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FIVE LECTURES ON PROJECTIVE INVARIANTS

Abstract. We introduce invariant rings for forms (homogeneous polynomials) and for d points on the projective space, from the point of view of representation theory. We discuss several examples, addressing some computational issues. We introduce the graphical algebra for the invariants of d points on the line. This is an expanded version of the notes for the School on Invariant Theory and Projective Geometry, Trento, September 17-22, 2012.

Le teorie vanno e vengono ma le formule restano.[5] *G.C. Rota*

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[*]member of GNSAGA-INDAM.

[5]The theories may come and go but the formulas remain

1. Introduction and first examples

1.1. What is invariant theory, and the more modest aim of these lectures

Invariant theory is a classical and superb chapter of mathematics. It can be pursued from many points of view, and there are several excellent introductions to the subject ([3,13,15,28,30,31,39,41,46,47,53] and many others).

Given a group G acting on a variety X, we want to describe the invariant subring $A(X)^G$ inside the coordinate ring $A(X)$. This framework is very general, in the spirit of Klein's Erlangen Program.

Most of the classical work on the topic was done on invariants of forms and invariants of sets of points. In the case of invariant of forms, V is a complex vector space, $G = SL(V)$ and X is the natural embedding of $\mathbb{P}V$ in $\mathbb{P}S^dV$, which is called the d-Veronese variety (S^dV is the d-th symmetric power of V). In the case of invariants of points, there are two interesting situations. When the points are *ordered*, we have $G = SL(V)$ acting on d copies of $\mathbb{P}V$, that is on the Segre variety $\mathbb{P}V \times \ldots \times \mathbb{P}V$. When the points are *unordered*, we quotient the Segre variety by the symmetric group Σ_d. Note that when dim $V = 2$, d unordered points are described in equivalent way by a homogeneous polynomial of degree d in two variables, and we reduce again to the case of forms, here X is the rational normal curve. Note that, in the dual description, points correspond to hyperplanes and we get the well known "arrangement of hyperplanes".

In XIX century invariants were constructed by means of the two fundamental theorems, that we review in §3.10 for invariants of forms and in §5.1 for invariants of points. The First Fundamental Theorem (1FT for short) claims that all the invariants can be constructed by a clever combinatorial procedure called "symbolic representation" (see §3.9). Our approach is close to [31,47], where the 1FT is obtained as a consequence of Schur-Weyl duality. The classical literature is rich of interesting examples and computations. As main textbooks from the classical period we recommend [21]. Also several parts from [17,50] are developed in the setting of invariant theory. Hilbert

lectures [24] deserve a special mention. They are the translation of handwritten notes of a course held by Hilbert in 1897 at Göttingen. The reading of such master work is particularly congenial to understand the modern development of invariant theory. The "symbolic representation" of XIX century was a hidden way to introduce Schur functors and representation theory, not yet having a formal setting for them.

In the case of forms, the invariants of degree m for d-forms are the $SL(V)$-invariant subspace of $S^m(S^d V)$. The decomposition of this space as a sum of irreducible representations is a difficult problem called *plethysm*. Although there are algorithms computing this decomposition, for any m, d, a simple description is missing.

The case $\dim V = 2$ is quite special. The elements of $S^d V$ are called binary forms of degree d (ternary forms correspond to $\dim V = 3$ and so on) and most of classical results regarded this case. The common zero locus of all the invariant functions is called the "nullcone", and it coincides with the nilpotent cone of Manivel's lectures [37] when X is the Lie algebra of G and the action is the adjoint one.

In these lectures we try to give some tools to apply and use projective invariants, possibly in related fields, involving algebraic geometry or commutative algebra. This aim should guide the volunteered reader into useful and beautiful mathematics.

There are several approaches to invariant theory. Some examples look "natural" and "easy" just from one point of view, while they look more sophisticated from other points of view. So it is important to have a plurality of descriptions for the invariants, and to look for several examples. During lectures, understanding is more important than efficiency, so we may prove the same result more than once, from different points of view.

Computers opened a new era in invariant theory. Anyway, there are basic cases where the needed computations are out of reach, even with the help of a computer, and even more if the computer is used in a naive way. We will try to sketch some computational tricks that we found useful in mathematical practice. Our basic computational sources are [13, 53].

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1.2. The Veronese variety and its equations

Let V be a (complex) vector space of dimension $n + 1$. We denote by $S^d V$ the d-th symmetric power of V. The d-Veronese variety embedded in $\mathbb{P}S^d V$ is the image of the

map

$$
\begin{array}{ccc}
I\!\!PV & \to & I\!\!PS^dV \\
v & \mapsto & v^d
\end{array}
$$

We denote it by $v_d(I\!\!PV)$, it consists of all homogeneous polynomials of degree d in $n+1$ variables which are the d-th power of a linear form. Historically, this construction gave the main motivation to study algebraic geometry in higher dimensional spaces. Its importance is due to the fact that hypersurfaces of degree d in $I\!\!PV$ are cut out by hyperplanes in the Veronese embedding.

These elementary remarks are summarized in the following.

THEOREM 1.

- *(i) A linear function on S^dV is uniquely determined by its restriction to the Veronese variety.*

- *(ii) Linear functions over S^dV correspond to homogeneous polynomials of degree d over V.*

Proof. Let H be a linear function which vanishes on the d-th Veronese variety. It induces a homogeneous polynomial of degree d which vanishes for all the values of the variables. Hence the polynomial is zero, proving (i). (ii) is proved in the same way, since both spaces have the same dimension. □

In equivalent way, Theorem 1 says that $I\!\!PS^dV$ is spanned by elements lying on the Veronese variety. To make effective the previous Theorem, compare a general polynomial

$$
f = \sum_{i_0+\ldots+i_n=d} \frac{d!}{i_0!\ldots i_n!} a_{i_0,\ldots,i_n} x_0^{i_0}\ldots x_n^{i_n}
$$

with the d-th power

$$
(b_0 x_0 + \ldots + b_n x_n)^d = \sum_{i_0+\ldots+i_n=d} \frac{d!}{i_0!\ldots i_n!} b_0^{i_0}\ldots b_n^{i_n} x_0^{i_0}\ldots x_n^{i_n}
$$

getting the correspondence

$$
(1) \qquad\qquad b_0^{i_0}\ldots b_n^{i_n} \mapsto a_{i_0,\ldots,i_n}
$$

It will be the basic tool for the symbolic representation of an invariant that we will see in §3.9.

The conormal space (which is the annihilator of the tangent space) at a point $x \in v_d(I\!\!PV)$ can be identified with the space of hyperplanes in $I\!\!PS^dV$ containing the tangent space at x. These hyperplanes correspond to the hypersurfaces of degree d which are singular at x, which give the vector space $H^0(I_{x^2}(d))$. This can be summarized with

PROPOSITION 1 (Lasker Lemma). *The conormal space of $v_d(I\!\!PV)$ at $[x]$ is isomorphic to $H^0(I_{x^2}(d))$.*

THEOREM 2. *The Veronese variety is defined as scheme by the quadrics which are the* 2×2*-minors of the contraction*

(2) $$V^\vee \xrightarrow{\ C_f\ } S^{d-1}V$$

for $f \in S^d V$.

Proof. We have to prove that f is a power of a linear form if and only if rk $C_f = 1$. The elements in $V^\vee$ can be seen as differential operators of first order. If $f = l^d$ then $\frac{\partial f}{\partial x_i} = d \frac{\partial f}{\partial x_i} l^{d-1}$ so that ImC_f is spanned by l^{d-1}. It follows rk $C_f = 1$. Conversely, assume that rk $C_f = 1$ and let $l \in V$ be a generator of the one dimensional annihilator of kerC_f. We may assume $l = x_0$. Then $\frac{\partial f}{\partial x_i} = 0$ for $i > 0$ implies that f is a multiple of x_0^d. This proves the result set-theoretically. The proof can be concluded by a infinitesimal computation. If $f = x^d$, then it is easy to check that ker$C_x = H^0(I_x(1))$, and $(\text{Im}C_f)^\perp = H^0(I_x(d-1))$.

The conormal space of the determinantal locus at $[x]$ is given by the image of the natural map
$$H^0(I_x(1)) \otimes H^0(I_x(d-1)) \to H^0(I_{x^2}(d)).$$

It is easy to check that this map is surjective, and from Proposition 1 the result follows.
□

The quadratic equations which define the Veronese variety can be considered as a $SL(V)$-module inside $S^2(S^d V)$.

We have the decomposition

$$S^2(S^d V) = \oplus_{i=0}^{\lfloor \frac{d}{2} \rfloor} S^{2d-2i,2i}$$

so that the quadratic part of the ideal I of the Veronese variety is

$$I_2 = \oplus_{i=1}^{\lfloor \frac{d}{2} \rfloor} S^{2d-2i,2i},$$

I_2 corresponds indeed to the 2×2 minors of C_f in (2). A stronger and nontrivial result is true.

THEOREM 3. *The ideal I of the Veronese variety is generated by its quadratic part* I_2.

This Theorem is a special case of result, due to Kostant, holding for any rational homogeneous variety G/P. For a proof in the setting of representation theory see [32] Theorem 16.2.2.6, or [47] chap. 10 §6.6. For a somewhat different approach, generalized to flag varieties, see [55] Prop. 3.1.8.

The composition of two symmetric powers like $S^k(S^d V)$ is quite hard to be computed. The formula solving this problem in the case dim $V = 2$ is due to Cayley and Sylvester and it is one of the most beautiful achievements of XIX century invariant theory. We will review it in §3.4.

For $d = 3$ we have

$$S^2(S^3V)) \quad = \quad S^6V \quad \oplus \quad S^{4,2}V$$

and the equations consist of the irreducible module $S^{4,2}V$.

This is given by the 2×2-minors of the matrix

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{bmatrix}$$

In classical notation, there are "dual" variables y_i and the single Hessian covariant

(3)
$$H = \begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ y_0^2 & y_0y_1 & y_1^2 \end{vmatrix} = 0$$

For $d = 4$ we have

$$S^2(S^4V)) \quad = \quad S^8V \quad \oplus \quad S^{6,2}V \quad \oplus \quad S^{4,4}V$$

and the equations consist of the sum of the two last modules $S^{6,2}V \oplus S^{4,4}V$. The module structure is not evident at a first glance from the minors and it is commonly not considered when these equations are encountered. There is a single invariant quadric, which is called the *equianharmonic* quadric and it is classically denoted by I. It corresponds to 4-ples which are apolar to themselves (we will describe apolarity in §4.3).

We have, given a binary quartic 4

$$f = \sum_{i=0}^{4} \binom{4}{i} f_i x^{4-i} y^i$$

the expression

(4)
$$I = f_0 f_4 - 4f_1 f_3 + 3f_2^2$$

This expression gives the most convenient way to check if a given f is anharmonic. Note that from this expression the invariance is not at all evident.

We will prove in Theorem 25 that the ring of invariants for a binary quartic is generated by I and another invariant J, which has a simpler geometric construction. It is the equation of the 2nd secant variety $\sigma_2(v_4(\mathbb{P}^1))$, that is, it is the equation of the locus spanned by the secant lines at the rational normal quartic curve $v_4(\mathbb{P}^1)$.

We get

$$(5) \qquad\qquad J = \det \begin{bmatrix} f_0 & f_1 & f_2 \\ f_1 & f_2 & f_3 \\ f_2 & f_3 & f_4 \end{bmatrix}$$

Indeed the above matrix has rank 1 on $v_4(\mathbb{P}^1)$, hence it has rank ≤ 2 on any secant line.

The weight of a monomial $f_0^{v_0} \ldots f_4^{v_4}$ is by definition $\sum_{i=0}^{4} i v_i$. A (homogeneous) polynomial is called isobaric if all its monomials have the same weight. The monomials $f_0 f_4, f_1 f_3, f_2^2$ are all the monomials of degree 4 and weight 2, and we will check (see the Proposition 7) that they are the only ones that can appear in (4).

But why the coefficients in the expression (4) have to be proportional at $(1, -4, 3)$? During these lectures, we will answer three times to this question, respectively in §4.3, §3.3, in §3.7.

These answers follow different approaches that are useful ways to look at invariants.

REMARK 1. The dual variety to the Veronese variety is the discriminant hypersurface of degree $(n+1)(d-1)^n$ in $\mathbb{P}S^d V$, parametrizing all singular hypersurfaces of degree d in $\mathbb{P}V$. More generally, the dual variety to the k-secant variety (see [32] 5.1) to the d-Veronese variety (which is denoted by $\sigma_k(v_2(\mathbb{P}^n))$) consists of all hypersurfaces of degree d in $\mathbb{P}V$ with at least k double points. This follows by Terracini Lemma ([32] 5.3).

For example the dual to $\sigma_k(v_2(\mathbb{P}^n))$ (symmetric matrices of rank $\leq k$) is given by $\sigma_{n+1-k}(v_2(\mathbb{P}^n))$ (symmetric matrices of rank $\leq n+1-k$).

The dual to $\sigma_2(v_3(\mathbb{P}^2))$ is given by plane cubic curves with two double points, that is by reducible cubics. The dual to $\sigma_3(v_3(\mathbb{P}^2))$ is given by cubics with three double points, that is by triangles. This is called a split variety, and we will consider it in next subsection.

1.3. The split variety and its equations

The split variety in $\mathbb{P}S^d V$ consists of all polynomials which split as a product of linear factors. This subsection is inspired by §8.6 in [32], where the split variety is called Chow variety. The first nontrivial example is the variety of "triangles" in $\mathbb{P}S^3\mathbb{C}^3$, which is a 6-fold of degree 15.

We consider the natural map

$$S^k S^d V \to S^d S^k V$$

constructed by dividing V^{dk} represented by a $d \times k$ rectangle, first by rows (in the source) and then first by columns (in the target).

The above map takes $(x_1^d \ldots x_k^d)$ to $(x_1 \ldots x_k)^d$, so it is nonzero on the coordinate ring of the split variety.

THEOREM 4 (Brion [4]). *The kernel of the above map gives the degree k part of the ideal of the split variety of d-forms on* $\mathbb{P}V$.

Note that in case $\dim V = 2$ we have that all d-forms split and indeed the previous map is an isomorphism (Hermite reciprocity).

EXAMPLE 1. $S^2 S^3 \mathbb{C}^3 = S^6 \mathbb{C}^3 \oplus S^{4,2} \mathbb{C}^3$,
$S^3 S^2 \mathbb{C}^3 = S^6 \mathbb{C}^3 \oplus S^{4,2} \mathbb{C}^3 \oplus S^{2,2,2} \mathbb{C}^3$.

