

J. Vives\*

## MALLIAVIN CALCULUS FOR LÉVY PROCESSES: A SURVEY

**Abstract.** Since Itô (1956) it is known that Lévy processes enjoy the chaotic representation property in a certain generalized form. In other words, the space of square integrable functionals of a certain independent random measure associated to a Lévy process has Fock space structure. The Fock space structure gives the possibility to develop a formal calculus where a gradient and a divergence operators, that are dual between them, are the main tools. On every space of random functionals with Fock space structure we can interpret probabilistically these operators and develop an stochastic calculus of Malliavin - Skorohod type. In this survey I present, first of all, a probabilistic interpretation of these operators in the case of functionals of a Lévy process. This interpretation generalizes the well-known interpretation for the standard Poisson process presented in Nualart and Vives (1990 and 1995) and, of course, the genuine Malliavin - Skorohod calculus for the Wiener process. As an application I obtain an anticipating Itô formula that extends both the usual adapted formula for Lévy processes and the anticipative version of the Itô formula on the Wiener space.

### 1. Introduction

This paper is a survey of Malliavin Calculus for Lévy processes since the point of view developed mainly in Solé, Utzet and Vives [15], that is strongly based on Itô [7], where the fact that square integrable functionals adapted to the filtration of a certain independent random measure associated to a Lévy process enjoy the chaotic representation property is proved. Of course, being Wiener process a particular example of Lévy process, Malliavin calculus for Lévy processes is an extension of Malliavin calculus for the Wiener process. Good references of Malliavin calculus for the Wiener process and for Gaussian processes in general are Sanz-Solé [13] and Nualart [8].

The fact that a process enjoys the chaotic representation property can be described also saying that the space of square integrable functionals has Fock space structure. This structure gives the possibility to develop a formal calculus where a gradient and a divergence operators (dual between them) are the main tools. On every space of random functionals with Fock space structure we can interpret probabilistically these operators and develop an stochastic calculus of Malliavin - Skorohod type. See Nualart-Vives [9] and Applebaum [5] for details.

In this paper, the probabilistic interpretation of these operators in the case of functionals of a Lévy process is presented following Solé, Utzet and Vives [15]. Previously, a canonical space for Lévy processes is constructed following the ideas developed by Neveu [11] for the standard Poisson case. This interpretation of the operators generalizes the interpretation given by Nualart and Vives in [9] and [10] for the standard Poisson case.

As an application I present an anticipating Itô formula, based on Alòs, León and Vives [1], that extends both the usual adapted formula for Lévy processes (see for

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example Cont and Tankov [6]) and the anticipative version of the Itô formula on the Wiener space developed in Alòs and Nualart [3]. Another recent application that can be found in Alòs, León, Pontier and Vives [2], is a Hull and White formula (pricing formula) for plain vanilla options based on a stochastic volatility jump diffusion price model. We have no space here to present this nice financial application.

Section 2 is devoted to Fock space structure. In section 3 we give the construction of the canonical space for a Lévy process. In section 4 we present the probabilistic interpretation of the operators. Finally, Section 5 is devoted to the anticipative Itô formula.

## 2. Formal calculus based on the Fock space structure

Let  $H$  be a real separable Hilbert space. For any  $n \geq 0$  we consider the tensor products  $H^{\otimes n}$ . Recall that  $H^{\otimes 0} = \mathbb{R}$  and  $H^{\otimes 1} = H$ . We define the Hilbert subspaces  $H^{\odot n} \subseteq H^{\otimes n}$  given by the symmetric elements with the scalar product

$$\langle f_n, g_n \rangle_{\odot n} := n! \langle f_n, g_n \rangle_{\otimes n}.$$

The Fock space associated to  $H$  is defined by the Hilbert space

$$\Phi(H) := \bigoplus_{n=0}^{\infty} H^{\odot n}$$

with the scalar product  $\langle f, g \rangle = \sum_{n=0}^{\infty} \langle f_n, g_n \rangle_{H^{\odot n}}$ , where  $f = \sum_{n=0}^{\infty} f_n$  and  $g = \sum_{n=0}^{\infty} g_n$ .

