line $l_0 \subset V \subset \mathbb{P}_4$ is defined over k_1. We choose homogeneous coordinates $(x : y : z : u : v)$ in $\mathbb{P}_4$ such that the line l_0 is given by the equations

$$x = y = z = 0. \tag{1}$$

Then the equation of V is

$$
\begin{aligned}
&a_{02}(x,y,z)u^2 + 2a_{11}(x,y,z)uv + a_{20}(x,y,z)v^2 + \\
&+2a_{12}(x,y,z)u + 2a_{21}(x,y,z)v + a_{22}(x,y,z) = 0
\end{aligned} \tag{2}
$$

with $a_{ij}(x,y,z) \in k[x,y,z]$ homogeneous polynomials of degree $i+j$ with coefficients in a field $k \supset k_1$ (so k is a field of definition for V). The coefficients of these polynomials can be considered as coordinates in some projective space $\mathbb{P}_N$ which parametrizes the quartic 3-folds with a double line (of course this is not the "moduli" space). "Otherwise general" means that there is a k_1-Zariski open subset $U \subset \mathbb{P}_N$ such that for V in U the theorem is true. In particular we take U such that the $V_3(4) \in U$ are smooth outside the double line $l_0 \subset V_3(4)$ but there are more restrictions which could - but are cumbersome to - be made explicit.

1.3. The 2-planes through l_0

Take in $\mathbb{P}_4$ a 2-dimensional linear space (a *2-plane* for short) N such that $N \cap l_0 = \emptyset$. To be specific let us take N with equations

$$u = v = 0. \tag{3}$$

Take a point $T = (a : b : c : 0 : 0) \in N$ and let $L_T = \langle l_0, T \rangle$ be the linear span. N parametrizes the 2-planes through l_0. Consider the intersection

$$V \cdot L_T = 2l_0 + K_T \tag{4}$$

with K_T a *conic* in L_T.

To make things explicit: a point $P \in L_T$ has coordinates $P = (ta : tb : tc : u : v)$ in $\mathbb{P}_4$ and (homogeneous) coordinates $(t : u : v)$ in L_T and the equation of the conic K_T in L_T is

$$
\begin{aligned}
&a_{02}(a,b,c)u^2 + 2a_{11}(a,b,c)uv + a_{20}(a,b,c)v^2 + \\
&+2a_{12}(a,b,c)ut + 2a_{21}(a,b,c)vt + a_{22}(a,b,c)t^2 = 0.
\end{aligned} \tag{5}
$$

So K_T is defined over the field $k(a,b,c)$.

In L_T we have the intersection

$$K_T \cdot l_0 = R_1 + R_2. \tag{6}$$

Note that in general R_1 and R_2 are not rational in $k(a,b,c)$, but only conjugate over $k(a,b,c)$.

1.4. The conic bundle associated with $V_3(4)$

Starting from $V = V_3(4) \subset \mathbb{P}_4$ with the double line l_0 we have now also the variety

$$V_1 = \{P \in K_T; \quad T \in N\} \tag{7}$$

with the morphism $p : V_1 \to N$ with $p(P) = T$. The fibres of p are the conics K_T and V_1 is a *conic bundle* over N. If V is general then V_1 is smooth, in fact V_1 is obtained by blowing up V along l_0 (see [9], prop. 1.17). Clearly V_1 is birational with V over k and hence *it suffices to prove the theorem for* V_1.

1.5. Two special curves in N

Consider in N the curve

(8) $$\Delta = \{T \in N; \quad K_T = l'_T + l''_T\}.$$

Over Δ the conic K_T degenerate in two lines; Δ is called the *discriminant curve* for V_1. Note that in general l'_T and l''_T are only conjugate over, and not defined in, the field $k(T)$.

Consider also the curve (see (6))

(9) $$\nabla = \{T \in N; \quad R_1 = R_2\}$$

and put

(10) $$\mathcal{B} = \Delta \cap \nabla.$$

∇ is the curve for which the K_T is tangent to l_0. From the equation (5) we deduce easily the following facts ([9], prop. 1.17 and lemma 1.5 and 1.7).
For V general the Δ and ∇ are irreducible and smooth over $k(T)$, $deg(\Delta) = 8$ hence $g(\Delta) = 21$, $deg(\nabla) = 4$ hence $g(\nabla) = 3$. For $T \in \Delta$ we have $l'_T \neq l''_T$, $l'_T \neq l_0 \neq l''_T$. $\mathcal{B}$ consists of 32 different points and for $T \in \mathcal{B}$ we have $R_1 = R_2 \in l'_T \cap l''_T \cap l_0$.

1.6. The associated Prym variety

Consider in V the lines l different from l_0 but meeting l_0, i.e.

(11) $$\Delta^* = \{l \subset V,\ l \cap l_0 \neq \emptyset,\ l \neq l_0\}.$$

Δ^* consists clearly of the lines l'_T and l''_T from (8) and is therefore a curve and

(12) $$q :\ \Delta^* \to \Delta$$

is a $2:1$ covering. In fact since $l'_T \neq l''_T$ the q is an *étale* $2:1$ cover and if V is general then Δ^* is irreducible; from Hurwitz we get $g(\Delta^*) = 40$. Let σ be the *involution* exchanging l'_T and l''_T

(13) $$\sigma(l'_T) = l''_T.$$

From (12) we get for the Jacobians

(14) $$q_* :\ J(\Delta^*) \to J(\Delta),\ \ q^* :\ \ J(\Delta) \to J(\Delta^*),\ \ q_* \cdot q^* = q.$$

Mumford has studied this situation in great detail. The involution σ gives also an involution σ on $J(\Delta^*)$ (by abuse of language denoted by the same letter). Mumford proved the following

THEOREM 2. ([36]):
This involution gives on $J(\Delta^*)$ a decomposition

$$J(\Delta^*) = J(\Delta) + Pr(\Delta^*/\Delta) \tag{15}$$

with σ operating as $+1$ on $J(\Delta)$ and as -1 on $Pr(\Delta^*/\Delta)$ and $Pr(\Delta^*/\Delta)$ is itself also an abelian variety, the so-called Prym variety *of $q: \Delta^* \to \Delta$. Furthermore $J(\Delta) \cap Pr(\Delta^*/\Delta) = \{0, a\}$ where a is a 2-torsion point. Moreover the Θ-divisor on $J(\Delta^*)$ induces a polarization 2Ξ on $Pr(\Delta^*/\Delta)$ with Ξ a principal polarization.*

Moreover Mumford studied the couple $(Pr(\Delta^*/\Delta), \Xi)$ carefully and did give a complete list of cases for which this couple is itself a jacobian variety of a curve with corresponding theta-divisor or a product of such a situation.

In our case $\dim J(\Delta^*) = 40$, $\dim J(\Delta) = 21$, hence $\dim Pr(\Delta^*/\Delta) = 19$. From Mumford's list we conclude

FACT. ([9], section 6, lemma 6.1 and cor. 6.2):
In our case $(Pr(\Delta^*/\Delta), \Xi)$ is as principally polarized abelian variety neither a jacobian of a curve, nor a product of such.

1.7. A double cover V_1' of the conic bundle V_1

Introduce first the double cover $f: N' \to N$ where

$$N' = \{(T,R);: T \in N, R \in K_T \cap l_0\} \tag{16}$$

and next the double cover $f_1: V_1' \to V_1$ given by

$$V_1' = V_1 \times_N N'.$$

Clearly we have a commutative diagram

$$\begin{tikzcd} V & V_1 \arrow[l] \arrow[d, "p"'] & V_1' \arrow[l, "f_1"'] \arrow[d, "p'"] \\ & N & N' \arrow[l, "f"] \end{tikzcd}$$

Let $K_{T,R} = p'^{-1}(T,R)$, clearly $K_{T,R} = K_T$ and hence

$$V_1' = \{(T,R,P);\, T \in N, R \in K_T \cap l_0, P \in K_{T,R} = K_T\}. \tag{17}$$

Clearly N' is a surface, a double cover of the plane N, branched over the smooth curve ∇ of degree 4, hence it is well-known that N' is a *smooth rational surface*. Furthermore V_1' is a conic bundle over N', but now the conics $K_{T,R}$ have a rational point R. Therefore V_1' is a *rational threefold* and hence:

PROPOSITION 1. *V_1, and therefore also V itself, is* unirational.

From the 2 : 1 covering $f_1 : V_1' \to V_1$ given by $f_1(T,R,P) = (T,P)$ we get also an *involution* τ on V_1'. Namely

(18) $$\tau(T,R_1,P) = (T,R_2,P)$$

where $K_{T,R_1} \cdot l_0 = K_T \cdot l_0 = R_1 + R_2$ (see 16).

Furthermore consider the curve

$$\Delta' = f^{-1}(\Delta)$$

on N', clearly this is a 2 - 1 covering of Δ in N. For $(T,R) \in \Delta'$ the fibre $p'^{-1}(T,R) = l_T' \cup l_T''$ with one of the lines, say l_T', going through R. To be explicit, let us introduce the notation $l_{T,R}^* = l_T'$ and $l_{T,R}^{**} = l_T''$ so that we can write for the fibre

(19) $$p'^{-1}(T,R) = \{l_{T,R}^* \cup l_{T,R}^{**}, \text{ with } R \in l_{T,R}^*\}.$$

Finally we have clearly a *birational morphism*

$$\alpha : \Delta^* \to \Delta'.$$

Concerning α^{-1}, since Δ' is smooth outside the points $\mathcal{B}' = f^{-1}(\mathcal{B})$, the α^{-1} is defined outside this finite set of points.

1.8. Another model V_1'' birational with V_1'

Consider the variety

(20) $$V_1'' = \{(T,R,m);\ T \in N, R \in K_T \cap l_0, m \text{ a line in } L_T \text{ with } R \in m\}.$$

Clearly we have a morphism $p'' : V_1'' \to N'$ and also a *birational map* $h : V_1'' \overset{\sim}{\dashrightarrow} V_1'$ defined by

(21) $$h(T,R,m) = (T,R,P)$$

where $P \in K_T$ is defined by

(22) $$K_{T,R} \cdot m = R + P$$

(recall $K_{T,R} = K_T$). Clearly we have a commutative diagram

$$\begin{tikzcd} V_1' \arrow[dr, "p'"] & & V_1'' \arrow[ll, dashed, "h"'] \arrow[dl, "p''"] \\ & N & \end{tikzcd}$$

Note that V_1'' is a *projective bundle* over the rational surface N', hence rational. Let us denote the "inverse" of h by g, so

$$g(T,R,P) = (T,R,\langle RP\rangle),$$

where $\langle -,-\rangle$ means - as usual - the linear span.

Let $\mathcal{F}_h'' \subset V_1''$ be the *fundamental locus* of h, i.e. the points where h is not defined.

LEMMA 1. *$\mathcal{F}_h'' = \Lambda'' \cup \mathcal{B}_1''$, where $\Lambda'' \subset V_1''$ is the* curve

$$\Lambda'' = \{(T,R,m = l_T'),\ T \in \Delta,\ R \in l_0 \cap l_T'\} \tag{23}$$

and $\mathcal{B}_1''$ is the finite set of points $\mathcal{B}_1'' = \{(T,R,m = l_T'')\}$, *with* $(T,R) \in \mathcal{B}' = f^{-1}(\mathcal{B}) \subset N'$ *with $\mathcal{B} \subset N'$ from* (10).

Proof. This follows immediately from the definition of h in (21). ☐

LEMMA 2. *The curve Λ'' is birational with Δ^* and Δ' and non-singular (hence isomorphic with Δ^*). In particular $Pr(\Delta^*/\Delta) = Pr(\Lambda''/\Delta)$.*

Proof. The birationality with Δ' (and hence with Δ^*) is immediately clear from the description in Lemma 1. For the non-singularity we refer to Lemma (3.6) of [9].

Now we blow up $\mathcal{F}_h''$ and we get $\tilde{V}_1 = B_{\mathcal{F}_h''}(V_1'')$ with maps $\tilde{h} : \tilde{V}_1 \dashrightarrow V_1'$, $j : \tilde{V}_1 \to V_1''$ all over N' and $\tilde{h}$ is a *birational* map with inverse $\tilde{g} : V_1' \dashrightarrow \tilde{V}_1$.

We have now the following big diagram

$$\begin{tikzcd}
 & & & \tilde{V}_1 \arrow[dl, dashed, "\tilde{h}"'] \arrow[d, "j"] \\
V & V_1 \arrow[l] \arrow[d, "p"'] & V_1' \arrow[l, "f_1"] \arrow[d, "p'"'] & V_1'' \arrow[l, dashed, "h"] \arrow[dl, "p''"] \\
 & N & N' \arrow[l, "f"] &
\end{tikzcd}$$

and $h^{-1} = g$ and $\tilde{h}^{-1} = \tilde{g}$.

Recall that from the $2:1$ covering f_1 we get an involution τ on V_1' and hence we have also involutions say τ_1 on V_1'' and $\tilde{\tau}$ on $\tilde{V}_1$ (all over V_1). ☐

1.9. The behavior of the involutions on the Chow groups

Recall that if X is a smooth, irreducible, projective variety then we have the Chow groups $CH_{d-i}(X) = CH^i(X)$ of algebraic cycles on X of codimension i modulo rational equivalence. We are interested in the subgroups $A^i(X) := CH^i_{alg}(X) \subset CH^i(X)$ of the cycle classes algebraically equivalent to zero.

Since V_1'' is a projective bundle over the rational surface N' (see 7) we have $A^2(V_1'') = 0$ and then by general theorems of blowing up (see [26],section 9)

$$A^2(\tilde{V}_1) = J(\Lambda'') = J(\Delta^*).$$

From the birational transformation $\tilde{h} : \tilde{V}_1 \overset{\sim}{\dashrightarrow} V_1'$ and its inverse (and the analysis of the fundamental loci) we have $A^2(V_1') = A^2(\tilde{V}_1)$ (see [9], section 3 for details). Therefore

$$A^2(V_1') = J(\Delta^*) = J(\Delta) + Pr(\Delta^*/\Delta) \tag{24}$$

on which the involutions τ and σ operate.

The *key point* is the *comparison of these two actions*. $J(\Delta)$ is the invariant part under σ and $Pr(\Delta^*/\Delta)$ the anti-invariant part (see the theorem in section 1.6).

KEY LEMMA. Let $\eta \in J(\Delta^*) \simeq A^2(V_1')$. Then $\tau(\eta) = -\sigma(\eta)$.

Proof. (outline)

We have the fibration $p' : V_1' \to N'$ and we consider the fibres $p'^{-1}(T,R)$ for the points $(T,R) \in \Delta' = f^{-1}(\Delta)$ (see sections 1.5 and 1.7). Because of the moving lemma we can restrict our attention to the points $(T,R) \in \Delta_0'$ with $\Delta_0' = \Delta' - \mathcal{B}'$ (i.e. $T \notin \mathcal{B} = \Delta \cap \nabla$, see (10)).

We have by (19)

$$p'^{-1}(T,R) = l^*_{TR} + l^{**}_{TR} = K_{TR} (= K_T) \tag{25}$$

with $R \in l^*_{TR}$. We examine now the action of τ on the 1-cycle $l^*_{TR} \in CH^2(V_1')$.

For $(T,R,P) \in l^*_{TR}$, i.e. $P \in l^*_{TR} = l'_T$ we get (see (18)):

$$\tau(T,R,P) = (T, \sigma(R), P). \tag{26}$$

Therefore as element of $CH^2(V_1')$

$$\tau(l^*_{TR}) = l^{**}_{T\sigma(R)} = K_{T\sigma(R)} - l^*_{T\sigma(R)}. \tag{27}$$

Next take a (certain) fixed point $(T_0,R_0) \in N' - \Delta'$.

Since N' is a *rational surface* we can move by *rational equivalence on* N' every point (T,R) to (T_0,R_0). Therefore

$$K_{TR} = K_{T_0R_0} \tag{28}$$

in $CH^2(V_1')$.

Now let $\eta = \sum_i (T_i,R_i) \in J(\Delta^*) \subset CH^1(\Delta^*)$, with $\deg(\eta) = 0$. As element of $A^2(V_1')$ (see (24)) it corresponds with the class of the 1-cycle $\sum_i l^*_{T_iR_i}$ and by (27) we get for the action of τ in $A^2(V_1')$

$$\tau(\sum_i l^*_{T_iR_i}) = \sum_i (K_{T_i\sigma(R_i)} - l^*_{T_i\sigma(R_i)}.$$

Now using (28) we get

$$\tau(\sum_i l^*_{T_iR_i}) = \sum_i (K_{T_0R_0}) - l^*_{T_i\sigma(R_i)} = -\sum_i l^*_{T_i\sigma(R_i)}$$

because since $\deg(\eta) = 0$ the terms with $K_{T_0R_0}$ cancel. Hence finally

(29) $$\tau(\eta) = -\text{class}(\sum_i l^*_{T_i\sigma(R_i)}) = -\sigma(\eta)$$

which proves the lemma. □

1.10.

Using the key lemma we obtain

MAIN THEOREM ON CHOW GROUP. ([9], thm 5.10)

$$A^2(V_1) = Pr(\Delta^*/\Delta)$$

Proof. (indication)

From the (2 : 1)-cover $f_1 : V_1' \to V_1$ (see section 1.7) we get homomorphisms

(30) $$f_1^* : A^2(V_1) \to A^2(V_1') \qquad \text{and} \qquad f_{1*} : A^2(V_1') \to A^2(V_1)$$

and moreover

(31) $$(f_1)_* \circ f_1^* = 2\, id_{A(V_1)}.$$

Clearly $\text{Im}(f_1^*) \subset \text{Inv}(\tau)$ (invariant part) and using the divisibility of the groups $A^2(-)$ we get in fact $\text{Im}(f_1^*) = \text{Inv}(\tau)$. Then using the key lemma of 1.9 we get

$$\text{Im}(f_1^*) = \text{Inv}(\tau) = Pr(\Delta^*/\Delta).$$

Using 31 we get then first $A^2(V_1) = Pr(\Delta^*/\Delta) + B$ with B some 2-torsion group but in fact (see [37] pages 201-202) we get a direct sum $A^2(V_1) = Pr(\Delta^*/\Delta) \oplus B$ and finally using divisibility again we get $B = 0$, hence the theorem. □

1.11. The non-rationality of V

It suffices clearly to consider V_1 instead of V. As already remarked in section 1.6 the $Pr(\Delta^*/\Delta)$ carries a principal polarization Ξ. From this one gets a Riemann form $l^{\Xi}(\xi,\eta)$ for $\xi, \eta \in T_l(Pr(\Delta^*/\Delta))$, with $T_l(Pr(\Delta^*/\Delta))$ the Tate group (see [35], p. 186). This Riemann form is closely related to the Riemann forms coming from $H^3(V_1) \simeq H^1_-(\Delta^*)$ where $H^1_-(\Delta^*)$ is the anti-invariant part of $H^1(\Delta^*)$, namely $l^{\Xi}(\xi,\eta) = -\xi \cup \eta$ where $\xi \cup \eta$ is the cup product of $H^3(V_1)$ (see [39], p 148 and [9], p 172).

Using this one can show (similar as in [6], Cor. 3.26) that if V_1 should be rational then $(Pr(\Delta^*/\Delta), \Xi)$ as principally polarized abelian variety is isomorphic to a jacobian variety of a curve polarized by its theta divisor or to a product of such pairs (see [38], thm 3.11).

However we have already remarked in section 1.6 that in our case this does not happen. Therefore we have

THEOREM 3. ([9], thm 6.3)
Let $V = V_3(4) \subset \mathbb{P}_4$ *be a quartic 3-fold with a double line but otherwise general, then* V *is unirational but* not *rational.*

1.12. Further developments

It turned out afterwards that in the same period (1977) Beauville in Paris wrote a beautiful thesis on "fibrés en quadriques". This paper is a standard work and fundamental on this subject and our result on the quartic 3-fold with a double line is a very special case of his results. Of course the works were done independent of each other. Beauville's method is different from ours. See [2] for his results.

2. Hodge conjecture for 4-folds covered by rational curves.

2.1. Hodge conjecture

Let us first recall the Hodge conjecture. Let X_d be a smooth projective variety defined over the complex numbers $\mathbb{C}$ and of dimension d. Consider the (Betti) cohomology groups $H^i(X,Q)$ and $H^i(X,\mathbb{C})$ $(0 \leq i \leq 2d)$. Then there is the Hodge decomposition

$$H^i(X,\mathbb{C}) = \bigoplus_{r+s=i} H^{r,s}(X) \tag{32}$$

with $H^{r,s}(X) = H^s(X,\Omega^r_X)$. On the other hand there is the *cycle map* from the group $\mathcal{Z}^p(X)$ of algebraic cycles on X of dimension $(d-p)$ (i.e. of codimension p)

$$\gamma^p: \ \mathcal{Z}^p(X)\otimes Q \to \{H^{2p}(X,Q)\cap H^{p,p}(X)\} \subset H^{2p}(X,\mathbb{C}). \tag{33}$$

Hodge made the *conjecture* that γ^p is *onto* $H^{2p}(X,Q)\cap H^{p,p}(X)$. This is so-called *Hodge (p,p)-conjecture*.

2.2. Known cases

For $p = 1$ the Hodge (1,1)-conjecture is known by the so-called (1,1) theorem of Lefschetz. By another theorem of Lefschetz, the strong Lefschetz theorem, it then follows that also the Hodge ($d-1$, $d-1$)-conjecture is true. The Hodge conjecture is therefore true for 3-folds (and for surfaces and curves).

The first open case is in dimension 4, namely the Hodge (2, 2)-conjecture.

Here we have the following result:

2.3. Theorem. [10]

Hodge (2,2) is true for fourfolds which are covered by rational curves.

For the proof, which is not difficult, we refer to the original paper.

2.4. Applications.

Hodge (2,2) is true for unirational fourfolds ([40]), in particular for the cubic fourfold (this was also proven by different methods by Zucker [48]). In the paper [11] we did give many more examples, including in particular $V_4(4) \subset \mathbb{P}_5$ (covered by lines), $V_4(5) \subset \mathbb{P}_5$, (covered by conics), $V_4(2,4)$ and $V_4(3,3)$ in $\mathbb{P}_6$ and many more cases. Also the theorem applies to *all uniruled fourfolds*.

2.5. Some further developments.

In 1992 Campana [4] and Kollar-Miyaoka-Mori [27] independently proved that Fano varieties are rationally connected, therefore our theorem from 2.3 above applies *to all Fano fourfolds*. From this result of rationally connectedness follows also a completely different proof of Hodge (2,2) for Fano 4-folds by theorem of Bloch-Srinivas in 1983 ([3]) (see also section 9 of [42]).

3. On Fano's theorem on threefolds which have as hyperplane sections Enriques surfaces.

3.1.

