

Ya. Belopolskaya * and W.A. Woyczynski

SDES, FBSDES AND FULLY NONLINEAR PARABOLIC SYSTEMS

Abstract. In this article we describe two probabilistic approaches to construction of the Cauchy problem solution for a class of nonlinear parabolic systems. Namely, we describe probabilistic models associated with classical and viscosity solutions and use them to state conditions on the problem data that ensure the existence and uniqueness of the required solution of the PDE system.

Introduction

Systems of nonlinear second order parabolic equations appear in various fields of control theory, differential geometry, financial mathematics and others. Here we consider a class of nonlinear PDEs of the form

$$(1) \quad \frac{\partial u_l}{\partial s} + [\mathcal{B}u]_l + g = 0, \quad u_l(T, x) = u_{0l}(x), \quad l = 1, \dots, d_1, \text{ where}$$

$$[\mathcal{B}u]_l = a_i \nabla_i u_l + \frac{1}{2} \text{Tr} A^* \nabla^2 u_l A + B_{lm}^i \nabla_i u_m + c_{lm} u_m$$

and all coefficients a, A, B, c and a scalar function g depend on $x, u, \nabla u$ and $\nabla^2 u^l$. A is an invertible operator and $*$ denotes the transposition.

Here and below we assume a common convention about summation over all repeating indices if the contrary is not mentioned.

We call a system (1) semilinear when a, A, B, c and g depend on x, u , quasilinear when these parameters depend on $x, u, \nabla u$ and fully nonlinear when they depend on $x, u, \nabla u$ and $\nabla^2 u$.

A construction of a stochastic problem associated with (1) strictly depends on our understanding of a solution to a system, namely, on our intention to construct a strong, weak or viscosity solution. In this paper we give some new results concerning strong and viscosity solutions of (1) constructed via stochastic approaches.

1. Probabilistic approach to a strong solution of the Cauchy problem for a PDE system

Let (Ω, \mathcal{F}, P) be a probability space, $w(t) \in \mathbb{R}^d$ be a Wiener process defined on it and \mathcal{F}_t be a flow generated by $w(t)$.

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To construct a strong solution to (1) in a semilinear case we consider a stochastic problem of the form

$$(2) \quad d\xi(t) = a(\xi(t), u(t, \xi(t)))dt + A(\xi(t), u(t, \xi(t)))dw(t), \quad \xi(s) = x \in \mathbb{R}^d,$$

$$(3) \quad d\eta(t) = c(\xi(t), u(t, \xi(t)))\eta(t)dt + C(\xi(t), u(t, \xi(t)))\eta(t), dw(t), \quad \eta(s) = h \in \mathbb{R}^{d_1},$$

$$(4) \quad \langle h, u(s, x) \rangle = E[\langle \eta(T), u_0(\xi(T)) \rangle + \int_s^T \langle \eta(\theta), g(\xi(\theta), u(\theta, \xi(\theta))) \rangle],$$

where $B_{im}^l = C_{km}^l A_i^k$ and $\langle h, u \rangle = \sum_{k=1}^{d_1} h_k u_k$ denotes the inner product in \mathbb{R}^{d_1} .

Actually, we can set $\gamma(t) = (\xi(t), \eta(t))$ and present (2),(3) in the form

$$(5) \quad d\gamma(t) = n_u(\gamma(t))dt + N_u(\gamma(t))dW(t), \quad \gamma(s) = \gamma,$$

where $W(t) = (w(t), w(t))^*$, $n_u(\gamma(t)) = n(\gamma(t), u(t, \gamma(t)))$, $N_u(\gamma(t)) = N(\gamma(t), u(t, \gamma(t)))$, and

$$N_u(x, h) = \begin{pmatrix} A(x, u) & 0 \\ 0 & C(x, u)h \end{pmatrix}, \quad n_u(x, h) = \begin{pmatrix} a(x, u) & 0 \\ 0 & c(x, u)h \end{pmatrix}.$$