If  $(S, \mathcal{B}(S), \mu_S)$  is a certain measure space we can consider  $H = L^2(S)$ . In this case we have  $H^{\odot n} = L_s^2(S^n)$ , that is the space of  $n$ -dimensional and symmetric square integrable functions, with the modified scalar product. So, if  $F \in \Phi(H)$ , we have  $F = \sum_{n=0}^{\infty} f_n$  with  $f_n \in L_s^2(S^n)$ .

We define the gradient or annihilation operator  $D$  as an application that maps an element  $F \in \Phi(H)$  to an element  $DF \in \Phi(H) \times H \cong L^2(S, \Phi(H))$  such that

$$D_t F = \sum_{n=1}^{\infty} n f_n(\cdot, t), t - a.e.,$$

of course provided that  $DF \in L^2(S, \Phi(H))$ , that is equivalent to

$$\sum_{n=1}^{\infty} n n! \|f_n\|_{L^2(S^n)}^2 < \infty.$$

It is easy to see that this operator is densely defined and closed. Its domain is denoted by  $\text{Dom} D$ .

Let  $u \in L^2(S, \Phi(H))$ . Of course we have  $u_t = \sum_{n=0}^{\infty} u_n(t, \cdot)$ ,  $\mu_S - a.e.$  where  $u_n \in L^2(S^{n+1})$  is symmetric with respect to the  $n$  last variables. Denote by  $\tilde{u}_n$  be the symmetrization in all  $n+1$  variables. Then we define the divergence or creation operator of  $u$  by

$$\delta(u) = \sum_{n=0}^{\infty} \tilde{u}_n,$$

provided this series is in  $\Phi(H)$ , that is equivalent to assume

$$\sum_{n=0}^{\infty} (n+1)! \|\tilde{u}_n\|_{L^2(S^{n+1})}^2 < \infty.$$

We denote by  $\text{Dom}\delta$  its domain. This operator is also densely defined and closed.

Operators  $D$  and  $\delta$  are dual. Concretely we have that if  $F \in \text{Dom}D$  and  $u \in \text{Dom}\delta$  then

$$\langle u, DF \rangle_{L^2(S, \Phi(H))} = \langle F, \delta(u) \rangle_{\Phi(H)}.$$

This is the basis of a calculus on the Fock space, that we can name Malliavin-Skorohod calculus without probability, and that can be largely developed, obtaining abstract formulas such as a Clark-Ocone type one (see Nualart and Vives [9]).

### 3. Lévy processes

In all the paper  $X$  will be a Lévy process with triplet  $(\gamma, \sigma^2, \nu)$  where  $\gamma \in \mathbb{R}$ ,  $\sigma^2 > 0$  and  $\nu$  is a Lévy measure. Good references for Lévy processes are Sato [14] and Cont and Tankov [6]. Recall that Lévy processes can be usefully represented by the so called Lévy-Itô representation  $X_t = \gamma t + \sigma W_t + J_t$ , where  $W$  is the standard Wiener process and  $J$  is a *pure jump* Lévy process, independent of  $W$ , such that

$$J_t := \int_0^t \int_{\{|x|>1\}} x dN(s, x) + \lim_{\varepsilon \downarrow 0} \int_0^t \int_{\{\varepsilon < |x| \leq 1\}} x d\tilde{N}(s, x),$$

where  $N(B) = \#\{t : (t, \Delta X_t) \in B\}$ , for  $B \in \mathcal{B}((0, \infty) \times \mathbb{R}_0)$ , is the jump measure of the process,  $d\tilde{N}(t, x) := dN(t, x) - dt d\nu(x)$  is the compensated jump measure and the limit is *a.s.* uniform in  $t$  on every bounded interval. Recall also that for every  $t \geq 0$ ,  $\mathcal{F}_t^X = \mathcal{F}_t^W \vee \mathcal{F}_t^J$ .

From Itô [7], a Lévy process  $X$  can be associated to a centered and independent random measure  $M$  on  $\mathbb{R}_+ \times \mathbb{R}$ . We consider the continuous measure  $\mu(dt, dx) = \eta(dx)dt$ , where  $\eta(dx) := \sigma^2 \delta_0(dx) + x^2 \nu(dx)$ . More explicitly, we have, for any  $E \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ ,

$$\mu(E) = \sigma^2 \int_{E(0)} dt + \iint_{E'} x^2 d\nu(x)dt,$$

where  $E(0) = \{t \in \mathbb{R}_+ : (t, 0) \in E\}$  and  $E' = E - \{(t, 0) \in E\}$ . Then, for  $E \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$  with  $\mu(E) < \infty$ , we define the measure

$$M(dt, dx) = \sigma W(dt) \delta_0(dx) + x \tilde{N}(dt, dx),$$

that is,

$$M(E) = \sigma \int_{E(0)} dW_t + \iint_{E'} x d\tilde{N}(t,x),$$

and it is a centered independent random measure such that  $E[M(E_1)M(E_2)] = \mu(E_1 \cap E_2)$ , for  $E_1, E_2 \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$  with  $\mu(E_1) < \infty$  and  $\mu(E_2) < \infty$ .

