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A ONE-DAY TOUR OF REPRESENTATIONS AND INVARIANTS OF QUIVERS

Abstract. These notes are taken from an introductory lecture on representations and invariants of quivers that I gave at the *School (and workshop) on invariant theory and projective geometry*, Trento, September 2012. All the results mentioned in this draft have appeared before, and are surveyed here unfortunately without proofs, or with only rough sketches of them. A great proportion of this material has been taken freely from one of the many excellent sets of lecture notes available on the web. I borrowed most of the ideas from [28]. A good source of inspiration, and of challenging exercises, is available at Derksen's webpage, [21]. Let me mention three more exhaustive lecture notes: those by Crawley-Boevey, [13, 14], those by Brion, [12] and by Ringel, [50]. A very nice example worked out in detail appears in [32].

We will sometimes appeal to some basic projective geometry, see [33] for notation and background.

Contents

1. Representations of quivers 4
    1.1. Quivers 4
    1.2. The category of representations of a quiver 5
    1.3. Path algebra 6
    1.4. Standard resolution 7
    1.5. Dimension vectors, Euler and Cartan forms 8
    1.6. Action by change of basis, invariants 9
2. Quivers of finite and infinite type, theorems of Gabriel and Kac 12
    2.1. Roots 12
    2.2. Quivers of finite type: Gabriel's theorem 16
    2.3. Tame quivers: Euclidean graphs 17
    2.4. Finite-tame-wild trichotomy 19
    2.5. Kac's theorem 20
3. Derksen-Weyman-Schofield semi-invariants 21
4. Quivers with relations 22
    4.1. Tilting bundles 22
    4.2. Homogeneous bundles 23
5. Moduli spaces of quiver representations 26
    5.1. Stability of quiver representations 27
    5.2. Moduli space and ring of semi-invariants 27
    5.3. Schofield's result on birational type of quiver moduli 28
    5.4. More reading 29
References 30

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1. Representations of quivers

The main object of these notes are quivers and their representations. Instead of just a linear map of vector spaces, these consist of a collection of linear maps, indexed by a graph, whose vertices correspond to the vector spaces involved, and whose edges carry an arrow pointing in the direction of the map.

1.1. Quivers

A *quiver* is a *finite directed graph*. The finiteness hypothesis is sometimes omitted. We will comment on a couple of situations that require infinite quivers, however for us a quiver will be finite unless otherwise stated.

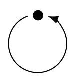
We write a quiver Q as $Q = (Q_0, Q_1)$, where Q_0 is the set of *vertices* and Q_1 is the set of *arrows*. Each arrow $a \in Q_1$ has a head $ha \in Q_0$ and a tail $ta \in Q_0$. In other words, a quiver consists of a quadruple (Q_0, Q_1, t, h) where t, h are functions $Q_1 \to Q_0$. Many authors prefer to write (s,t), for source and target, instead of (t,h). We depict a as an arrow pointing from ta to ha.

EXAMPLE 1. Let us consider some basic examples.

(1) The 1-arrow quiver.

$$\bullet \longrightarrow \bullet$$

(2) The loop quiver.



(3) The oriented straight quiver $\vec{A}_n$ (here $\vec{A}_4$):

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

(4) The n-Kronecker Θ_n quiver (here the 2-Kronecker quiver Θ_2) with 2 vertices $1,2$ and n arrows from 1 to 2.

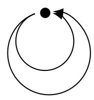
$$\overset{1}{\bullet} \rightrightarrows \overset{2}{\bullet}$$

The Kronecker quiver will play a prominent role in these notes.

(5) The n-star quiver (here the 4-star quiver):

$$\begin{array}{ccccc} & & \overset{1}{\bullet} & & \\ & & \downarrow & & \\ \overset{2}{\bullet} & \longrightarrow & \overset{3}{\bullet} & \longleftarrow & \overset{4}{\bullet} \\ & & \uparrow & & \\ & & \overset{5}{\bullet} & & \end{array}$$

(6) The n-loop quiver (here the double loop quiver):



(7) A disconnected quiver.

$$\bullet \leftrightarrows \bullet \qquad \bullet \leftarrow \bullet \leftleftarrows \bullet \leftrightarrows \bullet \rightrightarrows \bullet \rightarrow \bullet$$

1.2. The category of representations of a quiver

Let us a fix a field k. A *k-representation* V of a quiver Q (over k) is a collection of k-vector spaces $(V(x))_{x\in Q_0}$ and a collection of k-linear maps $(V(a))_{a\in Q_1}$.

If V and W are representations of Q, a morphism $\varphi : V \longrightarrow W$ is a collection of linear maps $\varphi(x) : V(x) \longrightarrow W(x)$, one for each $x \in Q_0$, such that, for all $a \in Q_1$ the diagram commutes:

$$\begin{array}{ccc} V(ta) & \xrightarrow{\varphi(ta)} & W(ta) \\ {\scriptstyle V(a)}\downarrow & & \downarrow{\scriptstyle W(a)} \\ V(ha) & \xrightarrow[\varphi(ha)]{} & W(ha). \end{array}$$

We write $\mathrm{Hom}_Q(V,W)$ for the set of these morphisms. This is in fact a k-vector space, while $\mathrm{Hom}_Q(V,V)$ is a k-algebra.

In many cases, representations will be assumed to have finite dimension, i.e., the vector spaces V_x are finite-dimensional for all $x \in Q_0$. In this case we can speak of the *dimension vector* $\underline{\dim}(V)$ of V, i.e., the assignment $x \mapsto \dim V(x)$, as an element of $Q_0^{\mathbb{N}}$. When the quiver Q is not finite, usually representations are assumed to have finite support, where the support of a representation V is the subset of $x \in Q_0$ such that $V_x \neq 0$.

The category of representations of Q over k is denoted by $\mathrm{Rep}_k(Q)$, or simply by $\mathrm{Rep}(Q)$. This is a *k-linear* category, in the sense that, given representations U,W,V of Q, composition of maps $\mathrm{Hom}_Q(V,W) \times \mathrm{Hom}_Q(W,U) \to \mathrm{Hom}_Q(V,U)$ is a k-bilinear map.

EXAMPLE 2. Looking at Example 1, we see the following.

(1) A representation of the 1-arrow quiver (see (1) in Example 1) with dimension vector (m,p) is a linear map from an m-dimensional vector space to a p-dimensional one. After fixing basis in these spaces, the representation is identified with a matrix $M : k^m \to k^p$. Giving a morphism φ from a representation with dimension

vector (m,p) to another representation, with dimension vector (m',p') amounts to the datum of a commuting diagram:

$$\begin{array}{ccc} k^m & \longrightarrow & k^p \\ \big\downarrow & & \big\downarrow \\ k^{m'} & \longrightarrow & k^{p'}. \end{array}$$

(2) A representation of the loop quiver (see (2) in Example 1) with dimension vector m is an endomorphism of a vector space over k of dimension m.

(3) For the Kronecker quiver Θ_n (see (4) in Example 1) we can identify a representation of dimension vector (m,p) with an n-tuple of $m \times p$ matrix $M_1,\dots,M_n$.

A very useful way to think about this is to consider a projective space $\mathbb{P}^{n-1}$ over k, with ambient variables $x_1,\dots,x_n$, and to combine the matrices $M_1,\dots,M_n$ in:

$$M = x_1 M_1 + \cdots + x_n M_n.$$

The resulting object M is matrix of linear forms in n variables. Since each linear form is a global section of $\mathscr{O}_{\mathbb{P}^{n-1}}(1)$, i.e., a map $\mathscr{O}_{\mathbb{P}^{n-1}}(-1) \to \mathscr{O}_{\mathbb{P}^{n-1}}$, the matrix M can be written as:

$$M : \mathscr{O}_{\mathbb{P}^{n-1}}(-1)^m \longrightarrow \mathscr{O}_{\mathbb{P}^{n-1}}^p.$$

(4) A representation of the 4-star quiver (see (5) in Example 1) with dimension vector (n_1,n_2,n_3,n_4,n_5) is the following datum. Set $n = n_3$ and consider a vector space U_3 of dimension n, and vector spaces U_i of dimension n_i: then V is given by 4 linear maps $U_i \to U_3$. If all maps are injective, then, up to change of basis in the U_i's, this amounts to a configuration of 4 linear subspaces of U_3.

It is immediate to construct the kernel and cokernel of a morphism of representations $V \to W$, and to check that $\mathrm{Rep}(Q)$ is an *abelian category*. We have the obvious notions of *monomorphism* and *epimorphism* of representations, i.e., of subrepresentation and quotient representation. A representation V of Q is *irreducible* (or *simple*) if V has no nontrivial subrepresentations. One also defines the direct sum representation in the obvious way, and V is said to be *indecomposable* if it is not isomorphic to a direct sum of two nontrivial representations.

1.3. Path algebra

The category of representations of a (finite) quiver Q over k is a category of modules over a certain algebra, called the path algebra, that we will now describe. So let us fix a quiver Q and a field k. A *path* p in Q is a sequence $p = a_1,\dots,a_m$ with a_i in Q_1 such that $ha_i = ta_{i-1}$ for all i (we write paths as compositions). We also write $tp = ta_m$ and $hp = ha_1$, and we say that p is a path from tp to hp. Given two vertices $x,y \in Q_0$, denote by $[x,y]$ the set of paths from x to y. We write $\langle [x,y] \rangle$ for the vector space freely generated by elements of $[x,y]$. Given a vertex $x \in Q_0$, the *trivial path based at* x, denoted by e_x, is the path of length zero, with head and tail at x.

The *path algebra* kQ is the associative algebra generated by paths from x to y, for $x,y \in Q_0$. The unit e of kQ is the empty path, i.e., the product of all trivial paths based at $x \in Q_0$: $e = \prod_x e_x$. In other words, the set $\{e_x \mid x \in Q_0)\}$ forms a family of orthogonal idempotents in kQ. Multiplication in kQ is defined by:

$$a_1 \dots a_m \times b_1 \dots b_p \mapsto a_1 \dots a_m b_1 \dots b_p,$$

if $ta_m = hb_1$, or zero if $ta_m \neq hb_1$ (product by composition).