Let $X = \mathbb{R}^d$, $Y = \mathbb{R}^{d_1}$, $M^d = \mathbb{R}^d \otimes \mathbb{R}^d$, $a(x, u) \in \mathbb{R}^d$, $A(x, u) \in M^d$, $c(x, u) \in M^{d_1}$, $C(x, u) \in M^{d_1} \otimes \mathbb{R}^d$ provided $x \in X$, $u \in Y$. We say $\mathbf{C}^{1,k}$ holds if

- i) a, A have sublinear growth in $x \in \mathbb{R}^d$, c, C, g and u_0 are bounded in x and all of them but u_0 have polynomial growth in $u \in \mathbb{R}^{d_1}$;
- ii) a, A, c, C and u_0, g are C^k -smooth in all arguments in correspondent norms.

THEOREM 1. *Assume $\mathbf{C}^{1,1}$ holds. Then there exists a unique solution of (2)-(4).*

Let $\mathbf{C}^{1,k}$ hold, $k \geq 1$. Along with (2),(3) we consider

$$(6) \quad d\zeta(t) = \nabla a_u(\xi(t))\zeta(t)dt + \nabla A_u(\xi(t))(\zeta(t), dw(t)), \quad \zeta(s) = I,$$

$$(7) \quad d\kappa(t) = c_u(\xi(t))\kappa(t)dt + C_u(\xi(t))(\kappa(t), dw(t)) + \nabla c_u(\xi(t))(\zeta(t), \eta(t))dt + \nabla C_u(\xi(t))(\zeta(t), \eta(t), dw(t)), \quad \kappa(s) = 0,$$

$$(8) \quad \langle h, \nabla_y u(s, x) \rangle = E[\langle \eta(T), \nabla_{\zeta(T)} u_0(\xi(T)) \rangle + \langle \kappa(T), u_0(\xi(T)) \rangle | \mathcal{F}_s] + E \left[\int_s^T [\langle \eta(\theta), \nabla_{\zeta(\theta)} g(\xi_{s,x}(\theta), u(\theta, \xi_{s,x}(\theta))) \rangle + \langle \kappa(\theta), g(\xi(\theta), u(\theta, \xi(\theta))) \rangle] d\theta | \mathcal{F}_s \right].$$

THEOREM 2. *Let $\mathbf{C}^{1,3}$ hold. Then there exists an interval $[T_1, T]$ on which the Cauchy problem (1) has a unique solution $u(s, x)$ which admits a probabilistic representation of the form (4).*

In addition, if $\mathcal{N}^{1,2}$ is the set of scalar functions of the form

$$\Psi(s, x, h) = \langle h, u(s, x) \rangle$$

defined on $[0, T] \times X \times Y$, differentiable in $s \in [0, T]$, and twice differentiable in $z \in X \times Y$, then a solution to (2) belongs to $\mathcal{N}^{1,2}$ if $\Psi_0 \in \mathcal{N}^{1,2}$.

Provided $u(s, x)$ is a classical solution of (1), we can check that $\Phi(s, x, h) = \langle h, u(s, x) \rangle$ given by (4) satisfies the scalar Cauchy problem,

$$(9) \quad \frac{\partial \Phi}{\partial s} + \langle n(\gamma), \nabla \Phi \rangle + \frac{1}{2} Tr N^*(\gamma) \nabla^2 \Phi N(\gamma) + G(\gamma, u) = 0, \quad \Phi(T, \gamma) = \langle h, u_0(x) \rangle$$

w.r.t. $\Phi(t, \gamma) = \langle h, u(s, x) \rangle, \gamma = (x, h)$, where $G = \langle h, g \rangle$.