Indeed conics which split in two lines have a single invariant in degree 3, which is the determinant of the symmetric matrix defining them, while cubics which split in triangles have no equations in degree two.

Even the case $k = d$ is interesting, the natural map

$$S^d S^d V \to S^d S^d V$$

obtained by reshuffling between rows and columns, turns out to be a isomorphism for $d \leq 4$, but is is degenerate for $d = 5$. When $d = 5$ get $\binom{9}{5} = 126$ and $\binom{126+4}{5} = 286,243,776$, so the question corresponds to the rank computation for a square matrix of this size and it is already a computational challenge. It has been performed by Müller and Neunhöffer in [40]. It should be interesting to understand theoretically this phenomenon.

So there are equations of degree 5 for the split variety of "pentahedra" in $\mathbb{P}^4$.

It is interesting the split variety of triangles in the plane, which has quartic equations. These equations correspond to the proportionality between a cubic form $f \in S^3 \mathbb{C}^3$ and its Hessian $H(f)$.

REMARK 2. The next interesting variety for invariant theory is certainly the Grassmannian. We give for granted its description and the fact that its ideal is generated by the Plücker quadrics. For a proof, like in the case of Veronese variety, we may refer again to [32] Theorem 16.2.2.6. Let just remind the shape of the Plücker quadrics. The coordinates in the Plücker embedding of the Grassmannian of $k+1$ linear subspaces of V are indexed by sequences $[i_0 \ldots i_k]$ where $0 \leq i_0 < i_1 < \ldots < i_k \leq n$. Fix a subset of $k+2$ elements $i_0 \ldots i_{k+1}$ and a set of k elements $j_0 \ldots j_{k-1}$. Then the Plücker relations are

$$\sum_{s=0}^{k+1} (-1)^s [i_0 \ldots \hat{i}_s \ldots i_{k+1}][i_s j_0 \ldots j_{k-1}] = 0$$

which hold for any subsets of respectively $k+2$ and k elements.

2. Facts from representation theory

2.1. Basics about representations

In this section we recall basic facts about representation theory, that can be found for example in [19]. From a logical point of view, the facts in this section are the foundations of the following sections. Anyway, the reader may find useful reading the section 3 for a better understanding of the use of representations, and going back when needed.

We will need to study representations of two basic groups, namely the finite symmetric group Σ_d of permutations on d elements, and the group $SL(n+1)$ of $(n+1) \times (n+1)$ matrices having $\det = 1$. Both are reductive groups.

A representation of a group G is a group morphism $G \xrightarrow{\rho} GL(W)$, where W is a complex vector space. We say that W is a G-module, indeed we may write $g \cdot w = \rho(g)(w)$ for any $g \in G, w \in W$. This notation underlines that G acts over W. This action satisfies the following properties, which follow immediately from the definitions

$g \cdot (w_1 + w_2) = g \cdot w_1 + g \cdot w_2, \forall g \in G, w_1, w_2 \in W,$

$g \cdot \lambda w = \lambda g \cdot w, \forall g \in G, w \in W, \lambda \in \mathbb{C},$

$(g_1 g_2) \cdot w = g_1 \cdot (g_2 \cdot w).$

These properties resume the fact that a representation is a *linear* action.

Given two G-modules V, W, then $V \oplus W$ and $V \otimes W$ are G-modules in a natural way, namely

$g \cdot (v + w) := (g \cdot v) + (g \cdot w),$

$g \cdot (v \otimes w) := (g \cdot v) \otimes (g \cdot w).$

The m-th symmetric power $S^m W$ is a G-module, satisfying $g \cdot (v^m) := (g \cdot v)^m$.

A morphism between two G-modules V, W is a linear map $f : V \to W$ which is G-equivariant, namely it satisfies $f(g \cdot v) = g \cdot f(v) \ \forall g \in G, v \in V$.

Every G-module V corresponding to $\rho : G \to GL(V)$ has a character $\chi_V : G \to \mathbb{C}$ defined as $\chi_V(g) = \text{trace } \rho(g)$.

It satisfies the property $\chi_V(h^{-1}gh) = \chi_V(g)$ hence the characters are defined on conjugacy classes in G.

Moreover $\chi_{V \oplus W} = \chi_V + \chi_W, \chi_{V \otimes W} = \chi_V \chi_W$.

Characters are the main tool to work with representations and to identify them.

Every G-module has a G-invariant submodule

$$V^G = \{v \in V | g \cdot v = v \quad \forall g \in G\}.$$

In other words G acts trivially over V^G. The character of the trivial representation of dimension r is constant, equal to r on every conjugacy classes.

For any G-module V, G acts on the graded ring $\mathbb{C}[V] = \oplus_{m=0}^{\infty} S^m V$.

Since the sum and the product of two invariant elements are again invariant, we have the invariant subring $\mathbb{C}[V]^G = \oplus_{m=0}^{\infty} (S^m V)^G$.

2.2. Young diagrams and symmetrizers

A Young diagram denoted by $\lambda = (\lambda_1, \ldots \lambda_k)$, where $\lambda_1 \geq \lambda_2 \geq \ldots$ consists of a collection of boxes ordered in consecutive rows, where the i-th row has exactly λ_i boxes. The number of boxes in λ is denoted by $|\lambda|$.

All Young diagrams with d boxes correspond to the partitions of d, namely $\lambda = (\lambda_1, \ldots \lambda_k)$ corresponds to $|\lambda| = \lambda_1 + \ldots + \lambda_k$.

The following are the Young diagram corresponding to $(2,1,1)$ and $(4,4)$:
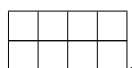
Any filling of λ with numbers is called a *tableau*.

Just to fix a convention, for a given Young diagram, number consecutively the boxes like

1	2	3	4
5	6	7	8

Here we used all numbers from 1 to d to fill d boxes. More generally, a tableau can allow repetitions of numbers, as we will see in the sequel.

Let Σ_d be the symmetric group of permutations over d elements. Due to the filling, we can consider the elements of Σ_d as permuting the boxes. Let $R_\lambda \subseteq \Sigma_d$ be the subgroup of permutations preserving each row.

Let $C_\lambda \subseteq \Sigma_d$ be the subgroup of permutations preserving each column.

DEFINITION 1. *The element*

(6)
$$c_\lambda = \sum_{\sigma \in R_\lambda} \sum_{\tau \in C_\lambda} \varepsilon(\tau) \sigma \tau \in \Sigma_d$$

is called the Young symmetrizer corresponding to λ.

Note immediately that it depends on λ but also on the filling of λ, see Remark 3.

2.3. Representations of finite groups and of Σ_d

Let G be a finite group.

PROPOSITION 2. *There are exactly $n(G)$ irreducible representations of G, where $n(G)$ is the number of conjugacy classes of G.*

Proof. [19] Prop. 2.30. □

If V_i $i = 1,\ldots,n(G)$ are the irreducible representations of G and g_j, $j = 1,\ldots,n(G)$ are representatives in the conjugacy classes of G then the square matrix $\chi_{V_i}(g_j)$ is called the *character table* of G.

EXAMPLE 2. The group $G = \Sigma_2$ has two elements $1, -1$, each one is a conjugacy class. Besides the trivial representation V_2, we have another representation $V_{1,1}$ defined on its basis element w by $1 \cdot w = w, (-1) \cdot w = -w$.

The character table is

	1	−1
V_2	1	1
$V_{1,1}$	1	−1

Note that $V_2^G = V_2, V_{1,1}^G = 0$.

We will see in section 5.4 the character table of Σ_6.

The proof of the following proposition is straightforward

PROPOSITION 3. *There is a G- surjective morphism $R: V \to V^G$ which is defined as $R(v) = \frac{1}{|G|} \sum_{g \in G} g \cdot v$ which satisfies $R(v) = v \ \forall v \in V^G$.*

The main theorem on finite groups is the following

THEOREM 5. *Let G be a finite group*.

- *(i) Given any G-module V, there are uniquely determined nonnegative a_i for $i = 1,\ldots n(G)$ such that $V = \oplus V_i^{\oplus a_i}$.*

- *(ii) The a_i are the unique solution to the square linear system*

$$\chi_V(g_j) = \sum_i a_i \chi_{V_i}(g_j), \quad j = 1,\ldots,n(G).$$

In particular, from the character χ_V, it can be computed the dimension of the invariant part V^G.

Proof. (i) is [19], Prop. 1.8. (ii) follows from [19] Coroll. 2.14, Prop. 2.30. □

In the case of the symmetric group $G = \Sigma_d$ its irreducible representations are in bijective correspondence with the conjugacy classes of Σ_d, which correspond to the cycle structures of permutations and so they are described by partitions λ of d.

Recall that for any finite group G, the group algebra $\mathbb{C}G$ is defined in the following way. The underlying vector space of dimension $|G|$ has a basis e_g corresponding to the elements $g \in G$ and the algebra structure is defined by the rule $e_g \cdot e_h = e_{gh}$.

Σ_d acts on the vector space $\mathbb{C}\Sigma_d$ by $\sigma \cdot e_\alpha = e_{\alpha\sigma}$, which extends by linearity to $\sigma: \mathbb{C}\Sigma_d \to \mathbb{C}\Sigma_d$.

DEFINITION 2. *The Young symmetrizer c_λ defined in (6) acts by right multiplication on $\mathbb{C}\Sigma_d$, its image is a Σ_d-module that we denote by V_λ.*

We remark that the notations we have chosen in Example 2 are coherent with this definition.

Let T be a tableau corresponding to a filling of the Young diagram λ with d boxes with the numbers $\{1,\ldots,d\}$. A tableau is called *standard* if the filling is strictly increasing on rows and columns. A tableau corresponds to a permutation $\sigma_T \in \Sigma_d \subset \mathbb{C}\Sigma_d$. By definition V_λ is spanned by $c_\lambda \sigma_T$ for any tableau T.

REMARK 3. Σ_d acts by conjugation over $\mathbb{C}\Sigma_d$. Any conjugate $\sigma^{-1}c_\lambda\sigma$ acts by right multiplication on $\mathbb{C}\Sigma_d$, its image is isomorphic to V_λ, although they may be embedded in different ways. These different copies can be obtained, in equivalent way, starting by a different tableau in Def. 1, see [47] chap. 9, remark 2.2.5.

THEOREM 6 ([47] chap. 9, §2.4 and §9.2).

- *(i) V_λ is an irreducible representation of Σ_d.*

- *(ii) All irreducible representations of Σ_d are isomorphic to V_λ for some Young diagram λ.*

- *(iii) If λ and μ are different partitions, then $c_\lambda c_\mu = 0$ and c_λ^2 is a scalar multiple of c_λ.*

- *(iv) A basis of V_λ is given by $c_\lambda \sigma_T$ where $\sigma_T \in \Sigma_d \subset \mathbb{C}\Sigma_d$ corresponds to standard tableau T. In particular $\dim V_\lambda$ is equal to the number of standard tableaux on λ.*

THEOREM 7 ([19] Prop. 3.29). *We have the isomorphism of algebras*

(7) $$\mathbb{C}\Sigma_d = \oplus_{\{\lambda||\lambda|=d\}}\mathrm{End}(V_\lambda).$$

According to the isomorphism (7), any c_μ is a diagonalizable endomorphism of rank one (and nonzero trace) in $\mathrm{End}(V_\mu)$ and it is zero in $\mathrm{End}(V_\lambda)$ for $\lambda \neq \mu$.

Note the two extreme cases, when $\lambda = d$ we get the trivial one dimensional representation of Σ_d, while if $\lambda = \underbrace{(1,\ldots,1)}_{d} = 1^d$ (for short) we get the one dimensional representation given by sign, that is the action on a generator e is given by $\sigma \cdot e = \varepsilon(\sigma)e$.

2.4. Representations of $GL(n+1)$ and $SL(n+1)$, Schur functors

THEOREM 8. *Let $f\colon GL(n+1) \to \mathbb{C}$ be a polynomial function invariant by conjugation, that is $f(G^{-1}AG) = f(A)$ for every $G,A \in GL(n+1)$.*

Then f is a polynomial symmetric function of the eigenvalues of A.

Proof. Let $D(d_1, \ldots, d_{n+1})$ be the diagonal matrix having d_i on the diagonal. If $\tau \in \Sigma_{n+1}$ and M_τ is the corresponding permutation matrix, then

$$M_\tau^{-1} D(d_1, \ldots, d_{n+1}) M_\tau = D(d_{\tau(1)}, \ldots, d_{\tau(n+1)}).$$

It follows that $f(D(d_1, \ldots, d_{n+1}))$ is a polynomial symmetric function of $d_1, \ldots, d_{n+1}$. By the Main Theorem on symmetric polynomials (see [53] Theor. 1.1.1) there is a polynomial $g \in \mathbb{C}[x_1, \ldots x_{n+1}]$ such that

$$f(D(d_1, \ldots, d_{n+1})) = g(\sigma_1(d_1, \ldots, d_{n+1}), \ldots, \sigma_{n+1}(d_1, \ldots, d_{n+1})),$$

where σ_i is the i-th elementary symmetric polynomials. For any matrix A, denote $\det(A - tI) = \sum_{i=0}^{n+1} t^i (-1)^i c_{n+1-i}(A)$. Note that $\sigma_i(d_1, \ldots, d_{n+1}) = c_i(D(d_1, \ldots, d_{n+1}))$, and that $c_i(A)$ is the i-th elementary symmetric function of the eigenvalues of A. We have proved that $f(A) = g(c_1(A), \ldots, c_{n+1}(A))$ for every diagonal matrix A. Both sides are invariant by conjugation, then the equality is satisfied by any diagonalizable matrix. Since diagonalizable matrix form a dense subset, it follows that $f(A) = g(c_1(A), \ldots, c_{n+1}(A))$ for any matrix A, $\qquad\square$

The same argument works for polynomial functions $f \colon SL(n+1) \to \mathbb{C}$ which are invariant by conjugation. See [54] for an extension to invariants of several matrices.

COROLLARY 1. *Characters of $GL(n+1)$ (and of $SL(n+1)$) are symmetric functions of the eigenvalues.*

Let $V = \mathbb{C}^{n+1}$, the group Σ_d acts on $\otimes^d V$ by permuting the factors.

We define the Schur projection $c_\lambda \colon \otimes^d V \to \otimes^d V$ from the Young symmetrizer c_λ defined in (6).

We fill the Young diagram with numbers from 1 to $n+1$, allowing repetitions. After a basis of V has been fixed, any such tableau T gives a basis vector $v_T \in \otimes^d V$.

A tableau is called *semistandard* if it has nondecreasing rows and strictly increasing columns.