FACT 1. The category of left kQ-modules is equivalent to $\mathrm{Rep}_k(Q)$.

To see why this is true, consider a k-representation V of Q and define the vector space $M = \bigoplus_{x \in Q_0} V(x)$. This is equipped with an action of kQ, defined first for trivial paths and length-1 paths, and extended to kQ by linearity. For trivial paths, one sets $e_x.v = v$ for all $x \in Q_0$ and $v \in V(x)$ and $e_x.v = 0$ for $v \in V(y)$ with $x \neq y$. As for length-1 paths, for all $a \in Q_1$ one defines $a.v = V(a)v$ if $v \in V(ta)$, and $a.v = 0$ if $v \in V(y)$ with $y \neq ta$.

Conversely, with a kQ-module M we associate the representation V having $V(x) = e_x M$ for all $x \in Q_0$ and $V(a) : V(x) \to V(y)$ defined as the multiplication $e_x M \to e_y M$ when this makes sense, or zero otherwise. So, if $x = ta, y = ha$, and for all $v \in e_x M$, we get $V(a)v = a.v \in e_y M$.

EXAMPLE 3. The path algebra of the loop quiver is $k[t]$. For the n-loop quiver, we get the free associative algebra in n variables. One sees also that kQ is finite-dimensional if and only if Q has no oriented cycles.

The path algebra of the straight quiver $\vec{A}_n$ (see (3) in Example 1) is the algebra of lower triangular matrices of size n.

1.4. Standard resolution

Let us fix a field k and our quiver Q. Given a vertex $x \in Q_0$, we define the *standard representation* P_x, by:

$$P_x(y) = \langle [x,y] \rangle, \qquad P_x(a) : p \mapsto ap.$$

This representation corresponds to the submodule of kQ spanned by the idempotent e_x. Let V be a k-representation of Q. We define $P(V)$ and $\Omega(V)$ as:

$$P(V) = \bigoplus_{x \in Q_0} P_x \otimes_k V(x), \qquad \Omega(V) = \bigoplus_{a \in Q_1} P_{ha} \otimes_k V(ta).$$

The representations $P(V)$ and $\Omega(V)$ fit together to give the so-called *standard resolution* of V, which is an exact sequence of the form:

$$0 \longrightarrow \Omega(V) \xrightarrow{\Psi} P(V) \xrightarrow{\Phi} V \longrightarrow 0.$$

Exercise: guess the maps Ψ and Φ.

Next, we would like to point out an important feature of the category $\mathrm{Rep}(Q)$. Recall that an abelian category $\mathscr{A}$ is *hereditary* if, for all objects A, B of $\mathscr{A}$, we have:

$$\mathrm{Ext}^n_{\mathscr{A}}(A,B) = 0, \qquad \text{when } n \geq 2.$$

FACT 2. The category $\mathrm{Rep}(Q)$ is an abelian k-linear hereditary category.

A proof of this fact can be obtained by the standard resolution, or from the fact that the algebra $\mathrm{Hom}_Q(V,V)$ is local, see [13,50]. Looking at the first method, given a representation V, one observes that the new representations $\Omega(V)$ and $P(V)$ are projective, i.e., by definition, they correspond to projective modules over kQ. Therefore, it is clear that $\mathrm{Ext}^n_Q(\Omega(V),W) = 0$ and $\mathrm{Ext}^n_Q(P(V),W) = 0$ for all $n \geq 1$ and all representations W of Q. Now, by applying the functor $\mathrm{Hom}_Q(-,W)$ to the standard resolution, we obtain $\mathrm{Ext}^n_Q(V,W) = 0$ for all $n \geq 2$. The property just mentioned justifies the notation $\mathrm{Ext}_Q(V,W) = \mathrm{Ext}^1_Q(V,W)$.

Moreover, this way we also get the *canonical exact sequence* attached to a pair of representations V, W. This is an exact sequence of the following form.

$$0 \to \mathrm{Hom}_Q(V,W) \longrightarrow \bigoplus_{x \in Q_0} \mathrm{Hom}_k(V(x),W(x)) \xrightarrow{d^V_W}$$
$$\xrightarrow{d^V_W} \bigoplus_{a \in Q_1} \mathrm{Hom}_k(V(ta),W(ha)) \longrightarrow \mathrm{Ext}_Q(V,W) \to 0.$$

The arrow d^V_W is defined by:

$$d^V_W(\varphi(x)_{x \in Q_0}) = (\varphi(ha)V(a) - W(a) \circ \varphi(ta))_{a \in Q_1}.$$

1.5. Dimension vectors, Euler and Cartan forms

Let us fix a quiver Q and the field k. We already described the dimension vector of a k-representation of Q as an element of $\mathbb{N}^{Q_0}$. Any assignment $\alpha : Q_0 \to \mathbb{N}$ will thus be called a *dimension vector* of Q. The set of dimension vectors of Q is a subset of $\Gamma_Q = \mathbb{Z}^{Q_0}$.

Euler form

The Euler form (or Ringel form) is the bilinear map on Γ_Q defined by:

$$\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha), \qquad \forall \alpha, \beta \in \Gamma_Q.$$

Let $(b_{i,j})$ be the matrix of $\langle \cdot, \cdot \rangle$ in the coordinate basis of $\Gamma_Q = \mathbb{Z}^{Q_0}$. Then:

$$b_{i,j} = \delta_{i,j} - \#\{a \in Q_1 \mid ta = i, ha = j\}|.$$

Let V, W be representations of Q over k. We define the Euler characteristic:

$$\chi(V,W) = \dim_k \mathrm{Hom}_Q(V,W) - \dim_k \mathrm{Ext}_Q(V,W).$$

If V, W have dimension vectors respectively α and β, then:

$$\chi(V,W) = \langle \alpha, \beta \rangle.$$

In particular we have:

$$\chi(V,V) = \dim_k \mathrm{End}_Q(V) - \dim_k \mathrm{Ext}_Q(V,V) = \langle \alpha, \alpha \rangle.$$

Cartan form

We can symmetrize the Euler form to obtain the *Cartan* (or *Cartan-Tits*) symmetric bilinear form on Γ_Q:

$$(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle, \qquad \forall \alpha, \beta \in \Gamma_Q.$$

This does not depend on the orientation of the arrow of Q. The matrix of $(c_{i,j})_{i,j}$ of $(\cdot,\cdot)$ in the coordinate basis of Γ_Q is called Cartan matrix.

EXAMPLE 4. Consider the quiver Q:

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet$$

The underlying graph of Q is A_4. The Euler and Cartan matrices of Q read:

$$(b_{i,j})_{1\leq i,j\leq 4} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (c_{i,j})_{1\leq i,j\leq 4} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

1.6. Action by change of basis, invariants

Let Q be a fixed quiver and k be a field. Let $\alpha \in \Gamma_Q$ be a dimension vector of Q. Once chosen α, the representations of Q of dimension vector α form a vector space, acted on by a product of general or special linear groups via change of basis. The invariants or semi-invariants relevant to this situation are the polynomial functions on this space, invariant under this group. Let us sketch this situation.

The space of all representations with fixed dimension vector

The k-vector space of all representations of dimension vector α is denoted by $\mathrm{Rep}(Q,\alpha)$. By choosing a basis for all vector spaces $(V(x))_{x\in Q_0}$ we get an identification:

$$\mathrm{Rep}(Q,\alpha) \simeq \prod_{a\in Q_1} \mathrm{Hom}_k(k^{\alpha(ta)}, k^{\alpha(ha)})$$

So a representation $V \in \mathrm{Rep}(Q,\alpha)$ can be seen as $V = (V(a))_{a\in Q_1}$ where $V(a)$ is a matrix with $\alpha(ta)$ columns and $\alpha(ha)$ rows.

The set of polynomial function f on $\mathrm{Rep}(Q,\alpha)$ is denote by $k[\mathrm{Rep}(Q,\alpha)]$. This is a polynomial ring in as many variables as $\dim_k(\mathrm{Rep}(Q,\alpha)) = \sum_{a\in Q_1} \alpha(ta)\alpha(ha)$.

Characters of the linear group of given dimension vector

Given a dimension vector α of Q, the group $\mathrm{GL}(Q,\alpha)$ is defined as:

$$\mathrm{GL}(Q,\alpha) = \prod_{x\in Q_0} \mathrm{GL}_{\alpha(x)}(k).$$

A character $\theta : \mathrm{GL}(Q,\alpha) \to k^*$ is of the form:

$$\theta((g_x)_{x\in Q_0}) = \prod_{x\in Q_0} \det(g_x)^{\theta_x},$$

where each θ_x lies in $\mathbb{Z}$. So $\theta = (\theta_x)_{x\in Q_0}$ lies in $\Gamma = \Gamma_Q$.

We can also think of the *space of characters* $\mathscr{X}(\mathrm{GL}(Q,\alpha))$ as $\Gamma^* = \mathrm{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{Z})$. Indeed, if σ as an element of Γ^*, we define θ_σ by :

$$\theta_\sigma((g_x)_{x\in Q_0}) = \prod_{x\in Q_0} \det(g_x)^{\sigma(\delta_x)},$$

where the dimension vector δ_x is defined by $\delta_x(x) = 1$ and $\delta_x(y) = 0, \forall y \neq x$.

The general linear group $\mathrm{GL}(Q,\alpha)$ acts by conjugation on $\mathrm{Rep}(Q,\alpha)$, namely if $g \in \mathrm{GL}(Q,\alpha)$ and $V \in \mathrm{Rep}(Q,\alpha)$, then:

$$g.V = (g(ha) \circ V(a) \circ g(ta)^{-1})_{a\in Q_1}.$$

This action can be thought of as change of basis in the spaces $V(x)$. The kernel of the action Z is a normal subgroup of $\mathrm{GL}(Q,\alpha)$, consisting of transformations that act trivially on $\mathrm{Rep}(Q,\alpha)$. Note that the 1-dimensional torus $k^*\,\mathrm{id}$ is always contained in Z. We are naturally lead to consider the action of the group:

$$G(Q,\alpha) = \mathrm{GL}(Q,\alpha)/Z,$$

and in many cases we have $G(Q,\alpha) = \mathrm{GL}(Q,\alpha)/k^*\,\mathrm{id}$.