Moreover, we can show that systems (2)-(4), (6)-(8), and (2)-(4) have a similar structure. To this end we set $\Theta = Y \oplus X \diamond Y$, and let $\gamma = (\gamma_1, \gamma_2) \in \Theta$ have the form $\gamma_1 = \kappa, \gamma_2 = y \diamond h$. Then one can treat (2)-(4), (6)-(8) as a system that consists of (2) and

$$(10) \quad d\lambda(t) = m(\xi(t))\lambda(t)dt + M(\xi(t))(\lambda(t), d\tilde{W}(t)), \quad \lambda(s) = \lambda \in \Theta.$$

Here $\tilde{W}(t) = (w(t), w(t) \diamond w(t))^*$ and the coefficients have the form

$$m(x) \begin{pmatrix} \kappa \\ \xi \diamond \eta \end{pmatrix} = \begin{pmatrix} c(x) & \hat{\nabla}c(x) \\ 0 & \nabla a(x) \oplus c(x) \end{pmatrix} \begin{pmatrix} \kappa \\ \xi \diamond \eta \end{pmatrix},$$

$$M(x) \begin{pmatrix} \kappa \\ \xi \diamond \eta \end{pmatrix} = \begin{pmatrix} C(x) & \hat{\nabla}C(x) \\ 0 & \nabla \mathcal{M}(x) \oplus C(x) \end{pmatrix} \begin{pmatrix} \kappa \\ \xi \diamond \eta \end{pmatrix},$$

where $\hat{\nabla}c(x)\xi \diamond \eta = \nabla c(x)(\xi, \eta)$, $\hat{\nabla}C(x)\xi \diamond \eta = \nabla C(x)(\xi, \eta)$.

Along with (1) we will consider the Cauchy problem

$$(11) \quad \frac{\partial v_i^l}{\partial s} + [\mathcal{B}(x)v_i^l] + [[\nabla_i \mathcal{B}]u^l] + \nabla g^l(x, u) = 0, \quad v_i^l(T, x) = \nabla_i u_0^l(x)$$

w.r.t. $v_i^l = \nabla_i u^l$, where $[[\nabla_i \mathcal{B}]u^l] = v_k^l \nabla_i a^k(x) + Tr \nabla_i A(x) \nabla v^l A(x) + \nabla_i B_m^{lk} v_k^m + \nabla_i c_m^l u^m$. Here $\nabla_i g(x, u) = g_i^1(x, u) + g_m^2(x, u) \nabla_i u^m$ and, given $\alpha = (\alpha_1, \dots, \alpha_k)$, we use notation $g_m^j(\alpha) = \frac{\partial g_m(\alpha)}{\partial \alpha_j}$, $j = 1, \dots, k$. Actually, the system (1),(11) is a semilinear system w.r.t. $V(t, x) = (u(t, x), \nabla u(t, x))$. This together with (10) allows us to apply the above theorems to construct solutions both to SDEs (2), (10) and to the Cauchy problem (1), (11).

Note that another useful way to view (2),(3), (6),(7) is to consider them as an SDE system w.r.t. components of a process $\beta(t) = (\chi(t), v(t))$, where the processes $\chi(t)$ and $v(t)$ satisfy SDEs

$$(12) \quad d\chi(t) = b(\chi(t))dt + B(\chi(t))dW(t), \quad \chi(s) = \chi = (x, y) \in H_1,$$

$$(13) \quad dv(t) = q(\chi(t))v(t)dt + Q(\chi(t))v(t)dW(t), \quad v(s) = v = (0, h) \in H_2,$$

where

$$q(\chi) \begin{pmatrix} \kappa \\ h \end{pmatrix} = \begin{pmatrix} c(x)\kappa \\ \nabla_y c(x)h \end{pmatrix}, \quad Q(\chi) \begin{pmatrix} \kappa \\ h \end{pmatrix} = \begin{pmatrix} C(x)\kappa \\ \nabla_y C(x)h \end{pmatrix},$$

and $b(\chi) = (a(x), \nabla_y A)^*$, $B(\chi) = (A(x), \nabla_y A(x))^*$.

All the above constructions can be extended to the case when coefficients a , A , c , C , and the function g , depend on $x, u, \nabla u$, and even $\nabla^2 u$. This allows us to include a quasilinear or fully nonlinear system of the form (1) into a semilinear system with a similar structure w.r.t. a function $U = (u, \nabla u, \nabla^2 u)$ or $U = (u, \nabla u, \nabla^2 u, \nabla^3 u)$, respectively, and to prove the existence and uniqueness of its solution on a small interval $[\tau, T]$ depending on coefficients and functions u_0 and g , when they satisfy $\mathbf{C}^{1,k}$ with $k = 5$ or $k = 6$. One can see the detailed proof of the above results in [4], [5].