THEOREM 9.

- *The image of c_λ is a irreducible $GL(n+1)$-module, which is nonzero if and only if the number of rows is $\leq n+1$, denoted by $S^\lambda V$.*

- *All irreducible $GL(n+1)$-modules are isomorphic to $S^\lambda V$ for some Young diagram λ with number of rows $\leq n+1$.*

- *Any $GL(n+1)$-module is a sum of irreducible ones.*

- *The images $c_\lambda(v_T)$ where T is a semistandard tableau give a basis of $S^\lambda V$.*

Proof. [19] Prop. 15.47. $\qquad\square$

A particular case, very useful in the sequel, is when λ consists of g columns of length $n + 1$. This happens if and only if $S^\lambda V$ is one dimensional.

The theory of $SL(n+1)$-representations is very similar. The basic fact is that if $S^{\lambda_1,\lambda_2...,\lambda_{n+1}}V$ and $S^{\lambda_1-\lambda_{n+1},\lambda_2-\lambda_{n+1}...,\lambda_n-\lambda_{n+1},0}V$ are isomorphic as $SL(n+1)$-modules, indeed

$$S^{\lambda_1,\lambda_2...,\lambda_{n+1}}V \simeq \wedge^{n+1}V \otimes S^{\lambda_1-\lambda_{n+1},\lambda_2-\lambda_{n+1}...,\lambda_n-\lambda_{n+1},0}V.$$

In other words, all columns of length $n+1$ correspond to the one dimensional determinantal representation, which is trivial as $SL(n+1)$-module. Removing these columns, we get another Young diagram with the number of rows $\leq n$.

THEOREM 10.

- *All irreducible $SL(n+1)$-modules are isomorphic to $S^\lambda V$ for some Young diagram λ with number of rows $\leq n$.*

- *Any $SL(n+1)$-module is a sum of irreducible ones.*

Proof. [19]Prop. 15.15, Theor. 14.18 , [47] chap. 9 §8.1 □

REMARK 4. The construction $W \mapsto S^\lambda W$ is indeed functorial, in the sense that a linear map $W_1 \to W_2$ induces a linear map $S^\lambda W_1 \to S^\lambda W_2$ with functorial properties, see [47] chap. 9 §7.1. See also [55] chap. 2.

DEFINITION 3. *The character of $S^\lambda V$ is a symmetric polynomial s_λ which is called a Schur polynomial.*

There is a way to write s_λ as the quotient of two Vandermonde determinants ([36] Prop. 1.2.1), and more efficient ways to write down explicitly s_λ which put in evidence the role of tableau ([36] Theor. 1.4.1).

If $\lambda = d$, then $s_\lambda(t_1,\dots t_n)$ is the sum of all possible monomials of degree d in $t_1,\dots,t_n$. If $\lambda = \underbrace{1,\dots,1}_{i}$, then $s_\lambda(t_1,\dots t_n)$ is the i-th elementary symmetric polynomial in $t_1,\dots,t_n$.

$SL(n+1)$ contains the torus $(\mathbb{C}^*)^n$ of diagonal matrices

$$T = \{D\left(t_1,\dots,t_n,\frac{1}{(t_1\cdots t_n)}\right) \mid t_i \neq 0\}.$$

Given $(a_1,\dots,a_n) \in \mathbb{Z}^n$ we have the one dimensional (algebraic) representation $\rho\colon T \to \mathbb{C}^*$ defined by

$$\rho\left(D\left(t_1,\dots,t_n,\frac{1}{(t_1\cdots t_n)}\right)\right) := t_1^{a_1}\cdots t_n^{a_n}.$$

We denote it by $V_{a_1,\dots,a_n}$.

PROPOSITION 4.

- *Every irreducible (algebraic) representation of T is isomorphic to $V_{a_1,\ldots,a_n}$ for some $(a_1,\ldots,a_n) \in \mathbb{Z}^n$.*

- *Every (algebraic) T-module is isomorphic to the direct sum of irreducible representations.*

Proof. [47] chap. 7, §3.3 □

THEOREM 11 (Cauchy identity).

$$S^p(V \otimes W) = \oplus_\lambda S^\lambda V \otimes S^\lambda W$$

where the sum is extended to all partitions λ of p.

Proof. By using characters, the proof reduces to a nontrivial identity on symmetric functions, see [47] (6.3.2) or [36] 1.4.2. □

2.5. The Lie algebra $\mathfrak{sl}(n+1)$ and the weight structure of its representations

We denote by $\mathfrak{sl}(n+1)$ the Lie algebra of $SL(n+1)$. It corresponds to the traceless matrices of size $(n+1)$, where the bracket is $[A,B] = AB - BA \ \forall A,B \in \mathfrak{sl}(n+1)$. The tangent space at the identity of $SL(n+1)$ is naturally isomorphic to $\mathfrak{sl}(n+1)$.

A representation of the Lie algebra $\mathfrak{sl}(n+1)$ is a Lie algebra morphism $\mathfrak{sl}(n+1) \to \mathfrak{sl}(W)$. The derivative (computed at the identity) of a group representation $SL(n+1) \to SL(W)$ is a representation of the Lie algebra $\mathfrak{sl}(n+1)$.

Since $SL(n+1)$ is simply connected, there is a natural bijective correspondence between $SL(n+1)$-modules and $\mathfrak{sl}(n+1)$-modules, in the sense that every Lie algebra morphism $\mathfrak{sl}(n+1) \to \mathfrak{sl}(W)$ is the derivative of a unique group morphism $SL(n+1) \to SL(W)$. In particular all $\mathfrak{sl}(n+1)$-modules are direct sum of irreducible ones.

This definition behaves in a different way when applied on direct sums and tensor products. Let $\mathfrak{g}$ be a Lie algebra. If V and W are two $\mathfrak{g}$-modules then $V \oplus W$ and $V \otimes W$ are G-modules in a natural way, namely

$$g \cdot (v+w) := (g \cdot v) + (g \cdot w),$$
$$g \cdot (v \otimes w) := (g \cdot v) \otimes (w) + (v) \otimes (g \cdot w).$$

The m-th symmetric power $S^m W$ is a G-module, satisfying

$$(8) \qquad\qquad g \cdot (v^m) := m(g \cdot v)v^{m-1}.$$

A morphism between two $\mathfrak{g}$-modules V, W is a linear map $f \colon V \to W$ which is $\mathfrak{g}$-equivariant, namely it satisfies $f(g \cdot v) = g \cdot f(v) \ \forall g \in \mathfrak{g}, v \in V$.

Every $\mathfrak{g}$-module has a $\mathfrak{g}$-invariant submodule

$$V^{\mathfrak{g}} = \{v \in V | g \cdot v = 0 \quad \forall g \in \mathfrak{g}\}.$$

The adjoint representation of $SL(n+1)$ is the group morphism

$$
\begin{aligned}
SL(n+1) &\rightarrow GL(\mathfrak{sl}(n+1)) \\
g &\mapsto (h \mapsto g^{-1}hg).
\end{aligned}
$$

Its derivative is the Lie algebra morphism

$$
\begin{aligned}
\mathfrak{sl}(n+1) &\rightarrow End(\mathfrak{sl}(n+1)) \\
g &\mapsto (h \mapsto [h,g]).
\end{aligned}
$$

The diagonal matrices $H \subset \mathfrak{sl}(n+1) = \mathfrak{g}$ make a Lie subalgebra which can be identified with Lie (T). It is abelian, that is $[H,H] = 0$, and it is called a Cartan subalgebra. Write generators as

$$
H = \{D(t_1,\ldots,t_{n+1}) \mid \sum_i t_i = 0\}.
$$

A basis of H is given by $\{D(1,0,\ldots,0,-1), D(0,1,0,\ldots,0,-1),\ldots\}$. In dual coordinates, we have a basis $h_i = (0,\ldots,0,\underset{\underset{i}{\uparrow}}{1},0,\ldots,0,-1) \in H^\vee$ for $i = 1,\ldots,n$. We can consider h_i as Lie algebra morphisms $h_i \colon H \to \mathbb{C}$.

Representations of $\mathfrak{sl}(n+1)$, when restricted to H, satisfy the analogous to Proposition 4.

PROPOSITION 5. *Let W be a Lie algebra representation of $sl(n+1)$. When restricting the representation to H, it splits in the direct sum of irreducible representations, each one isomorphic to an integral combination $\sum_{i=1}^{n} a_i h_i$ with $a_i \in \mathbb{Z}$. These representations are the derivative of the representations $V_{a_1,\ldots,a_n}$ of Prop. 4.*

The strictly upper triangular matrices

$$
N^+ := \{g \in \mathfrak{sl}(n+1) \mid g_{ij} = 0 \text{ for } i < j\}
$$

make a nilpotent subalgebra, in the sense that the descending chain

$$
\mathfrak{g}^+ \supset [\mathfrak{g}^+,\mathfrak{g}^+] \supset [[\mathfrak{g}^+,\mathfrak{g}^+],\mathfrak{g}^+] \supset \ldots
$$

terminates to zero. It holds $[N^+,H] \subset H$, which means that N^+ is an invariant subspace for the adjoint representation restricted to H, indeed it splits as the sum of one dimensional representations, which are spanned by the elementary matrices e_{ij} which are zero unless one upper triangular entry which is 1.

More precisely, there are certain $\alpha = \sum_{i=1}^{n} a_i h_i \in H^*$ as in Prop. 5 and $n_\alpha \in N^+$ such that

$$
(9) \qquad\qquad [h,n_\alpha] = \alpha(h)n_\alpha \qquad \forall h \in H.
$$

Such α are called (positive) roots, and N^+ has a basis of eigenvectors n_α.

In the same way we can define the subalgebra of strictly lower triangular matrices N^-, which has a similar basis of eigenvectors. The corresponding roots are called negative.

THEOREM 12 (Weight Decomposition). *Let W be a $\mathfrak{sl}(n+1)$-module. W is also a H-module, and it splits as the sum of H-representations $W_{(\lambda_i)}$ where $\lambda_i \in H^\vee$ is called a weight and satisfies the following property.*

$$W_{(\lambda_i)} = \{w \in W | h \cdot w = \lambda_i(h)w, \forall h \in H\}.$$

The elements in the weight space $W_{(\lambda_i)}$ are called H-eigenvectors with weight λ_i.

THEOREM 13. *Let W be a irreducible $\mathfrak{sl}(n+1)$-module. Then there is a unique (up to scalar multiples) vector $w \in W$ satisfying $N^- \cdot w = 0$. w is called a maximal vector and it is an H-eigenvector.*

The representation W is spanned by repeated applications of elements $g \in N^+$ to w. More precisely, $e_{i_1 j_1} \ldots e_{i_s j_s} w$ span W for convenient $e_{i_k j_k} \in N^+$.

The basic principle is that the generators of N^+ (as well as the generators of $N-$) act on the weight decomposition of W.

PROPOSITION 6. *Let $n_\alpha \in N^+$ be an eigenvector with eigenvalue the root α as in (9). Let $\lambda \in H^\vee$ be a weight of a representation W. Then*

$$n_\alpha(W_{(\lambda)}) \subseteq W_{(\alpha+\lambda)}.$$

Proof. Let $w \in W_\lambda$. For any $h \in H$ we have

$$h(n_\alpha(w)) = [h,n_\alpha](w) + n_\alpha(h(w)) = \alpha(h)n_\alpha(w) + n_\alpha\lambda(h)(w) = (\alpha(h)+\lambda(h))(n_\alpha(w)).$$

This proves that $n_\alpha(w) \in W_{(\alpha+\lambda)}$ as we wanted. $\qquad\square$

It is instructive to draw pictures of eigenspaces decomposition, denoting any weight space as a vertex, identifying the action of elements n_α as in Proposition 6 with arrows from $W_{(\lambda)}$ to $W_{(\alpha+\lambda)}$.

We begin with $\mathfrak{sl}(2)$, which has dimension three, spanned by $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$,
satisfying
$$[h,x] = 2x, \quad [h,y] = -2y, \quad [x,y] = h.$$

Each irreducible $\mathfrak{sl}(2)$-module is isomorphic to $S^m\mathbb{C}^2$ for an integer $m \in \mathbb{Z}_{\geq 0}$. If v is the maximal vector, the space $S^m\mathbb{C}^2$ is isomorphic to $\oplus_{i=0}^m < x^i \cdot v >$ for $i = 0, \ldots m$ as in the following picture

$$x^m \cdot v \xleftarrow{x} x^{m-1} \cdot v \xleftarrow{x} \ldots x \cdot v \xleftarrow{x} v.$$

If we want to emphasize the dimensions of the weight spaces, we just write

$$1 \longleftarrow 1 \longleftarrow 1 \longleftarrow \cdots \longleftarrow 1 \longleftarrow \mathbf{1}$$

The exterior skeleton of $S^p\mathbb{C}^2$ has the following pattern (the maximal vector is marked)

$$\bullet \xleftarrow{\hspace{4cm} p \hspace{4cm}} \odot$$
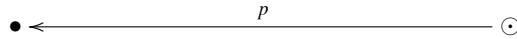
Note that $x^{m+1}v = 0$. We have $h \cdot v = mv$, and in general $h \cdot (x^i \cdot v) = (2m - 2i)(x^i \cdot v)$. The natural construction in the setting of invariant theory is the following. Each H-eigenspace is generated by the monomial $s^{m-i}t^i$. The monomial s^m corresponds to the maximal vector. According to (8) we compute that $(x^i \cdot s^m)$ is a scalar multiple of the monomial $s^{m-i}t^i$.

Each irreducible $\mathfrak{sl}(3)$-module is isomorphic to $S^{a,b}\mathbb{C}^3$ for a pair of integers $a, b \in \mathbb{Z}_{\geq 0}$ with $a \geq b$.

We have the following pictures, which describe the general pattern that "weights increase by one on hexagons and remain constant on triangles", see [19] chapter 6. The maximal vectors are marked in bold.