Let  $S := [0, \infty) \times \mathbb{R}$  endowed with the Borel  $\sigma$ -algebra and the measure  $\mu$  defined above. Then we can consider

$$H^{\otimes n} = L_n^2 := L^2\left((\mathbb{R}_+ \times \mathbb{R})^n, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})^n, \mu^{\otimes n}\right).$$

For  $f_n \in L_n^2$ , following Itô [7], we can define a multiple stochastic integral  $I_n(f_n)$  with respect  $M$ , through the same steps as in the Wiener case, and prove that  $L^2(\Omega, \mathcal{F}^X)$  has Fock space structure, that is,

$$L^2(\Omega, \mathcal{F}^X) = \bigoplus_{n=0}^{\infty} I_n(L_n^2).$$

Then, we can represent any functional  $F \in L^2(\Omega, \mathcal{F}^X)$  via the expansion

$$F = \sum_{n=0}^{\infty} I_n(f_n), \quad f_n \in L_n^2.$$

This expansion is unique if we take every  $f_n$  symmetric.

This fact makes possible to apply the machinery of annihilation and creation operators in a Fock space as presented before.

If  $F \in L^2(\Omega)$ , with chaotic representation  $F = \sum_{n=0}^{\infty} I_n(f_n)$ , ( $f_n$  symmetric) and such that  $\sum_{n=1}^{\infty} n n! \|f_n\|_{L_n^2}^2 < \infty$ , we define its gradient as

$$D_z F = \sum_{n=1}^{\infty} n I_{n-1}\left(f_n(z, \cdot)\right), \quad z \in \mathbb{R}_+ \times \mathbb{R},$$

Recall that  $D_z F$  is an element of  $L^2(\mathbb{R}_+ \times \mathbb{R} \times \Omega, \mu \otimes \mathbb{P})$ .

In particular we can consider the two particular cases

$$D_{t,0} F = \sum_{n=1}^{\infty} n I_{n-1}\left(f_n((t,0), \cdot)\right), \quad t \in \mathbb{R}_+,$$

in  $L^2(\mathbb{R}_+ \times \Omega, dt \otimes \mathbb{P})$  and

$$D_{t,x} F = \sum_{n=1}^{\infty} n I_{n-1}\left(f_n((t,x), \cdot)\right), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}_0,$$

in  $L^2(\mathbb{R}_+ \times \mathbb{R}_0 \times \Omega, dt x^2 d\nu(x) \otimes \mathbb{P})$ .

If we define its domains analogously to previous cases and denote them by  $\text{Dom}D^0$  and  $\text{Dom}D^J$  respectively, we have that if  $\sigma > 0$  and  $\nu \neq 0$ ,  $\text{Dom}D = \text{Dom}D^0 \cap \text{Dom}D^J$ .

On other hand, let  $u \in L^2(\mathbb{R}_+ \times \mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}) \otimes \mathcal{F}^X, \mu \otimes \mathbb{P})$ . As before, we have the chaotic decomposition

$$u(t, x) = \sum_{n=0}^{\infty} I_n(u_n((t, x), \cdot))$$

where  $u_n \in L^2_{n+1}$  is symmetric in the  $n$  last variables. Then, if  $\tilde{u}_n$  denotes the symmetrization in all  $n + 1$  variables we have

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{u}_n),$$

in  $L^2(\Omega)$ , provided  $u \in \text{Dom} \delta$ , that means  $\sum_{n=0}^{\infty} (n + 1)! \|\tilde{u}_n\|_{L^2_{n+1}}^2 < \infty$ .