The general linear group $\mathrm{GL}(Q,\alpha)$ acts also on the ring of polynomial functions $k[\mathrm{Rep}(Q,\alpha)]$ by the formula $g.f(V) = f(g^{-1}.V)$.

Invariants and semi-invariants

Let G be a closed algebraic subgroup of $\mathrm{GL}(Q,\alpha)$.

DEFINITION 1. A *G-invariant* of (Q,α) is a polynomial function f over $\mathrm{Rep}(Q,\alpha)$ such that, for all $g \in G$ we have $g.f = f$. We write $\mathrm{Rep}(Q,\alpha)^G$ for the ring of all invariants of (Q,α). If θ is a character of $\mathrm{GL}(Q,\alpha)$, a *semi-invariant* of (Q,α) of weight θ is a polynomial function f over $\mathrm{Rep}(Q,\alpha)$ such that:

$$g.f = \theta(g)f, \qquad \forall g \in \mathrm{GL}(Q,\alpha).$$

We denote by $\mathrm{SI}(Q,\alpha)_\theta$ the space of semi-invariants of (Q,α) of weight θ.

FACT 3. The ring of $\mathrm{SL}(Q,\alpha)$-invariants functions over $\mathrm{Rep}(Q,\alpha)$ is the direct sum of the spaces of semi-invariant functions for $\mathrm{GL}(Q,\alpha)$:

$$k[\mathrm{Rep}(Q,\alpha)]^{\mathrm{SL}(Q,\alpha)} = \bigoplus_{\theta \in \mathscr{X}(\mathrm{GL}(Q,\alpha))} \mathrm{SI}(Q,\alpha)_\theta.$$

To understand this fact, consider a semi-invariant f for $\mathrm{GL}(Q,\alpha)$, relative to a character $\theta \in \mathscr{X}(\mathrm{GL}(Q,\alpha))$. We know that there are integers $(\theta_x)_{x\in Q_0}$ such that, for all $(g_x)_{x\in Q_0}$ lying in $\mathrm{GL}(Q,\alpha)$, we have $\theta(g) = \prod_{x\in Q_0} \det(g_x)^{\theta_x}$. So, for all $h \in \mathrm{SL}(Q,\alpha)$ we have $\theta(h) = 1$, hence $h.f = f$ so that f is $\mathrm{SL}(Q,\alpha)$-invariant. Conversely, one writes $\mathrm{GL}(Q,\alpha)$ as semi-direct product of $\mathrm{SL}(Q,\alpha)$ and $H = \prod_{x\in Q_0} k^*$, and the characters of $\mathrm{GL}(Q,\alpha)$ are given by those of the algebraic torus H. Then, once given a $\mathrm{SL}(Q,\alpha)$-invariant function f, we decompose it into eigenvectors for H, hence into semi-invariants for $\mathrm{GL}(Q,\alpha)$.

EXAMPLE 5. Given an integer m, an m-dimensional representation of the loop quiver Q is an endomorphism u of k^m, so the algebra $\mathrm{Rep}(Q,m)$ is $k[x_{1,1},\dots,x_{m,m}]$, with $\mathrm{GL}_m(k)$ acting by conjugation. According to Chevalley's theorem, we have:

$$\mathrm{Rep}(Q,m)^{\mathrm{SL}_k(m)} = k[\varepsilon_1,\dots,\varepsilon_m],$$

where $\varepsilon_1,\dots,\varepsilon_m$ are the elementary symmetric functions of the eigenvalues of u. We refer to [47] for an extremely interesting approach to invariant theory. Here $\varepsilon_i = \mathrm{tr}(\wedge^i u)$, so ε_i is a semi-invariant for $\mathrm{GL}_k(m)$ of weight i.

EXAMPLE 6. Set $Q = \vec{A}_2$ and let m, p and $r \le \min(m,p)$ be integers. A representation V of Q of dimension vector $\alpha = (m,r,p)$ is a pair of maps:

$$M \to R \to P, \qquad \text{with } \dim(M) = m, \dim(R) = r, \dim(P) = p.$$

Thus the algebra $B = \mathrm{Rep}(Q,\alpha)$ is the symmetric algebra over the vector space

$$\mathrm{Hom}(M,R) \oplus \mathrm{Hom}(R,P).$$

We consider the action of the subgroup $G = \mathrm{GL}(R) \simeq \mathrm{GL}_r(k)$ of $\mathrm{GL}(Q,\alpha)$ only. Fixing basis of M and P, and given any two indexes i, j with $1 \le i \le m$ and $1 \le j \le p$, we have the G-invariant functions $[\cdot_i, \cdot_j]$ on B defined on $\mathrm{Hom}(M,R) \oplus \mathrm{Hom}(R,P)$ by the (j,i)-th coefficient of the composition matrix:

$$(f,g) \mapsto [f_i, g_j] = (g \circ f)_{j,i}.$$

Consider also a polynomial ring $A = k[x_{1,1}, \dots, x_{m,p}]$, and the $m \times p$ matrix X of indeterminates. The ring $k[\mathrm{Hom}(M,P)]$ can be identified with A. The First and Second Fundamental Theorems of invariant theory for $\mathrm{GL}_r(k)$ give:

i) a surjective ring homomorphism $\phi : A \mapsto B^G$ given by $x_{i,j} \mapsto [\cdot_i, \cdot_j]$,

ii) an equality $\ker(\phi) = (\wedge^{r+1}(X))$.

This says that the GIT quotient $\mathrm{Hom}(M,R) \oplus \mathrm{Hom}(R,P) /\!\!/ \mathrm{GL}(R)$ is the affine subvariety of $\mathrm{Hom}(M,P)$ consisting of matrices of rank at most $\dim(R)$, which is cut by all minors of order $r+1$ of X. We refer to [47].

2. Quivers of finite and infinite type, theorems of Gabriel and Kac

We fix a field k and a (finite) quiver Q. It is natural to ask if there are finitely or infinitely many indecomposable representations of Q over k.

DEFINITION 2. If there are only finitely many isomorphism classes of indecomposable finite-dimensional representations of Q over k, then Q is said to be of *finite type* (over k), or of *finite representation type*.

For instance, we take $Q = \vec{A}_2$. For $m, p > 0$, a representation of dimension vector (m, p), corresponding to a matrix $k^m \to k^p$ or rank $r \leq \min(m, p)$ is equivalent, up to changing basis in k^m and k^p, to the direct sum of the identity I_r and of the zero matrix. Of course this matrix corresponds to a non-zero indecomposable representation if and only if $r = 1$, so $\vec{A}_2$ is of finite type. The only dimension vectors giving indecomposable representations are $(1,0)$, $(0,1)$ and $(1,1)$.

Similarly, we can look at the quiver $Q = \vec{A}_3$. In this case, representations of Q are given by maps $M \xrightarrow{u} R \xrightarrow{v} P$. Besides the obvious dimension vectors $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, we can build an indecomposable representation with dimension vector $(1,1,0)$ by setting $M = R = k$ and $u = \mathrm{id}_k$, and for $(0,1,1)$ by setting $R = P = k$ and $v = \mathrm{id}_k$. One more case arises taking the dimension vector $(1,1,1)$, so $M = P = R = k$ and setting $v = u = \mathrm{id}_k$.

On the other hand, the loop quiver, for instance, is of infinite type, actually of tame type. Indeed the isomorphism classes of representations of this quiver are given by conjugacy classes of matrices, i.e. (at least over $\bar{k}$) by Jordan normal forms. Therefore, for any dimension m, there is a one-dimensional family (parametrized by $\lambda \in k$) of non-isomorphic indecomposable representations whose normal form is:

$$\begin{pmatrix} \lambda & 1 & & & \\ 0 & \lambda & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & \lambda & 1 \\ & & & 0 & \lambda \end{pmatrix}$$

2.1. Roots

Let us sketch briefly here the notion of Kac-Moody algebra and the related idea of root system. We refer to [36, 38] for a complete treatment, and to [53] for some lecture notes.

Kac-Moody algebras

For this part we work over the field $\mathbb{C}$ of complex numbers. Kac-Moody algebras are an infinite-dimensional analogue of simple complex Lie algebras, and are associated with generalized Cartan matrices.

A square matrix $C = (c_{i,j})_{i,j}$ of size n is a *generalized Cartan matrix* if:

i) on the diagonal we have $c_{i,i} = 2$;

ii) off the diagonal we have $-c_{i,j} \in \mathbb{N}$ and $c_{i,j} = 0 \Leftrightarrow c_{j,i} = 0$.

There are two more conditions that one can add to C. The first one is that C is indecomposable, which means that C is not equivalent under change of basis to a block matrix $\begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}$, unless $C_i = C$. The second is that C is *symmetrizable*, namely that there are a symmetric matrix $S = (s_{i,j})_{i,j}$ and a diagonal non-degenerate matrix D, such that $C = DS$. This decomposition is unique for C indecomposable if we assume that $s_{i,i}$ are relatively prime integers and that $-2s_{i,j} \in \mathbb{N}$.

Irreducible symmetric Cartan matrices are in one-to-one correspondence with connected graphs without loops. Indeed, with a graph without loops we associate the Cartan matrix as we have already did. The absence of loops is translated into the value 2 along the diagonal, connectedness corresponds to irreducibility, and the matrix is symmetric by definition. On the other hand, having an $n \times n$ symmetric generalized Cartan matrix $C = (c_{i,j})_{i,j}$, in order to construct the graph, we draw n vertices and we join the i-th and j-th vertices with $|c_{i,j}|$ vertices.