In these pictures

(10)   the arrow $\bullet \searrow$ corresponds to the action of $A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(11)   the arrow $\bullet \downarrow$ corresponds to the action of $A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

and the arrow $\swarrow \bullet$ corresponds to the action of $[A_1, A_2] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
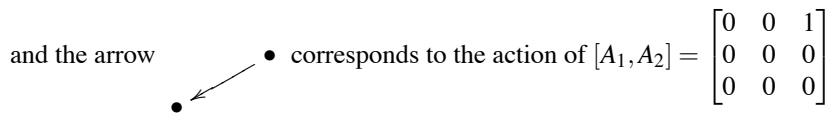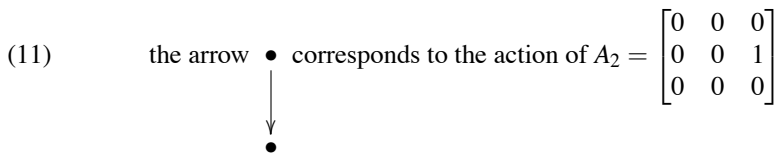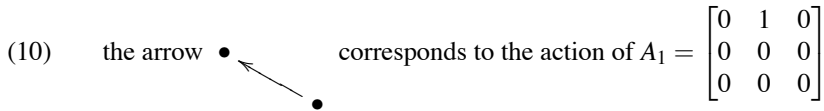
$S^3\mathbb{C}^3 = S^{3,0}\mathbb{C}^3$
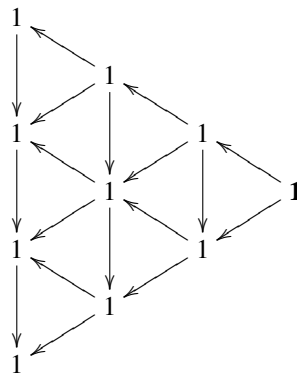
$$\left(S^3\mathbb{C}^3\right)^\vee = S^{3,3}\mathbb{C}^3$$
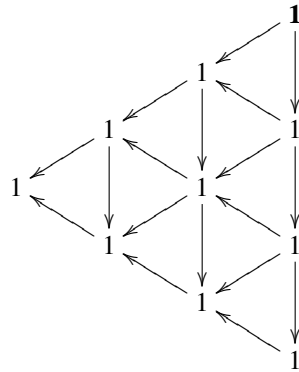
$$S^{2,1}\mathbb{C}^3$$
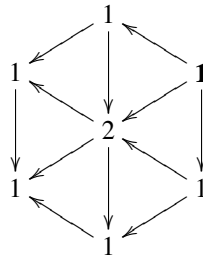
$$S^{5,1}\mathbb{C}^3$$
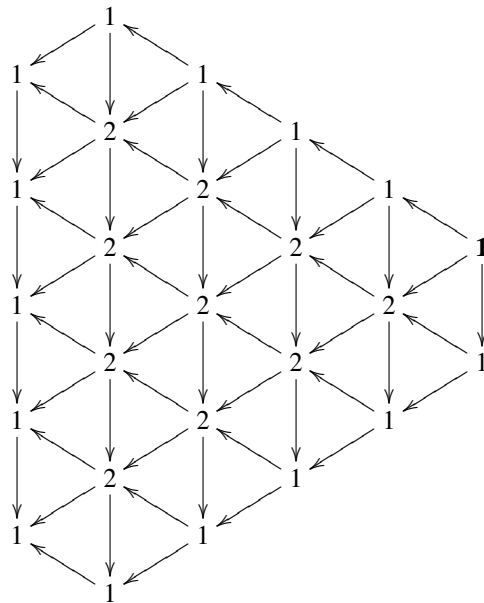
$S^{4,2}\mathbb{C}^3$
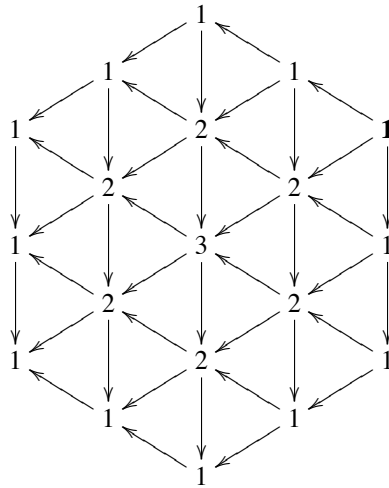
The exterior skeleton of $S^{p,q}\mathbb{C}^3$ has the following pattern (the maximal vector is marked differently)

$S^{p,q}\mathbb{C}^3$
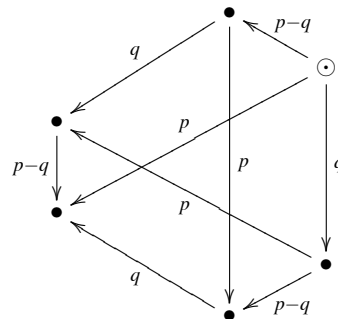
2.6. Schur-Weyl duality

THEOREM 14 (Schur-Weyl duality). *There is a $\Sigma_d \times SL(V)$-decomposition*

$$V^{\otimes k} = \oplus_\lambda V_\lambda \otimes S^\lambda V$$

where the sum is extended on all Young diagrams with k boxes, V_λ has been defined in Def. 2 and $S^\lambda V$ has been defined in Theorem 9.

Proof. [47] chap. 9 (3.1.4). □

Now fill the Young tableau with numbers from 1 to $n+1$ in such a way that they are nondecreasing on rows and strictly increasing on columns. For example we have the following

$$
\boxed{\begin{smallmatrix}1&1\\2\end{smallmatrix}}\ ,\ \boxed{\begin{smallmatrix}1&2\\2\end{smallmatrix}}\ ,\ \boxed{\begin{smallmatrix}1&3\\2\end{smallmatrix}}\ ,\ \boxed{\begin{smallmatrix}1&1\\3\end{smallmatrix}}\ ,\ \boxed{\begin{smallmatrix}1&2\\3\end{smallmatrix}}\ ,\ \boxed{\begin{smallmatrix}1&3\\3\end{smallmatrix}}\ ,\ \boxed{\begin{smallmatrix}2&2\\3\end{smallmatrix}}\ ,\ \boxed{\begin{smallmatrix}2&3\\3\end{smallmatrix}}\ .
$$

Each filling describes a vector in $V^{\otimes d}$. The image of these vectors through c_λ give a basis of $S^\lambda V$. In other words, the elements $c_\lambda(v_T)$ where T is any tableau span $S^\lambda V$.

Composing with a permutation of all the boxes, we may find different isomorphic copies of the same representation V_λ inside $V^{\otimes d}$.

This construction is quite important and make visible that $\mathrm{Im}\,c_\lambda$ defines just one copy of the representation $S^\lambda V$ inside $V^{\otimes d}$, which is not intrinsic but it depends on the convention we did in the definition of c_λ. For example by swapping in the definition 6 of c_λ the order of R_λ and C_λ we get another copy of $S^\lambda V$, in general skew with respect to the previous one.

But the main reason is that the order we chose in the d copies of V is arbitrary, so acting with Σ_d we can find other embeddings of $S^\lambda V$, all together spanning $V_\lambda \otimes S^\lambda V$ which is canonically embedded in $V^{\otimes d}$, and it is $\Sigma_d \times SL(V)$-equivariant.

For example in in $\mathbb{C}^{2\otimes 4}$ there is, corresponding to $\lambda = (2,2)$ the invariant subspace $\mathbb{C}^2_{2,2} \otimes S^{2,2}\mathbb{C}^2$.

Note that $\dim \mathbb{C}^2_{2,2} = 2$, while $\dim S^{2,2}\mathbb{C}^2 = 1$.

This 2-dimensional space of invariants is spanned by the three functions that take

$$
x_1 \otimes x_2 \otimes x_3 \otimes x_4 \in \left(\mathbb{C}^{2\vee}\right)^{\otimes 4}
$$

respectively in $(x_1 \wedge x_2)(x_3 \wedge x_4),\ (x_1 \wedge x_3)(x_2 \wedge x_4),\ (x_1 \wedge x_4)(x_2 \wedge x_3)$.

Note that we have the well known relation

$$
(x_1 \wedge x_2)(x_3 \wedge x_4) - (x_1 \wedge x_3)(x_2 \wedge x_4) + (x_1 \wedge x_4)(x_2 \wedge x_3) = 0.
$$

The image of the Schur symmetrizer $c_{2,2}$ is a scalar multiple of $f = (x_1 \wedge x_4)(x_2 \wedge x_3) + (x_1 \wedge x_3)(x_2 \wedge x_4)$. Note that applying the permutation $(12) \in \Sigma_4$, we get $(12) \cdot f = f$, while applying the permutation $(13) \in \Sigma_4$ we get an independent $SL(2)$-invariant element $(13) \cdot f$, which together with f spans the $\Sigma_4 \times SL(2)$-invariant subspace $\mathbb{C}^2_{2,2} \otimes S^{2,2}\mathbb{C}^2$.

This example generalizes to the following

THEOREM 15. *If $\sigma \in \Sigma_d$, the elements $\sigma c_\lambda(v_T)$ where T is any tableau, span a $SL(n+1)$-module isomorphic to $S^\lambda V$, lying in the subspace $V_\lambda \otimes S^\lambda V$.*

The whole subspace $V_\lambda \otimes S^\lambda V$ is spanned by these copies of $S^\lambda V$, for any $\sigma \in \Sigma_d$.

On the other hand, if T is a fixed tableau such that $c_\lambda(v_T) \neq 0$, the elements $\sigma c_\lambda(v_T)$ for $\sigma \in \Sigma_d$ span a Σ_d-module isomorphic to V_λ lying in the subspace $V_\lambda \otimes S^\lambda V$.

3. Invariants of forms and representation theory

Again we denote $V = \mathbb{C}^{n+1}$. The subject of this section is the action of $SL(V)$ over $\oplus_m S^m(S^d V)$. The invariant subspace $S^m(S^d V)^{SL(V)}$ is, by definition, the space of invariants of degree m for forms of degree d. We have from §2.5 the decomposition $\mathfrak{sl}(V) = H \oplus N^+ \oplus N^-$ and we study separately the actions of H and N^+.

3.1. Invariance for the torus

Denote the coefficients of a form $f \in S^d V$ of degree d in $n+1$ variables in the following way

$$ f = \sum_{i_0 + \ldots + i_n = d} \frac{d!}{i_0! \ldots i_n!} f_{i_0 \ldots i_n} x_0^{i_0} \ldots x_n^{i_n}. $$

The space $S^m S^d V$ is spanned by monomials $f_{i_{0,1} \ldots i_{n,1}} \ldots f_{i_{0,m} \ldots i_{n,m}}$.
The *weight* of the monomial $f_{i_{0,1} \ldots i_{n,1}} \ldots f_{i_{0,m} \ldots i_{n,m}}$ is the vector

$$ \left(\sum_{j=1}^m i_{0,j}, \sum_{j=1}^m i_{1,j}, \ldots, \sum_{j=1}^m i_{n,j} \right) \in \mathbb{Z}_{\geq 0}^{n+1}. $$

As we will see in the proof of Proposition 7, the subspaces of monomial of a given weight, are exactly the H-eigenspaces for the action of $\mathfrak{sl}(n+1)$ on $S^m S^d \mathbb{C}^{n+1}$ seen in Theorem 12.

A monomial is called *isobaric* if its weight has all equal entries ("democratic"). Consider the double sum $\sum_{k=0}^n \sum_{j=1}^m i_{k,j} = \sum_{j=1}^m d = md$. So the weight of an isobaric monomial in $S^m S^d V$ is $(\underbrace{\frac{md}{n+1}, \ldots, \frac{md}{n+1}}_{n+1})$, in particular isobaric monomials exist if and only if $n+1$ divides md.

An polynomial $I \in S^m S^d V$ has degree m in the coefficients $f_{i_0 \ldots i_n}$.

PROPOSITION 7. *$I \in S^m S^d V$ is invariant for the action of the torus of diagonal matrices $(\mathbb{C}^*)^n \subset SL(V)$ if an only if it is sum of isobaric monomials.*

Note that it is enough to check if a monomial of given degree is isobaric for n places in the $(n+1)$-dimensional weight vector. In particular for binary forms it is enough to check the condition just for one place.

Proof. Consider the diagonal matrix with entries $(\frac{1}{t_1 \ldots t_n}, t_1, \ldots, t_n)$ which acts on $f_{i_0 \ldots i_n}$ by multiplying for $(t_1 \ldots t_n)^{-i_0} t_1^{i_1} \ldots t_n^{i_n}$.

Acting on the monomial

$$ f_{i_{0,1} \ldots i_{n,1}} \ldots f_{i_{0,m} \ldots i_{n,m}}, $$

we multiply it for $(t_1 \ldots t_n)^{-\sum_j i_{0,j}} t_1^{\sum_j i_{1,j}} \ldots t_n^{\sum_j i_{n,j}}$ and this is equal to 1 if and only if $\sum_j i_{k,j}$ does not depend on k. $\square$

COROLLARY 2. *A necessary condition for the existence of a nonzero $I \in S^m S^d V$ which is invariant for $SL(V)$ is that $n+1$ divides md.*

Proof. If I is $SL(V)$- invariant then it is also invariant for the torus of diagonal matrices. $\square$

An equivalent way to express the fact that the polynomial I has to be isobaric is through the action of the Lie algebra H of diagonal matrices. This translates to the fact that I satisfies the system of differential equations (see [53] Theor. 4.5.2)

$$(12) \qquad \sum_{i_0 \ldots i_n} i_j f_{i_0 \ldots i_n} \frac{\partial I}{\partial f_{i_0 \ldots i_n}} = \frac{md}{n+1} I \quad \forall j = 0, \ldots, n.$$

There is a second set of differential equations for the action of the triangular part $N^+ \subset \mathfrak{sl}(n+1)$, see Propositions 9, 12.

EXAMPLE 3. In $S^3(S^4\mathbb{C}^3)$ there are 23 isobaric monomials among 680 monomials. In $S^6(S^3\mathbb{C}^3)$ there are 103 isobaric monomials among 5005. In Prop. 8 we will give a generating function that allows to compute these numbers.

3.2. Counting monomials of given weight

We compute now the number of monomials with a given weight.

Let $H_{g,d,p_0,\ldots,p_n}$ be the space of monomials in $S^g(S^d\mathbb{C}^{n+1})$ of weight $(p_0, \ldots, p_n)$. Since $p_0 + \ldots + p_n = dg$, it is enough to record the last n entries of weight vector.

We have

PROPOSITION 8.

$$\sum_{g=0}^{\infty} \sum_{p_0+\ldots+p_n=dg} H_{g,d,p_0,\ldots,p_n} x_1^{p_1} \ldots x_n^{p_n} y^g = \prod_{i_1+\ldots+i_n \leq d} \frac{1}{1 - x_1^{i_1} \ldots x_n^{i_n} y}.$$

Proof. The variable $a_{i_0,\ldots,i_n}$ has degree 1 and weight $(i_0, \ldots, i_n)$. The left hand side is the Hilbert series for the multigraded ring $K[a_{d,0,\ldots,0}, \ldots, a_{0,\ldots,0,d}]$. The graded ring in just the variable $a_{i_0,\ldots,i_n}$ has Hilbert series $\frac{1}{1-x_1^{i_1} \ldots x_n^{i_n} y} = 1 + x_1^{i_1} \ldots x_n^{i_n} y + x_1^{2i_1} \ldots x_n^{2i_n} y^2 +$ $\ldots$. Taking into account all the variables, we have the product of the corresponding Hilbert series. $\square$

Recall that for invariants of weight $(p_0, \ldots, p_n)$ we have $\sum_i p_i = dg$.