The duality property, in this case can be written in the following way: If  $u \in \text{Dom} \delta$  and  $F \in \text{Dom} D$  we have

$$E[\delta(u)F] = E \iint_{\mathbb{R}_+ \times \mathbb{R}} u(t, x) D_{t,x} F \mu(dt, dx).$$

#### 4. Probabilistic interpretation of gradient and divergence operators

##### 4.1. A canonical space for Lévy processes

The usual canonical Lévy process is built on the space of measurable functions from  $\mathbb{R}_+$  to  $\mathbb{R}$  or on the space of càdlàg functions, in both cases with the  $\sigma$ -field generated by the cylinders and using the Kolmogorov extension theorem. In order to have a probabilistic interpretation of the operator  $D$ , in Solé, Utzet and Vives [15] a different canonical Lévy process is constructed. This construction is an extension of the canonical Poisson process defined by Neveu [11] and is done in several steps. First of all we construct a canonical space for a compound Poisson process in a finite time interval, then we extend it to  $\mathbb{R}$  and after this, we construct the canonical space for a pure jump Lévy case. In fact, in this last case, the probability space is the set of all finite or infinite sequences of pairs  $(t_i, x_i)$  such that for every  $T > 0$ , there is only a finite number of  $t_i \leq T$ , including the empty sequence. Finally, for a general Lévy process we consider the canonical Wiener space  $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W, \{\bar{W}_t, t \geq 0\})$  and the canonical pure jump Lévy space  $(\Omega_J, \mathcal{F}_J, \mathbb{P}_J, \{\bar{J}_t, t \in \mathbb{R}_+\})$ . Then we define

$$(\Omega_W \times \Omega_J, \mathcal{F}_W \otimes \mathcal{F}_J, \mathbb{P}_W \otimes \mathbb{P}_J)$$

with  $W_t(\omega, \omega') := \bar{W}_t(\omega)$  and  $J_t(\omega, \omega') := \bar{J}_t(\omega')$ . The process  $X_t = \gamma t + \sigma W_t + J_t$  is the canonical Lévy process.

**4.2. Probabilistic interpretation of the operator  $D_{t,0}$**

We are going to see that  $D_{t,0}$  turns to be the derivative with respect to the Wiener part of  $X$  and that the usual rules of classical Malliavin Calculus apply

Recall that we have the isometry  $L^2(\Omega_W \times \Omega_J) \simeq L^2(\Omega_W; L^2(\Omega_J))$  and then we can apply the theory of Malliavin calculus for Hilbert space valued random variables as it is developed for example in Nualart [8].

Let be  $D^W$  the classical Malliavin derivative and denote by  $\text{Dom} D^W$  its domain. Given a real separable Hilbert space  $\mathcal{H}$ , we can extend this notion to  $\mathcal{H}$ -valued random variables. We write  $D^{W*}$  to denote the extended notion and  $\text{Dom} D^{W*}$  to denote its domain. In this case we have  $\text{Dom} D^{W*} \simeq \text{Dom} D^W \otimes \mathcal{H}$ . In the particular case of  $\mathcal{H} = L^2(\Omega')$ , for a certain probability space  $(\Omega', \mathcal{F}', \mathbb{P}')$ , such that  $L^2(\Omega')$  is separable, we have,

$$\text{Dom} D^{W*} \simeq \text{Dom} D^W \otimes L^2(\Omega') \simeq L^2(\Omega'; \text{Dom} D^W).$$

As a consequence, if  $F \in L^2(\Omega \times \Omega')$  such that for all  $\omega' \in \Omega'$ ,  $\mathbb{P}'$ -a.s.,  $F(\cdot, \omega') \in \text{Dom} D^W$ , then  $F \in \text{Dom} D^{W*}$  and

$$D_t^{W*} F(\omega, \omega') = D_t^W F(\cdot, \omega')(\omega), \ell \otimes \mathbb{P} \otimes \mathbb{P}' - \text{a.e.}$$

In our particular case we have  $L^2(\Omega') = L^2(\Omega_J)$ , which is a separable Hilbert space, and so  $L^2(\Omega_W \times \Omega_J) \simeq L^2(\Omega_W; L^2(\Omega_J))$ . Therefore we can compute both  $D_{t,0}F$  and  $D_t^{W*}F$ , and to obtain  $\text{Dom} D^{W*} \subset \text{Dom} D^0$ , and for  $F \in \text{Dom} D^{W*}$ , we have  $D_{t,0}F = \frac{1}{\sigma} D_t^{W*}F$ . This gives the probabilistic interpretation of  $D_{t,0}$ .