2. Probabilistic approach to a viscosity solution of the Cauchy problem for a nonlinear PDE system

In this section we construct a viscosity solution of a fully nonlinear version of the Cauchy problem (1) based on the BSDE theory developed in [6], [7] in combination with the constructions described in the previous section.

To be more precise we first develop a modification of the approach of [7] that allows us to construct a viscosity solution of a system of quasilinear parabolic equations of the form (1) with coefficients depending on $x, u, \nabla u$ and $g = g(x, u, A\nabla u)$, and then apply a differential prolongation procedure to a system of fully nonlinear parabolic equations to include it into a system of quasilinear parabolic equations. This makes it possible for us to apply the BSDE technique to construct a viscosity solution to a system of fully nonlinear parabolic equations. The details of the corresponding construction can be found in [4], [5].

Let us consider the Cauchy problem (1) in a larger system consisting of (1) and

$$(14) \quad \frac{\partial v_i^l}{\partial s} + [\mathcal{B}(x)v]_i^l + [[\nabla_i \mathcal{B}]u]^l + \nabla g^l(x, u, A\nabla u, \nabla^2 u_l) = 0, \quad v_i^l(T, x) = \nabla_i u_0^l(x)$$

w.r.t. $v_i^l = \nabla_i u^l$, where $[[\nabla_i \mathcal{B}]u]_l = v_k^l \nabla_i a^k + Tr \nabla_i A \nabla v_l A + \nabla_i B_m^{lk} v_{mk} + \nabla_i c_m^l u_m$,

$$\begin{aligned} \nabla_i g(x, u, A\nabla u, \nabla^2 u_l) &= g_i^1(x, u, A\nabla u, \nabla^2 u_l) + g_m^2(x, u, A\nabla u, \nabla^2 u_l) \nabla_i u_m + \\ &g_{jm}^3(x, u, A\nabla u, \nabla^2 u_l) \nabla_i (A\nabla u)_{jm} + g_{lkj}^4(x, u, \nabla u, \nabla^2 u_l) \nabla_k^2 u_{lj}. \end{aligned}$$

At this point we need to examine a fully coupled system of forward-backward SDEs (FBSDEs) associated with (1), (14), state conditions on their coefficients and functions g and u_0 to ensure the existence and uniqueness of a solution to the resulting FBSDE system and, finally, check that our results lead to construction of a viscosity solution of (1).

Let $V(s, x, y) = (u(s, x), p(s, x, y))$, $p(s, x, y) = \langle y, \nabla u(s, x) \rangle$. Then (1), (14) may be rewritten as

$$(15) \quad \frac{\partial V_m}{\partial s} + \mathcal{G}V_m + \hat{C}_{lm}^i \mathcal{M}_i^k \nabla_k V_l + \hat{c}_{lm} V_l + G_m(x, y, V, \nabla V) = 0, \quad \text{where}$$

$$(16) \quad \mathcal{G}V_m = \frac{1}{2} \text{Tr} \mathcal{M}^*(x, V, \nabla V) \nabla^2 V_m \mathcal{M}(x, V, \nabla V) + \langle m(x, V, \nabla V), \nabla V_m \rangle,$$

$$m = \begin{pmatrix} a \\ a \end{pmatrix}, \quad \frac{1}{2} [\mathcal{M}^* \mathcal{M}]_{jk} = \begin{pmatrix} A_{jk} + \frac{\partial g}{\partial q_{jk}} & 0 \\ 0 & A_{jk} + \frac{\partial g}{\partial q_{jk}} \end{pmatrix},$$

$$G(x, y, V, \nabla V_l) = \begin{pmatrix} g(x, u, p, \nabla p_l) \\ g^1(x, y, u, p, \nabla p^l) \end{pmatrix},$$

and \hat{C} , \hat{c} depend on C, c, a, A and their derivatives.