Fano threefolds of the so-called principal series, i.e. 3-folds V for which $-K_V$ is *very* ample, have, when they are embedded by this anticanonical system, K_3-surfaces as hyperplane sections (this is a classical fact, but see for instance [41], p.43). Therefore it is natural to ask whether there exist threefolds having as hyperplane sections Enriques surfaces. Indeed such threefolds exist and they were studied in 1933 by Godeaux and in much deeper detail and depth by Fano in 1938. In his paper [22] Fano obtained the following remarkable result.

THEOREM (FANO). Let V a threefold in $\mathbb{P}_N$ such that a general hyperplane section $S = V \cap H$ is a *smooth Enriques surface*. If V is "otherwise general" then V has exactly 8 singular points $P_1, P_2, \ldots, P_8$. Each P_i is a quadruple point on V, and the tangent cone to V in such a point P_i is isomorphic to the cone over the Veronese surface.

In the above paper [22] Fano claimed also a classification of such threefolds and he described - essentially - 4 different types (see below section 3.3).

3.2.

Fascinated by this striking result we (Conte and the author) started around 1980 to study Fano's paper. His "proof" contains many gaps. For many steps Fano did not give an argument at all, apparently he did – by his fabulous geometric insight and knowledge - simply "see" that this should be true! By an often case by case examination with long calculations we filled in these steps in our paper [13] and it turned out that Fano was almost always right at the end!

At the Varenna conference on Algebraic Threefolds in 1981 Conte did give a survey of our proof in [7] and in [8] he discussed two types of the four types mentioned by Fano; these examples are full of beautiful classical geometry.

3.3. Further developments

The classification problem was taken up again from a modern point of view by T. Sano [45] and by L. Bayle [1]. The point of view of Bayle is as follows. First he shows that such a threefold V with Enriques hyperplane section can be obtained from a pair (W,τ) consisting of a Fano threefold W together with an involution τ on W with a finite number of fixed points. Then using the classification of Fano 3-folds W by Iskovskikh and Mori-Mukai he studied which W admit such an involution. In this way he gets a complete classification. Bayle found that there are 6 different types; hence Fano missed two cases. We refer for these results to the nice paper of Bayle ([1]).

4. On the definition and on the nature of the singularities of Fano threefolds.

The original point of view of Fano of what nowadays are called *Fano threefolds* is that these are three dimensional varieties V which are embedded in projective space $\mathbb{P}_N$ with the property that the general linear section $C = V \cap H_1 \cap H_2$ is a *canonically embedded curve* (here H_1 and H_2 are sufficiently general hyperplanes). If V is smooth and embedded by $|-K_V|$ then this is equivalent with the modern definition (Iskovskikh) namely that the anticanonical class is ample [24].

In the paper [13] Conte and the author studied threefolds V with the property that $C = V \cap H_1 \cap H_2$ is canonically embedded but without the assumption that V is smooth. It turned out that the above condition puts severe restrictions on the nature of the singularities. In particular the singularities are Gorenstein singularities. Furthermore if there are only isolated singularities then there can be at most one "non-rational singularity" and in fact such a singularity must be necessarily "elliptic".

In our paper we used heavily the works of Du Val, Merindol and Epema on surfaces which have as hyperplane sections canonical curves (see [20]).

5. On Morin's work on the unirationality of hypersurfaces.

5.1. Introduction

The question of unirationality for hypersurfaces was studied by U. Morin (University of Padova) in a remarkable series of papers rangings over the period 1936-55 (see [34]). In 1940 he proved the following nice theorem

THEOREM 4. ([32])
Let $V = V_{n-1}(d) \subset \mathbb{P}_n$ be a "general" hypersurface of degree d and dimension $(n-1)$. There exists a constant $\rho(d)$ such that if $n > \rho(d)$ then V is unirational.

"General" means as usual that in the parameter space of the $V_{n-1}(d)$ there exists a Zariski open set such that the theorem holds for the V in that set.

Morin's theorem was extended to complete intersections by Predonzan in 1949 [43]. Modern proofs for Morin's theorem were given by Ciliberto [5] in 1980, Ramero (1990) and Paranjape - Srinivas in 1992. In 1998 Harris - Mazur - Pandharipande relaxed the condition "general" to "smooth".

In all these proofs the $\rho(d)$ is *very* large and certainly far from the best bound.

In a paper in 1938 (resp. in 1952) Morin did give a much sharper bound for the case $d = 5$ [31] (resp. $d = 4$ [33]). These papers are for "modern" readers difficult to read, firstly because they are written in a classical style and secondly - more important - Morin's exposition is very concise. On the other hand Morin's ideas are very geometrical and nice and therefore Conte and myself, later also joined by Marina Marchisio, have given a modern treatment of these papers ([14], [17]). Although we have at some places somewhat simplified the proofs and also slightly generalized the result for $d = 5$, I want to stress that we have followed Morin's ideas and, by looking carefully, Morin's original proofs are certainly correct. See section 5.3 and 5.4 below for the theorems.

5.2. Generalization to double coverings

Following an idea of Ciliberto from his above mentioned paper [5] from 1980 we extended Morin's 1940 result for hypersurfaces to double covers on $\mathbb{P}_N$.

THEOREM 5. ([15])
Let $B = B_{n-1}(2d) \subset \mathbb{P}_n$ be a hypersurface of degree $2d$ in $\mathbb{P}_n$ and let $\pi : W = W_n[2d, B] \to \mathbb{P}_n$ be a double covering branched over B. Then there exists a constant $\rho(d)$ such that if $n > \rho(d)$ and B is general then W is unirational.

For the proof see [15].

5.3. Quartic hypersurfaces

THEOREM 6. ([33])

Let $V = V_{n-1}(4) \subset \mathbb{P}_n$ be a general quartic (in particular smooth) hypersurface. Then V is unirational if $n > 5$.

In [14] we did give a "modern" proof of this. We followed the basic idea of Morin but we could give some simplifications based on results of S. Lang [28] and B. Segre [46] which were not available to Morin in 1952.

Note that the problem of the *unirationality* of the *general quartic* hypersurface remains open for dimension 3 and 4.

Examples of *"special"* smooth quartic 3-folds which are unirational are given by B. Segre [47], Predonzan [44] and Marina Marchisio [29], [30]; Marina Marchisio constructed in fact families of such 3-folds in her thesis. Also note that a smooth quartic 3-fold is always *non-rational* as was shown by Iskovskikh-Manin in 1971 [25].

5.4. Quintic hypersurfaces

THEOREM 7. (Morin 1938 [31])

Let $V = V_{n-1}(5) \subset \mathbb{P}_n$ be a general quintic hypersurface. Then V is unirational if $n > 17$.

In 2007 we (CMM) did give a modern proof of this theorem (again based on Morin's ideas). In fact we obtained a more precise result which included Morin's theorem.

THEOREM 8. ([17])

Let $V = V_{n-1}(5) \subset \mathbb{P}_n$ be defined over a field k (always of characteristic zero for simplicity). Assume that there exists a 3-dimensional linear space $\Sigma \subset V$ and defined *also over k. Let V be otherwise general- Then V is unirational if $n \geq 7$.*

This generalizes indeed the above theorem of Morin because for $n > 17$ there exists such a linear 3-dimensional space Σ inside a general $V_{n-1}(5)$ (as is shown by standard arguments).

Proof. (Very rough outline in order to give the idea). We assume for simplicity that $n = 7$ (otherwise we need to pay some extra care).

The idea of the proof - due to Morin - is to use a famous theorem of Enriques ([19]), namely the theorem that the variety $V_3(2,3) \subset \mathbb{P}_5$ (i.e. the intersection of a quadric and a cubic in $\mathbb{P}_5$) is unirational. However it is necessary to use this theorem in a refined form (see details below)!

So taking $n = 7$, let $V = V_6(5) \subset \mathbb{P}_7$ be a variety containing a 3-space $\Sigma \subset V$ (so $\Sigma \simeq \mathbb{P}_3$) but otherwise general and let k be a field over which both V and Σ are defined. Since Σ is defined over k we can take homogeneous coordinates $(t_0 : t_1 : t_2 : \ldots : t_7)$ in

$\mathbb{P}_7$ such that Σ has equations

$$t_4 = t_5 = t_6 = t_7 = 0. \tag{34}$$

Let $Y = (1, y_1, y_2, y_3, 0, 0, 0, 0)$ be the generic point of Σ over k, i.e., y_1, y_2, y_3 are independent transcendental over k; let $k(y) = k(y_1, y_2, y_3)$.

Consider now in $\mathbb{P}_7$ the variety K_Y defined by

$$\begin{aligned} K_Y = K_Y(V) := \{P' \in \mathbb{P}_7;\ P' \in l \subset V \ \text{ with } l \text{ a line in } \mathbb{P}_7 \\ \text{such that } l \cap V = 4Y + P\} \end{aligned} \tag{35}$$

i.e. K_Y is a cone with vertex Y consisting of lines through Y which intersect V in Y with multiplicity at least four. Clearly the variety K_Y is defined over the field $k(y)$ and

$$K_Y(V) = T_Y(V) \cap Q_Y(V) \cap C_Y(V) \tag{36}$$

when $T_Y(V)$ is the tangent space in Y to V (given by a linear equation), $Q_Y(V)$ the (usual) tangent cone in Y to V (given by a quadratic equation) and $C_Y(V)$ the cone with the vertex Y of the lines with contact higher than three to V (given by a cubic equation); all these equations have their coefficients in the field $k(y)$.

Consider the variety

$$W_Y = W_Y(V) := K_Y(V) \cap V. \tag{37}$$

Clearly P from (35) is on W_Y. If $F(t) = 0$ is the equation of V then P satisfies the equations (symbolically):

$$\begin{cases} F(P) = 0 \\ \dfrac{\partial F}{\partial t}(Y)(P) = 0 \\ \dfrac{\partial^2 F}{\partial t^2}(Y)(P) = 0 \\ \dfrac{\partial^3 F}{\partial t^3}(Y)(P) = 0 \end{cases} \tag{38}$$

Note also that starting with a point P on V we find 24 such points Y.

LEMMA 3. *If P is generic on W_Y over $k(y)$ then P is generic on V over k.*

Proof. Easy dimension count. ☐

Hence we have

$$k(V) \cong k(P) \subset k(Y, P) = k(y)(P).$$

Hence: it suffices to prove that $W_Y(V)$ is *unirational over* $k(y)$ (i.e. not merely unirational, but *unirational over the field* $k(y)$ *itself*).

Let $H_\infty = \{t_0 = 0\} \subset \mathbb{P}_7$ be the hyperplane "at infinity" and consider in H_∞ the variety

$$W'_Y = W'_Y(V) := H_\infty \cap T_Y(V) \cap Q_Y(V) \cap C_Y(V). \tag{39}$$

LEMMA 4. *There is a birational transformation*

$$f: \quad W_Y \overset{\sim}{\dashrightarrow} W'_Y$$

given by $f(P) = P_\infty$ *where* $P_\infty = l \cap H_\infty$ *and* f *is defined over the field* $k(y)$.

Proof. This is immediately clear from the way the variety K_Y has been defined in (5.2), i.e., W'_Y is the projection to H_∞ of W_Y from the point Y and f is birational since on the "general" line l there is only one such point P (see (35)). $\square$

Hence: it is sufficient to prove that $W'_Y(V)$ is *unirational over the field* $k(y)$.

Now $W'_Y(V) \subset H_\infty \cap T_Y(V)$ and $H_\infty \cap T_Y(V)$ is a projective space $\mathbb{P}_{5,Y}$ (if $n = 7$), all over the field $k(y)$ and in $\mathbb{P}_{5,Y}$ the $W'_Y(V)$ is the intersection

$$W'_Y(V) = Q'_Y(V) \cap C'_Y(V) \tag{40}$$

with $Q'_Y(V) = Q_Y(V) \cap \mathbb{P}_{5,Y}$ and $C'_Y(V) = C_Y(V) \cap \mathbb{P}_{5,Y}$, i.e., $W'_Y(V)$ is a variety of type $V_3(2,3) \subset \mathbb{P}_5$ defined over the field $k(y)$, By the theorem of Enriques ([19]) it is indeed unirational. However we need the more precise result that it is *unirational over the field* $k(y)$ itself. Although Morin is very short and not explicit it seems clear (by reading between the lines!) that he was aware of this fact and also knew that in his case the Enriques theorem holds in this refined form due to his linear space Σ in V.

In our paper [16] we have checked all this carefully and proved the Enriques theorem in the following precise form.

THEOREM 9. *Let* $W = Q \cap C \subset \mathbb{P}_5$ *with* Q *(resp.* C*) a quadric (resp. cubic) hypersurface containing a 2-plane* Δ*, but otherwise "general". Let* Q, C and Δ *be defined over a field* K. *Then* W *is* unirational over K.

For the proof see [17].

By this theorem it follows from the above that the proof of the theorem for the quintic is complete if we take $W = W_Y$ and $\Delta = \Sigma \cap H_\infty$ and $K = k(y)$. $\square$

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AMS Subject Classification: 14J30, 14J45

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Lavoro pervenuto in redazione il 02.07.2013.

Rend. Sem. Mat. Univ. Politec. Torino
Vol. 71, 3–4 (2013), 493 – 504

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SEVERI'S RESULTS ON CORRESPONDENCES

Abstract. We analyze Severi's formula for the virtual number of fixed points of a correspondence T on a surface, and his notion of the rank of T. If the diagonal has valence zero, we verify Severi's formula with rank being the trace on the Neron-Severi group. Otherwise, we show that Severi's formula holds with a corrected notion of rank. We apply Severi's formula to complex surfaces with involution, both K3 surfaces and surfaces with $p_g = q = 0$.

1. Introduction

Severi developed a theory of correspondences in a series of papers which appeared in 1933, introducing the notions of *valences* and *indices*. One of the results achieved by Severi is a formula for the virtual number of fixed points of a correspondence on a smooth projective surface X. These papers are part of Severi's attempt to develop a theory of the series of equivalences on a surface. In fact Severi encountered (sometimes without being completely aware of what was going on!) the problem of not having a rigorous definition for the different equivalence relations among cycles, which are now known as rational, algebraic, homological and numerical equivalence. However as W. Fulton writes in [7, p.26]:

> *It would be unfortunate if Severi's pioneering works in this area were forgotten; and if incompleteness or the presence of errors are grounds for ignoring Severi's work, few of the subsequent papers on rational equivalence would survive.*

The above considerations indicate that Severi was often wrong and certainly too bold in making conjectures. However Severi was somehow able to perceive the *motivic* content of the matter, by considering correspondences and their action both on Chow groups and cohomology groups. In fact he was the first to relate the action of a correspondence $\Gamma \subset X \times X$ on the Chow group of 0-cycles on a smooth projective surface X to the cohomology class of Γ in $H^4(X \times X, \mathbb{C})$. In [15] (see also [4, 3.3]) he made a claim that in its original form is not correct but can be easily restated as what is now known as Bloch's conjecture.

CONJECTURE 1.1. *Let S be smooth projective surface over $\mathbb{C}$. If $p_g(S) = q(S) = 0$ the Chow group $CH_0(X)_0$ of 0-cycles of degree 0 vanishes.*

Bloch's conjecture is known to hold for all surfaces which are not of general type, see [2], and for many surfaces of general type, see [1] and [14].

In this note we will give a precise formulation of Severi's result on the virtual number of fixed points of a correspondence on a surface S (Theorem 4.7). We also provide a proof of what Severi claimed for the case when the diagonal Δ_S has valence

0 (Theorem 4.1). Then, in Sect. 5, we apply these results to a complex surface S with an involution, in the case $p_g(S) = q(S) = 0$ and in the case S is a K3 surface.

If X is a smooth projective variety, we will write $CH^i(X)$ for the Chow group of codimension i cycles on X, and write $A^i(X)$ for the Chow group with $\mathbb{Q}$-coefficients, $CH^i(X)\otimes\mathbb{Q}$. We write $NS(S)$ for the Nerón-Severi group of S, and $\rho(S)$ for the rank of $NS(S)$. If Γ is any correspondence on X, we write $\mathrm{trace}_{NS}(\Gamma)$ for the trace of Γ acting on the vector space $NS(X)\otimes\mathbb{Q}$. For example, $\mathrm{trace}_{NS}(\Delta_X)$ is the rank $\rho(X)$ of $NS(X)$.

2. Indices and valence of a correspondence

Let S be a smooth projective surface over C and let Γ be a correspondence in $CH^2(S\times S)$. The formula for the virtual number of fixed points of Γ is given in terms of the following numbers: the second Chern class $c_2(S)$, the trace of the action of Γ on the vector space $NS(S)\otimes\mathbb{Q}$, the *valence* v and the *indices* α, β of Γ (see Definition 2.2 below). The following formula appeared for the first time in 1933 in [17, p. 871]:

$$\deg(\Gamma\cdot\Delta_S) = \alpha(\Gamma)+\beta(\Gamma)+\delta(\Gamma)-v(\Gamma)(I+1), \tag{2.1}$$

where $\delta(\Gamma)$ is the *rank* of Γ (see Definition 4.6) and I is the Zeuthen-Segre invariant, $I = c_2(S)-4$. The same formula was reproduced in the first edition of Zariski's book on Algebraic Surfaces which appeared in 1935, and also in the second edition of it in 1971 (see [21, p. 246]).

The notion of the indices and valence of a correspondence Γ on a smooth projective variety X are about 100 years old and were well known to Severi and Lefschetz; see [11]. We give their precise definition below, following [7, §16], using the notion of degenerate correspondences in the Chow group of algebraic cycles modulo rational equivalence.

However, Severi's notion of the rank of Γ, as given in *op. cit.*, is rather obscure. Also, Severi's formula is based on the assumption that the correspondence Δ_S has rank 1, i.e., if Δ_S does not belong to the ideal $\mathcal{J}(S)$ of degenerate correspondences (see Definition 2.3). We will give the correct definition of the rank of a correspondence in Definition 4.6.

DEFINITION 2.2. *Let X be a smooth projective variety over a field k. The indices of a correspondence $\Gamma\subset X\times X$ are the numbers $\alpha(\Gamma)=\deg(\Gamma\cdot[P\times X])$ and $\beta(\Gamma)=\deg(\Gamma\cdot[X\times P])$, where P is any rational point on X; see [7, 16.1.4].*

The indices are additive in Γ, and $\beta(\Gamma)=\alpha({}^t\Gamma)$.

DEFINITION 2.3. *A correspondence is said of valence zero if it belongs to the ideal $\mathcal{J}(X)$ in $A^n(X\times X)$ of degenerate correspondences, i.e., the ideal generated by correspondences of the form $[V\times W]$, with V or W proper subvarieties of X. We say that a correspondence Γ has valence v if $\Gamma+v\Delta_X$ has valence 0.*

For example, Δ_X always has valence -1, but it may also have valence 0, as is the case for $X = \mathbb{P}^1$. If Γ_1, Γ_2 in $A^d(X \times X)$ have valences v_1, v_2 then $\Gamma = \Gamma_1 + \Gamma_2$ has valence $v_1 + v_2$, and $\Gamma_1 \circ \Gamma_2$ has valence $-v_1 v_2$ by [7, 16.1.5(a)].

If Δ_X does not have valence zero then the valence of a correspondence Γ is either unique or undefined. On the other hand, if Δ_X has valence zero and the valence of a correspondence Γ is defined, then $\Gamma \in \mathcal{J}(X)$, hence it has valence v for every $v \in \mathbb{Q}$. This is, for example, the case if X is a rational surface.

EXAMPLE 2.4 (Chasles-Cayley-Brill-Hurwitz). Let C be a curve of genus g. If $T \in A^1(C \times C)$ is a correspondence with valence v, then the Cayley-Brill formula is: $\deg(T \cdot \Delta_C) = \alpha(T) + \beta(T) + 2vg$. This is proven in [Fu 16.1.5(e)].

We thank the Referee for pointing out the following Lemma; cf. [19, 2.2.1].

LEMMA 2.5. *If Δ_X has valence zero, then rational, algebraic, homological and numerical equivalence coincide in $A^*(X)$.*

Proof. If $v(\Delta_S) = 0$ then the diagonal decomposes as $\sum_j c_j [V_j \times W_j]$. Hence for any cycle $Z \in A^*(X)$ we have

$$Z = \Delta_X \cdot Z = \sum_j c_j (Z \cdot V_j)[W_j]$$

It follows that numerically equivalent cycles are rationally equivalent. $\square$

3. The Chow motive of a surface

Let $\mathcal{M}_{rat}(k)$ be the (covariant) category of *Chow motives* with $\mathbb{Q}$-coefficients over an algebraically closed field k of characteristic 0 and let $h(X)$ be the motive associated to a smooth projective variety X. If S is a smooth projective surface then the motive $h(S)$ has a *reduced Chow-Künneth decomposition* as in [10, 7.2.2] of the form

$$h(S) = h_0(S) \oplus h_1(S) \oplus h_2(S) \oplus h_3(S) \oplus h_4(S),$$

where $h_i(S) = (S, \pi_i, 0)$; each π_i is a projector whose cohomology class is the $(i, 4-i)$ component in the Künneth decomposition of Δ_S in $H^4(S \times S)$. Here H^* means any classical Weil cohomology theory, such as Betti cohomology with $\mathbb{Q}$ coefficients when $k = \mathbb{C}$.

The motive $h_2(S)$ further decomposes as $h_2^{\mathrm{alg}}(S) \oplus t_2(S)$ where $h_2^{\mathrm{alg}}(S)$ is the algebraic part of $h_2(S)$ and $t_2(S)$ is the *transcendental motive*. The motive $h_2^{\mathrm{alg}}(S)$ may be constructed by choosing a basis $\{E_1, \cdots, E_\rho\}$ for the $\mathbb{Q}$-vector space $NS(S)_{\mathbb{Q}}$ which is orthogonal in the sense that $E_i \cdot E_j = 0$ for $i \neq j$ and the self-intersections E_i^2 are nonzero. The correspondences $\varepsilon_i = \frac{[E_i \times E_i]}{(E_i)^2}$ are orthogonal and idempotent, so their sum

$$\pi_2^{\mathrm{alg}} = \varepsilon_1 + \cdots + \varepsilon_\rho = \sum_{1 \le i \le \rho} \frac{[E_i \times E_i]}{(E_i)^2} \tag{3.1}$$

is also an idempotent correspondence, and $h_2^{\mathrm{alg}}(S) = (S, \pi_2^{\mathrm{alg}}, 0)$. Since $\{E_i/(E_i)^2\}$ is a dual basis to the $\{E_i\}$, π_2^{alg} and $h_2^{\mathrm{alg}}(S)$ are independent of the choice of basis.

Setting $M_i = (S, \varepsilon_i, 0)$, $h_2^{\mathrm{alg}}(S)$ is the direct sum of the M_i, and each M_i is isomorphic to the Lefschetz motive $\mathbb{L}$ (see [10, 7.2.3]), so $h_2^{\mathrm{alg}}(S) \cong \mathbb{L}^{\oplus\rho}$. Setting $H^2_{\mathrm{alg}}(S) = \pi_2^{\mathrm{alg}} H^2(S)$, we also have isomorphisms $H^2_{\mathrm{alg}}(S) \cong NS(S)_{\mathbb{Q}}$.