The most general chain rule is proved in Petrou [12]: If  $F = f(Z)$  with  $Z \in \text{Dom} D^{W*}$  and  $f$  in  $C_b^1(\mathbb{R})$ , then  $F \in \text{Dom} D^{W*}$  and  $D_t^{W*}F = f'(Z) D_t^{W*}Z$ .

**4.3. Probabilistic interpretation of  $D_{t,x}$  for  $x \neq 0$ .**

Consider now a pure jump Lévy process  $J$  with Lévy measure  $\nu$ . Given  $\omega \in \Omega^J$  and  $z = (t, x) \in \mathbb{R}_+ \times \mathbb{R}_0$ , we introduce in  $\omega$  a jump of size  $x$  at instant  $t$ , and call the new element  $\omega_z = ((t_1, x_1), (t_2, x_2), \dots, (t, x), \dots)$ .

For a  $\mathcal{F}^J$ -random variable  $F$ , we define the transformation  $(T_z F)(\omega) := F(\omega_z)$ , and the application  $TF: \mathbb{R}_+ \times \mathbb{R}_0 \times \Omega \rightarrow \mathbb{R}$ , that applies  $(z, \omega)$  to  $F(\omega_z)$  is  $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_0) \otimes \mathcal{F}^J$  measurable and if  $F = 0$ ,  $P$ -almost surely, then  $TF = 0$ ,  $\ell \otimes \nu \otimes P$  a.e.

Now we can define the increment quotient operator

$$\Psi_{t,x}F(\omega) := \frac{(T_{t,x}F)(\omega) - F(\omega)}{x}.$$

Thanks to the results given above,  $\Psi_{t,x}$  is a measurable operator from  $L^0(\Omega^J)$  to  $L^0(\mathbb{R}_+ \times \mathbb{R}_0 \times \Omega^J)$ . It is linear, closed and if  $F, G \in L^0(\Omega^J)$ ,

$$\Psi_{t,x}(FG) = G\Psi_{t,x}F + F\Psi_{t,x}G + x\Psi_{t,x}(F)\Psi_{t,x}(G).$$

Using the same ideas as in Nualart and Vives [10], given  $F \in L^2(\Omega^J)$ , we have

$$F \in \text{Dom}D^J \iff \Psi F \in L^2(\mathbb{R}_+ \times \mathbb{R}_0 \times \Omega^J),$$

and in this case  $D_{t,x}F = \Psi_{t,x}F$ ,  $\mu \otimes P$  – a.e. This gives the probabilistic interpretation of  $D_{t,x}$  for  $x \neq 0$ .

In the general case, given  $z = (t, x) \in \mathbb{R}_+ \times \mathbb{R}_0$ , for  $\omega = (\omega^W, \omega^J) \in \Omega_W \times \Omega_J$  define  $\omega_z = (\omega^W, \omega_z^J)$ , and for a random variable  $F \in L^0(\Omega_W \times \Omega_J)$  let  $(T_z^*F)(\omega) := F(\omega_z)$ . Define also the operator

$$\Psi_{t,x}^*F := \frac{F(\omega_{t,x}) - F(\omega)}{x}.$$

Then, for  $F \in L^2(\Omega)$  we have that  $F \in \text{Dom}D$  if and only if  $F \in \text{Dom}D^{W*}$  and  $\Psi^*F \in L^2(\Omega \times [0, \infty) \times \mathbb{R}_0)$ , and in this case,

$$D_{t,x}F = \mathbb{1}_{\{\sigma > 0\}} \mathbb{1}_{\{0\}}(x) \frac{1}{\sigma} D_t^{W*}F + \mathbb{1}_{\mathbb{R}_0}(x) \Psi_{t,x}^*F.$$

#### 4.4. Probabilistic interpretation of $\delta$

From now on, fix a finite time  $T > 0$  and consider the process  $\{X_t, t \in [0, T]\}$ . Consider the independent random measure  $M$  restricted to  $[0, T] \times \mathbb{R}$ . Assume also  $\int_{\mathbb{R}} x^2 d\nu(x) < \infty$ .

Following Applebaum [4], the random measure  $M$ , with the filtration  $\{\mathcal{F}_t^X, t \in [0, T]\}$ , induces a martingale-valued measure and allows to define an stochastic integral.