With a generalized Cartan matrix $C = (c_{i,j})_{i,j}$ of size n, one also associates naturally a free abelian group Λ_C with n generators, denoted by $(\alpha_1, \ldots, \alpha_n)$. We should think of Λ_C as the set of dimension vectors of the graph associated with C, or of any quiver with this underlying graph. For C symmetrizable, the matrix S gives a bilinear form on Λ_C, denoted by $(\cdot, \cdot)$. The matrix of the bilinear form is $\frac{1}{2}C$.

DEFINITION 3. The *Kac-Moody algebra* $\mathfrak{g}$ associated with C is the unique Λ_C-graded Lie $\mathbb{C}$-algebra $\mathfrak{g} = \oplus_{\alpha \in \Lambda_C} \mathfrak{g}_\alpha$ satisfying the following conditions:

i) for every Λ_C-graded ideal $\mathfrak{i}$ of $\mathfrak{g}$, we have $\mathfrak{i} \cap \mathfrak{g}_0 = 0 \Rightarrow \mathfrak{i} = 0$;

ii) the algebra $\mathfrak{g}$ is generated by $(e_i, f_i, h_i \mid i = 1, \ldots, n)$, where $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i$, $\mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i$ and $(h_1, \ldots, h_n)$ is a basis of $\mathfrak{g}_0$;

iii) for all $1 \leq i, j \leq n$, we have the Lie algebra relations induced by C:

$$[h_i, h_j] = 0, \quad [e_i, f_j] = \delta_{i,j} h_i, \quad [h_i, e_j] = c_{i,j} e_j, \quad [h_i, f_j] = -c_{i,j} f_j.$$

A *root* of $\mathfrak{g}$ is an element $\alpha \in \Lambda_C$ such that $\mathfrak{g}_\alpha \neq 0$. A root is *positive* if its expression in terms of the α_i involves only non-negative coefficients. In this context, a root α is *simple* if $\dim(\mathfrak{g}_\alpha) = 1$: in particular $\alpha_1, \dots, \alpha_n$ are simple. For $i = 1, \dots, n$, one defines the reflection r_i by $r_i(\alpha) = \alpha - \varphi_i(\alpha)\alpha_i$, where φ_i is defined extending linearly the function $\varphi_i(\alpha_j) = c_{i,j}$. The subgroup of $\mathrm{GL}_m(\mathbb{C})$ generated by the r_i is called the *Weyl group* W. This way we can define the *real* roots, namely the orbit of simple roots under W. All other roots are said to be *imaginary*. The terminology is justified by the fact that α is real if and only if $(\alpha,\alpha) > 0$ and imaginary if and only if $(\alpha,\alpha) \leq 0$.

We write $u \geq 0$ or $u > 0$ if all coefficients of u are non-negative or positive. For an indecomposable generalized Cartan matrix C, one and only case occurs:

Positive: C is non-degenerate and $Cu \geq 0$ implies $u = 0$ or $u > 0$. In this case C is symmetrizable with S positive definite, and $\mathfrak{g}$ is a simple Lie algebra.

Zero: $\mathrm{rk}(C) = n-1$, $Cu = 0$ for some $u > 0$, and $Cu \geq 0 \Leftrightarrow Cu = 0$. In this case C is symmetrizable with S positive semi-definite, and $\mathfrak{g}$ is an infinite-dimensional Lie algebra called *affine Lie algebra*: this is a central extension by $\mathbb{C}$ of $\mathfrak{g}_\circ \otimes \mathbb{C}[t,t^{-1}]$, where $\mathfrak{g}_\circ$ is a finite-dimensional Lie algebra.

Negative: there is $u > 0$ with $Au < 0$, and $Au \geq 0$ together with $u \geq 0$ implies $u = 0$.

Recall that simple Lie algebras (the *positive* case) are classified by Dynkin diagrams. The algebras coming from the *zero* case are also classified completely, they come from Euclidean diagrams (see [10]): we will review part of them in a minute, in connection with quivers of tame representation type. The algebras of the *negative* case are a much more unexplored territory.

Schur roots

The indecomposable representations of a quiver Q have some basic blocs, called Schurian representations, defined by the condition that their endomorphism algebra is constituted by homotheties. The dimension vectors of these representations are tightly connected with the positive roots of the Kac-Moody algebra associated with the graph underlying Q.

DEFINITION 4. A representation V of Q is called *Schurian* in case $\mathrm{End}_Q(V) \simeq k$. In particular a Schurian representation is indecomposable. If V has $\mathrm{Ext}_Q(V,V) = 0$, then V is said to be *rigid*. A dimension vector α is a *Schur root* if $\alpha = \underline{\dim}(V)$, for a Schurian representation V. Moreover, α is called a *real Schur root* if there exists a unique representation $V \in \mathrm{Rep}(Q,\alpha)$, up to isomorphism. On the other hand, α is called an *imaginary Schur root* if there exist infinitely many non-isomorphic representations $V \in \mathrm{Rep}(Q,\alpha)$.

REMARK 1. Here are some more comments on Schur roots.

i) If V is Schurian, then the tangent space at the point $[V]$ of the universal deformation space of V is identified with $\mathrm{Ext}_Q(V,V)$. Therefore the dimension of this space is $1-\langle\alpha,\alpha\rangle$.

ii) Asking that α is a Schur root is equivalent to ask that a *general* representation V is of dimension vector α is indecomposable (here a claim about a "general" element of a given parameter space, means that the elements that do not satisfy the claim form a proper (Zariski) closed subset of the parameter space, i.e., the set of these elements is defined by a finite set of nontrivial polynomial equation).

For instance, if $Q=\vec{A}_3$, then the Schur roots are $(1,0,0),(0,1,0),(0,0,1),(1,1,0)$, $(0,1,1)$ and $(1,1,1)$. Likewise, for any n, the Schur roots of the quiver $\vec{A}_n$ are the vectors with 0's, and adjacent 1's only. There are precisely the positive roots of the Lie algebra $\mathfrak{sl}_{n+1}$, whose Dynkin diagram is A_n. This is not a coincidence, as we will see in a minute.

As another example, we see that, for the loop quiver, only 1 is a Schur root. Note that an indecomposable representation of dimension m is given by matrices made of a Jordan block of size m. So these representations are parametrized by the eigenvalues of M, i.e. by the line k. However, this representation is not general: the general one will have n distinct eigenvalues and will thus decompose as direct sum of n indecomposable subrepresentations of dimension 1.

iii) We have said that a general representation V of Q having dimension vector α need not be indecomposable. However, it turns out that the dimension vectors $\alpha_1,\dots,\alpha_n$ of the indecomposable summands of V are determined by α, so one writes $\alpha=\alpha_1\oplus\cdots\oplus\alpha_n$. The α_i are Schur roots. This is called the *canonical decomposition* of α, see [27]. For instance for the loop quiver we have $n=1^{\oplus n}$.

EXAMPLE 7. Let $n\geq 3$ and consider the Kronecker quiver Θ_n (see (4) of Example 1):

Figure 1: Kronecker quivers Θ_3 and Θ_4.

Then, $\alpha=(m,p)$ is an imaginary Schur root if and only if:

$$\frac{m}{p}\in\left]\frac{n-\sqrt{n^2-4}}{2},\frac{n+\sqrt{n^2-4}}{2}\right[.$$

On the other hand, one defines the *generalized Fibonacci numbers*:

$$a_k=\frac{(n+\sqrt{n^2-4})^k-(n-\sqrt{n^2-4})^k}{2^k\sqrt{n^2-4}},$$

equivalently defined by the relations $a_0 = 0, a_1 = 1$ and $a_{n+1} = ka_k - a_{k-1}$. It turns out that $\alpha = (m,p)$ is a real Schur root if and only if:

$$m = a_{k-1}, \qquad p = a_k, \qquad \exists k \geq 0.$$

In terms of sheaves on $\mathbb{P}^{n-1}$, a representation V of dimension vector (m,p) amounts to a matrix $M : \mathscr{O}_{\mathbb{P}^{n-1}}(-1)^m \longrightarrow \mathscr{O}_{\mathbb{P}^{n-1}}^p$, and the datum of V is equivalent to that of the sheaf $E = \mathrm{cok}(M)$. Real Schur roots correspond to the case when E is *exceptional*, i.e., $\mathrm{Hom}_{\mathscr{O}_{\mathbb{P}^{n-1}}}(E,E) \simeq k$ and $\mathrm{Ext}^{>0}_{\mathscr{O}_{\mathbb{P}^{n-1}}}(E,E) = 0$. Imaginary Schur roots correspond to the case when E is a simple bundle with positive-dimensional deformation space. See also [11] for the tightly related study of Fibonacci bundles.

2.2. Quivers of finite type: Gabriel's theorem

A very nice classification of quivers of finite type is available. The story began with Gabriel's theorem, see [30], and was then developed by a great number of authors. For more general fields (including non-algebraically closed fields), see [7, Section 4].

THEOREM 1 (Gabriel). *A finite connected quiver Q is of finite type if and only if its underlying undirected graph is a Dynkin graph of type A, D, E.*

Moreover, in this case the indecomposable representations of Q are in bijection with the positive roots of the associated Lie algebra.

Recall that A_n has $n(n+1)/2$ positive roots, D_n has $n(n-1)$ positive roots and E_6, E_7, E_8, have respectively $36, 63, 120$ positive roots.

REMARK 2. The fact of being of finite type does not depend on the orientation of arrows, nor does the set of indecomposable representations of a finite quiver. Indeed, the indecomposable representations of a quiver Q of finite type are in bijection with the positive roots of the Lie algebra whose Dynkin graph is the undirected graph of Q.

After the original proof by Gabriel, [30,31], an argument using Coxeter functors was developed by Bernstein-Gelfand-Ponomarev in [8]. This allows to give a clean proof of one direction (perhaps the most difficult one) of Gabriel's theorem, namely that quivers obtained by Dynkin graphs of type A, D, E are of finite type.