The transcendental motive $t_2(S)$ is defined as $t_2(S) = (S, \pi_2^{\mathrm{tr}}, 0)$, where

$$\pi_2^{\mathrm{tr}} = \pi_2 - \pi_2^{\mathrm{alg}}.$$

Setting $H^2_{\mathrm{tr}}(S) = \pi_2^{\mathrm{tr}} H^2(S)$, we have $H^2(S) = H^2_{\mathrm{alg}}(S) \oplus H^2_{\mathrm{tr}}(S)$, so $H^2(t_2(S))$ is the "transcendental part" of $H^2(S)$. In addition, $A_0(t_2(S))$ is the Albanese Kernel $T(S)$; see [10, 7.2.3]. The Chow motive $t_2(S)$ does not depend on the choices made to define the refined Chow-Künneth decomposition, it is functorial on S for the action of correspondences, and it is a birational invariant of S (see [KMP]).

If $p_g(S) = 0$ then $H^2_{tr}(S) = 0$ and $t_2(S) = 0$ iff $T(S) = 0$, i.e., S satisfies Bloch's conjecture 1.1. The condition $t_2(S) = 0$ is also equivalent to the finite dimensionality of $h(X)$, see [8]. If $p_g(S) > 0$ then $T(S) \neq 0$, hence $t_2(S) \neq 0$.

Recall that $A^2(S \times S)$ is the endomorphism ring of $h(S)$ in $\mathcal{M}_{rat}$, with the diagonal correspondence Δ_S acting as the identity. Consider the ring projection

$$\Psi_S : A^2(S \times S) = \mathrm{End}_{\mathcal{M}_{rat}}(h(S)) \to \mathrm{End}_{\mathcal{M}_{rat}}(t_2(S)) \tag{3.2}$$

sending Γ to $\pi_2^{\mathrm{tr}} \circ \Gamma \circ \pi_2^{\mathrm{tr}}$. By construction, Ψ_S sends the class $[\Delta_S]$ of the diagonal to π_2^{tr}, which is the identity map of the motive $t_2(S)$. In fact, Ψ_S induces a ring isomorphism

$$\Psi_S : A^2(S \times S)/J_{\mathrm{nd}}(S) \xrightarrow{\cong} \mathrm{End}_{\mathcal{M}_{\mathrm{rat}}}(t_2(S)),$$

where $J_{\mathrm{nd}}(S)$ is the ideal of $A^2(S \times S)$ generated by the classes of correspondences which are not dominant over S by at least one of the two projections $S \times S \to S$, see [10, 7.4.3].

LEMMA 3.3. *Let S be a smooth projective surface with $q(S) = 0$. Then $\mathcal{J}(S) = J_{nd}(S)$ in $A^2(S \times S)$.*

Proof. From the definition of the ideals $\mathcal{J}(S)$ and $J_{\mathrm{nd}}(S)$ we get $\mathcal{J}(S) \subseteq J_{\mathrm{nd}}(S)$. Let $\Gamma \in J_{\mathrm{nd}}(S)$ such that Γ is not dominant over S under the first projection. We claim that Γ belongs to the ideal of degenerate correspondences. Γ vanishes on some $V \times S$, with V open in S, hence it has support on $W \times S$, with $\dim W \leq 1$. If $\dim W = 0$ then $\Gamma = \sum_i n_i [S \times P_i]$ in $A^2(S \times S)$, where P_i are closed points in S. Hence $\Gamma \in \mathcal{J}(S)$. If $\dim W = 1$ then $\Gamma \in A^1(W \times S)$, where $A^1(W \times S) = p_1^*(A^1(W)) \times p_2^*(A^1(S))$, with p_i the projections, because $H^1(S, O_S) = 0$ (see [9, p. 292]). Therefore $\Gamma \in \mathcal{J}(S)$. □

4. Severi's formula

In [16, p. 761], Severi claims that if on a surface S there exists a correspondence $\Gamma \in A^2(S \times S)$ with two distinct valences, i.e., $v(\Delta_S) = 0$, then S is "regular of genus 0', i.e $q(S) = p_g(S) = 0$. The following theorem verifies Severi's claim.

THEOREM 4.1. *Let S be a smooth projective surface. Then the following conditions are equivalent:*

(1) There exists a correspondence $\Gamma \in A^2(S \times S)$ with two distinct valences v and v';

(2) $v(\Delta_S) = 0$;

(3) $p_g(S) = q(S) = 0$ and S satisfies Bloch's conjecture;

(4) $A_0(S) \simeq \mathbb{Q}$.

Proof. Since both $\Gamma + v\Delta_S$ and $\Gamma + v'\Delta_S$ are in $\mathcal{J}(S)$, so is $(v - v')\Delta_S$. Therefore $\Delta_S \in \mathcal{J}(S)$, i.e., Δ_S has valence 0, which shows that (1) $\Longrightarrow$ (2). Clearly (2) $\Longrightarrow$ (1), by taking $\Gamma = \Delta_S$.

If $v(\Delta_S) = 0$ then $\Delta_S \in \mathcal{J}(S)$. Since $\mathcal{J}(S) \subseteq J_{\mathrm{nd}}(S)$ we get $\Psi_S(\Delta_S) = 0$, i.e., the identity map on $t_2(S)$ is 0 in $\mathcal{M}_{rat}$. This is equivalent to $t_2(S) = 0$ and hence $T(S) = A_0(t_2(S))$ equals 0. The condition $T(S) = 0$ forces $p_g(S) = 0$ and is a form of Bloch's conjecture. By Lemma 2.5, $A^1(S)$ injects into $H^2(S, \mathbb{Q})$, so $q(S) = 0$. This shows that (2) implies (3).The equivalence (3) $\Longleftrightarrow$ (4) is well known.

If $p_g(S) = q(S) = 0$ and Bloch's conjecture holds for S, then, by [3, Prop. 1], there exist a closed $V \subset S$ of dimension 0 and a divisor D on S such that $\Delta_S = \Gamma_1 + \Gamma_2$ in $A^2(S \times S)$, with Γ_1 supported on $V \times S$ and Γ_2 supported on $S \times D$. Hence $\Delta_S \in J_{\mathrm{nd}}(S)$. By Lemma 3.3 we get $\Delta_S \in \mathcal{J}(S)$, because $q(S) = 0$, so that $v(\Delta_S) = 0$, i.e., (3) $\Longrightarrow$ (2).

The equivalence (3) $\Longleftrightarrow$ (4) is well known. $\square$

REMARK 4.2. For a surface S, if $v(\Delta_S) = 0$, then $p_g(S) = q(S) = 0$ and also $t_2(S) = 0$. Therefore $h(S) = \mathbf{1} \oplus \mathbb{L}^{\oplus \rho(S)} \oplus \mathbb{L}^2$, so that $h(S)$ coincides with the motive of S in the category of numerical motives $\mathcal{M}_{num}$.

EXAMPLE 4.3. Let S be a hyperelliptic surface over $\mathbb{C}$, i.e., a smooth projective surface with $p_g(S) = 0$ and $q(S) = 1$, which is isomorphic to a quotient $E \times F/G$, with E, F elliptic curves and G a finite group. By [2] the Albanese kernel of S vanishes, hence, by [3, Prop.1], $\Delta_S = \Gamma_1 + \Gamma_2$ with $\Gamma_1 \subset V \times S$, $\Gamma_2 \subset S \times D$ and $V \neq S, D \neq S$. Therefore $\Delta_S \in J_{\mathrm{nd}}(S)$. Because $q(S) \neq 0$, Theorem 4.1 implies that $v(\Delta_S) \neq 0$, so that $\Delta_S \notin \mathcal{J}(S)$.

PROPOSITION 4.4. *Let S be a smooth projective surface. Then for every correspondence $T \in A^2(S \times S)$ of valence zero:*

$$\deg(T \cdot \Delta_S) = \alpha(T) + \beta(T) + \mathrm{trace}_{NS(S)}(T).$$

Proof. Since the correspondence T has valence 0, it belongs to the ideal $\mathcal{J}(S)$ of degenerate correspondences. Therefore we may write $T = T_0 + T_1$, where

$$T_0 = \sum p_j [P_j \times S] + \sum q_k [S \times Q_k], \qquad T_1 = \sum m_i [D_i \times D_i'];$$

here $D_i, D_i' \in A^1(S)$ and P_j, Q_j are points. We may move $[D_i \times D_i']$ within its class in

$A^2(S\times S)$ in such a way that it does not meet any of the $P_j\times S$ or $S\times Q_k$, so that

$$[D_i\times D_i']\cdot[P_j\times S]=[D_i\times D_i']\cdot[S\times Q_k]=0$$

for all i,j,k. With this reduction, we have $\alpha(T_1)=\beta(T_1)=0$, $\alpha(T_0)=\deg(T_0\cdot[P\times S])=\sum q_k$, and $\beta(T_0)=\deg(T_0\cdot[S\times P])=\sum p_j$. We also have

$$\deg(T_0\cdot\Delta_S)=\sum p_j+\sum q_k=\alpha(T)+\beta(T).$$

Now for any divisor C on S, we have $[P_j\times S]_*(C)=[S\times Q_k]_*(C)=0$. Thus T_0 acts as zero on $NS(S)$, so $\mathrm{trace}_{NS}(T_0)=0$. Therefore we may assume that $T=T_1$, and need to evaluate

$$\deg(T_1\cdot\Delta_S)=\deg(\sum m_i[D_i\times D_i']\cdot\Delta_S)=\sum m_i\,(D_i\cdot D_i').$$

Choose an orthogonal basis $\{E_\ell, 1\le\ell\le\rho(S)\}$ for the $\mathbb{Q}$-vector space $NS(S)\otimes\mathbb{Q}$. In terms of this basis,

$$D_i=\sum\nolimits_k a_{ik}\,E_k,\qquad D_i'=\sum\nolimits_\ell b_{i\ell}\,E_\ell.$$

Since $E_k\cdot E_\ell=0$ when $k\ne\ell$, we may expand $D_i\cdot D_i'$ to get

$$\deg(T\cdot\Delta_S)=\sum_i m_i\,(D_i\cdot D_i')=\sum_{i,k} m_i\,a_{ik}\,b_{ik}\,(E_k)^2.$$

Because $(D_i\times D_i')_*(E_k)=(D_i\cdot E_k)D_i'=a_{ik}(E_k)^2D_i'$,

$$T_*(E_k)=(\sum m_i[D_i\times D_i'])_*(E_k)=\sum m_i a_{ik} b_{i\ell}(E_k)^2E_\ell.$$

Thus $\mathrm{trace}_{NS(S)}(T_*)=\sum m_i a_{ik} b_{ik}(E_k)^2$, and the result follows. □

REMARK 4.5. If S is a smooth projective surface over $\mathbb{C}$ and Γ is a correspondence in $A^2(S\times S)$ then, by the Lefschetz fixed point formula (see [7, 16.1.15]):

$$\deg(\Gamma\cdot\Delta_S)=\sum_{0\le i\le 4}(-1)^i\,\mathrm{trace}_{H^i(S)}(\Gamma).$$

Note that $\alpha(\Gamma)$ and $\beta(\Gamma)$ are the traces of Γ acting on $H^4(S)$ and $H^0(S)$, respectively. This is immediate from Definition 2.2, since $\pi_0=[S\times P]$ and $\pi_4=[P\times S]$ in our covariant setting. If $v(\Gamma)=0$ then $\Psi_S(\Gamma)=0$, so Γ acts as 0 on $t_2(S)$ and on $H^2_{tr}(S,\mathbb{Q})$. Therefore Proposition 4.4 says that a correspondence of valence 0 has trace 0 on the odd cohomology of S.

In [17, p. 871], Severi gave a definition of the *rank* $\delta(T)$ of a correspondence T of valence 0 and gave an argument asserting that if T is a correspondence of valence 0 on a surface S, then

$$\deg(T\cdot\Delta_S)=\alpha(T)+\beta(T)+\delta(T).$$

Severi pointed out that $\delta(T)$ may be computed by taking any decomposition of T into the sum of degenerate correspondences. Proposition 4.4 shows that $\delta(T)=\mathrm{trace}_{NS}(T)$.

Severi then defines the *rank* of a correspondence Γ of valence v to be the number $\delta(\Gamma)$ such that $\delta(\Gamma)+v$ is the rank of the correspondence $T=\Gamma+v\Delta_S$ of valence 0. He also sets the rank of the the diagonal Δ_S to be 1, which is consistent when Δ_S has valence -1. Thus we may reinterpret Severi's definition as follows.

DEFINITION 4.6. *If $T \in A^2(X\times X)$ is a correspondence on a smooth projective variety X of valence 0, we define its rank $\delta(T)$ to be the trace $\mathrm{trace}_{NS}(T)$ of T acting on $NS(X)$. If Δ_X does not have valence 0, and Γ is a correspondence of valence v, we define the rank of Γ to be*

$$\delta(\Gamma)=\mathrm{trace}_{NS}(\Gamma)+v\,(\rho(X)-1).$$

With this definition of $\delta(\Gamma)$ we recover Severi's formula (2.1) for a surface S. If $v=v(\Gamma)$ then $T=\Gamma+v\Delta_S$ has valence 0 and by Proposition 4.4 we have

$$\begin{aligned}\delta(T)=\delta(\Gamma{+}v\Delta_S)&=\mathrm{trace}_{NS}(\Gamma+v\Delta_S)=\mathrm{trace}_{NS}(\Gamma)+v\,\mathrm{trace}_{NS}(\Delta_S)\\&=\mathrm{trace}_{NS}(\Gamma)+v\cdot\rho(S)=\delta(\Gamma)+v.\end{aligned}$$

Recall that the second Chern class of S satisfies $c_2(S)=\deg(\Delta_S\cdot\Delta_S)$ (see [7, 8.1.12]).

THEOREM 4.7. *Let S be a smooth projective surface. If Δ_S does not have valence 0 and $\Gamma\in A^2(S\times S)$ is a correspondence with valence v, then*

$$\deg(\Gamma\cdot\Delta_S)=\alpha(\Gamma)+\beta(\Gamma)+\mathrm{trace}_{NS}(\Gamma)+v\cdot(2+\rho(S)-c_2(S))\,.$$

Therefore

$$\deg(\Gamma\cdot\Delta_S)=\alpha(\Gamma)+\beta(\Gamma)+\delta(\Gamma)-v\cdot(c_2(S)-3)$$

as in (2.1).

Proof. By definition, $T=\Gamma+v\Delta_S$ has valence 0. From Proposition 4.4 we have

$$\deg(T\cdot\Delta_S)=\alpha(T)+\beta(T)+\mathrm{trace}_{NS}(T),$$

with $\alpha(T)=\alpha(\Gamma)+v$, $\beta(T)=\beta(\Gamma)+v$ and $\mathrm{trace}_{NS}(T)=\mathrm{trace}_{NS}(\Gamma)+v\,\rho(S)$ by additivity of the trace. We also have

$$\deg(T\cdot\Delta_S)=\deg(\Gamma\cdot\Delta_S)+v\cdot\deg(\Delta_S\cdot\Delta_S)=\deg(\Gamma\cdot\Delta_S)+v\cdot c_2(S).$$

Equating the formulas yields the desired formula for $\deg(\Gamma\cdot\Delta_S)$. □

5. Surfaces with an involution

We now consider the case of the correspondence on a smooth projective surface S over $\mathbb{C}$ which is the graph of an involution σ, i.e. $\Gamma_\sigma=\{(x,\sigma(x))\in S\times S\}$ and show that, if $p_g(S)=q(S)=0$, then $\deg(\Gamma_\sigma\cdot\Delta_S)$ is given by the same formula as in Proposition 4.4. We also apply Theorem 4.7 to the case of a K3 surface with an involution, see Example 5.5.

The fixed locus of σ consists of a 1-dimensional part D (possibly empty) and $k \geq 0$ isolated fixed points $\{P_1, \dots, P_k\}$. The images Q_i in S/σ of the P_i are nodes, and S/σ is smooth elsewhere. The blow-up X of S at the k set of isolated fixed points resolves these singularities, σ lifts to an involution on X (which we will still call σ), and the quotient $Y = X/\sigma$ is a desingularization of S/σ. The images $C_1, \dots, C_k$ in Y of the exceptional divisors of X are disjoint nodal curves, i.e., smooth rational curves with self-intersection -2. In summary, we have a commutative diagram

(5.1)
$$\begin{array}{ccc} X & \xrightarrow{h} & S \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & S/\sigma. \end{array}$$

If $p_g(S) = q(S) = 0$ then $k = K_S \cdot D + 4$, see [5, 3.2].
For brevity, we write t for $\mathrm{trace}_{NS}(\Gamma_\sigma)$, the trace of the action of σ on $NS(S)_{\mathbb{Q}}$.

The following result is immediate from the Lefschetz fixed point formula and Remark 4.5, but we prefer to give an elementary proof.

THEOREM 5.2. *Let S be smooth projective surface over $\mathbb{C}$, with $p_g(S) = q(S) = 0$. Let σ be an involution on S and let $\Gamma_\sigma = (1 \times \sigma)\Delta_S$. Then*

$$\deg(\Gamma_\sigma \cdot \Delta_S) = \alpha(\Gamma_\sigma) + \beta(\Gamma_\sigma) + t = 2 + t = 4 - D^2.$$

Moreover σ acts as the identity on $H^2(S, \mathbb{Q})$ (i.e., $t = \rho(S)$) iff

$$K_S^2 = D^2 + 8.$$

Proof. Since S has no odd cohomology, the motive $h(S)$ has a Chow-Künneth decomposition

$$h(S) = h_0(S) \oplus h_2^{\mathrm{alg}}(S) \oplus t_2(S) \oplus h_4(S),$$

where $\pi_0 = [S \times P]$, $\pi_4 = [P \times S]$, with P a rational point on S, so that

$$\Delta_S = [S \times P] + \pi_2^{\mathrm{alg}} + \pi_2^{\mathrm{tr}} + [P \times S].$$

Since $h_2^{\mathrm{alg}} = (S, \pi_2^{\mathrm{alg}}, 0)$, where $\pi_2^{\mathrm{alg}} = \sum \varepsilon_i$ is as in (3.1), the action of σ on h_2^{alg} is determined by $\Gamma_\sigma(\varepsilon_i) = \frac{[E_i \times \sigma(E_i)]}{(E_i)^2}$. Let a_{ij} be such that $\sigma(E_i) = \sum_j a_{ij} E_j$. Then

$$(1 \times \sigma)\pi_2^{\mathrm{alg}} \cdot \Delta_S = \sum_{1 \leq i \leq \rho} \frac{[E_i \times \sigma(E_i)]}{(E_i)^2} \cdot \Delta_S = \sum_{1 \leq i \leq \rho} a_{ii}.$$

Therefore $\deg(\Gamma_\sigma \cdot \pi_2^{\mathrm{alg}}) = \mathrm{trace}_{NS}(\sigma)$. We have $\pi_2^{\mathrm{tr}} H^2(S) = H^2_{\mathrm{tr}}(S) = 0$, because $p_g(S) = 0$, hence $((1 \times \sigma)\pi_2^{\mathrm{tr}}))H^2(S) = 0$. By [7, 19.2]

$$cl((1 \times \sigma)\pi_2^{\mathrm{tr}} \cdot \Delta_S) = cl((1 \times \sigma)\pi_2^{\mathrm{tr}}) \cdot cl(\Delta_S) = 0$$

in $H_0(S\times S)$, hence the 0-cycle $(\Gamma_\sigma\cdot\pi_2^{\mathrm{tr}})$ has degree 0 in $A_0(S\times S)$. We also have

$$\beta(\Gamma_\sigma)=\deg(\Gamma_\sigma\cdot\pi_0)=1\ ;\ \alpha(\Gamma_\sigma)=\deg(\Gamma_\sigma\cdot\pi_4)=1.$$

Summing up we get

$$\deg(\Gamma_\sigma\cdot\Delta_S)=\deg(\Gamma_\sigma\cdot\pi_0)+\deg(\Gamma_\sigma\cdot\pi_2^{\mathrm{alg}})+\deg(\Gamma_\sigma\cdot\pi_4)=2+t.$$

From [6, 4.2], we get $t=2-D^2$, hence $\deg(\Gamma_\sigma\cdot\Delta_S)=4-D^2$.
Finally the trace $t=2-D^2$ of σ on $H^2(S,\mathbb{Q})$ is at most the rank $\rho(S)$ of $NS(S)$. By Noether's formula $c_2(S)=12-K_S^2$ because $q(S)=p_g(S)=0$, hence $\rho(S)=10-K_S^2$. Therefore $D^2\geq K_S^2-8$, with equality iff $t=\rho(S)$, i.e., iff σ acts as the identity on $H^2(S,\mathbb{Q})$. □

REMARK 5.3. Theorem 5.2 gives a simplified version of a formula that appears in [17, p. 874] and also in [7, 16.2.4], showing that in this case $\deg(\Gamma_\sigma\cdot\Delta_S)$ only depends on D^2.

EXAMPLE 5.4. (1) Let S be a minimal surface of general type with $p_g(S)=0$ and $K_S^2=8$ with an involution. By [6, 4.4] σ acts as the identity on $H^2(S,\mathbb{Q})$, $\rho(S)=t=2$ and $D^2=0$. Therefore $\deg(\Gamma_\sigma\cdot\Delta_S)=4$. If $D=0$ the number k of isolated fixed points of σ is 4, otherwise k is even and $6\leq k\leq 12$.
(2) Let S be a numerical Godeaux surface with an involution σ. S is a minimal surface of general type with $p_g(S)=q(S)=0$ and $K_S^2=1$. By [5, 4.5] σ has $k=5$ isolated fixed points and $-7\leq D^2\leq 1$. If $D^2=-7$ then σ acts as the identity on $H^2(S,\mathbb{Q})$, $t=\rho(S)=9$ and $\deg(\Gamma_\sigma\cdot\Delta_S)=11=c_2(S)$. If $D^2=1$ then $t=1$ and $\deg(\Gamma_\sigma\cdot\Delta_S)=3$.