Let  $u$  be a predictable process such that  $E \iint_{[0, T] \times \mathbb{R}} u^2(z) \mu(dz) < \infty$ . We can define a stochastic integral  $\iint_{[0, T] \times \mathbb{R}} u(z) dM_z$  such that for  $u$  and  $v$  square integrable predictable processes we have

$$E \left[ \iint_{[0, T] \times \mathbb{R}} u dM \cdot \iint_{[0, T] \times \mathbb{R}} v dM \right] = E \left[ \iint_{[0, T] \times \mathbb{R}} uv d\mu \right].$$

An explicit expression for the integral  $\iint_{[0, T] \times \mathbb{R}} u(z) dM_z$  is given by

$$\iint_{[0, T] \times \mathbb{R}} u(z) dM_z = \sigma \int_0^T u(t, 0) dW_t + \iint_{[0, T] \times \mathbb{R}_0} xu(t, x) d\tilde{N}(t, x).$$

As in the Wiener case, the Skorohod integral restricted to predictable processes coincides with the integral with respect to the random measure  $M$ .

In fact, if  $\delta^0$  is the dual operator of  $D_{t,0}$  and  $\delta^J$  is the dual operator of  $D_{t,x}$  for  $x \neq 0$ , we have

$$\delta(u) = \delta^0(u_{\cdot, 0}) + \delta^J(u \mathbb{1}_{\mathbb{R}_0}(x)).$$

In particular  $\delta^0$  coincides with  $\sigma\delta^W$  and  $\delta^J$  coincides with the path by path integral with respect to  $x\tilde{N}$  over predictable processes.

Next result will play a key role in the application:

LEMMA 1. *Let  $F \in \text{Dom}D$  be a bounded random variable and  $u \in \text{Dom}\delta$  such that*

$$E \int_{[0,T] \times \mathbb{R}} (u(t,x)(F + xD_{t,x}F))^2 \mu(dt, dx) < \infty.$$

*Then  $u(t,x)(F + xD_{t,x}F) \in \text{Dom}\delta$  if and only if*

$$F\delta(u) - \int_{[0,T] \times \mathbb{R}} u(t,x)D_{t,x}F\mu(dt, dx) \in L^2(\Omega)$$

*and in this case  $\delta(Fu) = F\delta(u) - \delta(xuDF) - \int_{[0,T] \times \mathbb{R}} u(t,x)D_{t,x}F\mu(dt, dx)$ .*

#### 4.5. The space $\mathbb{L}^F$

To go further we need some structure into the space  $\text{Dom}\delta$ . We follow Alòs and Nualart [3]. It is known that  $L_a^2([0, T] \times \mathbb{R} \times \Omega)$ , the space of square integrable and adapted processes, is included in  $\text{Dom}\delta$ . So, we search for a Hilbert space included in the domain of  $\delta$  but that includes adapted and square integrable processes.

We define  $\mathbb{L}^{1,2,f}$  as the space of processes  $u \in L^2([0, T] \times \mathbb{R} \times \Omega)$  such that  $D_{s,x}u_{t,y}$  exists a.e. for  $s \geq t$ . and belongs to  $L^2([0, T] \times \mathbb{R})^2 \times \Omega$ . Observe that  $\mathbb{L}^{1,2,f}$  is a Hilbert space with the norm

$$\|u\|_{\mathbb{L}^{1,2,f}}^2 := \|u\|_{L^2([0,T] \times \mathbb{R} \times \Omega)}^2 + \|D_{s,x}u_{t,y}\mathbb{1}_{\{s \geq t\}}\|_{L^2([0,T] \times \mathbb{R})^2 \times \Omega}^2$$

and  $L_a^2([0, T] \times \mathbb{R} \times \Omega) \subseteq \mathbb{L}^{1,2,f} \subseteq L^2([0, T] \times \mathbb{R} \times \Omega)$ .