An argument due to Tits allows us to understand the other direction of Gabriel's theorem, namely that the underlying undirected graph of a finite connected quiver of finite type must be a simply laced Dynkin diagram. Indeed, let V be an indecomposable representation of Q and set $\alpha = \underline{\dim}(V)$. Note that $\mathrm{GL}(Q,\alpha)$ must act with a finite number of orbits on $\mathrm{Rep}(Q,\alpha)$, otherwise there would be infinitely many non-isomorphic representations already with dimension vector α. Now, if V is Schurian then the stabilizer of the $\mathrm{GL}(Q,\alpha)$-action on $\mathrm{Rep}(Q,\alpha)$ is k^*. So we have the inequality:

$$\dim(\mathrm{Rep}(Q,\alpha)) - \dim(\mathrm{GL}(Q,\alpha) - 1) \leq 0,$$

which means:

$$\langle \alpha, \alpha \rangle \geq 1.$$

Therefore, the Cartan matrix of Q is positive definite, and one is reduced to the classifications of simple Lie algebras, hence the graph must be of Dynkin type. Also, the graph must be simply-laced, for the loop quiver and the Kronecker quivers Θ_n with $n \geq 2$ are of infinite type as we have already seen.

One can also argue that, if V is indecomposable and not Schurian, then we can find a subrepresentation W of V which is Schurian, and that satisfies $\mathrm{Ext}_Q(W,W) \neq 0$. Since $\mathrm{Ext}_Q(W,W)$ is the tangent space to the deformations of a Schurian representation W, and since this space is unobstructed ($\mathrm{Rep}(Q)$ is hereditary), we get that Q is of infinite type as soon as V is not Schurian. To understand why W exists, one notes that V having non-trivial endomorphisms allows the construction of a nilpotent endomorphism f. Choose f to be of minimal rank so that $f^2 = 0$. Then, choose an indecomposable summand W of $\ker(f)$ intersecting non-trivially $\mathrm{Im}(f)$. It is easy to check that W has $\mathrm{Ext}_Q(W,W) \neq 0$. If W is again not Schurian, we get to the desired W by iterating this process (see [12] for this proof).

2.3. Tame quivers: Euclidean graphs

We have just seen that quivers of finite type are classified. There is another class of quivers, which are of infinite type, but still exhibit a very moderate behavior, as far as the families of their representations are concerned. This is the class of quivers of tame type.

DEFINITION 5. If Q is of infinite type, and, for each dimension vector α, all isomorphism classes of finite-dimensional indecomposable representations of Q of dimension vector α form a finite number of families of dimension at most 1, then Q is said to be of *tame type*.

EXAMPLE 8. The Kronecker quiver Θ_2 is of tame type over any algebraically closed field. We have said that a representation V of Θ_2 of dimension vector (m,p) is a pair of matrices M_1, M_2, with $M_i : k^m \to k^p$, and that this amounts to a matrix of linear forms:

$$M = x_1 M_1 + x_2 M_2 : \mathscr{O}_{\mathbb{P}^1}(-1)^m \to \mathscr{O}_{\mathbb{P}^1}^p.$$

These pencils are classified up to linear coordinate change on k^m and k^p according to their Kronecker-Weierstrass canonical form, going back to an observation of Weierstrass [54], taken up again by Kronecker in [40]. Let us sketch this here, and refer to [5, Chapter 19.1] or [7] for a comprehensive treatment. We write $M \simeq M' \boxplus M''$ if M is equivalent to a block matrix having M' and M'' on the diagonal and zero elsewhere. Given positive integers u, v, one defines:

$$\mathfrak{C}_u = \begin{pmatrix} x_1 & & & \\ x_2 & x_1 & & \\ & x_2 & \ddots & \\ & & \ddots & x_1 \\ & & & x_2 \end{pmatrix}, \qquad \mathfrak{B}_v = \begin{pmatrix} x_1 & x_2 & & & \\ & x_1 & x_2 & & \\ & & \ddots & \ddots & \\ & & & x_1 & x_2 \end{pmatrix},$$

where $\mathfrak{C}_u$ has size $(u+1)\times u$ and $\mathfrak{B}_v$ has size $v\times(v+1)$. Also, given $a\in k$ and a positive integer n one defines:

$$J_{a,n}=\begin{pmatrix} a & 1 & & \\ & \ddots & \ddots & \\ & & a & 1 \\ & & & a \end{pmatrix}\in k^{n\times n},\qquad \text{and:}\qquad \mathfrak{J}_{a,n}=x_1 I_n + x_2 J_{a,n}.$$

Up to a change of variables in $\mathbb{P}^1$, we may assume that $\infty=(0:1)\in\mathbb{P}^1$ is not critical for M, i.e. that M has no *infinite elementary* divisors.

PROPOSITION 1. *Up to possibly changing basis in $\mathbb{P}^1$, M is equivalent to:*

$$\mathfrak{C}_{u_1}\boxplus\cdots\boxplus\mathfrak{C}_{u_r}\boxplus\mathfrak{B}_{v_1}\boxplus\cdots\boxplus\mathfrak{B}_{v_s}\boxplus\mathfrak{J}_{n_1,a_1}\boxplus\cdots\boxplus\mathfrak{J}_{n_t,a_t}\boxplus\mathfrak{Z}_{a_0,b_0},$$

for some integers r,s,t,a_0,b_0 and u_i,v_j,n_k, and some $a_1,\ldots,a_t\in k$, where $\mathfrak{Z}_{a_0,b_0}$ is the zero matrix of size $a_0\times b_0$.

To have an insight on the Kronecker-Weierstrass normal form and of a matrix pencil, it is useful to note that:

$$\mathrm{cok}(\mathfrak{C}_u)\simeq\mathscr{O}_{\mathbb{P}^1}(u),\qquad \ker(\mathfrak{B}_v)\simeq\mathscr{O}_{\mathbb{P}^1}(-v-1),$$

Note that these are all the indecomposable torsionfree sheaves on $\mathbb{P}^1$, if we allow $u,v\geq 0$. All of these sheaves are exceptional, and the dimension vectors $(u+1,u)$ and $(v,v+1)$ are real Schur roots.

On the other hand, $\mathrm{cok}(\mathfrak{J}_{a,n})\simeq\mathscr{O}_{n[-a,1]}$, the structure sheaf of the point $[-a,1]\in\mathbb{P}^1$, counted with multiplicity n. This sheaf is indeed indecomposable and varies in the $\mathbb{P}^1$ parametrized by a.

We refer to [22,24,44] for the proof of the result analogous to Gabriel's theorem, for quivers of tame type.

THEOREM 2. *Let Q be a finite connected quiver. Then Q is of tame type if and only if its underlying undirected graph is Euclidean of type $\widehat{A}$, $\widehat{D}$, $\widehat{E}$.*

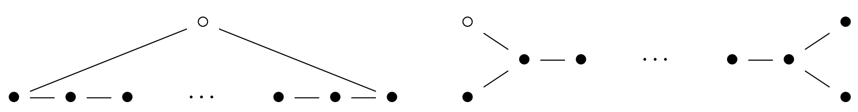


Figure 2: Euclidean graphs of type $\widehat{A}_n$ and $\widehat{D}_n$.

In this picture, we see the 3 exceptional simply-laced Euclidean graphs.

Figure 3: Euclidean quivers of type $\widehat{E}_6, \widehat{E}_7, \widehat{E}_8$.

Similarly to the case of quivers of finite type, for tame quivers one is reduced to classify positive semidefinite (but not definite) generalized Cartan matrices.

2.4. Finite-tame-wild trichotomy

We have sketched the classification of quivers of finite and tame types. One can now ask what is the behavior of families of indecomposable representations over the remaining quivers. The first answer is that these families are *large*, in the sense that there is no integer bounding the dimension of families of indecomposable representations. The second answer will be provided by Kac's theorem, that says that the dimension vectors of indecomposable representations are still controlled by the roots of a Lie algebra. A third answer will be the construction of the moduli space of representations: this allows to describe the set of representations, at least those that are semi-stable, as an algebraic variety.

Looking at the first of these topics, we say that a quiver Q is of *wild type* (over k) if the category $\mathrm{Rep}_k(Q)$ contains $\mathrm{Rep}_k(L)$ as a full subcategory, where L is the *double loop quiver*. We refer to [50, Lecture 6] (in those notes, *wild* is replaced by *strictly wild*). If Q is of wild type, then there are arbitrarily large families of non-isomorphic finite-dimensional indecomposable representations.

FACT 4 (Finite-tame-wild trichotomy). A connected quiver Q which is neither of finite nor of tame type is of wild type.

To understand this, one can construct explicitly a finite number of graphs admitting a dimension vector α with $(\alpha, \alpha) = -1$. These graphs include a 1-edge extension of the Euclidean graphs.

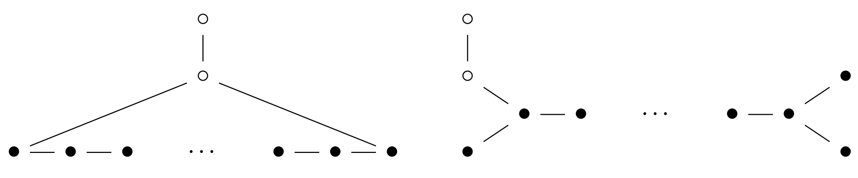
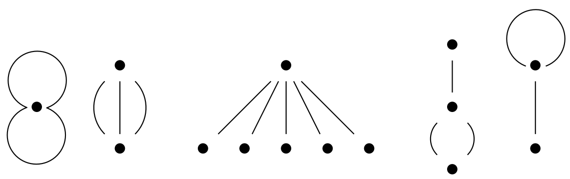


Figure 4: Extended graphs of type $\widehat{\widehat{A}}_n$ and $\widehat{\widehat{D}}_n$.

In this picture, we see the 3 simply-laced extended graphs.