EXAMPLE 5.5. (*K3 Surfaces with an involution*) Let S be a complex K3 surface with an involution σ. We have $p_g(S)=1$ and $q(S)=0$, hence $H^{1,0}(S)=0$ and $H^{2,0}(S)\simeq\mathbb{C}$. Also $c_2(S)=24$, $H^2(S,\mathbb{Q})=NS(X)_{\mathbb{Q}}\oplus H^2_{\mathrm{tr}}(S,\mathbb{Q})$ with $\dim H^2(S,\mathbb{Q})=22$, $\dim H^2_{\mathrm{tr}}(S,\mathbb{Q})=22-\rho(S)$ and $\rho(S)=\dim(NS(S)_{\mathbb{Q}})\leq 20$. If σ is an involution on S then $\sigma(\omega)=\pm\omega$, where ω is a generator of the vector space $H^{2,0}(S)$. Then the same argument as in [20, 3.10] shows that σ either acts as +1 or as -1 on $H^2_{tr}(S,\mathbb{Q})$. The correspondence Γ_σ induces an involution $\pi_2^{tr}\circ\Gamma_\sigma\circ\pi_2^{tr}$ on $t_2(S)$, which we will still denote by σ. Let $\pi=1/2(\pi_2^{tr}-\sigma)$. Then $\pi\in A^2(S\times S)$ is a projector of $t_2(S)$. π_* acts either as 0 or as the identity on $H^{2,0}(S)\simeq\mathbb{C}$. In the first case $\ker(\pi_*)_{|H^2_{tr}(S)}$ is a sub-Hodge structure with $(2,0)$-component equal to $H^{2,0}(S)$. Its orthogonal complement is then contained in $NS(S)_{\mathbb{Q}}$, which implies that $\pi_*=0$ on $H^2_{tr}(S,\mathbb{Q})$. Therefore σ acts as the identity on $H^2_{tr}(S,\mathbb{Q})$. In the second case π_* equals the identity on $H^2_{tr}(S,\mathbb{Q})$, so that σ acts as - 1 on $H^2_{tr}(S,\mathbb{Q})$.

Now suppose that $\sigma(\omega)=-\omega$, so that σ acts as -1 on $H^2_{\mathrm{tr}}(S)$. By [22, 1.2], the quotient surface S/σ is an Enriques surface iff $S^\sigma=\emptyset$, while S/σ is a rational surface if $S^\sigma\neq\emptyset$. In either case $t_2(S/\sigma))=0$, by [2], hence $t_2(S)\neq t_2(S/\sigma))$ because $p_g(S)\neq 0$. Let Ψ_S be the map defined in (3.2). Then, by [13, Prop. 1(iv)], $\Psi_S(\Gamma_\sigma)=-id_{t_2(S)}=\Psi_S(-\Delta_S)$. Since $q(S)=0$ the kernel of Ψ_S is $\mathcal{J}(S)$ by Lemma 3.3. Hence $\Gamma_\sigma+\Delta_S\in\mathcal{J}(S)$, i.e., Γ_σ has valence 1. Since $p_g(S)\neq 0$, Δ_S cannot have valence 0 by

Theorem 4.1. The correspondence Γ_σ has indices $\alpha(\Gamma_\sigma) = \beta(\Gamma_\sigma) = 1$. By Theorem 4.7, if $t = \mathrm{trace}_{NS(S)}(\Gamma_\sigma)$ we get

$$\deg(\Gamma_\sigma \cdot \Delta_S) = 2 + t + \rho(S) - 22 = 2 + t - \dim\ H^2_{\mathrm{tr}}(S, \mathbb{Q}).$$

By [22, Th.3], σ has no isolated fixed points and a 1-dimensional (possibly empty) fixed locus D. Let τ be the trace of σ on $H^2(S,\mathbb{C})$. By the topological fixed point formula (see [6, 4]) $\tau + 2 = e(D)$, where $e(D) = -D^2 - D \cdot K_S$ is the topological Euler characteristic of D. By the holomorphic fixed point formula, see [6, 4]

$$-D \cdot K_S = 4(\mathrm{trace}_{H^0(S,O_S)}(\sigma) + \mathrm{trace}_{H^2(S,O_S)}(\sigma)) = 0,$$

because σ acts as -1 on $H^2(S, O_S)$. Therefore $e(D) = -D^2$ and

$$t = \tau - \mathrm{trace}_{H^2_{\mathrm{tr}}(S)}(\Gamma_\sigma) = \tau + \dim\ H^2_{\mathrm{tr}}(S) = -D^2 - 2 + \dim\ H^2_{\mathrm{tr}}(S),$$

because σ acts as -1 on $H^2_{\mathrm{tr}}(S)$, so that

$$\deg(\Gamma_\sigma \cdot \Delta_S) = -D^2.$$

If $S^\sigma = \emptyset$, in which case $S \to S/\sigma$ is the canonical unramified cover of an Enriques surface, then $D = 0$ and $(\Gamma_\sigma \cdot \Delta_S) = 0$.

If $S^\sigma \neq \emptyset$ then, by [22, Th. 3], the fixed locus D of σ is the disjoint union of m smooth curves C_i, with $1 \leq m \leq 10$, so that $D^2 = \sum_{1\leq i\leq m}(2g(C_i) - 2)$, by the adjunction formula. If $m = 10$ and the curves C_i are rational then $D^2 = -20$ and $\rho(S) = 20$, see [22, Th. 3′(1)]. Therefore $\deg(\Gamma_\sigma \cdot \Delta_S) = 20 = \rho(S)$, dim $H^2_{\mathrm{tr}}(S) = 2$ and $t = 20 = \rho(S)$. Hence σ acts as the identity on $NS(S)_\mathbb{Q}$. S is the unique (up to isomorphism) complex K3 surface described in [18, Th. 1].

If $\sigma^*(\omega) = \omega$, with $\omega \in H^{2,0}(S)$, then σ is a *Nikulin involution*. A Nikulin involution has 8 isolated fixed points, no 1-dimensional fixed locus and the desingularization Y of the quotient surface S/σ is a K3 surface, see [12, 5.2]. Therefore $c_2(S) = c_2(Y) = 24$, and $\rho(S) = \rho(Y)$ because σ acts as the identity on $H^2_{tr}(S)$. From the Lefschetz fixed point formula (see Remark 4.5) we get

$$\deg(\Gamma_\sigma \cdot \Delta_S) = \tau + 2 = 8$$

where τ is the trace of σ on $H^2(S,\mathbb{C})$. In order to compute $\deg(\Gamma_\sigma \cdot \Delta_S)$ using Theorem 4.7 we should know that the correspondence Γ_σ has a a valence v. By [13, Th.4] this is equivalent to $t_2(S) \simeq t_2(Y)$ in which case $v(\Gamma_\sigma) = -1$. Then we get

$$\deg(\Gamma_\sigma \cdot \Delta_S) = 2 + t - (\rho(S) - 22)$$

where

$$t = \mathrm{trace}_{NS}(\Gamma_\sigma) = \tau - \dim H^2_{tr}(S) = \tau - (22 - \rho(S)) = \rho(S) - 16,$$

so that $\deg(\Gamma_\sigma \cdot \Delta_S) = 8$. The condition $t_2(S) \simeq t_2(Y)$ is satisfied if the K3 surface S has a finite dimensional motive, which is in particular the case if $\rho(S) = 19$ or $\rho(S) = 20$, see [13, Thms. 2 and 3].

Acknowledgements

The authors would like to thank Ciro Ciliberto for his help in understanding some of Severi's results. We also thank the referee for several corrections and suggestions on the original version of the manuscript.

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AMS Subject Classification: Primary 14E10, Secondary 01A60, 14J99

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Lavoro pervenuto in redazione il 02.07.2013.

Rend. Sem. Mat. Univ. Politec. Torino
Vol. 71, 3–4 (2013), 505 – 534

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SIX DIMENSIONAL PRYMS AND CONTE-MURRE THREEFOLDS

Abstract. The singular locus $\mathcal{S}$ of the universal Prym theta divisor over the Prym moduli space $\mathcal{R}_7$ of genus 7 curves is considered. An irreducible component of $\mathcal{S}$ is constructed which is unirational and dominates $\mathcal{R}_7$. This result relies on several classical constructions, related to the geometry of Enriques surfaces, offering a tour through them as a byproduct.

1. Introduction

The aim of this paper is to investigate some of the multiple relations between two, quite different, classes of complex algebraic varieties. From one side we consider Prym varieties of dimension 6 and, from the other side, the family of Enriques-Fano threefolds of genus 6.

An Enriques-Fano threefold X is a normal threefold containing an ample Cartier divisor H which is a minimal Enriques surface with at most Du Val singularities. Moreover it is assumed that X is not a generalized cone over H, that is, X is not a contraction of $\mathbf{P}(\mathcal{O}_H \oplus \mathcal{O}_H(H))$. The genus of X is $p := \frac{H^3}{2} + 1$.

The families of these threefolds were studied by Fano and by Godeaux in the 30's of the last century, when H is very ample. The adjunction map often provides a birational model of them which is a singular Fano threefold with special properties. In the 80's they were reconsidered in modern terms by Conte and Murre in the paper [6]. This was a starting point for a renewed long series of investigations, reaching several classification results and the sharp bound $p \leq 17$ for the genus. See for instance: [2], [3], [6], [20], [23], [17] among many other papers.

The family of threefolds of genus 6, more precisely their birational Fano models in $\mathbf{P}^5$, is the initial family studied in detail in [6]. Throughout all the paper, the members of this family will be called Conte-Murre threefolds.

From another point of view, Enriques-Fano threefolds appear also as quotients of Fano threefolds endowed with an involution having exactly 8 fixed points, [2]. Hence they are endowed with a quasi étale double covering branched at 8 points. If T is a Conte-Murre threefold, then T is a general linear section of the space $Q^{[2]} \subset |\mathcal{O}_{\mathbf{P}^3}(2)|$, parametrizing pairs of planes of $\mathbf{P}^3$, and the covering of T is induced by the natural map $\mathbf{P}^{3^*} \times \mathbf{P}^{3^*} \to Q^{[2]}$. Note that T is embedded in $\mathbf{P}^6$ as a projectivitely normal threefold whose general hyperplane sections are smooth Enriques surfaces of degree 10. On the other hand $\mathbf{P}^6$ is the canonical space for curves of genus 7. Assume

$$C \subset T - Sing\ T \subset \mathbf{P}^6$$

is such a curve. Then the covering of T induces an étale double covering $\pi : \tilde{C} \to C$

which is defined by a non trivial 2-torsion η of Pic^0C. Hence the pair (C,η) is a Prym curve. This relates Prym curves of genus 7 and Conte-Murre threefolds.

Let us see why this relation is rich and interesting. More in general let (C,η) be a smooth Prym curve of genus g and let (P,Ξ) be its associated Prym variety. Then P is an abelian variety of dimension $g-1$, principally polarized by its theta divisor Ξ. We can comment on three related arguments from the theory of Prym varieties:

1) *The study of the Prym Brill-Noether loci* $P^r(C,\eta)$,

2) *the Taylor expansion of* Ξ *at* $o \in Sing\ \Xi$ *and the Prym-Torelli problem,*

3) *the birational structure of the Prym moduli space* $\mathcal{R}_g$.

1) $P^r(C,\eta)$ is the family of line bundles $\tilde{L} \in Pic^{2g-2}\tilde{C}$ such that $Nm\ \tilde{L} \cong \omega_C$, $h^0(\tilde{L}) \geq r+1$ and $h^0(\tilde{L}) = r+1 \mod 2$. Assume (C,η) is general. Then $P^r(C,\eta)$ is smooth of dimension $\rho = g-1-\binom{r+1}{2}$ if $\rho \geq 0$ and connected if $\rho \geq 1$, [25].

We can also consider the universal Prym Brill-Noether locus $\mathcal{P}^r_g$ over $\mathcal{R}_g$ that is the moduli space of triples $(C,\eta,\tilde{L})$. If $\rho \geq 0$ then $\mathcal{P}^r_g$ dominates $\mathcal{R}_g$ via the forgetful map. A unique irreducible component of $\mathcal{P}^r_g$ dominates $\mathcal{R}_g$ if $\rho \geq 1$. If $\rho = 0$, this latter result, though plausible, is not yet available in the literature.

2) Continuing with a general Prym curve, then $Sing\ \Xi$ is biregular to $P^3(C,\eta)$. A general point $o \in Sing\ \Xi$ is an ordinary double point. To give o is equivalent to give a line bundle $\tilde{L} \in Pic^{2g-2}\tilde{C}$ such that $h^0(\tilde{L}) = 4$ and $Nm\ \tilde{L} \cong \omega_C$, cfr. [1] App. C.

3) Finally various steps have been realized in the global study of $\mathcal{R}_g$. Farkas and Ludwig recently proved that $\mathcal{R}_g$ is of general type for $g \geq 14$, with the exception of $g = 15$ which is unknown, [13]. Nothing seems to be known on the Kodaira dimension of $\mathcal{R}_g$ for $9 \leq g \leq 13$. On the other hand the unirationality of $\mathcal{R}_g$ is well known for $g \leq 6$. See [15] and [21] for some general accounts on all this matter.

Coming to genus 7 the unirationality of $\mathcal{R}_7$ is known, see [14]. The case $g = 7$ is also the case where $Sing\ \Xi$ is finite for a general (C,η): Debarre proves in [7] that then $Sing\ \Xi$ consists of 16 ordinary double points. Since $Sing\ \Xi = P^3(C,\eta)$ these are the 16 elements $\tilde{L}$ of $P^3(C,\eta)$. We will see along this paper that the Petri map

$$\mu : H^0(\tilde{L}) \otimes H^0(\omega_{\tilde{C}} \otimes \tilde{L}^{-1}) \to H^0(\omega_{\tilde{C}})$$

is surjective and that $Ker\ \mu$ is the invariant eigenspace of the involution induced by $\pi : \tilde{C} \to C$. To summarize the situation, let $\mathbf{P}^3 := \mathbf{P}H^0(\tilde{L})^*$. Then $Ker\ \mu$ defines a 6-dimensional linear space $\mathbf{P}^6 \subset |\mathcal{O}_{\mathbf{P}^3}(2)|$ so that, with the previous notations,

$$C \subset T = \mathbf{P}^6 \cap Q^{[2]} \subset |\mathcal{O}_{\mathbf{P}^3}(2)|.$$

C is canonically embedded and T is a Conte-Murre threefold. Conversely, an embedding $C \subset T$ reconstructs a Prym curve (C,η) and two elements $\tilde{L}$ and $\omega_{\tilde{C}} \otimes \tilde{L}^{-1} \in P^3(C,\eta)$. Building on the geometry of this construction, and of Enriques surfaces as well, we explicitely describe an irreducible component $\mathcal{P}$ of $\mathcal{P}^3_7$ and show that

THEOREM $\mathcal{P}$ *is unirational and dominates* $\mathcal{R}_7$.

As an immediate consequence, we obtain a new proof that

COROLLARY $\mathcal{R}_7$ *is unirational.*

Since $\rho = 0$ the existence of other irreducible and dominant components of $\mathcal{P}^3_7$ is a priori possible. It seems plausible, with more substantial effort, to exclude them. Assuming this property, the ideal geometric picture for $g = 7$ appears to be as follows: let $C \subset \mathbf{P}^6$ be a general canonical curve. Consider the map

$$t : Sing\ \Xi / < i^* > \to T(C) := \{ \text{ Conte-Murre threefolds through } C \}$$

defined by the condition $t(\tilde{L}, i^*\tilde{L}) = T$. One expects that t is bijective and that, over an open neighborhood of C in its Hilbert scheme, the monodromy of $T(C)$ is irreducible.

To conclude this introduction we want to outline a second natural step of the program started in this paper. To this purpose let us stress once more that a Conte-Murre threefold T is defined by a quadratic singularity o of the theta divisor Ξ of a general 6-dimensional Prym P. Our claim is that the Fano model of T, described in [6], is exactly the complete intersection, in the projectivized tangent space to P at o, of the quadratic and the cubic terms of the Taylor expansion of Ξ at o. We intend to inquire this claim and possibly prove it in a future paper.

In the 70's several young persons, among them the author of this paper, were introduced to beautiful algebraic varieties, like Enriques surfaces or Pryms or conic bundles, thanks to Alberto Conte and his passion for geometry. Time is passing, but the beautiful geometry learned at that time is not passing. This paper is dedicated to Alberto for his 70th birthday, with gratitude.

2. Preliminary results and basic notations

We will work over the complex field. A *Prym curve of genus g* is a pair (C, η) such that C is a smooth, integral projective curve of genus g and η is a non trivial 2-torsion element of $Pic\ C$. We fix from now on the following notations:

- $\pi : \tilde{C} \to C$ *is the étale double covering defined by* η.
- $i : \tilde{C} \to \tilde{C}$ *is the involution exchanging the two sheets of* π.

Then i is fixed point free and $\tilde{C}$ is a smooth integral curve of genus $2g - 1$. As is well known the Prym variety of (C, η) is a principally polarized abelian variety of dimension $g - 1$, naturally associated to (C, η). It is a pair (P, Ξ) such that P is an abelian variety of dimension $g - 1$ and Ξ is a principal polarization on it. To construct P consider the Norm map $Nm : Pic\ \tilde{C} \to Pic\ C$ sending $\mathcal{O}_{\tilde{C}}(d)$ to $\mathcal{O}_C(\pi_* d)$. $\forall m \in \mathbb{Z}$ each fibre of

$$Nm : Pic^m\ \tilde{C} \to Pic^m\ C$$

is the disjoint union of two copies of the same abelian variety of dimension $g - 1$. By definition this is P. To define Ξ we previously introduce the Prym Brill-Noether loci.

DEFINITION 1. *The r-th Prym Brill-Noether locus of* (C,η) *is*

$$P^r(C,\eta) := \{\tilde{L} \in Pic^{2g-2}\tilde{C} \;/\; Nm\,\tilde{L} \cong \omega_C,\; h^0(\tilde{L}) \geq r+1,\; h^0(\tilde{L}) \equiv r+1 \mod 2\}.$$

$P^r(C,\eta)$ has the scheme structure defined in [25]. Note that i^* acts on $P^r(C,\eta)$ and that $i^*\tilde{L} \cong \omega_{\tilde{C}} \otimes \tilde{L}^{-1}$. Let $\tilde{L} \in P^r(C,\eta)$, we fix the natural identification

$$H^0(\tilde{L}) := i^*H^0(\tilde{L}) = H^0(i^*\tilde{L}).$$

Then i^* acts on $H^0(\tilde{L}) \otimes H^0(\tilde{L})$ by exchanging the factors and the decomposition

$$H^0(\tilde{L}) \otimes H^0(\tilde{L}) = Sym^2 H^0(\tilde{L}) \oplus \wedge^2 H^0(\tilde{L}).$$

is the direct sum of the 1 and -1 eigenspaces of the involution i^*. Let

$$\mu_{\tilde{L}} : H^0(\tilde{L}) \otimes H^0(\tilde{L}) \to H^0(\omega_{\tilde{C}})$$

be the Petri map, that is $s \otimes t \to s i^* t$. Then $\mu_{\tilde{L}}$ is the direct sum of the maps

$$\mu^+_{\tilde{L}} : Sym^2 H^0(\tilde{L}) \to H^0(\omega_{\tilde{C}})^+ \;,\; \mu^-_{\tilde{L}} : \wedge^2 H^0(\tilde{L}) \to H^0(\omega_{\tilde{C}})^-$$

where $\pm$ denotes the $\pm$-eigenspace of i^*. We fix the identifications

$$H^0(\omega_{\tilde{C}})^+ = \pi^* H^0(\omega_C) = H^0(\omega_C) \text{ and } H^0(\omega_{\tilde{C}})^- = \pi^* H^0(\omega_C \otimes \eta) = H^0(\omega_C \otimes \eta).$$

Some frequently used conventions:
- $X \subset \mathbf{P}^n$, then $< X >$ is its linear span.
- X is not degenerate if $< X >= \mathbf{P}^n$.
- $X \subset Y$, then $I_{X/Y}$ is the ideal sheaf of X.
- V a vector space: $\mathbf{P}V$ is its projectivization, V^* the dual.
- $V^\perp := \{h \in V^* \;/\; h/V \text{ is zero}\}$.
- $\mathcal{L}$ a line bundle, $V \subset H^0(\mathcal{L})$: $|V|$ is the linear system defined by V.
- $|\mathcal{L}|$ is the linear system defined by $H^0(\mathcal{L})$.

3. Prym theory in genus 7 and Conte-Murre threefolds

From now on C will be a *general* curve of genus 7, which implies that (C,η) will be a *general* Prym curve for each η. In particular we can assume that $\omega_C \otimes \eta$ is very ample, see [5] 0.6. Moreover we can assume that ω_C is very ample and that the canonical model of C is generated by quadrics. $\tilde{C}$ has genus 13. Since C is general it follows

$$P^r(C,\eta) = \emptyset, r \geq 4$$

from Prym Brill-Noether theory, [25]. From Debarre, [7], we have

THEOREM 1. *Sing* $\Xi = P^3(C,\eta)$ *is a smooth* 0*-dimensional scheme of length* 16.