Assume that **C**^{1,2} holds. Then (1) and (15) have similar structures.

Consider the Cauchy problem for (15) with the Cauchy data

$$(17) \quad V(T, x, y) = V_0(x, y) = (u_0(x), \nabla_y u_0(x)).$$

Set $H_1 = X \times X$, $H_2 = Y \times Y$, $H_3 = M \times M_X$ and $\chi(t) = (\xi(t), \zeta(t)) \in H_1$, $\beta(t) = (\eta(t), \kappa(t)) \in H_2$, $Y(t) = (y(t), p(t)) \in H_2$, $Z(t) = (p(t), q(t)) \in H_3$.

Assume that $\lambda(t)$ satisfies an equation of the form (10) associated with (15), (17). Consider a stochastic process $\Pi(t) = \Phi(t, \beta(t))$, where

$$\Phi(t, \beta(t)) = \Phi_1^h(t, \xi(t)) + \Phi_2^h(t, \chi(t)) =$$

$$[\langle \eta(t), u_0(\xi(t)) \rangle] + [\langle \kappa(t), u_0(\xi(t)) \rangle + \langle \eta(t), \nabla_{\zeta(t)} u_0(\xi(t)) \rangle]$$

and notice that $\Phi_2^h(t, \chi(t))$ is linear in h and y . From Ito's formula, and (12), we deduce that the stochastic differential of the process $\tilde{Y}(t) = \Phi_2^h(t, \chi(t)) = \langle v(t), V(t, \chi(t)) \rangle$ has the form

$$(18) \quad d\tilde{Y}(t) = -\tilde{G}(\chi(t), V(t, \chi(t)), \nabla V(t, \chi(t)) \zeta(t)) dt + \langle \nabla \Phi_2^h(t, \chi(t)), \mathcal{M}(\chi(t)) dW(t) \rangle,$$

where $G(\chi, V, \nabla V) = (g(x, u, \nabla u, \nabla^2 u), \nabla_y g(x, u, \nabla u, \nabla^2 u))$,

$$\tilde{G}(\chi(t), V(t, \chi(t)), \nabla_{\zeta(t)} V(t, \chi(t))) = \langle \beta, \Xi^*(s, t) G(\chi(t), V(t, \chi(t)), \nabla_{\zeta(t)} V(t, \chi(t))) \rangle,$$

and

$$(19) \quad \langle \nabla \Phi_2(t, \chi(t)), \mathcal{M}(t, \chi(t)) dW(t) \rangle = \langle \nabla C(\xi(t))(\zeta(t), \eta(t), dw(t)), u(t, \xi(t)) \rangle + \langle C(\xi(t))(\kappa(t), dw(t)), u(t, \xi(t)) \rangle + \langle \eta(t), \nabla u(t, \xi(t)) \nabla \mathcal{M}(\xi(t))(\zeta(t), dw(t)) \rangle.$$

We deduce from (18), (19) that the process $\tilde{Y}(t)$ satisfies

$$(20) \quad d\tilde{Y}(t) = -\tilde{G}(\chi(t), \tilde{Y}(t), \tilde{Z}(t))dt + \langle \tilde{Z}(t), dW(t) \rangle, \quad \tilde{Y}(T) = \langle \beta(T), V_0(\chi(T)) \rangle,$$

and the processes $Y(t) = (Y^1(t), Y^2(t)), Z(t) = (Z^1(t), Z^2(t))$ defined by

$$\begin{aligned} \tilde{Y}(t) = \langle \beta(t), Y(t) \rangle &= \langle \beta, \Xi^*(s, t)Y(t) \rangle = \langle \kappa(t), Y^1(t) \rangle + \langle \eta(t), Y^2(t) \rangle = \\ &= \langle \kappa, \Xi_1^*(s, t)Y_1(t) \rangle + \langle h, \Xi_2^*(s, t)Y_2(t) \rangle, \quad \tilde{Z}(t) = \langle \beta, Z(t) \rangle \end{aligned}$$

satisfy the BSDE

$$(21) \quad dY(t) = -G(\chi(t), Y(t), Z(t))dt + ZdW(t), \quad Y(T) = \Xi^*(s, T)V_0(\chi(T)).$$