Figure 5: Extended graphs of type $\widehat{E}_6, \widehat{E}_7, \widehat{E}_8$.

There are three more minimal wild graphs:



Any quiver whose generalized Cartan matrix is indefinite will contain one of these quivers. Finally one constructs for all these quivers Q a subcategory of $\mathrm{Rep}(Q)$ equivalent to that of the double loop quiver. This is achieved by considering some Ext-quivers (see [7, Section 4.1]), and using Ringel's simplification process, [49].

2.5. Kac's theorem

An analogue of Gabriel's theorem for wild quivers is provided by fundamental work of Kac, [36,37]. Let Q be a finite quiver, and let C be the generalized Cartan matrix associated to the underlying undirected graph of Q. We denote by $\mathfrak{g}$ the Kac-Moody algebra associated with C.

THEOREM 3 (Kac). *Assume k is either finite, or algebraically closed, and that Q has no loops. Then we have the following.*

i) If V is an indecomposable representation with $\underline{\dim}(V) = \alpha$, then α is a positive root of $\mathfrak{g}$.

ii) If α is a real root of $\mathfrak{g}$, then, up to isomorphism, there exists a unique indecomposable representation V with $\underline{\dim}(V) = \alpha$.

iii) If α is an imaginary root of $\mathfrak{g}$ and k is algebraically closed, then there are families of dimension $\geq 1 - \langle \alpha, \alpha \rangle$ of non-isomorphic indecomposable representations V with $\underline{\dim}(V) = \alpha$. In particular Q is of infinite type.

REMARK 3. Here are some comments on this result.

i) Kac's theorem holds even for quivers with loops (but one has to define the corresponding root system).

ii) In the case of non-algebraically closed fields, we say that an indecomposable representation V of Q is *absolutely indecomposable* if V remains indecomposable over the algebraic closure of k. Kac proved that, if V corresponds to a real root, then V is absolutely indecomposable and is defined over the prime field of k.

iii) If $k = \mathbb{F}_q$ is finite, one can count the number $a_\alpha(q)$ of absolutely indecomposable $\mathbb{F}_q$-representations of Q with dimension vector α. Kac proved that $a_\alpha(q) \in \mathbb{Z}[q]$ and conjectured that $a_\alpha(q)$ should be in $\mathbb{N}[q]$. This was proved in [15] for indivisible α and in [34] for arbitrary α.

iv) In order to take into account the whole enveloping algebra of $\mathfrak{g}$, and not only the positive part, one has to deal with Nakajima's quiver varieties, [42,43].

3. Derksen-Weyman-Schofield semi-invariants

We will give an outline of the main result of [26]. Let Q be a quiver without oriented loops, and k be a field. Set $\Gamma = \Gamma_Q$. Let us also fix a dimension vector β of Q and a character σ of $\mathrm{GL}(Q,\beta)$.

For any $\alpha \in \mathbb{N}^\Gamma$ such that $\langle \alpha, \beta \rangle = 0$ and any pair or representations V, W Q over k with dimension vectors respectively α, β, we have a square matrix:

$$d_W^V : \bigoplus_{x \in Q_0} \mathrm{Hom}_k(V(x), W(x)) \longrightarrow \bigoplus_{a \in Q_1} \mathrm{Hom}_k(V(ta), W(ha)).$$

We define the semi-invariant c of the action of $\mathrm{GL}(Q,\alpha) \times \mathrm{GL}(Q,\beta)$ on $\mathrm{Rep}(Q,\alpha) \times \mathrm{Rep}(Q,\beta)$ by:

$$c(V,W) = \det(d_W^V).$$

This is well-defined, up to a scalar which will be fixed once chosen a basis for the vector spaces $V(x)$ and $W(x)$ for all $x \in Q_0$. For fixed $V \in \mathrm{Rep}(Q,\alpha)$, we get a semi-invariant c^V of the action of $\mathrm{GL}(Q,\beta)$ on $\mathrm{Rep}(Q,\beta)$, also well-defined up to a scalar. The weight of c^V is $\langle \alpha, - \rangle : \gamma \mapsto \langle \alpha, \gamma \rangle$.

THEOREM 4. *The ring* $\mathrm{SI}(Q,\beta)$ *is a k-linear span of the semi-invariants* c^V*, where V runs through all representations of Q with* $\langle \underline{\dim}(V), \beta \rangle = 0$.

REMARK 4. As an algebra, $\mathrm{SI}(Q,\beta)$ is generated by c^V such that V is indecomposable. This follows from the fact that, given representations V, W with $\langle \underline{\dim}(V), \underline{\dim}(W) \rangle = 0$, if we have an exact sequence of representations:

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0,$$

then:

$$\langle \underline{\dim}(V'), \underline{\dim}(W) \rangle = 0 \quad \Rightarrow \quad c^V(W) = c^{V'}(W) c^{V''}(W).$$

The proof of the theorem is delivered in three steps.

i) Reduce to the case that Q has one sink and one source, plus vertices with zero-weight.

ii) Eliminate vertices x with zero-weight, by replacing pairs (a,b) of arrows having $ta = hb = x$ with a single arrow c having $tc = tb$ and $hc = ha$, so that c represents ba. By the fundamental theorems of invariant theory (see Example 6) invariants of matrices for GL are given by such compositions.

iii) Treat the case of the Kronecker Θ_m quiver, with weights $(1,-1)$.

4. Quivers with relations

In many cases of interest, one should consider not all the k-representations of a given quiver Q, but only those that satisfy some relations. This leads to introduce relations in the path algebra kQ, and one speaks of quiver with relations, or *bound quiver*. This has drastic effects on the category $\mathrm{Rep}(Q)$, which is no longer hereditary. All moduli problems tend thus to be much more complicated, in particular the deformation space to a Schurian representation need no longer be smooth, for obstructions are present in general.

DEFINITION 6. A *relation* ρ of Q is a linear combination $\rho = \sum \lambda_i p_i$ of paths p_i, all having same head and tail. Given a representation V of Q, we can evaluate $\rho(V)$ by considering $\rho(V) = \sum \lambda_i p_i(V)$, where for each path $p = a_1, \dots, a_m$ we define $V(p) = V(a_1) \circ \cdots \circ V(a_m)$. Given a finite set R of relations of Q, a *k-representation V of (Q,R)* is a k-representation V of Q such that $\rho(V) = 0$, for all $\rho \in R$. The category $\mathrm{Rep}(Q,R)$ of k-representations of (Q,R) is also an abelian k-linear category.

Given a relation ρ, we get a two-sided ideal (ρ) of kQ. Likewise, R determines an ideal (R) of kQ.

FACT 5. The category of left modules over the quotient algebra $kQ/(R)$ is equivalent to the category of representations with relations $\mathrm{Rep}(Q)_R$.

For a proof, we refer to [3, Chapter III].

4.1. Tilting bundles

An interesting example of quiver with relations arises when looking at tilting bundles on a smooth projective variety X over an algebraically closed field k. One considers the derived category $\mathscr{D}^b(X)$ of bounded complexes of coherent sheaves on X, and defines, for a vector bundle $\mathscr{T}$ on X, the algebra $B = \mathrm{End}_X(\mathscr{T})$. This gives rise to a functor $\mathrm{coh}(X) \to B-\mathrm{mod}$ that sends $\mathscr{T}$ to B, and any other coherent sheaf $\mathscr{E}$ on X to $\mathrm{Hom}_X(\mathscr{T},\mathscr{E})$, seen as a B-module. The associated derived functor is denoted by $\Phi_{\mathscr{T}}$:

$\mathscr{D}^b(X) \to \mathscr{D}^b(B-\mathrm{mod})$, and $\mathscr{T}$ is called *tilting* if $\Phi_{\mathscr{T}}$ is an equivalence, see [4]. Equivalently, this means that the endomorphism algebra of $\mathscr{T}$ has finite global dimension (i.e., the projective dimension of any module over this algebra is bounded by a fixed integer), that $\mathscr{T}$ has no higher self-extensions ($\mathrm{Ext}^k_X(\mathscr{T},\mathscr{T}) = 0$ for $k > 0$) and that the indecomposable direct summands $\mathscr{T}_1,\dots,\mathscr{T}_n$ of $\mathscr{T}$ generate the derived category of X. A quiver with relations arises when considering a vertex o_i for each direct summands $\mathscr{T}_i$, and a basis of the vector space $\mathrm{Hom}_X(\mathscr{T}_i,\mathscr{T}_j)$ as the set of arrows $o_i \to o_j$. Relations are natural too: they are generated by linear combinations of paths corresponding to the kernel of the multiplication map $\mathrm{Hom}_X(\mathscr{T}_i,\mathscr{T}_k)\otimes \mathrm{Hom}_X(\mathscr{T}_k,\mathscr{T}_j) \to \mathrm{Hom}_X(\mathscr{T}_i,\mathscr{T}_j)$, for all i, j, k.

We refer to [17] for a nice treatment of tilting bundles in relation with quiver representations. A fundamental example is the following.

EXAMPLE 9. For any integer $n \geq 1$, the *Beilinson bound quiver* $\mathcal{B}_n$ for $\mathbb{P}^n$ has $n+1$ vertices $o_0,\dots,o_n$. From a vertex o_i to a vertex o_{i+1}, we draw $n+1$ arrows, labelled by $x_0,\dots,x_n$, and by the vertex i. For instance, $\mathcal{B}_2$ looks:

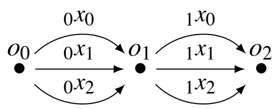


Figure 6: Beilinson quiver $\mathcal{B}_2$.