COROLLARY 1. *The forgetful map $f : \mathcal{P}_7^3 \to \mathcal{R}_7$ is generically étale of degree 16.*

Moreover each $o \in Sing\ \Xi$ is a stable singularity with respect to (C,η). This just means that o is a line bundle $\tilde{L} \in P^3(C,\eta)$ such that $Nm\ \tilde{L} \cong \omega_C$ and $h^0(\tilde{L}) = 4$. Let $V := H^0(\tilde{L})$, we want to study the Petri map

$$\mu_{\tilde{L}} : V \otimes i^*V \to H^0(\omega_{\tilde{C}}).$$

LEMMA 1. *Let (C,η) be any Prym curve of genus 7 and let $\tilde{L} \in P^3(C,\eta) - P^5(C,\eta)$. Assume that the Petri map of $\tilde{L}$ is surjective, then its Prym-Petri map is an isomorphism.*

Proof. Recall that $Coker\ \mu_{\tilde{L}}$ is the cotangent space at $\tilde{L}$ to the Brill-Noether scheme $W^3_{12}(\tilde{C}) := \{\tilde{M} \in Pic^{12}\tilde{C}\ /\ h^0(\tilde{M}) \geq 4\}$. Then $\mu_{\tilde{L}}$ surjective $\Rightarrow \tilde{L}$ is an isolated point of $W^3_{12}(\tilde{C})$, hence of $P^3(C,\eta)$. Hence $Coker\ \mu^- = 0$, that is, μ^- is an isomorphism. $\square$

Since $\tilde{C}$ is not general, the Petri map $\mu_{\tilde{L}}$ needs not to be injective. Actually, the size of this paper is due to the steps for proving that, for a general (C,η), the subset

$$\{\tilde{L} \in P^3(C,\eta)\ /\ \mu_{\tilde{L}} \text{ is surjective}\}$$

is not empty. Next we remark that the linear system $|Im\ \mu_{\tilde{L}}|$ defines the product map

$$\phi := f_{\tilde{L}} \times (f_{\tilde{L}} \cdot i) : \tilde{C} \to \mathbf{P}^3 \times \mathbf{P}^3,$$

where $\mathbf{P}^3 := \mathbf{P}V^*$ and $f_{\tilde{L}} : \tilde{C} \to \mathbf{P}^3$ is the map defined by $\tilde{L}$. Let

$$\mathbf{P}^3 \times \mathbf{P}^3 \subset \mathbf{P}^{15} := \mathbf{P}(V^* \otimes V^*),$$

be the Segre embedding. We consider the projective involution $I : \mathbf{P}^{15} \to \mathbf{P}^{15}$ induced by the linear map $s \otimes t \to t \otimes s$. The projectivized eigenspaces of I are

$$\mathbf{P}^{5-} := \mathbf{P}(\wedge^2 V^*)\ ,\ \mathbf{P}^{9+} := \mathbf{P}(Sym^2 V^*).$$

Moreover it is clear that $i : \tilde{C} \to \tilde{C}$ is induced by I, in other words we have:

$$I \circ \phi = \phi \circ i.$$

Notice also that $\mathbf{P}^{5-}$ is the Plücker space for the Grassmannian

$$\mathbf{G} := G(1, \mathbf{P}V) \subset \mathbf{P}^{5-}$$

of pencils of planes of $\mathbf{P}^3$. On the other hand $\mathbf{P}(Sym^2 V^*)$ is the linear system

$$\mathbf{P}^9 + = |\mathcal{O}_{\mathbf{P}V}(2)|$$

of quadrics of $\mathbf{P}V = \mathbf{P}^{3^*}$. In particular this space admits the rank stratification

$$\mathbb{Q}^{[1]} \subset \mathbb{Q}^{[2]} \subset \mathbb{Q}^{[3]} \subset \mathbf{P}^{9+},$$

where $\mathbb{Q}^{[r]}$ denotes the locus of quadrics of rank $\leq r$. Let us consider the linear projections $\pi^+ : \mathbf{P}^{15} \to \mathbf{P}^{9+}$ and $\pi^- : \mathbf{P}^{15} \to \mathbf{P}^{9+}$ respectively of centers $\mathbf{P}^{5-}$ and $\mathbf{P}^{9+}$. We are interested to the commutative diagram

$$\begin{array}{ccccc}
\mathbf{P}^{5-} & \xleftarrow{\pi^-} & \mathbf{P}^{15} & \xrightarrow{\pi^+} & \mathbf{P}^{9+} \\
\uparrow & & \uparrow & & \uparrow \\
\mathbf{G} & \xleftarrow{\pi^-/\mathbf{P}^3\times\mathbf{P}^3} & \mathbf{P}^3 \times \mathbf{P}^3 & \xrightarrow{\pi^+/\mathbf{P}^3\times\mathbf{P}^3} & \mathbb{Q}^{[2]} \\
\uparrow & & \uparrow \phi & & \uparrow \\
C^- & \xleftarrow{\pi^-\circ\phi} & \tilde{C} & \xrightarrow{\pi^+\circ\phi} & C^+
\end{array}$$

where the horizontal arrows are surjective and the top vertical ones injective, cfr. [22]. We observe that the quotient map of the involution $I/\mathbf{P}^3 \times \mathbf{P}^3$ is exactly

$$\pi^+/\mathbf{P}^3 \times \mathbf{P}^3 : \mathbf{P}^3 \times \mathbf{P}^3 \to \mathbb{Q}^{[2]}.$$

$\mathbb{Q}^{[2]}$ is a determinantal 6-fold of degree 10 parametrizing pairs of planes of $\mathbf{P}V$. As is well known the previous map is a quasi étale 2:1 cover of $\mathbb{Q}^{[2]}$, branched on $\mathbb{Q}^{[1]}$. Note that $\phi^* O_{\mathbf{P}^{15}}(1) \cong \omega_{\tilde{C}}$, because ϕ is defined by $Im\ \mu_{\tilde{L}} \subseteq H^0(\omega_{\tilde{C}})$. Notice also that

$$\tilde{L} \cong \phi^* O_{\mathbf{P}^3\times\mathbf{P}^3}(1,0) \text{ and } i^*\tilde{L} \cong O_{\mathbf{P}^3\times\mathbf{P}^3}(0,1).$$

Moreover $\pi^- \cdot \phi$ and $\pi^+ \cdot \phi$ are respectively defined by $|Im\ \mu^-_{\tilde{L}}|$ and $|Im\ \mu^+_{\tilde{L}}|$. In particular it follows that the linear spans $< C^+ >$ and $< C^- >$ are the orthogonal spaces:

$$< C^+ >= \mathbf{P}(Im\ \mu^+_{\tilde{L}})^{\perp} \subset \mathbf{P}H^0(\omega_C)^* \text{ and } < C^- >= \mathbf{P}(Im\ \mu^-_{\tilde{L}})^{\perp} \subset \mathbf{P}H^0(\omega_C \otimes \eta)^*.$$

Therefore we have:

LEMMA 2. *Assume that $\mu_{\tilde{L}}$ is surjective, then the projective models*

$$C^+ \subset < C^+ > \text{ and } C^- \subset < C^- >$$

are exactly the canonical and the Prym canonical embeddings of (C, η).

Proof. By lemma 3.3 the surjectivity of $\mu_{\tilde{L}}$ implies that the Prym-Petri map $\mu^-_{\tilde{L}}$ is an isomorphism. Then, since i^* acts on $Ker\ \mu_{\tilde{L}}$, it follows that $Ker\ \mu_{\tilde{L}} = Ker\ \mu^+_{\tilde{L}}$. This implies that $\mu^+_{\tilde{L}}$ is surjective. Then $|Im\ \mu^-_{\tilde{L}}| = |\omega_C \otimes \eta|$ and $|Im\ \mu^+_{\tilde{L}}| = |\omega_C|$, which implies the statement. □

Now it is the appropriate moment to introduce Conte-Murre threefolds:

DEFINITION 2. *A Conte-Murre threefold is a transversal 3-dimensional linear section of $Q^{[2]}$.*

More informations on the family of Conte-Murre threefolds are available in the introduction. In particular for such a threefold T we have $Sing\ T = T \cap Q^{[1]}$ and $Sing\ T$ consists of 8 points of multiplicity 4. The study of the family of triples $(C, \eta, \tilde{L})$ is one of the main goals of the next sections.

In what follows C is always a smooth, irreducible curve of genus 7 which is not hyperelliptic nor trigonal and (C, η) is a Prym curve. It is useful to fix the following

DEFINITION 3. *A triple $(C, \eta, \tilde{L})$ is a good triple if:*

1 $h^0(\tilde{L}) = 4$.

2 *$\tilde{L}$ is very ample.*

3 *The Petri map $\mu_{\tilde{L}}$ is surjective.*

4 *$< C^+ >$ is transversal to $Q^{[2]}$.*

Let $(C, \eta, \tilde{L})$ be a good triple. By (1) we have $h^0(\tilde{L}) = 4$. Since $\mu_{\tilde{L}}$ is surjective then $< C^+ >$ is a 6-dimensional space in $\mathbf{P}^{9+}$. Therefore C^+ is the canonical embedding of C and $< C^+ >$ is its canonical space. Moreover the map $\phi : \tilde{C} \to \mathbf{P}^3 \times \mathbf{P}^3$, considered in the previous diagram, is an embedding because $\tilde{L}$ is very ample. Let us also point out that $\tilde{L} \cong \phi^* O_{\mathbf{P}^3 \times \mathbf{P}^3}(1, 0)$.

Therefore a good triple defines an embedding of the canonical model of C in a Conte-Murre threefold. Conversely let us start from an embedding

$$C \subset T - Sing\ T,$$

where T is a Conte-Murre threefold and C is a smooth, integral curve of genus 7 which is canonically embedded in $< T >$. Then the map $\pi^+ : \mathbf{P}^3 \times \mathbf{P}^3 \to Q^{[2]}$ induces an étale double covering $\pi : \tilde{C} \to C$, where $\tilde{C} := \pi^{+^*} C$. The next lemma is standard.

LEMMA 3. *$\pi : \tilde{C} \to C$ is not the trivial étale double covering.*

Proof. For $m >> 0$ consider a general $S \in |O_T(m)|$ containing C. Then $\tilde{S} := \pi^* S$ is a complete intersection in $\mathbf{P}^3 \times \mathbf{P}^3$ of four very ample divisors $D, H_1, H_2 H_3$ such that D has bidegree (m, m) and H_i has bidegree $(1, 1)$, $i = 1, 2, 3$. In particular $\tilde{S}$ is a smooth, connected surface of general type. Its canonical sheaf is $O_{\tilde{S}}(m-1, m-1)$. Now assume π is trivial. Then $\tilde{C} = C_1 + C_2$, where C_1, C_2 are disconnected copies of C. Hence it follows $C_1^2 = C_2^2 = 12(2-m)$ and $C_1 C_2 = 0$: against Hodge index theorem. □

By the lemma $\pi : \tilde{C} \to C$ defines a Prym curve (C, η). Let $\tilde{L} := O_{\tilde{C}}(1, 0)$, then it is easy to see that $Nm\ \tilde{L} \cong \omega_C \cong Nm\ i^* \tilde{L}$. Starting from the embedding $C \subset T - Sing\ T$, we have constructed a triple $(C, \eta, \tilde{L})$ satisfying $h^0(\tilde{L}) \geq 4$ and conditions 3) and 4) of definition 3.6. $(C, \eta, \tilde{L})$ is expected to be a good triple.

DEFINITION 4. *An embedding $C \subset T - Sing\ T$ is good if $(C, \eta, \tilde{L})$ is a good triple.*

The next assumption is proved to be true in the last section of this paper, from now on we keep it:

ASSUMPTION 2. *Good triples* $(C,\eta,\tilde{L})$ *do exist.*

4. Genus 7 canonical curves in a Conte-Murre threefold

We will denote a good triple by $(C,\eta,\tilde{L})$. To simplify notations we will also set

$$\Lambda := < C^+ > \ , \ T := \mathbb{Q}^{[2]} \cdot \Lambda \text{ and } C^+ = C.$$

Since Λ is transversal to $Q^{[2]}$, then T is a Conte-Murre threefold and, moreover, the embedding $C \subset T - Sing\ T$ is good. Many properties of T have a classical flavor. We review some of them to be used, cfr. [9] 6.3. The orthogonal space of Λ is a plane

$$\Lambda^{\perp} \subset |O_{\mathbf{P}^{3*}}(2)|$$

in the space of quadrics of $\mathbf{P}^{3*} = \mathbf{P}V$. Therefore, up to projective equivalence, to give $\Lambda^{\perp}$ is equivalent to give an even spin curve (Q,θ), where $Q \subset \Lambda^{\perp}$ is a smooth plane quartic and θ is an even theta characteristic on Q. In other words the threefold

$$\tilde{T} := \pi^{+^*} T$$

is the base scheme of a net $N := |I_{\tilde{T}/\mathbf{P}^3 \times \mathbf{P}^3}(1,1)|$ of symmetric bilinear forms of $\mathbf{P}^3 \times \mathbf{P}^3$. Giving a plane $\Lambda^{\perp}$ is equivalent to give a net N. Let $(x,y) = (x_1 : \ldots : x_4) \times (y_1 : \ldots : y_4)$ be coordinates on $\mathbf{P}^3 \times \mathbf{P}^3$, this means that N is generated by three bilinear forms

$$q^k(x,y) = \sum_{1 \le i,j \le 4} q^k_{ij} x_i y_j, \ (k = 1,2,3),$$

such that (q^k_{ij}) is a 4×4 symmetric matrix. We can also view $\tilde{T}$ as the graph of a very well known Cremona involution. We denote such a birational map as

$$\phi_{\tilde{T}} : \mathbf{P}^3 \to \mathbf{P}^3$$

and summarize without proofs its realization. Let $\pi_i : \tilde{T} \to \mathbf{P}^3$ be the projection onto the i-th factor, $i = 1,2$. Then its fibre at $\bar{x}$, is defined by the linear equations

$$q^1(\bar{x},y) = q^2(\bar{x},y) = q^3(\bar{x},y) = 0.$$

Since $\tilde{T}$ is integral, it follows that $\pi_1^*(\bar{x})$ is a point for a general $\bar{x}$. Hence the map

$$\pi_1^{-1} : \mathbf{P}^3 \to \tilde{T}$$

is birational. The fundamental locus of p_1^{-1} is the rank 2 locus of the 3×4 matrix of linear forms $M_x := (q^k_{1j} x_1 + \cdots + q^k_{4j} x_4)$. Such a fundamental locus is the embedding

$$Q_1 \subset \mathbf{P}^3$$

of the plane quartic Q by the map associated to $\omega_Q(\theta)$. The 3×3 minors of M_x generate the ideal of Q_1. The same properties hold for $\pi_2 : \mathbf{P}^3 \to \tilde{T}$. Again its fundamental locus is the same embedding of Q. In the coordinates $(y_1 : \ldots : y_4)$ this is the curve

$$Q_2 \subset \mathbf{P}^3$$

whose ideal is generated by the 3×3 minors of $M_y := (q^k_{i1}y_1 + \cdots + q^k_{i4}y_4)$. In particular it follows that $\pi_i^{-1} : \mathbf{P}^3 \to \tilde{T}$ is the blowing of Q_i, $i = 1, 2$. By definition we will have

$$\phi_{\tilde{T}} := \pi_2 \cdot \pi_1^{-1} \text{ and } \phi_{\tilde{T}}^{-1} = \pi_1 \cdot \pi_2^{-1}.$$

THEOREM 3. *The birational maps $\phi_{\tilde{T}}$ and $\phi_{\tilde{T}}^{-1}$ are respectively defined by the linear systems of cubic surfaces $|I_{Q_1/\mathbf{P}^3}(3)|$ and $|I_{Q_2/\mathbf{P}^3}(3)|$.*

REMARK 1. It is well known that π_2, π_1 are respectively the contractions of the unions of the trisecant lines to Q_1 and to Q_2. Notice also that $\phi^2_{\tilde{T}} = id_{\mathbf{P}^3}$. By far $\phi_{\tilde{T}}$ is the most famous example of cubo-cubic birational involution of $\mathbf{P}^3$.

Given our good triple $(C, \eta, \tilde{L})$ and putting $\tilde{\Lambda} :=< \tilde{T} >$ we have, in the Segre embedding of $\mathbf{P}^3 \times \mathbf{P}^3$, the following situation:

$$\tilde{C} \subset \tilde{T} = \tilde{\Lambda} \cdot \mathbf{P}^3 \times \mathbf{P}^3 \subset \mathbf{P}^{15}$$

Since the triple is good, $Im\ \mu_{\tilde{L}} = H^0(\omega_{\tilde{C}})$. Hence the next proposition is immediate.

PROPOSITION 1. *$\tilde{C}$ is canonically embedded in $\tilde{\Lambda}$.*

Now our program is to link $\tilde{C}$ to a second curve $\tilde{B}$ by a complete intersection of two smooth surfaces in $\tilde{T}$. Then we will use the family of curves $\tilde{B}$ to construct a rational variety, which in turn dominates a family of good triples $(C, \eta, \tilde{L})$ such that (C, η) has general moduli.

To realize this program we work with the contractions $\pi_1, \pi_2 : \tilde{T} \to \mathbf{P}^3$. Let $H_i \in |\pi_i^* \mathcal{O}_{\mathbf{P}^3}(1)|$ and let E_i be the exceptional divisor of π_i, $i = 1, 2$. Then we have the following relations in $Pic\ \tilde{T}$:

$$3H_1 - E_1 \sim H_2\ ,\ 3H_2 - E_2 \sim H_1.$$

We consider the linear systems $|H_i + H_1 + H_2|$, $i = 1, 2$. Due to these relations we have

$$|H_i + H_1 + H_2| = |5H_i - E_i| = \pi_i^* |I_{Q_i/\mathbf{P}^3}(5)|.$$

Since the ideal of Q_i is generated by cubic forms, then $I_{Q_i}(5)$ is globally generated and hence $|5H_i - E_i|$ is base point free. It is easy to conclude that:

PROPOSITION 2. *A general $D \in |H_i + H_1 + H_2|$ is a smooth surface such that $\pi_i : D \to \mathbf{P}^3$ is an embedding of D as a smooth quintic surface passing through Q_i.*

REMARK 2. For completeness we mention that $|H_i + H_1 + H_2| = |7H_j - 2E_j|$ where $i \neq j$ and $i, j = 1, 2$. In particular let $\overline{D} = \pi_j(D)$, then $\overline{D}$ is a septic surface passing with multiplicity two through Q_j and $\pi_j : D \to \overline{D}$ is the normalization map of $\overline{D}$.

From now on let $\tilde{C}_i$ be the image of the embedding $\pi_i : \tilde{C} \to \mathbf{P}^3$, then we have

$$|I_{\tilde{C}/\tilde{T}}(H_i + H_1 + H_2)| = |I_{\tilde{C}/\tilde{T}}(5H_i - E_i)| = \pi_i^* |I_{Q_i \cup \tilde{C}_i}(5)|.$$

PROPOSITION 3. *The ideal sheaf $I_{\tilde{C}/\tilde{T}}(H_i + H_1 + H_2)$ is globally generated.*

Proof. We have $H^0(O_{\tilde{C}}(H_1)) = H^0(\tilde{L}$ and $H^0(O_{\tilde{C}}(H_2)) = H^0(i^*\tilde{L})$. Putting $W_k := H^0(O_{\tilde{C}}(H_k))$, $k = 1, 2$, we consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Ker\ m_1 & \longrightarrow & W_1 \otimes W_1 \otimes Sym^2 W_2 & \xrightarrow{m_1} & W_1 \otimes H^0(L \otimes \omega_{\tilde{C}}) & \longrightarrow & 0 \\
 & & \Big\downarrow n & & \Big\downarrow l & & \Big\downarrow m_3 & & \\
0 & \longrightarrow & Ker\ m_2 & \longrightarrow & Sym^2 W_1 \otimes Sym^2 W_2 & \xrightarrow{m_2} & H^0(\omega_{\tilde{C}}^{\otimes 2}) & \longrightarrow & 0,
\end{array}$$

where m_1, m_2, m_3 are the natural multiplication maps and $l(a \otimes b \otimes c) := (a \otimes b + b \otimes a) \otimes c$. Since the Petri map $\mu_{\tilde{L}} : W_1 \otimes W_2 \to H^0(\omega_{\tilde{C}})$ is surjective, the multiplication map

$$\mu_{\tilde{L}}^2 : W_1 \otimes W_1 \otimes W_2 \otimes W_2 \to H^0(\omega_{\tilde{C}}^{\otimes 2})$$

is surjective as well. This easily implies the surjectivity of m_1, m_2. Since l is clearly surjective, the surjectivity of m_3 also follows. Hence the diagram is exact and, by standard diagram chase, the map $n : Ker\ m_1 \to Ker\ m_2$ is surjective. Now let $I_{\tilde{C}}$ be the ideal sheaf of $\tilde{C}$ in $\mathbf{P}^3 \times \mathbf{P}^3$, then $Ker\ m_1 = W_1 \otimes H^0(I_{\tilde{C}}(H_1 + 2H_2))$. On the other hand we have $Ker\ m_2 = H^0(I_{\tilde{C}}(2H_1 + 2H_2))$ and n is surjective. Hence $I_{\tilde{C}}(H_1 + 2H_2)$ is globally generated if $I_{\tilde{C}}(2H_1 + 2H_2)$ is globally generated. Since the Segre embedding $\mathbf{P}^3 \times \mathbf{P}^3$ is generated by quadrics, $I_{\tilde{C}}(2H_1 + 2H_2)$ is globally generated if $\tilde{C}$ is not hyperelliptic, which is obvious, nor trigonal. Notice that $\tilde{C}$ is not trigonal because a non hyperelliptic trigonal curve of genus $g \geq 4$ has no fixed point free involution. Finally $\tilde{T}$ is a linear section of $\mathbf{P}^3 \times \mathbf{P}^3$, hence the sheaf $I_{\tilde{C}/\tilde{T}}(H_1 + 2H_2)$ is globally generated as well: we omit the easy details of this last step. The same proof works for $I_{\tilde{C}/\tilde{T}}(2H_1 + H_2)$. $\square$

The next proposition is a straighforward, though useful, corollary.

PROPOSITION 4. *A general $D \in |I_{\tilde{C}}(H_i + H_1 + H_2)|$ is a smooth, connected surface. Moreover $\pi_i : D \to \mathbf{P}^3$ is an embedding of D as a smooth quintic surface containing $Q_i \cup \tilde{C}_I$.*

Proof. The argument is standard, cfr. [22]. By the lemma $\tilde{C}$ is locally complete intersection of two elements of $\mathbb{I} := |I_{\tilde{C}/\tilde{T}}(H_i + H_1 + H_2)|$ at every point $o \in \tilde{C}$. Since $\tilde{C}$ is smooth, then $\mathbb{I}_o := \{D \in \mathbb{I}\ /\ o \in Sing\ D\}$ has codimension ≥ 2 in $\mathbb{I}$. Therefore, counting dimensions, it follows that a general $D \in \mathbb{I}$ is smooth along $\tilde{C}$. Then D is smooth by Bertini theorem and obviously connected because it is very ample. Finally

$\tilde{C}$ is smooth and $\pi_i/\tilde{C}$ is an embedding. Hence π_i is an embedding along the scheme $\tilde{C}\cdot E_i$. Since $I_{\tilde{C}}(D)$ is globally generated and D is general, we can then assume that π is an embedding along $D\cdot E_i$, so that $\pi_i : D \to \mathbf{P}^3$ is an embedding. □

Definitely we fix now $i=1$ for simplicity, then we consider a general smooth

$$\tilde{S} \in |I_{\tilde{C}/\tilde{T}}(2H_1+H_2)|.$$

DEFINITION 5. *$NS(\tilde{S})$ is the Neron-Severi lattice of $\tilde{S}$. For any line bundle $O_{\tilde{S}}(D)$ we set:*

$$d := \text{ numerical class of } D \,.$$

Using the classical description of $\tilde{T}$ it is easy to compute that

$$h_1^2 = 5 \,,\; h_2^2 = 7 \,,\; h_1h_2 = 9 \,,\; \tilde{c}^2 = \tilde{c}h_1 = \tilde{c}h_2 = 12.$$

in the lattice $NS(\tilde{S})$. Since $\iota^*\tilde{C} = \tilde{C}$, we will also consider the smooth surface

$$\iota^*\tilde{S} \in |I_{\tilde{C}/\tilde{T}}(H_1+2H_2)|.$$

PROPOSITION 5. *A general element of $|O_{\tilde{S}}(H_1+2H_2-\tilde{C})|$ is a smooth, connected curve of genus* 11 *and bidegree* $(11,11)$.