Finally, we deduce that one can associate with (15), (17) the following FBSDEs w.r.t. \mathcal{F}_t -measurable stochastic processes $\chi(t) = (\xi(t), \zeta(t)) \in H_1, Y(t) = (y(t), p(t)) \in H_2, Z(t) = (p(t), q(t)) \in H_3 = M \times M_X,$

$$(22) \quad d\chi(t) = b(\chi(t), Y(t), Z(t))dt + B(\chi(t), Y(t), Z(t))dW(t), \quad \chi(s) = \chi \in H_1,$$

$$(23) \quad dY(t) = -G(\chi(t), Y(t), Z(t))dt + Z(t)dW, \quad \alpha = Y(T) = (\alpha_1, \alpha_2) \in H_2.$$

Here $b = (a, \nabla A), B = (A, \nabla A), G = (g, g^1) \in H_2,$ and $Y(T)$ is \mathcal{F}_T -measurable.

Let $\mathcal{M}^2([0, T]; \mathbb{R}^d)$ denote the set of progressively measurable square integrable stochastic processes $\xi(t) \in \mathbb{R}^d, E \left[\int_0^T \|\xi(\tau)\|^2 d\tau \right] < \infty,$ and $\mathcal{S}^2([0, T], X)$ denote the set of semimartingales $\eta(t) \in \mathbb{R}^d,$ such that $E \left[\sup_{0 \leq t \leq T} \|\eta(t)\|^2 \right] < \infty.$

A solution to FBSDE (22),(23) is a triple of progressively measurable processes $(\chi(t), Y(t), Z(t))$ in $\mathcal{S}^2([0, T]; H_1) \times \mathcal{S}^2([0, T]; H_2) \times \mathcal{M}^2([0, T]; H_3)$ such that

$$(24) \quad \chi(t) = \chi + \int_s^t b(\chi(\tau), Y(\tau), Z(\tau))d\tau + \int_s^t B(\chi(\tau), Y(\tau), Z(\tau))dW(\tau),$$

and

$$(25) \quad Y(t) = \alpha + \int_s^t G(\chi(\tau), Y(\tau), Z(\tau))d\tau - \int_s^t Z(\tau)dW(\tau), \quad 0 \leq t \leq T,$$

with probability 1.

Now we are in the framework of the FBSDE theory and have to consider a fully coupled system of forward-backward stochastic equations. To prove the existence and uniqueness of a solution to (22), (23) we need some additional conditions that allow us to apply the technique of homotopy prolongation [8].

We say that \mathbf{C}^2 holds when $\mathbf{C}^{1,1}$ holds and the random function $F(t, Y, Z) = G(\chi(t), Y, Z) \in H_2$ satisfies the standard conditions of the BSDE theory [6] which ensure the existence and uniqueness of a solution to a BSDE equation

$$dY(t) = -F(t, Y(t), Z(t))dt + Z(t)dW, \quad \alpha = Y(T) \in H_2.$$

Let

$$\mathcal{H}_1 = \{\chi(t) \in H_1 : E \sup_{t \in [0, T]} \|\chi(t)\|^2 < \infty\}, \quad \mathcal{H}_2 = \{Y(t) \in H_2 : E \sup_{t \in [0, T]} \|Y(t)\|^2 < \infty\},$$

$$\mathcal{H}_3 = \{Z(t) \in H_3 : E \int_0^T |Z(t)|^2 dt < \infty\}, \quad \mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3$$

and $\|\cdot\|_{\mathcal{H}}$ denote the norm in \mathcal{H} , that is, if $\Theta = (\chi, Y, Z) \in \mathcal{H}$, then

$$\|\Theta\|_{\mathcal{H}}^2 = E \left[\sup_{[0, T]} \|\chi(t)\|^2 + \sup_{[0, T]} \|Y(t)\|^2 + \int_0^T |Z(t)|^2 dt \right].$$