The relations R in the path algebra $k\mathcal{B}_n$ are generated by the products of the following form, for all $0 \leq h \leq n$ and all i, j:

$$\rho_{i,j} = {}_{h+1}x_i \cdot {}_hx_j - {}_{h+1}x_j \cdot {}_hx_i.$$

The algebra $k\mathcal{B}_n$ is isomorphic to:

$$\mathrm{End}_{\mathbb{P}^n}(\mathscr{T}), \quad \text{with } \mathscr{T} = \bigoplus_{i=0,\dots,n} \mathscr{O}_{\mathbb{P}^n}(i).$$

In fact, the quiver associated with $\mathscr{T}$, for $n = 2$ would look like:

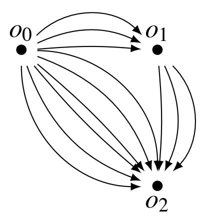


Figure 7: Beilinson quiver $\mathcal{B}_2$.

However, this gives rise to the same quiver with representations. The theorem of Beilinson asserts that the derived category of bounded complexes of coherent sheaves on $\mathbb{P}^n$ is equivalent to the derived category of finitely generated modules over this algebra, see [6]. We refer for instance to [35] for an account of Beilinson's theorem and several related topics.

4.2. Homogeneous bundles

An interesting application of quiver representations is the description of homogeneous vector bundles over rational homogeneous variety, we refer to [9,46]. For this section we work over $k = \mathbb{C}$.

A rational homogeneous variety is a product of projective varieties acted on transitively by simple Lie groups. Each of these factors is then of the form $X = G/P$, with G a simple affine algebraic group over $\mathbb{C}$, and P a parabolic subgroup. Recall that P is not reductive in general. In fact, one writes $P = LN$, where L is reductive (called a *Levi factor* of P) and N is a nilpotent group, non-trivial in general. In terms of Lie algebras, we have a parabolic subalgebra $\mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of G, and a decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$, where $\mathfrak{l}$ is a reductive Lie algebra and $\mathfrak{n}$ is the nilpotent radical of $\mathfrak{p}$.

These varieties are completely classified. Indeed, the choice of G corresponds to the choice of a Dynkin diagram of type A_n, B_n, C_n, D_n for some n, or E_6, E_7, E_8 or F_4 or G_2. Once this choice is made, the subgroup P corresponds to the choice of a subset of vertices of the Dynkin graph. The Levi part $\mathfrak{l}$ is obtained by removing the chosen vertices from the Dynkin diagram of $\mathfrak{g}$, and taking the associated Lie algebra.

A particularly simple class of rational homogeneous varieties is that of *cominuscule* varieties, also called *compact Hermitian symmetric varieties*: the variety X is of this kind, by definition, if $[\mathfrak{n},\mathfrak{n}] = 0$. In the notation of [10], the possible pairs of Dynkin diagram and vertex (only one vertex must be chosen) are (A_n, α_j) (Grassmannians); (B_n, α_1), (D_n, α_1) (quadrics); or (D_n, α_n), (D_n, α_{n-1}) (spinor varieties); or (C_n, α_n) (Lagrangian Grassmannians); or (E_6, α_1), (E_6, α_6) (the Cayley plane) or the E_7 variety (E_7, α_7). In the last two cases, X is said to be of *exceptional type*.

A vector bundle E on X is *homogeneous* if, for all $g \in G$, one has $g^*E \simeq E$. The category of homogeneous bundles on X is equivalent to the category of finite-dimensional representations of P, or of $\mathfrak{p}$. Indeed, starting with E one restricts to E to the point e of X corresponding to the unit of G, and gets a vector space with a P-action, and vice-versa starting with a P-module M we get a homogeneous bundle by $M \times_P G$. For instance, $\mathfrak{n}$ corresponds to Ω_X.

Given a homogeneous bundle E and the associated representation M of P, restricting the action to the Levi factor L gives a splitting of M into irreducible L-modules M_i. Letting N act trivially on these factors and taking the associated bundles F_i we get the graded $\mathrm{gr}(E) = \oplus F_i$; in fact E has a filtration (non-split in general) with successive quotients isomorphic to the F_i. The *irreducible* bundles F_i are in bijection with the weights of $\mathfrak{g}$ which are dominant for the semi-simple part $[\mathfrak{l},\mathfrak{l}]$ of $\mathfrak{p}$. If $P = P(\alpha_j)$, such weights are of the form $\sum n_i\lambda_i$ with $n_i \geq 0$ for all $i \neq j$, where $(\lambda_1, \ldots, \lambda_n)$ are the fundamental weights of G.

However, the P-module structure of M, and hence the vector bundle E, are recovered from $F = \mathrm{gr}(E)$ by the equivariant maps, induced from one another:

$$\theta : \Omega_X \otimes F \to F, \qquad \theta_e : \mathfrak{n} \otimes (\oplus_i M_i) \to \oplus_i M_i.$$

In fact θ_e lifts to a map of $\mathfrak{p}$-modules if and only if $\theta_e \wedge \theta_e = 0$, and in this case the lift is unique (θ_e gives the so-called structure of Higgs module). Similarly, $(\mathrm{gr}(E), \theta)$ is called a Higgs bundle if $\theta \wedge \theta = 0$.

It turns out that the representations of P can be conveniently described by a bound quiver (Q, R) associated with X. The *vertices of Q* are indexed by the irreducible representations of $\mathfrak{l}$. Given two such representations and the associated weights λ, ν, we draw an *arrow* from the corresponding vertices $x_\lambda \to x_\nu$ in Q if and only if $\mathrm{Ext}^1(E_\lambda, E_\nu)^G \neq 0$. It turns out that, when this space is non-zero, it is one-dimensional. The *relations R* of Q, if X is not of exceptional type, are generated by the following two basic kinds:

i) for all weights λ, ξ, and η, and all diagrams:

$$\begin{array}{ccc} E_{\lambda+\xi} & \xleftarrow{\;a\;} & E_\lambda \\ {\scriptstyle d}\big\downarrow & & \big\downarrow{\scriptstyle b} \\ E_{\lambda+\eta+\xi} & \xleftarrow[c]{} & E_{\lambda+\eta} \end{array}$$

we have, if $\mathrm{Ext}^2(E_\lambda, E_{\lambda+\eta+\xi})^G \neq 0$, a relation of the form $da - cb = 0$.

ii) for all weights as above, and all diagrams:

$$\begin{array}{ccc} & & E_\lambda \\ & & \big\downarrow{\scriptstyle b} \\ E_{\lambda+\eta+\xi} & \xleftarrow[c]{} & E_{\lambda+\eta} \end{array}$$

we have, if $\mathrm{Ext}^2(E_\lambda, E_{\lambda+\eta+\xi})^G \neq 0$, a relation of the form $cb = 0$.

These relations are called *commutativity of all squares*. Note that Q is an infinite quiver; however, we only need representations of Q with finite support. Moreover, Q is disconnected, has no loops, and two vertices are joined by one arrow at most. For instance, for $X = \mathbb{P}^2$, Q has three connected components. The vertices of the component of Q containing $\mathscr{O}_{\mathbb{P}^2}$ are in bijection with the pairs of integers (i, j) with $i \geq 0$ and $j \leq 0$, and the point (i, j) corresponds to $\mathrm{Sym}^{i-j}\mathscr{Q}(i+2j)$, where $\mathscr{Q} = \mathscr{T}_{\mathbb{P}^2}(-1)$ is the rank-2 *quotient bundle*. The arrows go from (i,j) to $(i+1,j)$ and $(i,j-1)$. This component begins as follows, and continues indefinitely in the lower right quadrant:

$$\begin{array}{ccccc} \mathscr{O}_{\mathbb{P}^2} & \longleftarrow & \mathscr{Q}(1) & \longleftarrow & \mathrm{Sym}^2\mathscr{Q}(2) \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{Q}(-2) & \longleftarrow & \mathrm{Sym}^2\mathscr{Q}(-1) & \longleftarrow & \mathrm{Sym}^3\mathscr{Q} \end{array}$$

THEOREM 5. *Let X be a compact Hermitian symmetric space, not of exceptional type. The following categories are equivalent:*

i) G-homogeneous bundles on X;

ii) finite-dimensional representations of $\mathfrak{p}$;

iii) Higgs bundles (F,θ) on X;

iv) finite-dimensional representations of (Q,R).

More results are available in this direction, let us mention two of them, from [46]. For instance, a combinatorial way to compute cohomology of a homogeneous bundle can be derived from the above result, with the aid of Borel-Bott-Weil theorem (cf. for instance [55] for more on cohomology of homogeneous bundles). Moreover, the equivalence of categories just mentioned carries over to define equivalent stability conditions, in terms of equivariant slope-semi-stability for homogeneous bundles, or in terms of GIT for representations of (Q,R). Therefore, one can speak of moduli space of homogeneous bundles in terms of moduli spaces of representations of (Q,R).

5. Moduli spaces of quiver representations

Let us fix a quiver Q, and an algebraically closed field k. We have seen that, once fixed a dimension vector α of Q, all representations of dimension vector α of Q form a vector space, acted on by a product of general linear groups. One would like to think of the quotient space as variety of isomorphism classes of representations of dimension vector α, or as the *moduli space* of these representations. As usual in this situation, one has to throw away some representations in order to put a structure of variety on the quotient, so that geometric invariant theory comes into play to select an open subset where the quotient behaves well. We will sketch the construction here, following the fundamental paper of King, [39].

5.1. Stability of quiver representations

Let us fix a dimension vector α and a character θ of $G(Q,\alpha) = \mathrm{GL}(Q,\alpha)/k^* \,\mathrm{id}$. The character $\theta = (\theta_x)_{x \in Q_0}$ is just a character of $\mathrm{GL}(Q,\alpha)$, and it is non-trivial if and only if not all θ_x equal a fixed integer.

DEFINITION 7. Let V be a representation of dimension vector α. Then V is θ-*semi-stable* if there is $f \in \mathrm{SI}(Q,\alpha)_{t\theta}$, with $t>0$, such that $f(V) \neq 0$. If moreover the orbit $G.V$ is closed of dimension $\dim(G)$, then V is *stable*. It turns out that a semi-stable representation admits a filtration with stable factors. The direct sum of these factors is the *graded object* associated with a such filtration. Two semi-stable representations with filtrations giving rise to the same graded object are said to be *S-equivalent*.