In case of binary forms, let $H_{g,p,d}$ be the space of homogeneous polynomials of degree g and weight $(p, dg-p)$ in $a_0, \ldots a_d$.

THEOREM 16 (Cayley-Sylvester).

(13)
$$\sum_p \dim H_{g,p,d} x^p = \frac{(1-x^{d+1})\dots(1-x^{d+g})}{(1-x)\dots(1-x^g)}.$$

Proof. Write (after Proposition 8)

$\phi_d(x,y) := \sum_{p,g} dim H_{g,p,d} x^p y^g = \prod_{i=0}^d \frac{1}{1-x^i y} = \sum_{g=0}^\infty C_{g,d}(x) y^g$, so that the expression to be computed is $C_{g,d}(x)$.

We get

$$(1-y)\phi(x,y) = \prod_{i=1}^d \frac{1}{1-x^i y} = (1-x^{d+1}y)\prod_{i=1}^{d+1} \frac{1}{1-x^i y} =$$

$$(1-x^{d+1}y)\prod_{i=0}^d \frac{1}{1-x^{i+1}y} = (1-x^{d+1}y)\phi(x,xy).$$

Hence

$$(1-y)\sum_{j=0}^\infty C_{j,d}(x)y^j = (1-x^{d+1}y)\sum_{j=0}^\infty C_{j,d}(x)x^j y^j.$$

Comparing the coefficients of y^j we get

$$C_{j,d} - C_{j-1,d} = C_{j,d}x^j - C_{j-1,d}x^{d+j},$$

hence

$$C_{j,d}(x) = \frac{1-x^{d+j}}{1-x^j}C_{j-1,d}(x).$$

Since $C_{0,d}(x) = 1$, by induction on j we get the thesis.   □

Let's state the proposition 8 in the case of $SL(3)$, for future reference. We denote by H_{g,d,p_0,p_2,p_2} be the space of monomials in $S^g(S^d\mathbb{C}^3)$ of weight (p_1,p_2,p_2), and we get

(14)
$$\sum_{p_0+\dots+p_n=d,g} H_{g,d,p_0,p_1,p_2}x_1^{p_1}x_2^{p_2}y^g = \prod_{i_1=0}^d \prod_{i_2=0}^{d-i_1} \frac{1}{1-x_1^{i_1}x_2^{i_2}y}.$$

3.3. Lie algebra action on forms

We give a $SL(2)$ example, which illustrates the general situation. Recall that the generator $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2)$ integrates in the Lie group to the one parameter subgroup $e^{xt} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$

PROPOSITION 9.

(i) $I \in S^m S^d \mathbb{C}^2$ is invariant with respect to the subgroup $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ if and only if it is invariant with respect to the subalgebra N^+ if and only if $DI = 0$ where $D = \sum_{i=0}^{d-1}(i+1)a_i \frac{\partial}{\partial a_{i+1}} = a_0 \frac{\partial}{\partial a_1} + 2a_1 \frac{\partial}{\partial a_2} + \dots$

(ii) $I \in S^m S^d \mathbb{C}^2$ is SL(2)-invariant if and only if it is isobaric and $DI = 0$.

Proof. $I(x,y) = I(x'+ty', y') = a_0(x'+ty')^n + a_1 n(x'+ty')^{n-1}y' + a_2 \binom{n}{2}(x'+ty')^{n-2}y'^2 \dots = a_0 x'^n + n(a_0 t + a_1)x'^{n-1}y' + \binom{n}{2}(a_0 t^2 + 2a_1 t + a_2)x'^{n-2}y'^2 + \dots$

So $a'_k = \sum_{i=0}^k \binom{k}{i}a_i t^{k-i} = a_k + k a_{k-1}t + \dots$

Now the condition is $0 = \frac{d}{dt}I_{|t=0} = \sum \frac{\partial I}{\partial a'_i}\frac{\partial a'_i}{\partial t}_{t=0}$ and the proof of (i) is concluded by the fact that $\frac{\partial a'_i}{\partial t}_{t=0} = i a_{i-1}$. (ii) follows because $\mathfrak{sl}(2)$ is generated by h and x, see §2.5. $\square$

The main application of the previous proposition is that it allows to compute explicitly invariants. Remind the equianharmonic quadric $I = f_0 f_4 - 4 f_1 f_3 + 3 f_2^2 \in S^2 S^4 \mathbb{C}^2$ and our question posed in §1.2, namely why the coefficients $(1, -4, 3)$?

Now the coefficients can be computed by Prop. 9.

Call α, β, γ unknown coefficients and apply $D(\alpha f_0 f_4 + \beta f_1 f_3 + \gamma f_2^2) = f_0 f_3(4\alpha + \beta) + f_1 f_2(3\beta + 4\gamma) = 0$. We get that (α, β, γ) is proportional to $(1, -4, 3)$.

In the same way we can prove the following dual version.

PROPOSITION 10.

(i) $I \in S^m S^d \mathbb{C}^2$ is invariant with respect to the subgroup $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ if and only if it is invariant with respect to the subalgebra N^- if and only if $\Delta I = 0$ where

$$\Delta = \sum_{i=0}^{d-1}(d-i)a_{i+1}\frac{\partial}{\partial a_i} = d a_1 \frac{\partial}{\partial a_0} + (d-1)a_2 \frac{\partial}{\partial a_1} + \dots$$

(ii) $I \in S^m S^d \mathbb{C}^2$ is SL(2)-invariant if and only if it is isobaric and $\Delta I = 0$.

The following Proposition is a computation contained in [24], it is interesting because gave a motivation to study Lie algebras.

PROPOSITION 11.
(i) $D\Delta - \Delta D = \sum_{i=0}^d (d-2i)a_i \frac{\partial}{\partial a_i}$.
(ii) $(D\Delta - \Delta D)(a_0^{v_0} \dots a_d^{v_d}) = \sum_{i=0}^d (d-2i)v_i(a_0^{v_0} \dots a_d^{v_d}) = (dg - 2p)(a_0^{v_0} \dots a_d^{v_d})$.

Hilbert proved from this proposition that an isobaric polynomial $F \in S^g(S^d \mathbb{C}^2)$ (all its monomials have weight p where $dg - 2p = 0$) satisfying $DF = 0$ must satisfy also $\Delta F = 0$, which is nowadays clear from the structure of $\mathfrak{sl}(2)$-modules (see §2.5 and also §3.7). Indeed their weight are segments centered around $(\frac{dg}{2}, \frac{dg}{2})$.

Note that we have in the case of the quadrics generating the twisted cubic (see (22))

$$
a_1a_3 - a_2^2 \;\underset{\Delta}{\overset{D}{\rightleftarrows}}\; a_0a_3 - a_1a_2 \;\underset{\Delta}{\overset{D}{\rightleftarrows}}\; a_0a_2 - a_1^2.
$$

For ternary forms we introduce the differential operators

$$
D_1 = \sum_{i_0+\ldots+i_2=d} i_1 f_{i_0+1,i_1-1,i_2} \frac{\partial}{\partial f_{i_0\ldots i_2}} = 0,
$$

$$
D_2 = \sum_{i_0+\ldots+i_2=d} i_2 f_{i_0,i_1+1,i_2-1} \frac{\partial}{\partial f_{i_0\ldots i_2}} = 0.
$$

Note that D_1 adds $(1,-1,0)$ to the weight, while D_2 adds $(0,1,-1)$ to the weight.

They correspond to A_1 and A_2 in §2.5 and give the action on the directions depicted in (10) and (11). The following result is the natural extension to $\mathfrak{sl}(3)$ of Prop. 9, the proof is the same.

PROPOSITION 12.
(i) $I \in S^m S^d \mathbb{C}^3$ is invariant with respect to the subalgebra N^+ if and only if $D_1 I = D_2 I = 0$.
(ii) $I \in S^m S^d \mathbb{C}^3$ is $SL(3)$-invariant if and only if it is isobaric and $D_1 I = D_2 I = 0$.

3.4. Cayley-Sylvester formula for the number of invariants of binary forms

We recall now Cayley-Sylvester computation of the dimension of invariants and covariants for binary forms.

We have already seen the Hessian $f_{xx}f_{yy} - f_{xy}^2$ of a binary form $f \in S^d \mathbb{C}^2$, which can be considered as a module $S^{2d-4}\mathbb{C}^2 \subset S^2(S^d\mathbb{C}^2)$. Indeed it is a polynomial of degree $2d-4$ in x,y with coefficients of degree 2 in f_i. This is an example of a *covariant* of f. In general a covariant of degree g of f is any module $S^e\mathbb{C}^2 \subset S^g(S^d\mathbb{C}^2)$, so it corresponds to a ($SL(2)$-invariant) polynomial of degree e in x,y with coefficients of degree g in f_i.

One of the most advanced achievement of classical period was the following computation of the number of covariants, which is a nontrivial example of $SL(2)$-plethysm.

We recall from §3.1 that $H_{g,p,d}$ is the space of monomials in $S^g S^d \mathbb{C}^2$ of weight $(p, dg-p)$.

Let D be the differential operator defined in Proposition 9.

Let $I_{g,p,d}$ be the kernel of the map $H_{g,p,d} \xrightarrow{D} H_{g,p-1,d}$ of degree g and weight p in $a_0, \ldots a_d$.

THEOREM 17 (Cayley-Sylvester). *(i) Let* $p \leq \frac{dg}{2}$. *Then* $\dim I_{g,p,d}$ *is the degree p coefficient in*

$$\frac{(1-x^{d+1})\ldots(1-x^{d+g})}{(1-x^2)\ldots(1-x^g)}.$$

(ii) Let $2p = dg$. *Then* $\dim I_{g,\frac{dg}{2},d}$ *is the dimension of the space of invariants so it is the coefficient of degree* $\frac{dg}{2}$ *in*

$$\frac{(1-x^{d+1})\ldots(1-x^{d+g})}{(1-x^2)\ldots(1-x^g)}.$$

(iii) More generally we have the $SL(2)$*-decomposition*

$$S^g(S^d\mathbb{C}^2) = \oplus_e S^e\mathbb{C}^2 \otimes I_{g,\frac{dg-e}{2},d}$$

where $\dim I_{g,\frac{dg-e}{2},d}$ *is equal to the coefficient of degree* $\frac{dg-e}{2}$ *in*

$$\frac{(1-x^{d+1})\ldots(1-x^{d+g})}{(1-x^2)\ldots(1-x^g)}.$$

Proof. (i) Any $\mathfrak{sl}(2)$ representations splits as a sum of irreducible representations, each one centered around the weight $(\frac{dg}{2}, \frac{dg}{2})$ (this is meaningful even if $\frac{dg}{2}$ is not an integer). Hence for $p \leq \frac{dg}{2}$ the differential D is surjective and the result follows from Theorem 16.

(ii) follows from (i) and 9.

(iii) By the description in (i), the number of irreducible representations isomorphic to S^e can be computed looking at the space of monomials of weight $(\frac{dg-e}{2}, \frac{dg+e}{2})$ which are killed by D. □

REMARK 5. For $p \leq \frac{dg}{2}$, $\dim I_{g,p,d}$ may be computed by Proposition 8 also as the coefficient of $x^p y^g$ in

$$(1-x)\prod_{i=0}^{d}\frac{1}{1-x^i y}.$$

The following proof is borrowed from [24]. We include it to show that concrete applications of Cayley-Sylvester formula can be painful. We will give an alternative proof in Corollary 7.

COROLLARY 3. *Let* $d = 3$. *Let* $I_{g,\frac{3g}{2},3}$ *be the dimension of the space of invariants of degree g of the binary cubic. Then*

$$\sum_{g=0}^{\infty} I_{g,\frac{3g}{2},3} x^g = \frac{1}{1-x^4}.$$

The ring of invariants is freely generated by the discriminant D, *that is*

$$\oplus_m S^m(S^3(\mathbb{C}^2))^{SL(2)} = \mathbb{C}[D].$$

Proof. We have from Theorem 17 that

$I_{g,\frac{3g}{2},3}$ is the degree $\frac{3g}{2}$ coefficient of

$$\frac{(1-x^{3+1})\dots(1-x^{3+g})}{(1-x^2)\dots(1-x^g)} = \frac{(1-x^{g+1})(1-x^{g+2})(1-x^{g+3})}{(1-x^2)(1-x^3)}$$

and we write it as

$$\left\{ \frac{(1-x^{g+1})(1-x^{g+2})(1-x^{g+3})}{(1-x^2)(1-x^3)} \right\}_{\frac{3g}{2}}.$$

We can remove the terms which do not change the coefficient of $x^{\frac{3g}{2}}$, so getting

$$\left\{ \frac{(1-x^{g+1}-x^{g+2}-x^{g+3})}{(1-x^2)(1-x^3)} \right\}_{\frac{3g}{2}} = \left\{ \frac{1}{(1-x^2)(1-x^3)} \right\}_{\frac{3g}{2}} - \left\{ \frac{x(1+x+x^2)}{(1-x^2)(1-x^3)} \right\}_{\frac{g}{2}} =$$

(by using that $(1-x^3) = (1+x+x^2)(1-x)$)

$$= \left\{ \frac{1}{(1-x^2)(1-x^3)} \right\}_{\frac{3g}{2}} - \left\{ \frac{x}{(1-x)(1-x^2)} \right\}_{\frac{g}{2}} =$$

$$= \left\{ \frac{1}{(1-x^2)(1-x^3)} - \frac{x^3}{(1-x^3)(1-x^6)} \right\}_{\frac{3g}{2}} =$$

$$= \left\{ \frac{1-x^6-x^3+x^5}{(1-x^2)(1-x^3)(1-x^6)} \right\}_{\frac{3g}{2}} = \left\{ \frac{(1-x^2)(1-x^3)+x^2-x^6}{(1-x^2)(1-x^3)(1-x^6)} \right\}_{\frac{3g}{2}} =$$

$$= \left\{ \frac{1}{(1-x^6)} \right\}_{\frac{3g}{2}} + \left\{ \frac{x^2(1+x^2)}{(1-x^3)(1-x^6)} \right\}_{\frac{3g}{2}} =$$

$$= \left\{ \frac{1}{(1-x^4)} \right\}_{g} + \left\{ \frac{x^4(1+x^4)}{(1-x^6)(1-x^{12})} \right\}_{3g}.$$

The second summand contains only terms of the form x^{4m+3n} with $m = 1$ or 2, hence none of the form x^{3g}, and so does not contribute anything. $\qquad\square$

3.5. Counting partitions and symmetric functions

The following combinatorial interpretation is interesting.