Denote by $D = H_1 \times H_2 \times H_3$, $\mathcal{D} = \mathcal{M}^2(0, T; D) \cap \mathcal{H}$ and, for $\Theta = (\chi, \kappa, \nu) \in D$, let $Y(\Theta) = (-F(\Theta), b(\Theta), B(\Theta))$. We say that **C³** holds if there exists a constant $C > 0$ such that functions $Y : D \rightarrow D$ and V_0 satisfy the estimates

$$\|Y(\Theta) - Y(\Theta_1)\|_D \leq C \|\Theta - \Theta_1\|_D, \quad \forall \Theta, \Theta_1 \in D, \quad P - \text{a.s.}$$

$$\|V_0(\chi) - V_0(\chi_1)\| \leq C \|\chi - \chi_1\|, \quad \forall \chi, \chi_1 \in H_1 \quad P - \text{a.s.}$$

We say that **C⁴** holds if there exists a constant $C_1 > 0$ such that

$$\langle \langle Y(\Theta) - Y(\Theta_1), \Theta - \Theta_1 \rangle \rangle \leq -C_1 \|\chi - \chi_1\|^2, \quad \forall \chi, \chi_1 \in H_1, P - \text{a.s.},$$

where $\langle \langle \cdot, \cdot \rangle \rangle$ is an inner product in \tilde{D} and

$$\langle V_0(\chi) - V_0(\chi_1), N[\chi - \chi_1] \rangle \geq C_1 \|\chi - \chi_1\|^2 \quad \chi, \chi_1 \in H_1, P - \text{a.s.}$$

Let us start with a simple case as the starting point in the homotopy construction.

LEMMA 1. *Let $(b^0, F^0, B^0) \in \mathcal{D}$, $\kappa^0 \in L^2(\Omega, \mathcal{F}_T, P)$. Then there exists a unique solution $(\chi, Y, Z) \in \mathcal{D}$ of FBSDE*

$$(26) \quad d\chi(t) = [Y(t) - b^0(t)]dt + [Z(t) - B^0(t)]dw(t), \quad \chi(0) = \chi.$$

$$(27) \quad dY(t) = -[F^0(t) - \chi(t)]dt + Z(t)dw(t), \quad Y(T) = \chi(T) + \alpha, \quad 0 \leq t \leq T.$$

Next, for a given $\mu \in [0, 1]$, denote by

$$b^\mu(\chi, Y, Z) = (1 - \mu)Y - \mu b(\chi, Y, Z), \quad B^\mu(\chi, Y, Z) = (1 - \mu)z - \mu B(\chi, Y, Z),$$

$$F^\mu(\chi, Y, Z) = (1 - \mu)\chi - \mu F(\chi, Y, Z), \quad V_0^\mu(\chi) = \mu V_0(\chi) + (1 - \mu)\chi.$$

From general results of BSDE theory and Lemma 1 we can deduce that at least for $\mu = 0$ there exists a unique solution of the FBSDE

$$(28) \quad \chi(t) = \chi + \int_0^t [b^\mu(\Theta(\tau)) - b^0(\tau)]d\tau + \int_0^t [B^\mu(\Theta(\tau)) - B^0(\tau)]dW(\tau),$$

$$(29) \quad Y(t) = (V_0^\mu(\chi(T)) + \kappa^0) - \int_0^t [F^\mu(\Theta(\tau)) - F^0(\tau)]d\tau - \int_0^t Z(\tau)dW(\tau).$$

LEMMA 2. Assume that \mathbf{C}^2 - \mathbf{C}^4 hold, $(b^\mu, B^\mu, F^\mu) \in \mathcal{D}$ and, for $\mu = \mu_0 \in [0, 1]$, there exists a unique solution $\Theta^{\mu_0}(t) = (\chi^{\mu_0}(t), Y^{\mu_0}(t), Z^{\mu_0}(t)) \in \mathcal{D}$ of (28), (29). Then there exists a constant $\delta_0 \in [0, 1]$, depending on C_1, C_2 and T such that there exists a unique solution $(\chi^\mu(t), Y^\mu(t), Z^\mu(t)) \in \mathcal{D}$ of (28), (29) for $\mu = \mu_0 + \delta$, where $\delta \in [0, \delta_0]$.