Then we consider the space  $\mathbb{L}^F$  that it is defined in the following way:  $u \in \mathbb{L}^F$  if and only if  $u \in L^{1,2,f}$  and  $D_{r,w}D_{s,x}u_{t,y}$  exists a.e. for  $r \vee s \geq t$  and belongs to  $L^2([0, T]^3 \times \mathbb{R}^3 \times \Omega)$ .  $\mathbb{L}^F$  is a Hilbert space with the norm

$$\|u\|_{\mathbb{L}^F}^2 := \|u\|_{\mathbb{L}^{1,2,f}}^2 + \|D_{r,w}D_{s,x}u_{t,y}\mathbb{1}_{\{r \vee s \geq t\}}\|_{L^2([0,T] \times \mathbb{R})^3 \times \Omega}^2.$$

and  $L_a^2([0, T] \times \mathbb{R} \times \Omega) \subseteq \mathbb{L}^F \subseteq \mathbb{L}^{1,2,f} \cap \text{Dom}\delta \subseteq L^2([0, T] \times \mathbb{R} \times \Omega)$ . Moreover,

$$\mathbb{E}(\delta(u)^2) \leq 2\|u\|_{\mathbb{L}^F}^2.$$

Observe that this inequality allow to control convergence of  $\delta(u)$  by convergence with respect the norm of  $\mathbb{L}^F$  and apply, when necessary, dominated convergence theorem.

## 5. An anticipating Itô formula

In Alòs, León and Vives [1] we use the techniques presented before to obtain an anticipative version of the Itô formula for Lévy processes, where the coefficients are assumed to be in  $\mathbb{L}^F$ . Our Itô formula is not only an extension of the usual adapted formula for Lévy processes, but also an extension of the anticipative version of the Itô formula on the Wiener space, obtained by Alòs and Nualart (2008).

Consider the semimartingale

$$\begin{aligned} X_t &= X_0 + \int_0^t u_s dW_s + \int_0^t v_s ds \\ &+ \int_0^t \int_{|y|>1} z_1(s-, y) y N(ds, dy) + \int_0^t \int_{|y|\leq 1} z_2(s-, y) y \tilde{N}(ds, dy) \end{aligned}$$

where  $u$  and  $z_2(s-, y)y$  are adapted and have  $L^2$  trajectories a.s. and  $v$  is adapted and has  $L^1$  trajectories a.s. This is in fact a generalization of a generic Lévy process.

In this case (see Cont and Tankov [6] for example) it is well known that

$$\begin{aligned} F(X_t) &= F(X_0) + \int_0^t F'(X_{s-}) u_s dW_s \\ &+ \int_0^t F'(X_{s-}) v_s ds + \frac{1}{2} \int_0^t F''(X_{s-}) u_s^2 ds \\ &+ \int_0^t \int_{|y|>1} [F(X_s) - F(X_{s-})] N(ds, dy) \\ &+ \int_0^t \int_{|y|\leq 1} [F(X_s) - F(X_{s-}) - F'(X_{s-}) z_2(s-, y) y] N(ds, dy) \\ &+ \int_0^t \int_{|y|\leq 1} F'(X_{s-}) z_2(s-, y) y \tilde{N}(ds, dy). \end{aligned}$$

Our purpose is to obtain an analogous formula changing Itô stochastic integrals by Skorohod versions, that is, an anticipative version of this formula. Recall that if  $u$ ,  $v$ ,  $z_1$  and  $z_2$  are anticipating processes, the Itô integral with respect to  $W$  is not defined, so we need the Skorohod extension. Moreover, the integrals with respect to  $\tilde{N}$  are well defined path by path, but they are not zero expectation integrals, so we are also interested in an Skorohod type version for this case. Coefficients will be assumed to be in the domain of the gradient operator in the future sense. So, this application includes also the Lévy extension of the corresponding domains in the Wiener case as presented in Alòs and Nualart [3].

We introduce the space  $\mathbb{L}_-^{1,2,f}$ . A random field  $u = \{u(s, y) : (s, y) \in [0, T] \times \mathbb{R}\}$  in  $\mathbb{L}^{1,2,f}$  belongs to the space  $\mathbb{L}_-^{1,2,f}$  if there exists  $D^-u$  in  $L^2(\Omega \times [0, T] \times \mathbb{R})$  such that

$$\int_0^T \int_{\mathbb{R}} \sup_{(s-\frac{1}{n}) \vee 0 \leq r < s, y \leq x \leq y + \frac{1}{n}} E[|D_{s,y} u(r, x) - D^-u(s, y)|^2] \mu(ds, dy)$$

converges to zero as  $n$  goes to infinity.