PROPOSITION 13. *The number $H_{g,p,d}$ is the number of partitions of p as a sum of at most g summands from $\{1,\dots,d\}$, which is equal to the number of Young diagrams with p boxes, at most d rows and at most g columns.*

Proof. $H_{g,p,d}$ counts monomials $a_0^{i_0} \ldots a_d^{i_d}$ with $i_0 + \ldots + i_d = g$ and $i_1 + 2i_2 + \ldots + di_d = p$ which correspond to the partition

$$\underbrace{1 + \ldots + 1}_{i_1} + \underbrace{2 + \ldots + 2}_{i_2} + \ldots + \underbrace{d + \ldots + d}_{i_d} = p.$$

$\square$

COROLLARY 4 (Hermite reciprocity).

$$H_{g,p,d} = H_{d,p,g}, \qquad S^g(S^d\mathbb{C}^2) = S^d(S^g\mathbb{C}^2).$$

In equivalent way the number $H_{g,p,d}$ counts also the number of partitions of p as a sum of at most d summands from $\{1, \ldots, g\}$.

Proof. Taking transposition of Young diagrams. $\square$

For $d \geq p$, $H_{g,p,d}$ stabilizes to $H_{g,p}$ which is the number of partitions of p as a sum of at most g summands, or also the number of Young diagrams with p boxes and at most g rows. The coefficient $H_{g,p}$ is equal also to the number of partitions of p with summands from $\{1, \ldots, g\}$, indeed it is the number of Young diagrams with p boxes and at most g columns, which are the transpose of the previous ones.

For $d \geq p$, the numerator of (13) has no role in the computation and we get

$$(15) \qquad \sum_{p=0}^{\infty} H_{g,p}x^p = \prod_{i=1}^{g} \frac{1}{(1-x^i)}.$$

This formula has a more elementary interpretation, indeed we have

$$\prod_{i=1}^{g} \frac{1}{(1-x^i)} = \left(\sum_{i_1 \geq 0} x^{i_1}\right)\left(\sum_{i_2 \geq 0} x^{2i_2}\right) \cdots \left(\sum_{i_g \geq 0} x^{gi_g}\right) = \sum_{i_1, \ldots, i_g} x^{i_1 + 2i_2 + \ldots + gi_g}$$

and the coefficient of x^p counts all indexes i_j which make $i_1 + 2i_2 + \ldots + gi_g = p$. Cayley-Sylvester argument can be interpreted as a refinement of this elementary computation. The formula (15) can be considered as the Hilbert series of the ring of symmetric functions polynomials in g variables, having Schur polynomials s_λ as a basis.

Note from the Proposition 8 also the identity

$$(16) \qquad \sum_{p,g} dimH_{g,p}x^p y^g = \prod_{i=0}^{\infty} \frac{1}{1-x^i y}$$

which is meaningful because for each power of x only finitely many factors are taken into account in the right-hand side.

For any fixed p, for $g \geq p$ $H_{g,p}$ stabilizes to H_p, the number of partitions of p, and from (15) we can write the basic formula

$$\sum_{p=0}^{\infty} H_p x^p = \prod_{i=1}^{\infty} \frac{1}{(1-x^i)}$$

which is again meaningful because for each power of x only finitely many factors are taken into account in the right-hand side. The formula (16) can be considered as the Hilbert series of the ring of symmetric functions polynomials in infinitely many variables.

Hardy and Ramanujam proved the remarkable asympotic formula

$$H_p \sim \frac{1}{4p\sqrt{3}} \exp \pi \sqrt{\frac{2p}{3}} \text{ for } p \to \infty$$

which is one of the many wonderful and "not expected" appearances of π in combinatorics. For an interesting proof in the probabilistic setting, see [5].

3.6. Generating formula for the number of invariants of ternary forms

We saw in Theorem 17 that the dimension of the space of invariants in $S^g S^d \mathbb{C}^2$ can be obtained as $\dim H_{g,p,d} - \dim H_{g,p-1,d}$. The following Theorem perform an analogous computation for $SL(3)$-invariants, and gives a clue how to perform such computations in general.

THEOREM 18 (Sturmfels [53] Algorithm 4.7.5, Bedratyuk [7,8]). *, Let $dg = p_0 + p_1 + p_2$. Let $h_{g,d,p_0,p_1,p_2} = \dim H_{g,d,p_0,p_1,p_2}$. Let $I_{g,d,p,p,p}$ be the space of invariants in $S^g S^d \mathbb{C}^3$.*

(17) $\dim I_{g,d,p,p,p} = h_{g,d,p,p,p} - h_{g,d,p+1,p-1,p} - h_{g,d,p-1,p,p+1} +$
$$h_{g,d,p+1,p-2,p+1} + h_{g,d,p-1,p-1,p+2} - h_{g,d,p,p-2,p+2}.$$

So by (14) we get that $\dim I_{g,d,p,p,p}$ *is the coefficient of* $x_1^{p+2} x_2^p y^g$ *in*

$$(x_2 - x_1)(x_2 - 1)(x_1 - 1) \prod_{i_1=0}^{d} \prod_{i_2=0}^{d-i_1} \frac{1}{1 - x_1^{i_1} x_2^{i_2} y}.$$

Proof. $S^d S^g \mathbb{C}^2$ splits as the sum of irreducible representations. Representing their weights, they span hexagons or triangles like in §2.5, all centered around $(\frac{dg}{3}, \frac{dg}{3}, \frac{dg}{3})$. The right hand side counts 1 on every trivial summand in $S^g S^d \mathbb{C}^3$ and it counts 0 on every nontrivial summand in $S^g S^d \mathbb{C}^3$. The details are in [7,8]. $\qquad\square$

The six summands in (17) correspond to the action of the Weyl group Σ_3 for $SL(3)$, and are identified as vertices of a hexagon (see next table (18)). In [8] the formula is extended to any $SL(n+1)$.

The following M2 script shows how to get the whole decomposition of $h_{g,d,*}$ in cases of cubic invariant of plane quartics, other cases can be obtained by changing the values of d and m. For larger values, the size of the computation can be reduced by contracting separately with $y^m x1^{i_1} x2^{i_2}$ and bounding correspondingly the upper limit of the sum.

```
R=QQ[y,x1,x2]
d=4, m=3
sub(contract(y^m,product(d+1,i->
product(d+1-i,j->sum(m+1,k->y^k*x1^(k*i)*x2^(k*j))))),{y=>0})
```

The output of previous script can be recorded in the following table

(18)

```
1
    1
1       2
    2       3
2       4       4
    4       6       4
3       8       8       5
    6      11       9       4
4      11     (15)      9       4
    8     (16)    (15)      8       3
4      15      19      15       6       2
    9     (19)    (19)     11       4       1
5      15     ((23))     16       8       2       1
    9      19      19      11       4       1
4      15      19      15       6       2
    8      16      15       8       3
4      11      15       9       4
    6      11       9       4
3       8       8       5
    4       6       4
2       4       4
    2       3
1       2
    1
1
```

The sum of all the numbers in the triangle is $680 = \dim S^3(S^4\mathbb{C}^3)$. The rounded entries correspond to the six summands in Bedratyuk formula (17). According to the table (18) we compute from Theorem 18 $\dim I_{3,4,4,4,4} = 23 - 19 - 19 + 16 + 15 - 15 = 1$.

Note that in (18) it is enough to record the following "one third" part, and the others can be filled by symmetry.

```
4
    4
8           5
    9           4
15          9           4
    15          8           3
19          15          6           2
    19          11          4           1
(23)        16          8           2           1
    19          11          4           1
    15          6           2
        8           3
        4
```

In Sturmfels' book [53] there is the computation of Hilbert series for plane quartics up to degree 21. Shioda got in [52] the complete series, from where one can guess possible generators. There is an unpublished paper by Ohno, claiming a system of generators for the invariant subring of plane quartics (see [2]).

REMARK 6. L. Bedratyuk gives in [7,8] other formulas which extend Theorem 17 to n-ary forms. On his web page is available some software packages implementing these formulas.

3.7. The Reynolds operator and how to compute it. Hilbert finiteness theorem

When $G = SL(n+1)$ acts on a vector space V, we recall from §2.1 the notation

$$V^G = \{v \in V | g \cdot v = v \forall g \in G\}.$$

Since $SL(n+1)$ is reductive we have the direct sum $V = V^G \oplus \left(V^G\right)^\perp$, where $\left(V^G\right)^\perp$ is the sum of all irreducible invariant subspaces of V which are nontrivial (see Theorem 14).

The Reynolds operator is the projection

$$R \colon V \to V^G.$$

If G is a finite group, such a projection can be obtained just averaging over the group, that is $R(v) = \frac{1}{|G|} \sum_{g \in G} g \cdot v$. In the case of $G = SL(n+1)$, Weyl replaced the sum with the integration over a maximal compact real subgroup, which is $SU/n+1)$, this is the "Weyl unitary trick" [19]. It is difficult to compute the integral directly from the definition. We want to show, in an example, how the Reynolds operator can be explicitly computed by the Lie algebra techniques.

PROPOSITION 14. *Let* $f = a_0 x^4 + 4a_1 x^3 y + 6a_2 x^2 y^2 + 4a_3 xy^3 + a_4 y^4 \in S^4 \mathbb{C}^2$. *We consider* $R \colon S^2(S^4 \mathbb{C}^2) \to S^2(S^4 \mathbb{C}^2)^G$. *Then*

$$R(a_0 a_4) = \frac{2}{5}I, \qquad R(a_1 a_3) = -\frac{1}{10}I, \qquad R(a_2^2) = \frac{1}{15}I,$$

where $I = a_0 a_4 - 4a_1 a_3 + 3a_2^2 \in S^2(S^4 \mathbb{C}^2)^G$, *while* R *vanishes on all other monomials in* $S^2(S^4 \mathbb{C}^2)$.

Proof. The key is to consider the differential operator $D = \sum_{i=0}^{4}(i+1)a_i \frac{\partial}{\partial a_{i+1}}$ acting on $S^2(S^4 \mathbb{C}^2)$. The torus action has the eigenspaces $S^2(S^4 \mathbb{C}^2) = \oplus H_i$ where the weight of $a_i a_j$ is $2(i+j) - 8$. We write dimensions in superscripts

$$H_{-8}^1 \xleftarrow{D} H_{-6}^1 \xleftarrow{D} H_{-4}^2 \xleftarrow{D} H_{-2}^2 \xleftarrow{D} H_0^3 \xleftarrow{D} H_2^2 \xleftarrow{D} H_4^2 \xleftarrow{D} H_6^1 \xleftarrow{D} H_8^1$$

which come from the three representations

$$S^2(S^4 \mathbb{C}^2) = S^8 \mathbb{C}^2 \oplus S^4 \mathbb{C}^2 \oplus S^0 \mathbb{C}^2$$

namely

$$\circ \longleftarrow \circ \longleftarrow \circ \longleftarrow \circ \longleftarrow \circ \longleftarrow \circ \longleftarrow \circ \longleftarrow \circ \longleftarrow \circ$$
$$\circ \longleftarrow \circ \longleftarrow \circ \longleftarrow \circ \longleftarrow \circ$$
$$\circ$$

We refer to the three rows above. We have

$$a_0^2 \longleftarrow a_0 a_1 \longleftarrow \begin{matrix} a_0 a_2 \\ a_1^2 \end{matrix} \longleftarrow \begin{matrix} a_0 a_3 \\ a_1 a_2 \end{matrix} \longleftarrow \begin{matrix} a_0 a_4 \\ a_1 a_3 \\ a_2^2 \end{matrix} \longleftarrow \begin{matrix} a_1 a_4 \\ a_2 a_3 \end{matrix} \longleftarrow \begin{matrix} a_2 a_4 \\ a_3^2 \end{matrix} \longleftarrow a_3 a_4 \longleftarrow a_4^2$$

and the goal is to split these monomial spaces into the three previous rows. We compute the splitting as follows.

Since $D^5(a_3^2) = 720 a_0 a_1$, $D^5(a_2 a_4) = 720 a_0 a_1$, then $D^5(a_2 a_4 - a_3^2) = 0$ and $a_2 a_4 - a_3^2$ belongs to the second row.

Hence $D^2(a_2 a_4 - a_3^2) = 2(a_0 a_4 + 2a_1 a_3 - 3a_2^2)$ belongs again to the second row.

Moreover $D^4(a_4^2) = 48(a_0 a_4 + 16 a_1 a_3 + 18 a_2^2)$ belongs to the first row. Hence we have the decomposition

$$H_0^3 = <I> \oplus <a_0 a_4 + 2a_1 a_3 - 3a_2^2, a_0 a_4 + 16 a_1 a_3 + 18 a_2^2>$$

and R is the projection over $<I>$. We get the scalars stated by inverting the matrix

$$\begin{bmatrix} 1 & -4 & 3 \\ 1 & 2 & -3 \\ 1 & 16 & 18 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{2}{5} & * & * \\ -\frac{1}{10} & * & * \\ \frac{1}{15} & * & * \end{bmatrix}.$$

$\square$

From the above proof it should be clear that the same technique can be applied in order to compute the Reynolds operator in other cases. For an alternative approach, by using Casimir operator, see [13] 4.5.2. The computations performed by Derksen and Kemper are essentially equivalent to ours.

PROPOSITION 15. *Let $R = \oplus_i R_i$ be a graded ring where $G = SL(n)$ acts. If $f \in R_i^G$, $g \in R_j$, then*

$$R(fg) = fR(g).$$

Proof. Decompose $g = g_1 + g_2$ where $g_1 \in R_j^G$ and g_2 belongs to its complement $(R_j^G)^\perp$.

Then $fR_j^G \subset R_{i+j}^G$ and $f(R_j^G)^\perp \subset (R_{i+j}^G)^\perp$, indeed if f is nonzero, fR_j^G is a G-module isomorphic to R_j^G and $f(R_j^G)^\perp$ is a G-module isomorphic to $(R_j^G)^\perp$. □

We close this section by recalling the wonderful proof about the finite generation of the invariant ring, proved first by Hilbert in 1890. Hilbert result, together with the 1FT (which we will see in §3.10) implies that the ring of invariants is generated by finitely many products of tableau like in §3.8.