Proof. We know from proposition 4.6 that $I_{\tilde{C}/\tilde{T}}(\iota^*\tilde{S})$ is globally generated. Then the same remarks given to prove lemma 4.7 imply that $O_{\tilde{S}}(H_1+2H_2-\tilde{C})|$ is base point free. Since $(h_1+2h_2-\tilde{c})^2 > 0$ the first part of the statement follows from Bertini theorem. Let $\tilde{B} \in |O_{\tilde{S}}(H_1+2H_2-\tilde{C})|$. The canonical class of $\tilde{S}$ is h_1. Then, by adjunction formula, we have $2p_a(\tilde{B})-2 = (2h_1+2h_2-\tilde{c})(h_1+2h_2-\tilde{c}) = 20$. Hence $p_a(\tilde{B}) = 11$. Notice also that $h_1(h_1+2h_2-\tilde{c}) = h_2(h_1+2h_2-\tilde{c}) = 11$. □

The next result, though elementary, highlights a very interesting feature:

THEOREM 4. *For every $\tilde{B} \in |O_{\tilde{S}}(H_1+2H_2-\tilde{C})|$ there exists exactly one hyperplane of $\tilde{\Lambda}$ containing it.*

Proof. A general $\tilde{B}$ is smooth of genus 11. Then $O_{\tilde{B}}(1)$ is a non special line bundle for degree reasons and $h^0(O_{\tilde{B}}(1)) = 12$. Since we have $h^0(O_{\tilde{T}}(1)) = 13$ and $\tilde{\Lambda} =< \tilde{T} >$, it follows that $\tilde{B}$ is contained in a hyperplane. By semicontinuity this holds for every $\tilde{B}$. To prove the uniqueness result it suffices to show that $h^0(O_{\tilde{S}}(H_1+H_2-\tilde{B})) = 1$. For this notice that $h_1(h_1+h_2-\tilde{b}) = 3$. Since h_1 is very ample and $\tilde{S}$ of general type, every curve D satsfying $h_1d = 3$ is isolated; hence $h^0(O_{\tilde{S}}(H_1+H_2-\tilde{B})) = 1$. □

5. Quintic surfaces through a genus 3 sextic and a skew cubic

This section is a useful digression to study the linear system $|\tilde{C}|$ on $\tilde{S}$. We will use the notation $\tilde{S}_i$ for $\pi_i(\tilde{S})$, $i = 1,2$. In particular $\tilde{S}_1$ is a smooth quintic surface containing a genus 3 sextic Q_1. As we will see in a moment, $\tilde{S}_1$ also contains a skew cubic which is 4-secant to Q_1. In what follows a skew cubic is a reduced, connected curve of degree 3 and arithmetic genus 0, possibly reducible. By the latter theorem we can consider on $\tilde{S}$ the hyperplane section

$$\tilde{F} + \tilde{B} := < \tilde{B} > \cdot \tilde{S}.$$

for every $\tilde{B} \in |O_{\tilde{S}}(H_1 + 2H_2 - \tilde{C})|$.

PROPOSITION 6. *$\tilde{F}$ is a connected curve of arithmetic genus 0 and bidegree $(3,5)$ in $\mathbf{P}^3 \times \mathbf{P}^3$. It is isolated in $\tilde{S}$ and embedded by $\pi_1 : \tilde{F} \to \mathbf{P}^3$ as a skew cubic.*

Proof. We have $\tilde{f} = \tilde{c} - h_2$, hence $\tilde{f}^2 = -5$ and $\tilde{f}h_1 = 3$ so that $p_a(\tilde{F}) = 0$. Since $\tilde{f}h_2 = 5$, the bidegree is $(3,5)$. The map $p_1 : \tilde{S} \to \mathbf{P}^3$ embeds $\tilde{F}$ has a curve of degree 3. Then it is easy to deduce from $p_a(\tilde{F}) = 0$ that $\tilde{F}$ is reduced and connected. It is obviously isolated on $\tilde{S}$ and embedded in $\mathbf{P}^3$ as a skew cubic. □

DEFINITION 6. *$\tilde{F}_i$ is the image of the morphism $\pi_i : \tilde{F} \to \mathbf{P}^3$, $i = 1,2$.*

In particular $\tilde{F}_1$ is a skew cubic in the smooth quintic surface $\tilde{S}_1 := \pi_1(\tilde{S})$. Let us summarize the situation so far: from the previous remarks we have seen that

$$O_{\tilde{S}}(H_1 + H_2 - \tilde{F}) \cong O_{\tilde{S}}(H_1 + 2H_2 - \tilde{C}).$$

Therefore we have:

THEOREM 5. *Let $(C, \eta, \tilde{L})$ be a good triple. Then, with the previous notations, one has*

$$O_{\tilde{S}}(\tilde{C}) \cong O_{\tilde{S}}(\tilde{F} + H_2).$$

It is easy to see that a general element of $|O_{\tilde{S}}(H_2)|$ is a smooth curve of genus 9 and bidegree $(9,7)$ in $\mathbf{P}^3 \times \mathbf{P}^3$. Since the cubo-cubic Cremona transformation

$$p_2 \cdot p_1^{-1} : \mathbf{P}^3 \to \mathbf{P}^3$$

is defined by the linear system $|I_{Q_1/\mathbf{P}^3}(3)|$, we can see the elements of $|H_2|$ as follows.

LEMMA 4. *Let $\tilde{G} \in |H_2|$ be general and let $\tilde{G}_1 \subset \tilde{S}_1$ be its embedding via π_1. Then $\tilde{G}_1 \cup Q_1$ is a nodal complete intersection of $\tilde{S}_1$ and a general cubic surface $X \in |I_{Q_1/\mathbf{P}^3}(3)|$.*

REMARK 3. Taking a curve $\tilde{G}_1$ which is transversal to the skew cubic $\tilde{F}_1$ we obtain a nodal curve $\tilde{G}_1 + \tilde{F}_1$ which is linearly equivalent to $\tilde{C}$. We point out that

$$h_2 \tilde{f} = 5 \ \text{ and } \ e_1 \tilde{f} = 4,$$

where e_1 is the class of $\pi_1^* Q_1$ in $\tilde{S}$. The latter equality follows from $3h_1 - h_2 = e_1$. Then $\tilde{G}_1 \cap \tilde{F}_1$ is a set of 5 points. Counting multiplicities, $Q_1 \cap \tilde{F}_1$ is a set of 4 points.

So far we have seen that a good triple $(C,\eta,\tilde{L})$ defines a curve $\tilde{C}$ which moves in a linear system $|O_{\tilde{S}}(\tilde{F}+H_2)|$ as above, where $\tilde{S}$ is biregular to a smooth quintic surface in $\mathbf{P}^3$. Now we partially reverse the construction and study smooth quintic surfaces endowed with a linear system of this type and its properties.

Thus we start from a non hyperelliptic even spin curve (Q,θ) of genus 3 and from its embedding $Q_1\subset\mathbf{P}^3$ by $\omega_Q(\theta)$. We confirm, with the same meaning, all the previous notations. The spin curve (Q,θ) defines, via the cubo-cubic transformation defined by the linear system $|I_{Q_1/\mathbf{P}^3}(3)|$, a smooth threefold

$$\tilde{T}\subset\mathbf{P}^3\times\mathbf{P}^3$$

as above. Next we consider a general, smooth quintic surface $\tilde{S}_1\in|I_{Q_1/\mathbf{P}^3}(5)|$ and its pull-back $\tilde{S}:=\pi_1^*\tilde{S}_1\in|O_{\tilde{T}}(2H_1+H_2)|$. We know that $\pi_1:\tilde{S}\to\tilde{S}_1$ is biregular. Then we specialize $\tilde{S}$ paying attention to the latter remarks on skew cubics. We choose four independent points $o_1\dots o_4\in Q_1$ and a smooth skew cubic $\tilde{F}_1$ such that $Q_1\cup\tilde{F}_1$ is nodal and *Sing* $Q_1\cup\tilde{F}_1=\{o_1\dots o_4\}$. The proof of the next lemma is standard, see 4.7.

LEMMA 5. *A general element of* $|I_{Q_1\cup\tilde{F}_1}(5)|$ *is a smooth quintic surface.*

Let $\tilde{F}$ be the strict transform of $\tilde{F}_1$ by $\pi_1:\tilde{S}\to\tilde{S}_1$: it is now natural to consider

$$|O_{\tilde{S}}(\tilde{F}+H_2)|.$$

LEMMA 6. *A general curve* $\tilde{C}\in|O_{\tilde{S}}(\tilde{F}+H_2)|$ *is smooth, connected of genus* 13 *and bidegree* $(12,12)$.

Proof. It is standard to compute $p_a(\tilde{C})=13$ and that $\tilde{C}$ has bidegree $(12,12)$. Moreover $\tilde{c}^2=12$ is positive. To prove that $\tilde{C}$ is smooth and connected, we show that $|O_{\tilde{S}}(\tilde{F}+H_2)|$ is base point free. Consider the standard exact sequence

$$0\to O_{\tilde{S}}(H_2)\to O_{\tilde{S}}(H_2+\tilde{F})\xrightarrow{r} O_{\tilde{F}}(H_2+\tilde{F})\to 0$$

and its associated long exact sequence. The proof that $h^1(O_{\tilde{S}}(H_2)=0$ is standard, so we omit it. Then $h^0(r)$ is surjective. Since the system is clearly base point free on $\tilde{S}-\tilde{F}$, it suffices to check that $O_{\tilde{F}}(H_2+\tilde{F})$ is very ample. This is the very ample sheaf $O_{\mathbf{P}^1}(8)$. □

LEMMA 7. *Let* $\tilde{C}\in|O_{\tilde{S}}(\tilde{F}+H_2)|$ *be smooth, then* $O_{\tilde{C}}(H_1+H_2)\cong\omega_{\tilde{C}}$ *and moreover* $\tilde{C}$ *is canonically embedded in* $<\tilde{T}>$.

Proof. Since *deg* $O_{\tilde{C}}(H_1+H_2)=$ *deg* $\omega_{\tilde{C}}$, it suffices to show that $h^0(O_{\tilde{C}}(H_1+H_2))=h^0(\omega_{\tilde{C}})=13$ and that $\tilde{C}$ is not contained in a hyperplane of $<\tilde{T}>$. Since $O_{\tilde{S}}(\tilde{C})\cong O_{\tilde{S}}(\tilde{F}+H_2)$, we have the exact sequence

$$0\to O_{\tilde{S}}(H_1-\tilde{F})\to O_{\tilde{S}}(H_1+H_2)\xrightarrow{r} O_{\tilde{C}}(H_1+H_2)\to 0.$$

It suffices to show that $h^0(O_{\tilde S}(H_1-\tilde F)) = h^1(O_{\tilde F}(H_1-\tilde F)) = 0$. Then $h^0(r)$ is an isomorphism. We have $h^0(O_{\tilde S}(H_1-\tilde F)) = 0$ because $\tilde F_1$ is a skew cubic. By Serre duality we then have $h^2(O_{\tilde S}(\tilde F)) = 0$ and $h^1(O_{\tilde S}(H_1-\tilde F)) = h^1(O_{\tilde S}(\tilde F))$. Since $\tilde F = \mathbf{P}^1$, $\tilde f^2 = -5$ and $\tilde S$ is a regular surface, the vanishing of $h^1(O_{\tilde S}(\tilde F))$ then follows from the standard exact sequence

$$0 \to O_{\tilde S} \to O_{\tilde S}(\tilde F) \to O_{\mathbf{P}^1}(-5) \to 0.$$

$\square$

THEOREM 6. *For a smooth $\tilde C \in |O_{\tilde S}(\tilde F + H_2)|$ one has*

1 $h^0(\tilde L) = 4$.

2 *$\tilde L$ is very ample.*

3 *The Petri map $\mu_{\tilde L}$ is surjective.*

4 *$<\tilde C>$ is transversal to $\mathbf{P}^3 \times \mathbf{P}^3$.*

Proof. Every non hyperelliptic, smooth genus 3 sextic $Q_1 \subset \mathbf{P}^3$ defines a cubo-cubic Cremona transformation whose graph in $\mathbf{P}^3 \times \mathbf{P}^3$ is smooth. Since $<\tilde C>$ cuts such a graph on $\mathbf{P}^3 \times \mathbf{P}^3$, then 4) follows. Notice also that, by construction, $<\tilde C> = <\tilde T>$. To prove 3) we observe that $\omega_{\tilde C} \cong O_{\tilde C}(H_1+H_2)$, so that we have $\omega_{\tilde C} \otimes \tilde L^{-1}$ is $O_{\tilde C}(H_2)$. In particular it follows that $<\tilde C>$ is the linear space orthogonal to $Im\ \mu_{\tilde L}$. Then the equality $<\tilde C> = <\tilde T>$ implies 3). 2) follows because $\tilde C$ is smooth and $\pi_1 : \tilde S \to \mathbf{P}^3$ is an embedding. 3) Finally, to prove 1), consider the standard exact sequence

$$0 \to O_{\tilde S}(H_1 - \tilde F - D) \to O_{\tilde S}(H_1) \to O_{\tilde F \cup D}(H_1) \to 0$$

where $D \in |H_2|$ is smooth and transversal to $\tilde F$. Since $\tilde F$ is a skew cubic it follows $h^0(O_{\tilde S}(H_1 - \tilde F - D)) = 0$. Hence, the same exact sequence implies that $h^0(\tilde L) \geq 4$ for every $\tilde C \in |O_{\tilde S}(\tilde F + H_2)|$. On the other hand we have $h^1(O_{\tilde S}(H_1)) = 0$. We show that $h^0(O_{\tilde F \cup D}(H_1)) = 4$. Then this implies $h^1(O_{\tilde S}(\tilde F + D)) = 0$ and hence $h^0(O_{\tilde C}(H_1)) = 4$ for every $\tilde C \in |O_{\tilde S}(\tilde F + D)|$. We remark that $\tilde F$ is an integral skew cubic and that $\tilde F \cdot D$ is not contained in a plane. Hence it suffices to show that $O_D(H_1)) = 4$. But D is linked to Q_1 by the complete intersection of $\tilde S_1$ and a general, smooth cubic $X \in |I_{Q_1/\mathbf{P}^3}(3)|$. Let E be a plane section of X: we then have $O_X(D) \cong O_X(5E - Q_1)$. On the other hand consider the standard exact sequence

$$0 \to O_X(E - D) \to O_X(E) \to O_D(E) \to 0.$$

It is easy to check that $h^i(O_X(E-D)) = 0$ for $i = 0,1$. Hence, considering the associated long exact sequence, it follows $h^0(O_X(E)) = = h^0(O_D(E)) = 4$. $\square$

Now assume that a smooth $\tilde C \in |O_{\tilde S}(\tilde F + H_2)|$ is endowed with a fixed point free involution i, induced by the standard involution ι of $\mathbf{P}^3 \times \mathbf{P}^3$. Let $C := \tilde C / < i >$, $\eta \in Pic^0 C$ the line bundle defining the quotient map, $\tilde L := O_{\tilde C}(H_1)$. Then we have:

COROLLARY 2. *The triple $(C,\eta,\tilde{L})$ is a good triple.*

REMARK 4. The corollary explains why the construction realized in this section is not yet effective to produce good triples. We can now easily construct smooth curves $\tilde{C}$ of genus 13 endowed with a line bundle $\tilde{L}$, so that the previous conditions are satisfied. But we need to recognize in this family those curves $\tilde{C}$ such that $\iota/\tilde{C}$ defines a fixed point free involution. To realize this latter step, the very special feature observed in theorem 4.10 still has to be spoiled. We do this in the next section.

6. Symmetroids and Reye congruences in the play

To begin let us give the due motivations to the title of this section. A symmetroid is a very well known surface with a quartic birational model in $\mathbf{P}^3$, namely:

DEFINITION 7. *A symmetroid is a codimension four linear section $\tilde{W}$ of $\mathbf{P}^3\times\mathbf{P}^3$ which is complete intersection of four symmetric bilinear forms.*

By definition a bilinear form $q\in H^0(\mathcal{O}_{\mathbf{P}^3\times\mathbf{P}^3}(1,1))$ is symmetric if $\iota^*q=q$. Hence we have $q=\sum m_{ij}x_iy_j$ so that (m_{ij}) is a 4×4 symmetric matrix. Fixing a basis $q^1\dots q^4$ of $H^0(\mathcal{I}_{\tilde{W}/\mathbf{P}^3\times\mathbf{P}^3}(1,1))$, it follows that $\tilde{W}$ is the base locus of the linear system

$$z_1q^1+z_2q^2+z_3q^3+z_4q^4=0.$$

Restricting it to the diagonal $\{x=y\}$ we obtain a web of quadrics of $\mathbf{P}^3$ whose general member is smooth. To give $\tilde{W}$ is equivalent to give such a web. Let $\tilde{W}_x:=\pi_1(\tilde{W})$ and $\tilde{W}_y:=\pi_2(\tilde{W})$, then $\tilde{W}_x$, $\tilde{W}_y$ are quartic surfaces. It is easy to see that $\tilde{W}_x$ is defined by the equation *det* M_x, where M_x is a 4×4 symmetric matrix of linear forms. Replacing x by y in M_x, we obtain a matrix M_y such that $\tilde{W}_y=\{det\ M_y=0\}$. To give a symmetroid $\tilde{W}$ is therefore equivalent to give a quartic surface $\tilde{W}_x$ whose equation is the determinant of a symmetric 4×4 matrix of linear forms on $\mathbf{P}^3$. These quartic surfaces are known as quartic symmetroids, [9]. In a moment we will briefly recollect the main properties of symmetroids which are needed here.

Now let $(C,\eta,\tilde{L})$ be a good triple and $\tilde{C}\subset\tilde{T}$ the corresponding embedding. By proposition 4.7 a general $\tilde{S}\in|\mathcal{I}_{\tilde{C}/\tilde{T}}(2H_1+H_2)|$ is smooth and such that

$$\tilde{S}\cdot\iota^*\tilde{S}=\tilde{C}\cup\tilde{B}$$

where $\tilde{B}$ is a curve of arithmetic genus 11 and bidegree $(11,11)$. We also know that $<\tilde{B}>$ is a hyperplane of $\tilde{\Lambda}:=<\tilde{T}>$. We fix from now on the following notation

$$\tilde{Y}:=<\tilde{B}>\cdot\tilde{T}.$$

LEMMA 8. *$\tilde{Y}$ is a symmetroid.*

Proof. By proposition 4.9 a general $D\in|\tilde{B}|$ is smooth, connected of genus 11 and degree 22. This implies $h^1(\mathcal{O}_D(H_1+H_2))=0$ and $h^0(\mathcal{O}_D(H_1+H_2))=12$. Moreover we have $h^i(\mathcal{O}_{\tilde{S}}(H_1+H_2)) = 0$, $i \geq 1$. Then the standard exact sequence

$$0 \to \mathcal{O}_{\tilde{S}}(H_1+H_2-\tilde{B}) \to \mathcal{O}_{\tilde{S}}(H_1+H_2) \to \mathcal{O}_{\tilde{B}}(H_1+H_2) \to 0$$

implies $h^0(\mathcal{O}_{\tilde{S}}(H_1+H_2-\tilde{B})) = 1$ and $h^i(\mathcal{O}_{\tilde{S}}(H_1+H_2-\tilde{B})) = 0, i \geq 1$. Then it follows $h^1(\mathcal{O}_{\tilde{B}}(H_1+H_2)) = 0$ and $h^0(\mathcal{O}_{\tilde{B}}(H_1+H_2)) = 12$. Note that ι acts on $\tilde{B}$ as a fixed point free involution. Let $\pi_B : \tilde{B} \to B = \tilde{B}/<\iota>$ be the quotient map. Then $p_a(B) = 6$ and π_B is defined by a non trivial element $\eta_B \in Pic^0_2 B$. Let $M \in Pic^{11} B$ such that $\pi_B^* M \cong \mathcal{O}_{\tilde{S}}(H_1+H_2)$, then we have

$$H^i(\mathcal{O}_{\tilde{B}}(H_1+H_2)) = \pi_B^* H^i(M) \oplus \pi_B^* H^i(M \otimes \eta_B).$$

Here the summands are the eigenspaces of ι. Since $h^1(\mathcal{O}_{\tilde{B}}(H_1+H_2)) = 0$, it follows $h^1(M) = h^1(M \otimes \eta_B)) = 0$ and $h^0(M) = h^0(M \otimes \eta_B) = 6$. Finally note that the action of ι on $H^0(\mathcal{O}_{\tilde{T}}(H_1+H_2))$ has a $+$ eigenspace E^+ of dimension 7. Hence the restriction $E^+ \to \pi^* H^0(M)$ has 1-dimensional Kernel. Then $<\tilde{B}> \cdot \tilde{T}$ is a symmetroid. $\square$

The next assumption will be not restrictive to the goals of this paper.

ASSUMPTION 7. *$\tilde{Y}$ is a smooth and general symmetroid.*

The assumption implies that $\iota : \tilde{Y} \to \tilde{Y}$ is a fixed point free involution. Hence $Y := \tilde{Y}/<\iota>$ is a smooth Enriques surface. Note that $|\mathcal{O}_{\tilde{Y}}(\tilde{B})|$ is a ι-invariant linear system on $\tilde{Y}$ of curves of genus 11 and bidegree $(11,11)$. Moreover $\tilde{B}$ is a ι-invariant element of it. We have the diagram of projection maps

$$\mathbf{P}^{5-} \supset \mathbf{G} \supset Y_- \xleftarrow{\pi^-} \tilde{Y} \xrightarrow{\pi^+} Y_+ \subset T \subset \mathbf{P}^{9+}.$$

In particular $Y_- := \pi^-(\tilde{Y})$ and $Y_+ := \pi^+(\tilde{Y})$ are two copies of Y. Moreover we have

$$\mathbf{P}^{5-} \supset \mathbf{G} \supset B_- \xleftarrow{\pi^-} \tilde{B} \xrightarrow{\pi^+} B_+ \subset T \subset \mathbf{P}^{9+},$$

where $B_- := \pi^-(\tilde{B})$ and $B_+ := \pi^+(\tilde{B})$. B_+, B_- are two copies of the curve

$$B := \tilde{B}/<\iota>.$$

B is connected, $B^2 = 10$ so that $p_a(B) = 6$ and the quotient map $\pi_B : \tilde{B} \to B$ is an étale double covering. Both B_+ and B_- have degree 11. As in the proof of 6.2 we consider $M := \mathcal{O}_{B_+}(1)$ and η_B the line bundle defining π_B. Then it follows that $M \otimes \eta_B \cong \mathcal{O}_{B_-}(1)$. The proof of the next lemma is already part the proof of 6.2.

LEMMA 9. $h^0(M) = h^0(M \otimes \eta_B) = 6$ *and* $h^1(M) = h^1(M \otimes \eta_B) = 0$.

We need to add more informations on Y, which is an Enriques surface of special type, and on its linear system $|\mathcal{O}_Y(B)|$. Y_-, Y_- are copies of Y embedded as surfaces of degree 10. We fix the notations $H_+ \in |\mathcal{O}_{Y_+}(1)|$ and $H_- \in |\mathcal{O}_{Y-}(1)|$ for their general hyperplane sections. Both H_+, H_- are smooth curves of genus 6 Prym canonically embedded. Notice also that $H_+ - H_- \sim K_Y$.