We need also to precise the relationship between Skorohod and path by path integrals. Let  $z = \{z(s, x) : (s, x) \in [0, T] \times \mathbb{R}\}$  be a measurable random field such that:

- If  $s_n \uparrow s$  in  $[0, T]$  and  $y_m \rightarrow y$ ,  $y \neq 0$ , the limit  $z(s-, y) = \lim_{n, m \rightarrow \infty} z(s_n, y_m)$  is well-defined and belongs to  $\mathbb{L}_-^{1,2,f}$ .
- The random fields  $z(s-, y)$  and  $yD^-z(s-, y)$  belongs to  $\mathbb{L}^F$ .
- The random field  $z(s-, y)y$  is pathwise integrable with respect to  $\tilde{N}$ .

Then we have that for any interval  $(a, b]$  or  $(a, \infty)$  in  $(0, \infty)$ ,

$$\begin{aligned} & \int_0^t \int_{\{a < |y| \leq b\}} z(s-, y) y \tilde{N}(ds, dy) \\ &= \delta((z(s-, y) + yD^-z(s-, y)) \mathbb{1}_{\{a < |y| \leq b\}} \mathbb{1}_{[0, t]}(s)) \\ &+ \int_0^t \int_{\{a < |y| \leq b\}} D^-z(s-, y) \mu(ds, dy), \quad t \in [0, T]. \end{aligned}$$

Finally, consider the process

$$\begin{aligned} X_t &= X_0 + \delta^W(u \mathbb{1}_{[0, t]}) + \int_0^t v_s ds + \int_0^t \int_{\{|x| > 1\}} z_1(s-, x) x N(ds, dx) \\ &+ \int_0^t \int_{\{0 < |x| \leq 1\}} z_2(s-, x) x \tilde{N}(ds, dx), \quad t \in [0, T]. \end{aligned}$$

with the hypotheses

- $X_0 \in \text{Dom}D$ .
- $u \in \mathbb{L}^F$ ,  $\delta^W(u \mathbb{1}_{[0, t]})$  has continuous paths and  $\int_0^T u_s^2 ds$  is a.s. bounded by a constant.
- $v \in \mathbb{L}^{1,2,f}$  and  $\int_0^T v_s^2 ds$  is a.s. bounded by a constant.
- $z_1$  and  $z_2$  are bounded and satisfies the conditions of Theorem 1 on  $(1, \infty)$  and  $(0, 1]$  respectively. Moreover,  $D^-z_2 \in \mathbb{L}^{1,2,f}$

Then, if  $F \in C^2(\mathbb{R})$ , we have that

$$F'(X_{s-})(u_s \mathbb{1}_{\{y=0\}} + z_2(s-, y) \mathbb{1}_{\{0 < |y| \leq 1\}}) \mathbb{1}_{[0, t]}(s)$$

and

$$D^-(z_2(s-, y)F'(X_{s-}))(s, y)y\mathbb{1}_{\{0 < |y| \leq 1\}}\mathbb{1}_{[0, t]}(s)$$

belong to  $Dom\delta$  and

$$\begin{aligned} & F(X_t) - F(X_0) \\ &= \delta((F'(X_{s-})(u_s\mathbb{1}_{\{y=0\}} + z_2(s-, y)\mathbb{1}_{\{0 < |y| \leq 1\}})) \\ &+ y\mathbb{1}_{\{0 < |y| \leq 1\}}D_{(s, y)}^-(z_2(s-, y)F'(X_{s-})))\mathbb{1}_{[0, t]}(s) \\ &+ \frac{1}{2}\int_0^t F''(X_s)u_s^2 ds + \int_0^t F'(X_s)v_s ds + \int_0^t F''(X_s)D_{(s, 0)}^-X_s u_s ds \\ &+ \int_0^t \int_{\{0 < |y| \leq 1\}} D_{(s, y)}^-F'(X_{s-})z_2(s, y)\mu(ds, dy) \\ &+ \int_0^t \int_{\{0 < |y| \leq 1\}} [F(X_s) - F(X_{s-}) - F'(X_{s-})z_2(s-, y)y]N(ds, dy) \\ &+ \int_0^t \int_{\{|y| > 1\}} (F(X_s) - F(X_{s-}))N(ds, dy). \end{aligned}$$

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Josep VIVES,  
Departament de Probabilitat, Lògica i Estadística, Universitat de Barcelona  
Gran Via 585, 08007 Barcelona (Catalunya), SPAIN  
e-mail: josep.vives@ub.edu

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