THEOREM 19 (Hilbert). *Let $G = SL(n+1)$ (although any reductive group works as well). Let W be a finite dimensional G-module. Then the invariant subring $\mathbb{C}[W]^G$ is finitely generated.*

Proof. Let I be the ideal in $\mathbb{C}[W]$ generated by all the homogeneous invariants of positive degree. Since $\mathbb{C}[W]$ is Noetherian we get that I is generated by homogeneous invariants $f_1, \ldots, f_r$.

We will prove that for any degree d $\mathbb{C}[W]_d^G = \mathbb{C}[f_1, \ldots, f_r]_d$ by induction on d. The case $d = 0$ is obvious. Let $f \in \mathbb{C}[W]_d^G \subset I_d$. Then there exist $a_i \in R$ such that $f \in \sum_{i=0} a_i f_i$.

We get $f = R(f) = \sum_i R(a_i f_i) = \sum_i R(a_i) f_i$, the last equality by Proposition 15. By the inductive assumption, each $R(a_i)$ is a polynomial in f_i, hence the same is true for f.

□

3.8. Tableau functions. Comparison among different applications of Young diagrams

Let $V = \mathbb{C}^{n+1}$, more precisely for the following it is enough to consider a $(n+1)$-dimensional vector space with a fixed isomorphism $\wedge^{n+1} V \simeq \mathbb{C}$.

Then for every $v_1, \ldots v_{n+1} \in V$ the determinant $v_1 \wedge \ldots \wedge v_{n+1} \in \mathbb{C}$ is well defined and it is $SL(n+1)$-invariant for the natural action of $SL(n+1)$ on V. Every rectangular tableau T over a Young diagram of size $(n+1) \times g$ gives a tableau function which is constructed by taking the product of the determinant arising from each column.

So $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$ represents $x_1 \wedge x_2 \wedge x_3$, and $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline 3 & 4 \\ \hline \end{array}$ represents $(x_1 \wedge x_2 \wedge x_3)(x_1 \wedge x_3 \wedge x_4)$.

To define formally this notion, we set $[a] = \{1, \ldots, a\}$ for any natural number a and we notice that a tableau T is encoded in a function $t \colon [n+1] \times [g] \to [m]$, where $t(i, j)$ corresponds to the entry at the place (i, j) of the tableau.

DEFINITION 4 (From tableau to multilinear invariants). *For any tableau T over a Young diagram of size $(n+1) \times g$ filled with numbers from 1 appearing h_1 times until m appearing h_m times, so that $h_1 + \ldots + h_m = g(n+1)$, we denote by G_T the multilinear function $S^{h_1} V^\vee \times \ldots \times S^{h_m} V^\vee \to \mathbb{C}$ defined by*

$$G_T(x_1^{h_1}, \ldots, x_m^{h_m}) = \prod_{j=1}^m \left(x_{t(1,j)} \wedge \ldots \wedge x_{t(n+1,j)} \right).$$

G_T is well defined by Theorem 1, and it is called a tableau function.

PROPOSITION 16. *Every tableau function G_T is $SL(V)$-invariant.*

Proof. Immediate by the properties of the determinant. □

Every Young diagram λ defines (at least) four interesting objects, which give four similar theories, that is

- a representation $S^\lambda V$ of $SL(V)$ (see Theor. 9)

- a representation V_λ of the symmetric group (see Def. 2)

- a symmetric polynomial s_λ (see Def. 3)

- a Schubert cell X_λ in the Grassmannian (it will be defined in a while)

This comparison is carefully studied in [36]. The Schur functions s_λ give an additive basis of the ring of symmetric polynomials. There is a algorithm to perform, given a symmetric function f, the decomposition $f = \sum_\lambda c_\lambda s_\lambda$, described in the Algorithm 4.1.16 in [53]. Essentially, let $ct_1^{\nu_1 \ldots t_n^{\nu_n}}$ be the leading term in f, let λ be the unique partition of d such that the corresponding highest weight of s_λ is $t_1^{\nu_1 \ldots t_n^{\nu_n}}$. Then we output the summand cs_λ and we continue with $f - cs_\lambda$.

From the computational point of view, among the four similar theories listed above, the symmetric functions are the ones that can be better understood. For example, to compute $S^2(S^3 \mathbb{C}^3)$ consider first the 10 monomials $t_1^3, t_1^2 t_2, \ldots, t_3^3$. Then consider the sum of all product of two of these monomials, which is $t_1^6 + t_1^5 t_2 + \ldots + t_3^6$. This last polynomial can be decomposed as $s_6 + s_{4,2}$. It follows that $S^2(S^3 \mathbb{C}^3) = S^6 \mathbb{C}^3 + S^{4,2} \mathbb{C}^3$.

Often it is convenient, in practical computations, to consider Newton sum of powers, which behave better according to plethysm.

The Schubert cell X_λ is defined as

$X_\lambda = \{ m \in Gr(\mathbb{P}^k, \mathbb{P}^n) | \dim \ m \cap V_{n-k+i-\lambda_i} \geq i, \text{ for } i = 1, \ldots, k+1 \}$

where $e_0, \ldots, e_n$ is a basis of V and $V_i = <e_0, \ldots, e_i>$.

We have the following formulas which show the strong similarities among different theories

$$S^\lambda V \otimes S^\mu V = \sum_\nu c_{\lambda\mu\nu} S^\nu V$$

for some integer coefficients $c_{\lambda\mu\nu}$, which repeat in the following

$$s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu\nu} s_\nu,$$

$$X_\lambda \cap X_\mu = \sum_\nu c_{\lambda\mu\nu} X_\nu.$$

The tensor product of Σ_d-modules V_λ behaves in a different way.

3.9. The symbolic representation of invariants

The symbolic representation is an economic way to encode and write down invariants. It works both for invariants of forms, that we consider here, and for invariants of points, that we consider in §5.

It was called by Weyl "the great war-horse of nineteenth century invariant theory". The reader should not lose the historical article [49]. The "symbolic calculus" is essential to understand the classical sources. In the words of Enriques and Chisini ([17] pag. 37, chap. 1):

"Ma a supplire calcoli laboriosi determinandone a priori il risultato, si può anche far uso del procedimento di notazione simbolica di Cayley-Aronhold, che risponde a questa esigenza economica porgendo un modo sistematico di formazione. L'idea fondamentale contenuta nella rappresentazione simbolica costituisce un fecondo principio di conservazione formale rispetto alle degenerazioni."[1]

Since every invariant corresponds to a one dimensional representation in $S^m S^d V$, they are spanned by tableau as in Theorem 6.

Every $F \in S^m S^d V$ corresponds to a multilinear function

$$F \colon \underbrace{S^d V \times \ldots \times S^d V}_{m} \to \mathbb{C}$$

and this last is determined by Theor. 1 by $F(x_1^d, \ldots, x_m^d)$ for any linear forms x_i, $i = 1, \ldots, m$, which is symmetric in the m entries corresponding to our label numbers.

Let $md = g(n+1)$. In the symbolic representation we start from a tableau T filling the Young diagram g^{n+1} with the numbers 1 repeated d times, 2 repeated d times and so on until d repeated d times. The main construction of the symbolic representation is to define an invariant $F_T \in S^m(S^d \mathbb{C}^{n+1})^{SL(n+1)}$, by using the above idea.

DEFINITION 5. *[From tableau to polynomial invariants] Let $md = g(n+1)$. Let T be a tableau filling the Young diagram of rectangular size $(n+1) \times g$ with the numbers 1 repeated d times, 2 repeated d times and so on until m repeated d times. Let $G_T \colon \underbrace{S^d V^\vee \times \ldots \times S^d V^\vee}_{m} \to \mathbb{C}$ be the function introduced in Definition 4.*

---

[1] In order to avoid messy computations, by determining in advance their result, it may be used Cayley-Aronhold symbolic representation. It gives a systematic way to construct the invariants by answering the need of simplicity. The fundamental idea of symbolic representation gives a fruitful invariance principle with respect to degeneration.

We denote by $F_T \in S^m\left(S^d V\right)$ the polynomial obtained by symmetrizing G_T, that is $F_T(h) = G_T(\underbrace{h, \ldots, h}_{m})$ for any $h \in S^d V^\vee$. F_T is called a symmetrized tableau function.

THEOREM 20. *Any F_T as in Definition 5 is $SL(n+1) - invariant$.*

Proof. Let $g \in SL(n+1)$ and $h \in S^d V^\vee$. We have $F_T(g \cdot h) = G_T(\underbrace{g \cdot h, \ldots, g \cdot h}_{m}) = G_T(\underbrace{h, \ldots, h}_{m}) = F_T(h)$, the second equality by Proposition 16. $\qquad \square$

By the description of representations of $GL(V)$, we get an element F_T in the representation of dimension one $S^\lambda V \subset S^m(S^d V)$, possibly zero.

We emphasize that $F_T(x^d) = G_T(\underbrace{x^d, \ldots, x^d}_{m}) = 0$ for every $x \in V^\vee$, because we get a determinant with equal rows. Nevertheless the symmetrization of Def. 5 is meaningful for general $h \in S^d V^\vee$. It is strongly recommended to practice with some examples in order to understand how the construction of Def. 5 is powerful. Let's start with the example of the invariant J for binary quartics (compare also with the different example 4.5.7 in [53]).

Fill the Young diagram $\lambda = (6,6)$ with respectively four copies of each $1, 2, 3$, obtaining the following

(19)
$$T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 2 & 3 & 3 & 3 & 3 \\ \hline \end{array}.$$

It is defined the function $F_T(x^4, y^4, z^4) = (x \wedge y)^2 (x \wedge z)^2 (y \wedge z)^2$. In the classical literature, this representation was denoted sometimes as

(20)
$$[12]^2 [13]^2 [23]^2.$$

By developing $(x_0 y_1 - x_1 y_0)^2 (x_0 z_1 - x_1 z_0)^2 (y_0 z_1 - y_1 z_0)^2$, we get 19 monomials. The first monomial is $x_0^4 y_0^2 y_1^2 z_1^4$ and , according to the correspondence seen in (1), we get $x_0^4 \mapsto a_0, y_0^2 y_1^2 \mapsto a_2, z_1^4 \mapsto a_4$, so that the first monomial corresponds to $a_0 a_2 a_4$.

The following M2 script does automatically this job and it can be adapted to other symbolic expressions.

```
S=QQ[x_0,x_1,y_0,y_1,z_0,z_1,a_0..a_4]
f=(x_0*y_1-x_1*y_0)^2*(x_0*z_1-x_1*z_0)^2*(y_0*z_1-y_1*z_0)^2
symb=(x,h)->(contract(x,h)*transpose matrix{{a_0..a_4}})_(0,0)
fx=symb(symmetricPower(4,matrix{{x_0,x_1}}),f)
fxy=symb(symmetricPower(4,matrix{{y_0,y_1}}),fx)
fxyz=symb(symmetricPower(4,matrix{{z_0,z_1}}),fxy)
```

The result we get for F_T is (up to scalar multiples) the well known expression of J (compare with (5))

$$F_T = -a_2^3 + 2a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 + a_0a_2a_4.$$

In formula (19), the bracket [12] appears in two among the columns. If [12] appears in three columns, one is forced to repeat 3 on the same column, getting zero. By similar elementary arguments, the reader can easily convince himself that (19) gives the only nonzero invariant in $S^3S^4\mathbb{C}^2$.

For higher degree invariants, one meets quickly very large expressions. A computational trick, to reduce the size of the memory used, is to introduce the expression f step by step and to manage the symbolic reduction of any single variable correspondingly. In the following example, the variable x appears already in the first two square factors of f, and the symbolic reduction of x can be done just at this step.

```
S=QQ[x_0,x_1,y_0,y_1,z_0,z_1,a_0..a_4]
ff=(x_0*y_1-x_1*y_0)^2*(x_0*z_1-x_1*z_0)^2
symb=(x,h)->(contract(x,h)*transpose matrix{{a_0..a_4}})_(0,0)
fx=symb(symmetricPower(4,matrix{{x_0,x_1}}),ff)
--now we introduce the third square factor
fxy=symb(symmetricPower(4,matrix{{y_0,y_1}}),fx*(y_0*z_1-y_1*z_0)^2)
fxyz=symb(symmetricPower(4,matrix{{z_0,z_1}}),fxy)
```

Note that $\left(\mathbb{C}^2\right)^6$ contains for $\lambda = (3,3)$ a $SL(2)$-invariant subspace of dimension equal to dim $V_{3,3} = 5$. The dimension of $V_{m,m}$ is equal to $\frac{1}{m+1}\binom{2m}{m}$, the m-th Catalan number.

The main hidden difficulty in the application of the method of "symbolic representation" is that it is hard to detect in advance if a given symbolic expression gives the zero invariant.

3.10. The two Fundamental Theorems for invariants of forms

The First Fundamental Theorem for invariants of forms says that any invariant of forms is a linear combination of the invariants F_T in Definition 5.

THEOREM 21 (First Fundamental Theorem (1FT) for forms).

Let md = (n+1)g. The space of invariants of degree m for S^dV is generated by symmetrized tableau functions F_T, constructed by tableau T as in Definition 5.

We postpone the proof until §5, after 1FT for invariants of points (Theorem 28) will be proved.

The theorem extends in a natural way to covariants. For example the "symbolic representation" of the Hessian H defined in 3 is

1	1	1	2
2	2	y	y

or $H = [12]^2[1y][2y]$.

The second fundamental theorems describes the relations between these invariants.

THEOREM 22 (Second Fundamental Theorem (2FT) for forms).

The relations among the generating invariants F_T of Theorem 21 are generated by Plücker relations like in Remark 2.

More precisely, fix a subset of $k+2$ elements $i_0 \ldots i_{k+1}$ and a set of k elements $j_0 \ldots j_{k-1}$. Let $T_s = [i_0 \ldots \hat{i}_s \ldots i_{k+1}][i_s j_0 \ldots j_{k-1}]$ a $2 \times (k+1)$ tableau and assume the numbers appearing are as in Definition 5 (this does not depend on s because every T_s permutes the same numbers). Then the Plücker relations are

$$\sum_{s=0}^{k+1} (-1)^s F_{T_s} = 0,$$

which hold for any subsets of respectively $k+2$ and k elements.

This description gives unfortunately cumbersome computations.

4. Hilbert series of invariant rings. Some more examples of invariants

4.1. Hilbert series

In all the examples where a complete system of invariants (or covariants) is known, the following steps can be performed

- (i) compute the Hilbert series of the invariant ring.

- (ii) guess generators of the corrected degree.

- (iii) compute the syzygies among the generators of step (ii), hence compute the subalgebra generated by these generators.