Every Enriques surface admits projective embeddings in $\mathbf{P}^5$ as a surface of degree 10. They are called *Fano models* of the surface. For every Fano model the surface contains exactly 20 pair curves which are embedded in the Fano model as plane cubics. They subdivide in 10 pairs: the difference of the two curves of each pair is a canonical divisor.

Actually Y is a general member of a special family of Enriques surfaces: those containing smooth, integral rational curves. Since each of these curves can be contracted to a rational double point, Y is also said to be a nodal Enriques surfaces. The models Y_- and Y_- are special Fano models of Y.

Y_- is characterized by the condition that there exists a quadric, namely the Klein quadric $\mathbf{G}$, containing it. This is equivalent to the non quadratic normality of Y^-. The Fano models of this type are congruences of lines in the Grassmannian $\mathbf{G}$: they are known as *Reye congruences* and form an irreducible family. As is well known:

PROPOSITION 7. *A general Y_- does not contain smooth rational curves of degree ≤ 3.*

Since $H_+ \sim H_- + K_Y$, the same is true for a general Y_+. We recall that Fano models Y_+ are characterized by the condition that the family of their trisecant lines is 3-dimensional, [8]. Equivalently these Fano models are the smooth hyperplane sections of Conte-Murre threefolds.

For a Fano model Y_+ we will denote the 10 pairs of its plane cubics as E_n, E'_n, with $n = 1 \dots 10$. As for every Enriques surface they intersect as follows

$$E_m E_n = E_m E'_n = 1 - \delta_{mn},\ 1 \leq m,n \leq 10.$$

Every Enriques surface X admits morphisms $f : X \to \mathbf{P}^3$ whose schematic image is a sextic surface passing doubly through the edges of a tetrahedron Z. We say that $f(X)$ is an *Enriques model*, and $|f^*\mathcal{O}_{\mathbf{P}^3}(1)|$ is an *Enriques linear system*, if f is generically injective and Z is $\{z_1z_2z_3z_4 = 0\}$, where $z_1 \dots z_4$ are independent linear forms. Now let

$$B \subset Y_+$$

be any connected curve of degree 11 and arithmetic genus 6. Then we have:

THEOREM 8. *$|2H_+ - B|$ is an Enriques linear system. Moreover such a linear system is either $|E_a + E_b + E_c|$ or $|E_a + E_b + E_c + K_Y|$, for some (a,b,c) such that $1 \leq a < b < c \leq 10$.*

Proof. We have $(2H_+ - B)^2 = 6$. We claim that $|2H_+ - B|$ is base point free. Then, by the classification in [5] IV 9 and its corollary 1 p.283, it follows that either $|2H_+ - B|$ is an Enriques linear system or it is superelliptic. In the latter case $|2H - B|$ defines a morphism $f : Y \to f(X) \subset \mathbf{P}^3$ which is a $2:1$ cover of a 4-nodal cubic surface, [5] IV 7. Applying the classification for this case, [5] corollary IV 7.1, one can check that there exists $R_1 + R_2 \in |2H_+ - B|$ such that $p_a(R_1) = 0$ and $R_1(2H_+ - B) \leq 3$. But then Y_+ contains a smooth, connected rational curve of degree ≤ 3, which is excluded for a general Y_+. Let us prove the previous claim: assume that $|2H-B|$ has fixed components, then one of them is an integral, smooth rational curve R of degree $HR \geq 4$. Consider the standard exact sequence

$$0 \to \mathcal{O}_Y(H-B) \to \mathcal{O}_Y(2H-B) \xrightarrow{\rho} \mathcal{O}_H(2H-B) \to 0,$$

where $H \in |H_+|$ is smooth. Since $\tilde{Y}_+$ does not contain lines and $H(B-H)=1$, we have $|B-H| = |B-H+K_Y| = \emptyset$. This, by Riemann-Roch and Serre duality, implies $h^i(\mathcal{O}_Y(H-B)) = 0$, $i \geq 0$. Hence the map $h^0(\rho)$ is an isomorphis. Then $\mathcal{O}_H(H-R)|$ is linear series of dimension ≥ 3 and degree ≤ 6. Hence H is hyperelliptic by Clifford's theorem. This is excluded for a very ample divisor on an Enriques surface. In particular $2H-B$ is nef. Finally assume $|2H-B|$ has a base point. Then $|2H-B|$ is hyperelliptic and its classification is known, cfr. [5] IV 5.1. Applying it and the equality $H(2H-B)$ $=9$ one can check that then Y_+ contains a smooth, connected rational curve of degree ≤ 3, which is excluded for a general Y_+. $\square$

Keeping (a,b,c) as above, we fix the definition

$$\mathcal{E}(Y_+) := \{|D| \ / \ |D| = |E_a+E_b+E_c| \text{ or} |E_a+E_b+E_c+K_Y|\}.$$

We will say that $|D|$ is an Enriques linear system of Y_+. From $\mathcal{E}_{Y_+}$ we also define

$$\mathcal{F}(Y_+) := \{|2D-H_+| \ / \ |D| \in \mathcal{E}(Y_+)\}$$

and

$$\mathcal{B}(Y_+) := \{|2H_+-D| \ / \ \in \mathcal{E}(Y_+)\}.$$

REMARK 5. For any Fano model $X \subset \mathbf{P}^5$ of any Enriques surface one can define the previous sets. Let $H \in |\mathcal{O}_X(1)|$ and $D \in \mathcal{E}(X)$, then $p_a(2D-H)=0$. The non emptyness of $|2D-H|$ is discussed later.

7. Reye congruences and some unirationality results

Let $(C_k, \eta_k, \tilde{L}_k)$, $(k=1,2)$, be two good triples, they define the embeddings

$$\tilde{C}_k \subset \tilde{T}_k \subset \mathbf{P}^3 \times \mathbf{P}^3.$$

By definition they are isomorphic if there exists an isomorphism $\tilde{u}: C_1 \to C_2$ such that $i_2 \cdot \tilde{u} = \tilde{u} \cdot i_1$ and $\tilde{u}^* \tilde{L}_2 \cong \tilde{L}_1$, where i_k is the fixed point free involution defined on $\tilde{C}_k$. The isomorphism class of a good triple $(C, \eta, \tilde{L})$ will be denoted as $[C, \eta, \tilde{L}]$. We omit the very easy proof of the next lemma:

LEMMA 10. *Two good triples as above are isomorphic iff there exists* $\alpha \in$ *Aut* $\mathbf{P}^3 \times$ *Aut* $\mathbf{P}^3$ *such that* $\alpha(\tilde{C}_1) = \tilde{C}_2$.

We consider now good triples $(C, \eta, \tilde{L})$ satisfying the following conditions:

- $\tilde{C} \subset \tilde{S} \subset \tilde{T}$, *where* $\tilde{S} \in |I_{\tilde{B}/\tilde{T}}(2H_1 + H_2)|$.
- $\tilde{B} = \pi_+^* B \subset \tilde{Y}$, *where* $|B| \in \mathcal{B}(Y_+)$.
- $\tilde{Y}$ *is a general symmetroid.*
- $\tilde{S} \cdot \iota^* \tilde{S} = \tilde{B} \cup \tilde{C}$.
- $C = \tilde{C}/ < \iota >$, *the covering* $\pi : \tilde{C} \to C$ *is defined by* η, $\tilde{L} = \mathcal{O}_{\tilde{C}}(H_1)$.

Clearly, the 4-tuple $x := (\tilde{T}, \tilde{Y}, \tilde{B}, \tilde{S})$ uniquely defines the good triple $(C, \eta, \tilde{L})$.

DEFINITION 8. *x is a good 4-tuple, $\mathfrak{G}$ is the family of all good 4-tuples.*

Along this section we will study the moduli map

$$m : \mathfrak{G} \to \mathcal{P}_7^3$$

defined as follows: $m(x) := [C, \eta, \tilde{L}]$. To begin we point out once more that:

LEMMA 11. *Let $(C, \eta, \tilde{L})$ be a good triple, then $\tilde{L}$ is an isolated point of $P^3(C, \eta)$.*

Proof. Since the Petri map $\mu_{\tilde{L}}$ is surjective, the Prym-Petri map $\mu_{\tilde{L}}^-$ is an isomorphism. Then the tangent space at $\tilde{L}$ to the Prym Brill-Noether locus $P^3(C, \eta)$ is 0-dimensional, since it is isomorphic to *Coker* $\mu_{\tilde{L}}^-$. □

Let $\mathcal{Z}$ be an integral variety whose points are isomorphism classes of good triples. Let $f : \mathcal{Z} \to \mathcal{R}_7$ be the natural forgetful map sending $[C, \eta, \tilde{L}]$ to $[C, \eta]$. Then:

PROPOSITION 8. *The morphism $f : \mathcal{Z} \to \mathcal{R}_7$ is finite over its image.*

Proof. The tangent space to the fibre of $f : \mathcal{Z} \to \mathcal{R}_7$ at $[C, \eta]$ is the tangent space to $P(C, \eta)$ at $\tilde{L}$. Hence, by the lemma, it is 0-dimensional. □

LEMMA 12. *Let $x = (\tilde{T}, \tilde{Y}, \tilde{B}, \tilde{S}) \in \mathfrak{G}$ be a good 4-tuple, then*

$$dim\ m^{-1}(m(x)) \ \leq \ dim\ |I_{\tilde{C}/\tilde{T}}(2H_1 + H_2)| + dim\ Aut\ \mathbf{P}^3 \times Aut\ \mathbf{P}^3.$$

Proof. Let $x' = (\tilde{T}', \tilde{Y}', \tilde{B}', \tilde{S}') \in m^{-1}(m(x))$ and $m(x') = [C', \eta', \tilde{L}']$. Then there exists $\alpha \in Aut\ \mathbf{P}^3 \times Aut\ \mathbf{P}^3$ such that $\alpha(\tilde{C}) = \tilde{C}'$. Since $\tilde{T}' =< \tilde{C}' > \cdot \mathbf{P}^3 \times \mathbf{P}^3$, it follows that $\alpha(\tilde{T}) = \tilde{T}'$ and that $\alpha^* |I_{\tilde{C}'/\tilde{T}'}(2,1)| = |I_{\tilde{C}/\tilde{T}}(2,1)|$. Moreover $\tilde{Y}$ and $\tilde{B}$ are uniquely defined from $\tilde{S}$, because $\tilde{C} \cup \tilde{B} = \tilde{S} \cdot \iota^* \tilde{S}$ and $\tilde{Y} =< \tilde{B} > \cdot \tilde{T}$. Hence the fibre $m^{-1}(m(x))$ is dominated by the family of pairs $(\tilde{S}, \alpha)$ and the statement follows. □

LEMMA 13. $dim\ |I_{\tilde{C}/\tilde{T}}(2H_1 + H_2)| = 3$.

Proof. By propositions 4.6 and 4.7 we can choose a smooth $\tilde{S} \in |I_{\tilde{C}/\tilde{T}}(2H_1 + H_2)|$ such that $|O_{\tilde{S}}(2H_1 + H_2 - \tilde{C})|$ is base point free. Let $D \in |O_S(2H_1 + H_2 - \tilde{C})|$, then $D^2 = 3$ and a general D is smooth, connected of genus 6. Consider the standard exact sequence

$$0 \to O_{\tilde{S}} \to O_{\tilde{S}}(D) \to O_D(D) \to 0.$$

Its associated long exact sequence yelds $dim\,|D| \leq 2$ and hence $dim\,|I_{\tilde{C}/\tilde{T}}(2H_1 + H_2)| \leq 3$. Actually the equality holds. $\square$

LEMMA 14. *$dim\ I_{\tilde{B}}(2H_1 + H_2)| = 4$.*

Proof. The codimension in $|I_{\tilde{B}/\tilde{T}}(2H_1 + H_2)|$ of the linear system $\tilde{Y} + |O_{\tilde{T}}(H_1)|$ equals $h^0(O_{\tilde{Y}}(2H_1 + H_2 - \tilde{B}))$. So it suffices to show that it is 1. Let $\tilde{f} := c_1(O_{\tilde{Y}}(2H_1 + H_2 - \tilde{B}))$, we remark that $\tilde{f} + \iota^* \tilde{f}$ is $\pi_+^* c_1(O_{Y_+}(2D - H_+)$. It is shown in the next proposition 8.9 that $2D - H_+$ is the class of an isolated curve F of degree 8 and arithmetic genus 0. Hence $\pi_+^* F = \tilde{F} + \iota^* \tilde{F}$, where the two summands are isolated copies of F. This implies that $h^0(O_{\tilde{Y}}(2H_1 + H_2 - \tilde{B})) = h^0(O_{\tilde{Y}}(\tilde{F})) = 1$. $\square$

Now we can count dimensions:

(1) The family of pairs $(\tilde{T}, \tilde{Y})$ is naturally identified to the flag variety of pairs $< \tilde{T} >, < \tilde{Y} >$ of linear spaces of $\mathbf{P}^{9+}$. It is a rational variety of dimension 27. We denote it as $\mathfrak{F}$.

(2) We denote by $\mathfrak{P}$ the family of triples $(\tilde{T}, \tilde{Y}, O_{\tilde{Y}}(\tilde{B}))$. This latter family is a finite covering of the family of pairs $(\tilde{T}, \tilde{Y})$.

(3) The variety of triples $(\tilde{T}, \tilde{Y}, \tilde{B})$ is open in a projective bundle over $\mathfrak{P}$, with fibre $\pi_+^* |O_{Y_+}(B)|^+$ at $(\tilde{T}, \tilde{Y}, O_{\tilde{Y}}(\tilde{B}))$. In other words we only take in $|O_{\tilde{Y}}(\tilde{B})|$ those curves $\tilde{B}$ defined by a ι-invariant global section. We denote such a family as $\mathfrak{B}$.

(4) Finally $\mathfrak{G}$ embeds as an open set in the $\mathbf{P}^4$-bundle over $\mathfrak{B}$ whose fibre at $(\tilde{T}, \tilde{Y}, \tilde{B})$ is $|I_{\tilde{B}}(2H_1 + H_2)|$.

Due to the previous discussion we can conclude that $dim\ \mathfrak{G} = 36$ and moreover that $\mathfrak{G}$ is unirational if $\mathfrak{P}$ is irreducible and unirational. We will prove this latter property after some preparation. The proof relies on the special geometry of Reye congruences and of their Enriques models. As a preparation we consider a good 4-tuple $x = (\tilde{T}, \tilde{Y}, \tilde{B}, \tilde{S})$ and

$$Y_+ \subset \mathbf{P}^5.$$

From now on $Z := \{z_1 z_2 z_3 z_4 = 0\}$ will denote the fundamental tetrahedron of $\mathbf{P}^3$ and $e(Z)$ the union of its edges. Since we are assuming that $\tilde{Y}$ is general, every Enriques linear system $|D| \in \mathcal{E}(Y_+)$ defines a generically injective morphism $f : Y \to \mathbf{P}^3$ whose image is a sextic Enriques surface $Y' \in |I^2_{e(Z)/\mathbf{P}^3}(6)|$. In this sense the equation of Y' appears to be 'general':

$$z_1 z_2 z_3 z_4 q + a(z_1 z_2 z_3)^2 + b(z_1 z_2 z_4)^2 + c(z_1 z_3 z_4)^2 + d(z_2 z_3 z_4)^2 = 0.$$

The special feature of Y_+ is however present. Consider indeed $|2D - H_+| \in \mathcal{F}(Y_+)$. Differently from the case of a general Enriques surface, this linear system is in our case not empty: the next result is a private communication of I. Dolgachev, [10].

PROPOSITION 9. *$2H_+ - D \sim F$, where F is a connected curve of degree* 8 *and genus* 0*.*

Proof. Fano models Y_+ are also characterized by the non emptyness of $|H_+ - 2E_n| = |H_+ - 2E'_n|$, cfr. [11, 8]. Now $(2D - H_+) - (H_+ - 2E_n)$ is divisible by 2 in *Pic* Y. Then the existence of F follows from the next lemma. □

The next result is due to E. Loijenga:

LEMMA 15. *Let X be an Enriques surface, R a (-2)-curve on it of class r. Assume that $f - r$ is divisible by* 2 *in Pic X and that $f^2 = -2$. Then f is the class of an effective curve F.*

Proof. Putting $f = r + 2y$ consider the K3 cover $\pi : \tilde{X} \to X$. We have $\pi^* R = R_1 + R_2$, where R_1, R_2 are disjoint copies of R. Let r_i be the class of R_i, then $\pi^* f = r_1 + r_2 + 2\pi^* y = (r_1 + \pi^* y) + (r_2 + \pi^* y)$. Each summand is a -2 class on the K3 surface $\tilde{X}$. Hence it is effective. Then $\pi^* f$ is an effective class and the same is true for f. □

REMARK 6. In our situation we started with $x = (\tilde{T}, \tilde{Y}, \tilde{B}, \tilde{S})$. Then we can also observe directly that $\tilde{S} \cdot \tilde{Y} + \iota^* \tilde{S} \cdot \tilde{Y} = 2\tilde{B} + \tilde{F} + \iota^* \tilde{F} \in \mathcal{O}_{\tilde{Y}}(3H_1 + 3H_2)|$, where $\tilde{F}$ is effective. Putting $F = \pi_+(\tilde{F})$, it follows $2B + F \sim 3H_+$, that is, $F \sim 2D - H_+$.

Starting from our good 4-tuple $x = (\tilde{T}, \tilde{Y}, \tilde{B}, \tilde{S})$ let us consider the morphism

$$f : Y \to \mathbf{P}^3,$$

defined by $|D| = |2H_+ - B_+|$. Y_+ contains a connected curve $F \sim 2D - H_+$ such that $deg\, F = 8$ and $p_a(F) = 0$. Since Y_+ is general F does not contain components of degree ≤ 3. On the other hand $FD = 3$. The next lemma is almost immediate:

LEMMA 16. *$f : F \to \mathbf{P}^3$ embeds F as a skew cubic curve intersecting in one point each edge of the fundamental tetrahedron Z.*

Proof. Under our assumptions, either F is integral or it is the union of two integral smooth rational quartic curves. Assume that f/F is not an embedding. The, since $DF = 3$, the curve $f(F)$ is a plane cubic and hence $D - F$ is an effective divisorial class. Since $H_+(D - F) = 1$, then Y_+ contains a line: a contradiction. □

The lemma highlights a further characterization of a general Reye congruence, we briefly summarize. The family $\mathcal{Y}_-$ of Reye congruences Y_- is irreducible and a general member of it is constructed as follows. Let $X' \in |I^2_{Z/\mathbf{P}^3}(6)|$ be a general sextic among those containing a general skew cubic F' which intersects each edge of the tetrahedron Z in one point. Let $f : X \to X'$ be the normalization map and let

$$|D| := |\mathcal{O}_{X'}(1)|\ ,\ F := f^*F'.$$

Then X is a smooth Enriques surface and $\mathcal{O}_X(2D-F)$ defines an embedding of X

$$X_- \subset \mathbf{G} \subset \mathbf{P}^{5-}$$

as a Reye congruence. Actually the construction defines a stable rank two vector bundle $\mathcal{V}$ on X such that $c_1(\mathcal{V}) = \mathcal{O}_X(2D-F)$ and $deg\ c_2(\mathcal{V}) = 3$. It turns out that $h^0(\mathcal{V}) = 4$ and that X_- is the image of the classifying map of $\mathcal{V}$.

$\mathcal{V}$ is known as a *Reye bundle*, cfr. [11] and [8]. Now we construct a rational family which is specially useful to dominate the family $\mathfrak{P}$:

At first we consider the edges $Z_1 \dots Z_6$ of the fundamental tetrahedron Z of $\mathbf{P}^3$ and the rational family $\mathcal{S}$ of general 6-tuples $s := (o_1, \dots, o_6) \in Z_1 \times \cdots \times Z_6$. As is well known s uniquely defines an integral skew cubic curve F_s. It is very easy to show that the restriction map $\rho_s : H^0(I^2_{e(Z)/\mathbf{P}^3}(6)) \to H^0(\mathcal{O}_{F_s}(6))$ is surjective. Let $\mathbf{P}^{13} := |I_{e(Z)/\mathbf{P}^3}(6)|$; then, over the variety $\mathcal{S}$, we have the projective bundle

$$\mathcal{R} := \{(s, X') \in \mathcal{F} \times \mathbf{P}^{13}\ /\ F'_s \subset X'\},$$

whose fibre at s is $\mathbf{P}Ker\ \rho_s$. By the previous remarks the family $\mathcal{R}$ dominates the moduli space of Reye congruences. Indeed let (o, X') be general in $\mathcal{R}$ and let $f : X \to X'$ be the normalization map. Then, keeping the previous notations, X is an Enriques surface and $\mathcal{O}_X(2D - F + K_X)$ embeds X as a Reye congruence.

Secondly we construct, over a suitable open set U of $\mathcal{R}$, the natural universal family $p : \mathcal{X} \to U$. The fibre of p at (s, X') is the normalization X of X' and a smooth Enriques surface. Let $f : \mathcal{X} \to X'$ be the normalization map, over $\mathcal{X}$ there exist two line bundles $\mathcal{F}$ and $\mathcal{D}$ such that

$$\mathcal{D} \otimes \mathcal{O}_X := f^*\mathcal{O}_{X'}(1) \text{ and } \mathcal{F} := \mathcal{O}_X(f^*F'_o).$$

As is well known the 1-dimensional space $H^1(\mathcal{O}_X(F)) \cong Ext^1(\mathcal{O}_X(D), \omega_X(D-F))$ uniquely reconstructs the Reye bundle $\mathcal{V}$ as an extension. Therefore, globalizing this construction, there exists a rank two vector bundle $\mathcal{V}_{\mathcal{X}}$ over $\mathcal{X}$, whose restriction to X is $\mathcal{V}$. We consider the rank 4 vector bundle $p_*\mathcal{V}^*_{\mathcal{X}}$, with fibre $H^0(\mathcal{V})^*$ at (s, X') and its Grassmann bundle $\mathcal{G} \to U$, with fibre the Grassmannian $G(2, H^0(\mathcal{V})^*)$ at (s, X'). Let

$$\phi : \mathcal{X} \to \mathcal{G} \subset \mathbf{P}p_*\mathcal{V}^*_{\mathcal{X}},$$

be the classifying map. Clearly ϕ embeds each fibre X as a Reye congruence. Up to shrinking U, we can assume that $\mathcal{G}$ is trivial, that is we can assume that

$$\mathcal{G} = U \times \mathbf{G} \subset U \times \mathbf{P}^{5-}.$$

Let $\mathcal{Y}_-$ be the family of Reye congruences of $\mathbf{G}$, now we consider the rational map

$$\psi : U \times Aut\ \mathbf{G} \to \mathcal{Y}_-$$

which is so defined: $\psi(o, X', a) := X_-$ is the surface $X_- = a \cdot \phi(X)$ of the family.

By construction U dominates the moduli of Reye congruences. Hence $U \times$ *Aut* $\mathbf{G}$ dominates $\mathcal{Y}_-$, which is therefore unirational. We can finally prove our theorem:

THEOREM 9. *$\mathfrak{P}$ is unirational so that $\mathfrak{G}$ is a unirational variety of dimension* 36.