As a result we can deduce the following statement.

THEOREM 3. Assume that $\mathbf{C}^{1,1}$ – \mathbf{C}^3 hold. Then there exists a unique solution (χ, Y, Z) of (24)-(25). In addition, the function $V(s, \chi) = Y(s)$ is a continuous viscosity solution of (15), $V(s, \chi) = (u(s, x), \nabla_y u(s, x))$ and its first component $u(s, x)$ is a viscosity solution of (1).

To verify that $V(s, \chi)$ is a viscosity solution to (15) and hence $u(s, x)$ is a viscosity solution to (1) one needs comparison theorems which are well known for scalar equations but are much less known for the case of nondiagonal systems. Actually, we can overcome this difficulty due to the special structure of the systems under consideration, and our ability to reduce them to scalar equations in a new phase space (described in Section 1).

Finally, applying Ito's formula, it is not difficult to check that the following inequalities hold

$$E \left(\int_s^\tau \Lambda^l(\theta, \chi(\theta), Y(\theta), Z(\theta)) d\theta \right) \geq 0 (\leq 0),$$

where

$$\Lambda^l(s, \chi, Y, Z) = \left[\frac{\partial \Phi^l}{\partial \theta} + \mathcal{A}\Phi^l \right](s, \chi) - F^l(\chi, Y, Z),$$

$\Phi = (\phi, \nabla_y \phi)$, and $\phi(s, x) \in \mathbb{R}^d$ is a C^3 - smooth function such that (s, χ) is a point, where a local maximum (minimum) of $V^l(s, \chi) - \Phi^l(s, \chi)$, $l = 1, \dots, d_2 = 2d_1$ is attained. Combining this with the comparison results we can prove the last statement of the theorem.

References

- [1] BELOPOLSKAYA YA. AND DALECKY YU. *Investigation of the Cauchy problem for systems of quasilinear equations via Markov processes* Izv. VUZ Matematika. N 12 (1978) 6–17.
- [2] BELOPOLSKAYA YA. *Probability approach to solution of nonlinear parabolic equations*, Problems of Math. Anal. **13** (1992) 21–35.
- [3] BELOPOLSKAYA YA. AND DALECKY YU. *Markov processes associated with nonlinear parabolic systems*, DAN USSR **250**, N1 (1980) 521–524.
- [4] BELOPOLSKAYA YA. *Probabilistic approaches to nonlinear parabolic equations in jet-bundles.*, Global and Stoch. Anal., N1 **1** (2011), 3–40.

- [5] BELOPOLSKAYA YA.I. AND WOYCZYNSKI W.A. *Probabilistic approach to viscosity solutions of the Cauchy problem for systems of fully nonlinear parabolic equations*, J. Math. Sci, N 6 **188** (2013), 655–672.
- [6] PARDOUX E. AND PENG S. *Backward Stochastic Differential Equations and Quasilinear Parabolic Partial Differential Equations*, Lecture Notes in CIS **176** (1992) Springer-Verlag.
- [7] PENG S. *Probabilistic interpretation for systems of quasilinear parabolic partial differential equations*, Stoch. and Stoch. Rep., **37** (1991), 61–74.
- [8] HU Y. AND PENG S. *Solution of forward-backward stochastic differential equations*, Probab. theory and Relat. Fields , **103** (1995), 273–283.

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Yana BELOPOLSKAYA,
Department of Mathematics, St. Petersburg State University
for Architecture and Civil Engineering
2-ja Krasnoarmejskaja 4, 190005, St. Petersburg, RUSSIA
e-mail: yana@yb1569.spb.edu

Wojbor A. WOYCZYNSKI,
Department of Mathematics, Applied Mathematics and Statistics
Case Western Reserve University
10900 Euclid Avenue, OH 44106, Cleveland, USA
e-mail: waw@case.edu

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