We write $\mathrm{Rep}(Q,\alpha)^{ss}$ and $\mathrm{Rep}(Q,\alpha)^{s}$ for the open subsets of semi-stable and stable representations (with respect to θ), respectively. The character θ is *generic* if all θ-semi-stable representations are θ-stable.

Let $\sigma \in \Gamma^*$ and assume $\theta = \theta_\sigma$. We see that, if $V \in \mathrm{Rep}(Q,\alpha)$ is θ-stable, then $\sigma(\alpha)=0$, indeed for any $0 \neq \lambda \in k^*$, since $\lambda\,\mathrm{id}$ acts trivially on V, the semi-invariant f satisfies:

$$0 \neq f(V) = f(\lambda \mathrm{id} V) = \theta(\lambda) f(V) = \lambda^{\sigma(\alpha)} f(V).$$

FACT 6. The representation V is θ-semi-stable (respectively, stable) if and only if, for all proper subrepresentations W of V, we have $\sigma(\underline{\dim}(W)) \leq 0$ (respectively, $\sigma(\underline{\dim}(W)) < 0$).

5.2. Moduli space and ring of semi-invariants

We have a quotient map:

$$\mathrm{Rep}(Q,\alpha)^{ss} \longrightarrow \mathrm{Proj}\left(\bigoplus_{t\geq 0} \mathrm{SI}(Q,\alpha)_{t\theta}\right).$$

DEFINITION 8. Let α be a dimension vector of Q and θ be a character of the group $G(Q,\alpha)$. Then the *moduli space* $\mathscr{M}_Q(\alpha,\theta)$ is defined as:

$$\mathscr{M}_Q(\alpha,\theta) = \mathrm{Rep}(Q,\alpha) /\!\!/_{\theta}\, G = \mathrm{Proj}\left(\bigoplus_{t\geq 0} \mathrm{SI}(Q,\alpha)_{t\theta}\right).$$

This space parametrizes S-classes of $G(Q,\alpha)$-orbits of θ-semi-stable representations of Q with dimension vector α. This space contains the open set $\mathscr{M}_Q(\alpha,\theta)^s$ of $G(Q,\alpha)$-orbits of θ-stable representations of Q with dimension vector α. Here, two representations V,W of Q are in the same S-class if they both admit filtrations of subrepresentations having θ-stable factors, in such a way that the two direct sums of these factors are isomorphic.

FACT 7 (King). The space $\mathscr{M}_Q(\alpha,\theta)$ is a projective variety, and if α is indivisible $\mathscr{M}_Q(\alpha,\theta)^s$ is a fine moduli space for families of θ-stable representations. In this case, the set of generic characters is dense, and for any such character θ, $\mathscr{M}_Q(\alpha,\theta)^s = \mathscr{M}_Q(\alpha,\theta)$ is smooth.

Projectivity follows from the GIT statement that there is a proper map:

$$\mathrm{Proj}\left(\bigoplus_{t\geq 0}\mathrm{SI}(Q,\alpha)_{t\theta}\right)\longrightarrow \mathrm{Spec}\left(\mathrm{Rep}(Q,\alpha)^G\right),$$

and from the fact that the second scheme is a single point. The fact that the moduli space is fine (i.e., that there exists a universal family of representations over it) follows from a descent argument of the universal sheaf to the quotient by G. Smoothness over the stable locus follows from the fact that $\mathrm{Rep}(Q)$ is hereditary.

5.3. Schofield's result on birational type of quiver moduli

One more interesting fact concerning moduli spaces is that, once fixed Q and α, all moduli spaces $\mathscr{M}_Q(\alpha,\theta)$ are birational (when non-empty). The birational transformation between $\mathscr{M}_Q(\alpha,\theta)$ and $\mathscr{M}_Q(\alpha,\theta')$ is called "wall-crossing". A very nice result on the birational type of the moduli spaces $\mathscr{M}_Q(\alpha,\theta)$ is due to Schofield, [51]. We would like to review it here. First of all, we have that, for a given quiver Q and a fixed dimension vector α, there is a stable representation for a given character θ if and only if α is a Schur root.

THEOREM 6. *Let α be Schur root for Q, and let θ be a character of* $\mathrm{GL}(Q,\alpha)$ *such that there exists a θ-stable $V\in\mathrm{Rep}(Q,\alpha)$. Then $\mathscr{M}_Q(\alpha,\theta)$ is birational to the moduli space $(k^{m\times m})^p /\!\!/ \mathrm{GL}_m(k)$ of p matrices of size m up to simultaneous conjugacy, for suitable m,p.*

Note that the rationality, and even the stable rationality of the above moduli space of matrices up to conjugacy, is unknown in general.

The proof of this result is carried out along these lines:

i) One shows the result for representations of the Kronecker quiver $Q=\Theta_n$, as follows: given a dimension vector (a,b) for Θ_n, with a,b coprime, one constructs two representations V_1 and V_2 such that:

   - for a general representation V of dimension vector (ma,mb) of Θ_n, we have $\dim\mathrm{Hom}_Q(V_1,V)=\dim\mathrm{Hom}_Q(V_2,V)=m$;
   - for the same V, we have $\mathrm{Ext}_Q(V_1,V)=\mathrm{Ext}_Q(V_2,V)=0$;
   - we have $\dim_k\mathrm{Hom}_Q(V_1,V_2)=1+p$, with $p=1-\langle(a,b),(a,b)\rangle=1-a^2-b^2+nab$.

   With this setup, we have that $\mathrm{Hom}_Q(V_1\oplus V_2,-)$ carries a general representation V to a representation of dimension vector (m,m) of Θ_{p+1}. Up to the action of change of basis, this gives a birational equivalence between the moduli space of Θ_n-representation of dimension vector (ma,mb) and $(k^{m\times m})^p /\!\!/ \mathrm{GL}_m(k)$.

ii) The general case is reduced to the case of a Kronecker quiver by showing that a Schur root α admits smaller Schur roots β and γ such that $\alpha = a\beta + b\gamma$. This allows reduction to a dimension vector (a,b) of a quiver with only two vertices (and only loops at the first vertex or arrows from the first to the second vertex).

5.4. More reading

Here are some suggestions on further topics tightly related to the above material. There is a lot more directions that one could be interested in, this is only an extremely partial and an subjective point of view.

Representation type of algebras and varieties

In view of the separation of quivers into the three different kinds finite, tame or wild, according to their representation theory, one can ask about the representation type of more general algebras, see for instance [23,25,45] for some early work in this direction The literature in this area is fairly vast.

For graded Cohen-Macaulay rings R, this question leads to study families of maximal Cohen-Macaulay (MCM) modules over R (see for instance [2]). If R is the homogeneous coordinate ring of a smooth projective variety X, these families can be seen as families of vector bundles over X, whose intermediate cohomology modules are all zero.

One result of great interest for algebraic geometers is the classification of varieties of finite type in this sense (there are only finitely many indecomposable MCM modules over their coordinate rings). These are: projective spaces, smooth quadrics, rational normal curves, the Veronese surface in $\mathbb{P}^5$, and the rational cubic scroll in $\mathbb{P}^4$, see [29] and references therein. Although the finite-tame-wild trichotomy is not exhaustive for all varieties, it still might be so for smooth varieties. As last remarks let us note that several varieties are known by now to be of wild representation type, see for instance [16] for Segre products. Quite interestingly, the main argument to prove such a statement is to construct families of vector bundles with few cohomology by using again the Kronecker quiver Θ_m, plus Kac's theorem to see that the induced bundles are simple.

Cohomological properties of quiver moduli

The Betti numbers of the moduli spaces $\mathscr{M}_Q(\alpha,\theta)$ have been computed in [48]. The method here goes through a Hall-algebra variation of the Harder-Narasimhan method used for vector bundles over projective curves, plus Weil conjectures to reduce to counting quiver representations over finite fields.

In a different spirit, a nice paper by Craw investigates the geometry and the derived category of the moduli spaces $\mathscr{M}_Q(\alpha,\theta)$ when α is indivisible, θ is generic and Q has no loops. These moduli spaces are towers of Grassmann bundles, and are equipped with a natural tilting bundle, see [19].

Toric varieties and quiver moduli

Toric varieties are closely related to moduli spaces of representations of quivers. In particular, Craw and Smith proved that a projective toric variety is a fine moduli space of representations of an appropriate bound quiver, [20], see also [18] for a comprehensive treatment. The quivers Q and dimension vectors α involved in this construction are so that α has all values equal to 0 or 1, hence the group $\mathrm{GL}(Q,\alpha)$ is already an algebraic torus.

Moduli of semistable sheaves as quiver moduli

Let X be a smooth projective variety over an algebraically closed field, and H be an ample divisor class on X. The moduli space of H-semi-stable sheaves on X with fixed Hilbert polynomial with respect to H has been constructed relying on ideas of Maruyama and Gieseker, and turns out to be a projective variety (see [52], and [41] for the positive characteristic case).

A result due to Álvarez-Cónsul and King, [1] ties in this construction with the moduli spaces $\mathscr{M}_Q(\alpha,\theta)$. Indeed, given X, L and the Hilbert polynomial $P(t)$, one can choose two integers m,p such that for any coherent sheaf E on X with Hilbert polynomial $P(t)$, one has $P(m) = \mathrm{h}^0(E(m))$, $P(p) = \mathrm{h}^0(E(p))$ and:

$$\mathrm{H}^0(X,E(m)) \otimes \mathscr{O}_X(-m) \twoheadrightarrow E,$$
$$\mathrm{H}^0(X,E(m)) \otimes \mathrm{H}^0(X,\mathscr{O}_X((p-m)) \twoheadrightarrow \mathrm{H}^0(X,E(p)).$$

Set $n = \mathrm{h}^0(X,\mathscr{O}_X((p-m))$. This way, the point E of the moduli space is identified with a representation of Θ_n of dimension vector $(P(m),P(p))$. One more amazing manifestation of the ubiquitous Kronecker quiver!

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