Proof. To prove the unirationality result we use the rational family $U \times$ *Aut* $\mathbf{G}$ and the preceeding construction. We keep everywhere the same notations. Over $U \times$ *Aut* $\mathbf{G}$ we have the projective bundle $\mathbb{T} \to U \times$ *Aut* $\mathbf{G}$ defined as follows. Let $(s, X', a) \in U \times$ *Aut* $\mathbf{G}$, consider the embedded Reye congruence $X_- = a \cdot \phi(X) \subset \mathbf{G}$. Via our usual diagram of linear projections we construct from X the symmetroid $\tilde{X} := \pi_-^{-1}(X_-) \subset \mathbf{P}^3 \times \mathbf{P}^3$. Then, by definition, the fibre of $\mathbb{T}$ at (s, X', a) is $\mathbb{T}_{(s,X',a)} := |I_{\tilde{X}/\mathbf{P}^3 \times \mathbf{P}^3}(1,1)|^*$.

We remark that a general element of this fibre is a smooth threefold $\tilde{T}$, complete intersection of three general symmetric bilinear forms vanishing on $\tilde{X}$. Moreover $\tilde{X}$ is endowed with the polarization $O_{\tilde{X}}(\tilde{B}) := \pi_+^* O_{X_+}(2H_+ - D)$ where $X_+ := \pi_+(\tilde{X})$, $|D| := |f^* O_{X'}(1)|$ and $H_+ \in |O_{X_+}(1)|$. Hence a general point of $\mathbb{T}$ defines a triple $(\tilde{T}, \tilde{X}, O_{\tilde{X}}(\tilde{B})) \in \mathfrak{P}$. This defines a rational map $\tau : \mathbb{T} \to \mathfrak{P}$ sending (s, X', a) to $(\tilde{T}, \tilde{X}, O_{\tilde{X}}(\tilde{B}))$.

The surjectivity of τ is clear and follows from the the previous results: a general triple $t := (\tilde{T}, \tilde{X}, O_{\tilde{X}}(\tilde{B})$ is in the image of τ as follows. Let $X = \tilde{X}/ < \iota >$, choose $X' \in |I^2_{e(Z)/\mathbf{P}^3}(6)|$ so that $X' = f(X)$, where $f : X \to \mathbf{P}^3$ is defined by $O_X(2H_+ - B_+)$. Furthermore consider $F \in |2D - H_+|$ and choose the 6-tuple s so that s defines the skew cubic curve $f(F)$. Then, we have $t = \tau(s, X', a)$ for some $a \in$ *Aut* $\mathbf{G}$. $\square$

We can finally conclude our discussion proving some new unirationality results, which are a goal of this paper. Let $\mathcal{P}$ be the image of the map $m : \mathfrak{G} \to \mathcal{P}_7^3$, we have:

THEOREM 10. *$\mathcal{P}$ is a unirational component of $\mathcal{P}_7^3$ which dominates $\mathcal{R}_7$.*

Proof. We have *dim* $\mathfrak{G} = 36$. Moreover the general fibre of m has dimension ≤ 18. Hence $\mathcal{P}$ has dimension ≥ 18. But $\mathcal{P}$ is a family of isomorphism classes of good triples. Hence, by proposition 8.4, the forgetful map $f : \mathcal{P} \to \mathcal{R}_7$ is finite onto its image. Since *dim* $\mathcal{R}_7 = 18$, it follows that $\mathcal{P}$ is 18-dimensional and dominates $\mathcal{R}_7$ via f. The unirationality of $\mathcal{P}$, and of $\mathcal{R}_7$, then follows from the previous theorem. $\square$

We believe that $\mathcal{P}$ is the unique irreducible component of $\mathcal{P}_7^3$. The theorem implies the unirationality of $\mathcal{R}_7$. As remarked, this latter result has been recently proved in [14] by a different, much simpler method.

8. Appendix: existence of good triples and good 4-tuples

Finally, to show that the work performed so far is not empty, we prove that good triples and 4-tuples do exist. We prove that there exists a flat family

$$\tilde{\mathcal{U}} := \{\tilde{C}_u,\ u \in U\}$$

of connected curves $\tilde{C}_u \subset \mathbf{P}^3 \times \mathbf{P}^3$ of arithmetic genus 13 such that:

(1) U is smooth, connected with a distinguished point $o \in U$.
(2) $\tilde{C}_u$ is canonically embedded in $< \tilde{C}_u >$, it is smooth for $u \neq o$.
(3) ι acts on each $\tilde{C}_u$ as a fixed point free involution $i_u := \iota/\tilde{C}_u$.
(4) $(C_u, \eta_u, \tilde{L}_u)$ is a good triple for $u \neq o$.

Here we set $\tilde{L}_u := O_{\tilde{C}_u}(1,0)$, $C_u := \tilde{C}_u/ < \iota >$. $\eta_u \in Pic^0 C$ defines the quotient cover $\pi_u : \tilde{C}_u \to C_u$. Moreover let $\tilde{T}_u :=< \tilde{C}_u > \cdot \mathbf{P}^3 \times \mathbf{P}^3$, we prove the existence of families $\tilde{\mathcal{U}}$ such that:

(5) $\forall u \in U,\ \tilde{T}_u = \tilde{T}$ where $\tilde{T}$ is general in its family.
(6) $\forall u \in U$, the linear system $| I_{\tilde{C}_u/\tilde{T}}(2H_1 + H_2)|$ is 3-dimensional.

Let $\{\tilde{S}_u, u \in U\}$ be any family such that $\tilde{S}_u \in | I_{\tilde{C}_u/\tilde{T}}(2H_1 + H_2)|$, then we have:

$$\tilde{S}_u \cdot \iota^* \tilde{S}_u := \tilde{B}_u \cup \tilde{C}_u \text{ and } \tilde{Y}_u :=< \tilde{B}_u > \cdot \tilde{T}.$$

In particular the latter family yelds the family of 4-tuples

$$x_u := (\tilde{T}, \tilde{Y}_u, \tilde{B}_u, \tilde{S}_u),\ u \in U.$$

For a family $\mathcal{U}$ as above, we will also see that:

(7) Y_u is a smooth and general symmetroid, provided $u \in U$ is general.

Since the previous conditions are satisfied, it follows:

THEOREM 11. *A general x_u is a good 4-tuple, defining the good triple $(C_u, \eta_u, \tilde{L}_u)$.*

Performing such a program is not difficult if a convenient curve $\tilde{C}_o$ is chosen to begin, we construct $\tilde{C}_o$ as follows.

We fix a smooth and general symmetroid $\tilde{Y}_o$ and a smooth 3-dimensional linear section $\tilde{T}$ of $\mathbf{P}^3 \times \mathbf{P}^3$ containing it. Let $P \subset \mathbf{P}^3$ be a general plane. By transversality we can assume that $P \times P$ is transversal to $\tilde{Y}_o$ and to $\tilde{T}$. This implies that

$$\tilde{E} := P \times P \cap \tilde{T}$$

is a smooth, elliptic sextic curve and that $\iota/\tilde{E}$ is a fixed point free involution on it. Furthermore $\tilde{Y}_o$ defines a transversal hyperplane section of $\tilde{E}$, we denote as

$$n := n_1 + n_2 + n_3 + \iota(n_1 + n_2 + +n_3),$$

We have $n = \tilde{Y}_o \cdot \tilde{E}$, let us point out that $\tilde{E}$ is uniquely constructed from the choice of n_1, n_2, n_3 in $\tilde{Y}_o$. Indeed the plane P is determined by $\pi_1(n_1)$, $\pi_1(n_2)$, $\pi_1(n_3)$. In particular we have the ι-invariant 4-dimensional linear space

$$N := < \tilde{Y}_o > \cap < \tilde{E} > = < n > .$$

The construction also yelds a line and a plane. Respectively they are:

$$N_+ := \pi_+(N) \text{ and } N_- := \pi_-(N).$$

Now we fix an Enriques linear system $|D_o| \in \mathcal{E}(\tilde{Y}_{o+})$ and then $\pi_+^*|D_o|$. A general element of $\pi_+^*|D_o|$ is a smooth connected curve of bidegree $(9,9)$ and genus 7 endowed with the fixed point free involution induced by ι. Since $dim\ \pi_+^*|D_o| = 3$ there exists $\tilde{D} \in \pi_+^*|D_o|$ passing through n_1, n_2, n_3 and hence through $\iota(n_1)$, $\iota(n_2)$, $\iota(n_3)$ as well. Up to moving n_1, n_2, n_3 in $\tilde{Y}_o$, we can assume that $\tilde{D}$ is smooth. This implies that:

PROPOSITION 10. *$\tilde{C}_o := \tilde{D} \cup \tilde{E}$ is a nodal connected curve of bidegree $(12,12)$. We have $p_a(\tilde{C}_o) = 13$ and Sing $\tilde{C}_o = \tilde{E} \cap \tilde{D}$; $i_o := \iota/\tilde{C}_o$ is a fixed point free involution on $\tilde{C}_o$.*

We fix the notations $C_o := \tilde{C}_o / < i_o >$ and η_o for the line bundle defining the quotient cover $\pi_o : \tilde{C}_o \to C_o$. The curves $C_{o-} := \pi_-(\tilde{C}_o)$ and $C_{o+} := \pi_+(\tilde{C}_o)$ are copies of C_o. We have also to consider the curves $E := \tilde{E} / < \iota >$, $D := \tilde{D} / < \iota >$, then

$$E_- := \pi_-(\tilde{E}) \ , \ E_+ := \pi_+(\tilde{E}) \ \text{ and } \ D_- := \pi_-(\tilde{D}) \ , \ D_+ := \pi_+(\tilde{D})$$

are respectively copies of E, D. Let $T = \pi_+(T)$, we have the usual diagram

$$\mathbf{P}^{5-} \ \supset \ \mathbf{G} \ \supset D_- \cup E_- \ \xleftarrow{\pi_-} \ \tilde{C}_o \ \xrightarrow{\pi_+} \ E_+ \cup D_+ \ \subset \ T - Sing\ T \ \subset < T > .$$

PROPOSITION 11.

(1) $\tilde{C}_o$ is canonically embedded in $< \tilde{T} >$.

(2) C_{o+} is canonically embedded in $< T >$ and Sing C_{o+} spans N_+.

(3) C_{o-} is Prym canonically embedded in $\mathbf{P}^{5-}$ and Sing C_{o-} spans N_-.

Proof. (1) Consider $\tilde{C}_o$: it follows from its construction that $h^0(\mathcal{O}_{\tilde{C}_o}(1)) = 13$. Moreover its two components $\tilde{D}$ and $\tilde{E}$ are glued along a hyperplane section of $\tilde{E}$, namely n. It is a standard property that then $\mathcal{O}_{\tilde{C}_o}(1) \cong \omega_{\tilde{C}_o}$. Since $< \tilde{E} >$ is not contained in $< \tilde{Y}_o >$, we have $< \tilde{C}_o > = < \tilde{T} >$ and $dim\ < \tilde{C}_o > = 12$. Hence (1) follows. (2) We have $Sing\, C_{o+} = E_+ \cap D_+$. Now $E_+ \cap D_+$ consists of the points $\pi_+(n_1), \pi_+(n_2), \pi_+(n_3)$. They span the line $N_+ = \pi_+(N)$ considered above, which is therefore trisecant to D_+. Then the same argument used in (1) implies that $C_{o+} = E_+ \cup D_+$ is canonically embedded in $< T >$. (3) The proof is very similar to the previous ones: we omit it. □

Consider the line bundle $\tilde{L}_o := \mathcal{O}_{\tilde{C}_o}(H_1)$, we have $\tilde{L} \otimes i_o^* \tilde{L} \cong \omega_{\tilde{C}_o}(1)$. $\tilde{C}_o$ is not smooth, however all the conditions in the definition of good triple are satisified:

PROPOSITION 12.

1 $h^0(\tilde{L}_o) = 4$.

2 *$\tilde{L}_o$ is very ample.*

3 *The Petri map of $\tilde{L}_o$ is surjective.*

4 *$\tilde{T}$ is smooth.*

Proof. Statements 2), 3), 4) follow easily from the definitions or from the proposition. To show 1) we recall that $\tilde{L}_o$ is $O_{\tilde{C}_o}(H_1)$. Let $\rho : H^0(\tilde{L}_o) \to H^0(O_{\tilde{D}}(H_1))$ be the natural restriction map. If $\rho(s) = 0$ then s is zero on $\tilde{D}$. Since $DE = 6$ and $\tilde{L}$ has degree 3 on $\tilde{E}$, it follows $s = 0$. Hence ρ is injective and it suffices to show that $h^0(O_{\tilde{D}}(H_1)) = 4$. Since $\tilde{D}$ lies in $\tilde{Y}_o$ we can use the standard exact sequence

$$0 \to O_{\tilde{Y}_o}(H_1 - \tilde{D}) \to O_{\tilde{Y}_o}(H_1) \to O_{\tilde{D}}(H_1) \to 0.$$

Passing to the long exact sequence, it suffices to show that $h^0(O_{\tilde{Y}_o}(H_1 - \tilde{D})) = 0$ and $h^1(O_{\tilde{Y}_o}(H_1 - \tilde{D})) = 0$. The former vanishing follows from $H_1(H_1 - \tilde{D}) = -1$. Moreover note that $(D - H_1)^2 = -2$. Hence $D - H_1 \sim R$ where R is effective and $p_a(R) = 0$. Since $H_2R = 3$ it follows that R is connected and isolated. Hence $h^1(O_{\tilde{Y}_o}(R)) = 0$. □

As it happens for a good triple $(C, \eta, \tilde{L})$ we have for the curve $\tilde{C}_o$:

LEMMA 17. *$|I_{\tilde{C}_o/\tilde{T}}(2H_1 + H_2)|$ is 3-dimensional.*

Proof. Let $\rho : H^0(I_{\tilde{C}_o/\tilde{T}}(2H - 1 + H_2)) \to H^0(O_{\tilde{Y}_o}(2H_1 + H_2 - \tilde{D}))$ be the natural restriction map. It is clear that there exists exists a unique reducible surface of class $2H_1 + H_2$ containing $\tilde{Y}_o \cup \tilde{C}_o$. This is $\tilde{Y}_o \cup X$, where $X = P \times \mathbf{P}^3 \cap \tilde{T}$ and P is the plane considered above. Hence $dim\ Ker\ \rho = 1$. Moreover $|O_{\tilde{Y}_o}(2H_1 - H_2 - \tilde{D})|$ is base point free of self intersection 2: we omit the standard proof of this property. But then $dim\ Im\ \rho \leq h^0(O_{\tilde{Y}_o}(2H_1 + H_2 - \tilde{D})) = 3$. Hence $h^0(I_{\tilde{C}_o/\tilde{T}}(2H_1 + H_2)) \leq 4$. On the other hand the opposite inequality is immediately checked. This implies the statement. □

Assume $\tilde{\mathcal{U}} := \{\tilde{C}_u,\ u \in U\}$ is the family we require. By semicontinuity and the lemma, it is not restrictive to assume that $dim\ |I_{\tilde{C}_u/\tilde{T}}(2H_1 + H_2)| = 3$, $\forall u \in U$. Therefore we can choose a family $\tilde{\mathcal{S}} := \{\tilde{S}_u, u \in U\}$ of surfaces $\tilde{S}_u \in |I_{\tilde{C}_u/\tilde{T}}(2H_1 + H_2)|$ so that $\tilde{S}_o = \tilde{Y}_o \cup X$ as in the latter proof. Let $u \neq o$, then we have $\tilde{S}_u \cdot \iota^* \tilde{S}_u = \tilde{C}_u \cup \tilde{B}_u$ as above. Moreover $\tilde{Y}_u := \tilde{T} \cdot < \tilde{B}_u >$ is a symmetroid, defined by $(\tilde{C}_u, \tilde{S}_u)$.

PROPOSITION 13. *Let $\tilde{U}$ and $\tilde{Y}_u$ be as above. Then $\tilde{Y}_u$ is smooth and general for a general u.*

Proof. Let $\tilde{B}' := \{(u,x) \in U \times \tilde{T}\ /\ x \in \tilde{S}_u \cdot \iota^* \tilde{S}_u - \tilde{C}_u\}$. Putting $\tilde{B}'_u := \{u\} \times \tilde{T} \cdot \tilde{B}'$, we have $\tilde{B}'_u = \tilde{S}_u \cdot \iota^* \tilde{S}_u - \tilde{C}_u$. Let $\tilde{B}_u$ be its closure, we know that $\tilde{S}_u \cdot \iota^* \tilde{S}_u = \tilde{C}_u \cup \tilde{B}_u$ for $u \neq o$. Moreover $\tilde{Y}_u := < \tilde{B}_u > \cdot \tilde{T}$ is the symmetroid defined by $(\tilde{S}_u, \tilde{C}_u)$. For $u = o$ the construction yelds $\tilde{B}_o = \tilde{Y}_o$. We have constructed a family $\{\tilde{Y}_u,\ u \in U\}$ of symmetroids such that $\tilde{Y}_o$ is smooth and general. Then the same is true for a general u. $\square$

Finally we can conclude our program: assume that the canonical curve

$$C_{o+} = E_+ \cup D_+ \subset T - Sing\ T$$

is smoothable in T. Then there exists a smooth, connected variety U as above and a flat family $\mathcal{U} := \{C_u\ /\ u \in U\}$ such that $C_o = E_+ \cup D_+$ and, for $u \neq o$, C_u is a smooth, connected curve in $T - Sing\ T$ which is canonically embedded in $< T >$. Lifting this family by π_+ we obtain a flat family of curves

$$\tilde{\mathcal{U}} := \{\tilde{C}_u := \pi_+^* C_u,\ u \in U\}$$

such that $\tilde{C}_o = \tilde{E} \cup \tilde{D}$. By semicontinuity, a general $\tilde{C}_u$ satisfies the same properties proved for $\tilde{E} \cup \tilde{D}$ in the previous propositions 8.4 and 8.3. Hence the latter family is the family we aimed to have. Therefore, to prove theorem 8.1 and so to complete this paper, we are left to show that:

THEOREM 12. *$D_+ \cup E_+$ is smoothable in the Conte-Murre threefold T.*

Proof. For simplicity we put $C = C_o, D = D_+, E = E_+$. We denote the normal bundle of l$X \subset T - Sing\ T$ by N_X. Following [16] snd [24] it suffices to show that $h^1(N_C) = 0$ and that the natural map $H^0(N_C) \to T^1$ is surjective, where $T^1 := Coker\ (T_T \otimes O_C \to N_C)$. Applying theorem 4.5 of [16] and the identical proof used there for curves in $\mathbf{P}^3$, the following conditions are sufficient to deduce these properties: (i) $h^1(N_E) = h^1(N_D) = 0$, (ii) there exists a surjective map $H^0(M) \otimes O_D \to O_{Sing\ C}$ which factors through N_D, cfr. [16] proof of 4.5. To show (i) recall that D is contained in the hyperplane section $Y_+ := \pi_+(\tilde{Y}_o)$ of T and consider the standard exact sequence

$$0 \to O_D(D) \to N_D \to O_D(H_+) \to 0.$$

Since $O_D(D)$ is a Prym canonical line bundle, $h^1(O_D(D)) = 0$. The same is true for $O_D(H_+)$ because its degree is 9. Passing to the associated long exact sequence we obtain $h^1(N_D) = 0$. The same argument works for proving $h^1(N_E) = 0$. To prove (ii) we choose, among different possible ones, a proof highlighting the nice geometry of Reye congruences.

Finally $t := Sing\ C$ consists of three points. They are exactly the points on the trisecant line N_+ to the curve $D = \pi_+(\tilde{D})$ embedded in $\mathbf{P}^5 :=< Y_+ >$. Due to the previous exact sequence we have a morphism $\phi : H^0(O_D(D)) \otimes O_D \to O_S$, factoring through N_D. Note that $h^0(O_D(D)) = 3$. Hence ϕ is surjective if $h^0(O_D(D-t)) = 0$. This is indeed true: among different possible proofs we choose one highlighting again the nice geometry of Reye congruences.

Observe that $D \subset \mathbf{P}^5$ is a curve of genus 4 and degree 9 endowed with the trisecant line $N_+ =< t >$. This implies that $O_{D_+}(H_+ - t) \cong \omega_D$. On the other hand $\eta_D := O_D(K_{Y_+})$ is not trivial and $O_D(D) \cong \omega_D \otimes \eta_D$. Moreover the condition $h^0(O_D(D-t)) \geq 1$ is easily seen to be equivalent to $h^0(\eta_D(t)) \geq 1$.

Assume $h^0(\eta_D(t)) \geq 1$ and consider $t' \in |\eta_D(t)|$. Then we have $\mathcal{O}_D(H_-) = \mathcal{O}_D(H_+ + K_{Y_+}) \cong \omega_D(t')$. This means that, in the adjoint embedding $D_- \subset Y_- \subset \mathbf{P}^{5-}$, the image t'_- of t' spans a trisecant line to D_-. But Y_- is a Reye congruence model and it is well known that its only trisecant lines are trisecant to one of the 20 plane cubics of Y_-, [8, 11]. Such a curve is also embedded in Y_+ as a plane cubic A' and contains t'. It is also well known that, since A' is a plane cubic and D defines an Enriques linear system, then $DA \leq 3$. This follows because the classes of H_+, D, A' are well known in *Pic* Y_+, cfr. [5] IV 9. Hence we have $A' \cdot D = t'$. and $\mathcal{O}_D(t) \cong \mathcal{O}_D(A)$, where A is the unique element of $|A' + K_{Y_+}|$. It is standard to check that $h^0(\mathcal{O}_D(A)) = 1$ and deduce that $< t >$ is the unique trisecant line to D.

We have shown that the non surjetivity of $\phi : H^0(\mathcal{O}_D(D)) \otimes \mathcal{O}_D \to \mathcal{O}_t$ implies that $t = A \cdot D$, where A is a plane cubic. Now it is well known that the family $Trisec(Y_+)$ of trisecant lines to Y_+ is integral of dimension 3 and that the family of 0-dimensional schemes $\{l \cdot Y_+,\ l \in Trisec(Y_+)\}$ covers the surface Y_+, [8, 5]. Since $dim\ |D| = 3$ and D is chosen to be general in $|D|$, it is not restrictive to assume that no plane cubic of Y_+ contains t. Hence $h^0(\mathcal{O}_D(D - t)) = 0$ and ϕ is surjective. □

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AMS Subject Classification: Primary 14K10, Secondary 14H10, 14H40

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Lavoro pervenuto in redazione il 02.07.2013.

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dell'Università e del Politecnico di Torino

Volume 71, N. 3–4 (2013)

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ISSN 0373-1243

UNIVERSITÀ DEGLI STUDI DI TORINO
DIPARTIMENTO DI MATEMATICA