- (iv) check if the subalgebra coincides at any degree with the algebra by comparing the two Hilbert series.

The algebra of covariants has the bigraduation

$$Cov(S^d\mathbb{C}^2) = \oplus_{n,e} Hom^{SL(2)}(S^n(S^d\mathbb{C}^2), S^e\mathbb{C}^2)$$

and its Hilbert series depends correspondingly on two variables z, w

$$F_d(z,w) = \dim Cov(S^d\mathbb{C}^2)_{n,e} z^n w^e$$

in such a way that the coefficient of $w^a z^b$ denotes the dimension of

$$Hom^{SL(2)}(S^b\mathbb{C}^2, S^a(S^d\mathbb{C}^2)).$$

The exponent of z is the degree, the exponent of w is called the order. The covariants of order zero coincide with the invariants.

In 1980 Springer has found an efficient algorithm for the computation of Hilbert series for binary forms, by using residues. Variations of this algorithm may be found in Procesi's book [47] chap. 15, §3 or in Brion's notes [3]. In the following we follow [3].

Denote $\Phi_j \colon \mathbb{C}[z] \to \mathbb{C}[z]$ the linear map given by

$$\Phi_j(z^n) = \begin{cases} z^{n/j} & \text{if } n \equiv 0 \bmod j \\ 0 & \text{otherwise} \end{cases}$$

$\Phi_1(z^n) = z^n$

$$\Phi_2(z^n) = \begin{cases} z^{n/2} & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

$$\Phi_3(z^n) = \begin{cases} z^{n/3} & \text{if } n \equiv 0 \bmod 3 \\ 0 & \text{otherwise} \end{cases}$$

We have the equality $\Phi_j(ab) = \Phi_j(a)\Phi_j(b)$, if $a \in \mathbb{C}[z^j]$ (or $b \in \mathbb{C}[z^j]$). The map Φ_j extends to a unique linear map

$$\Phi_j \colon \mathbb{C}(z) \to \mathbb{C}(z)$$

which again satisfies the above equality.

THEOREM 23. *Let* $M = \oplus_n Hom^{SL(2)}(S^n(S^d\mathbb{C}^2), S^e\mathbb{C}^2)$. *The Hilbert series is*

$$F_M(z) = \sum_{0 \le j < d/2} (-1)^j \Phi_{d-2j}\left((1-z^2)z^e \gamma_{d,j}(z)\right)$$

where

$$\gamma_{d,j}(z) = \frac{z^{j(j+1)}}{\prod_{k=1}^{j}(1-z^{2k})\prod_{l=1}^{d-j}(1-z^{2l})}.$$

Proof. [3]   □

COROLLARY 5.

$$F_d(z,w) = \sum_{0 \le j < d/2} (-1)^j \Phi_{d-2j}\left(\frac{1-z^2}{1-zw}\gamma_{d,j}(z)\right).$$

The formula of Corollary 5 can be used to compute the Hilbert series of covariant rings of binary forms of small degree.

We reproduce for completeness some of the results, although in the following we will reprove some of them with tools from representation theory (like in §5.3), when available.

We have

$$F_3(z,0) = \Phi_3\left((1-z^2)\gamma_{3,0}(z)\right) - \Phi_1\left((1-z^2)\gamma_{3,1}(z)\right) =$$

$$\Phi_3\left(\frac{1}{(1-z^4)(1-z^6)}\right) - \Phi_1\left(\frac{z^2}{(1-z^2)(1-z^4)}\right)$$

and get

$$F_3(z,0) = \frac{1}{1-z^2}\Phi_3\left(\frac{1}{1-z^4}\right) - \frac{z^2}{(1-z^2)(1-z^4)} =$$

$$\frac{1}{(1-z^2)(1-z^4)} - \frac{z^2}{(1-z^2)(1-z^4)} = \frac{1}{1-z^4}.$$

Moreover

$$F_4(z,0) = \Phi_4\left((1-z^2)\gamma_{4,0}(z)\right) - \Phi_2\left((1-z^2)\gamma_{4,1}(z)\right) =$$

$$\Phi_4\left(\frac{1}{(1-z^4)(1-z^6)(1-z^8)}\right) - \Phi_2\left(\frac{z^2}{(1-z^2)(1-z^4)(1-z^6)}\right)$$

and get

$$F_4(z,0) = \frac{1}{(1-z)(1-z^2)}\Phi_4\left(\frac{1}{1-z^6}\right) - \frac{z}{(1-z)(1-z^2)(1-z^3)} =$$

$$\frac{1}{(1-z)(1-z^2)(1-z^3)} - \frac{z}{(1-z)(1-z^2)(1-z^3)} = \frac{1}{(1-z^2)(1-z^3)}.$$

For the Hilbert series of covariants we get

$$(21) \qquad F_3(z,w) = \frac{1+z^3w^3}{(1-z^4)(1-zw^3)(1-z^2w^2)},$$

$$F_4(z,w) = \frac{1+z^3w^6}{(1-z^2)(1-z^3)(1-zw^4)(1-z^2w^4)}.$$

4.2. Covariant ring of binary cubics

Let $f = a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3$. From the series (21), computed also in [53] (4.2.17), we can guess the table of covariants, namely

		order			
		0	1	2	3
	1				1
degree	2			1	
	3				1
	4	1			

Indeed we know some covariants exactly of the expected degrees, namely

		order			
		0	1	2	3
	1				f
degree	2			H	
	3				Q
	4	Δ			

where $\Delta = [12]^2[13][24][34]^2$ is the discriminant, $H = (f,f)_2 = [12]^2[1x][2x]$ is the Hessian, $Q = (f,H)_1$. The Hessian vanishes identically on the twisted cubic, indeed its plain expression is (we divide by 36 for convenience)

$$(22) \quad H = \frac{1}{36}\left(f_{00}f_{22} - f_{12}^2\right) = (a_1a_3 - a_2^2)x^2 + (a_0a_3 - a_1a_2)xy + (a_0a_2 - a_1^2)y^2.$$

The condition $H = 0$ represents the two points such that, together with $f = 0$, make a equianharmonic 4ple (see §4.4).

The condition $Q = 0$ represents the three points such that, together with $f = 0$, make a harmonic 4ple (see §4.4).

To fix the scalars, we pose
$\Delta = 4(a_0a_2 - a_1^2)(a_1a_3 - a_2^2) - (a_0a_3 - a_1a_2)^2$.

Then there is a unique syzygy which is $36H^3 + 9\Delta f^2 + Q^2 = 0$, note that Q^2 is expressed rationally by the others. Indeed the Hilbert series is

$$F_3(z,w) = \frac{1 + z^3w^3}{(1-z^4)(1-zw^3)(1-z^2w^2)}.$$

The three factors at the denominator correspond respectively to Δ, f, H. The Hilbert series says that the subalgebra generated by these covariants coincides with the covariant ring. In particular the invariant ring is free and it is generated by Δ. We have proved

THEOREM 24. *The covariants of a binary cubic f are generated by Δ, H, f and Q, satisfying the single relation*

$$36H^3 + 9\Delta f^2 + Q^2 = 0.$$

We recommend reading lecture XXI in [24] 1.8, where the solution of the cubic equation is obtained from the relation of Theorem 24.

4.3. Apolarity and transvectants

Let V be a vector space with basis $x_0, \ldots, x_n$ and let

$$R = K[x_0, \ldots, x_n] = \oplus_{m=0}^{\infty} S^m V$$

be the polynomial ring. Let's recall that the dual ring

$$R^{\vee} = K[\partial_0, \ldots, \partial_n] = \oplus_{m=0}^{\infty} S^m V^{\vee}$$

can be identified with ring of polynomial differential operators, where $\partial_i = \frac{\partial}{\partial x_i}$.

The action of $R^{\vee}$ over R was classically called as *apolarity*. In particular for any integers $e \geq d \geq 0$ we have the apolar linear maps

$$S^d V^{\vee} \otimes S^e V \to S^{e-d} V.$$

When $\dim U = 2$, that is for polynomials over a projective line, the apolarity is well defined for f, g both in $S^d U$. This is due to the canonical isomorphism $U \simeq U^{\vee} \otimes \wedge^2 U$.

This allows to make explicit the $SL(2)$-decompositions

$$S^d \mathbb{C}^2 \otimes S^e \mathbb{C}^2 = \oplus_{i=0}^{\min(d,e)} S^{d+e-2i} \mathbb{C}^2, \qquad S^2(S^d \mathbb{C}^2) = \oplus_{i=0}^{\lfloor \frac{d}{2} \rfloor} S^{2d-4i} \mathbb{C}^2.$$

Let (x_0, x_1) be coordinates on U. If $f = (a_0 x_0 + a_1 x_1)^d$ and $g = (b_0 x_0 + b_1 x_1)^d$, then the contraction between f and g is seen to be proportional to $(a_0 b_1 - a_1 b_0)^d$. This computation extends by linearity to any pair $f, g \in S^d U$, because any polynomial can be expressed as a sum of d-th powers. The resulting formula for $f = \sum_{i=0}^{d} \binom{d}{i} f_i x^{d-i} y^i$ and $g = \sum_{i=0}^{d} \binom{d}{i} g_i x^{d-i} y^i$ is that f is apolar to g if and only if

$$\sum_{i=0}^{d} (-1)^i \binom{d}{i} f_i g_{d-i} = 0.$$

In order to prove this formula, by linearity and by Theorem 1 it is again sufficient to assume $= u^d$ and $g = v^d$. In particular

PROPOSITION 17. *Let* $p, l^d \in S^d U$. *p is apolar to l^d if and only if l divides p.*

PROPOSITION 18. *If d is odd, any $f \in S^d U$ is apolar to itself. Apolarity defines a skew nondegenerate form in $\wedge^2(S^d \mathbb{C}^2)$.*

If d is even, the condition that f is apolar to itself defines a smooth quadric in $\mathbb{P}S^d U$. Apolarity defines a symmetric nondegenerate form in $S^2(S^d \mathbb{C}^2)$.

From the geometric point of view, let $f = l_1 \ldots l_d \in S^d \mathbb{C}^2$ be the decomposition in linear factors and denote by $[f]$ the corresponding point in $\mathbb{P}S^d \mathbb{C}^2$. Let $P(f)$ be the hyperplane spanned by $\{l_1^d, \ldots l_d^d\}$. f is apolar to itself if and only if $[f] \in P(f)$.

The natural algebraic generalization of the apolarity is given by transvection.

If $u^d \in S^d U$, $v^e \in S^e U$ we define the *n*-th transvectant[2] to be

$$(u^d, v^e)_n = u^{d-n} v^{e-n} [uv]^n \in S^{d+e-2n} \mathbb{C}^2.$$

If $f \in S^d U$, $g \in S^e U$ and $0 \le n \le \min(d,e)$, the general *SL*(2)-invariant formula
is

$$(f,g)_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{\partial f}{\partial x^{n-i} \partial y^i} \frac{\partial g}{\partial x^i \partial y^{n-i}}.$$

Note that $(f,g)_1$ is the Jacobian, while $(f,f)_2$ is the Hessian.

For $f \in S^4 \mathbb{C}^2$ we can express the invariants I and J introduced in §1.2 in terms of transvectants. Indeed it is easy to check that $I = (f,f)_2$, $J = (f,(f,f)_2)_4$. This gives a recipe to compute the expressions of I and J, that can be extended to other situations. A Theorem of Gordan states that all invariants of binary forms can be expressed by the iterate application of transvectants. We will not pursue this approach here. Transvectants are close to symbolic representation of §3.9, see [41] example 6.26.

4.4. Invariant ring of binary quartics

A polynomial $f = f_0 x^4 + 4 f_1 x^3 y + 6 f_2 x^2 y^2 + 4 f_3 x y^3 + f_4 y^4 \in S^4 U$ is called *equianharmonic* if its apolar to itself. So f is equianharmonic if and only if

$$f_0 f_4 - 4 f_1 f_3 + 3 f_2^2 = 0,$$

which is the expression for the classical invariant I of binary quartics, see (4).

The symbolic expression is $[12]^4$ or

1	1	1	1
2	2	2	2

.

The invariant J (see (5)) has the symbolic expression $[12]^2 [13]^2 [23]^2$ or

1	1	1	1	2	2
2	2	3	3	3	3

.

It is equal to the determinant

$$J = \begin{bmatrix} f_0 & f_1 & f_2 \\ f_1 & f_2 & f_3 \\ f_2 & f_3 & f_4 \end{bmatrix}.$$

A binary quartic with vanishing J is called *harmonic*. A binary quartic f is harmonic if and only if f has an apolar quadratic form, if and only if f is sum of two 4th powers, instead of the three which are needed for the general f.

---

[2] translated *"scorrimento"* in [17], from the original German *"ueber-schiebung."*

THEOREM 25. *Let $d = 4$, let $I_{g,2g,4}$ be the dimension of the space of invariants of degree g of the binary quartic. Then*

$$\sum_{g=0}^{\infty} I_{g,2g,4} x^g = \frac{1}{(1-x^2)(1-x^3)}.$$

The ring of invariants is freely generated by I, J, that is $\oplus_m S^m(S^4(\mathbb{C}^2))^{SL(2)} = \mathbb{C}[I,J]$.

Proof. The series has been shown (without proof) in §4.1. We sketch the proof by following again [24], in a way similar to the proof of Corollary 3. We will see a different proof in Theorem 36.

We have from Corollary 17 that $I_{g,2g,4}$ is the degree $2g$ coefficient of

$$\frac{(1-x^{4+1})\dots(1-x^{4+g})}{(1-x^2)\dots(1-x^g)} = \frac{(1-x^{g+1})(1-x^{g+2})(1-x^{g+3})(1-x^{g+4})}{(1-x^2)(1-x^3)(1-x^4)}$$

and we write it as

$$\left\{ \frac{(1-x^{g+1})(1-x^{g+2})(1-x^{g+3})(1-x^{g+4})}{(1-x^2)(1-x^3)(1-x^4)} \right\}_{2g}.$$

We can remove the terms which do not change the coefficient of x^{2g}, so getting

$$\left\{ \frac{(1-x^{g+1}-x^{g+2}-x^{g+3}-x^{g+4})}{(1-x^2)(1-x^3)(1-x^4)} \right\}_{2g} =$$

$$= \left\{ \frac{1}{(1-x^2)(1-x^3)(1-x^4)} \right\}_{2g} - \left\{ \frac{x(1+x+x^2+x^3)}{(1-x^2)(1-x^3)(1-x^4)} \right\}_g =$$

(by using that $(1-x^4) = (1+x+x^2+x^3)(1-x)$)

$$= \left\{ \frac{1}{(1-x^2)(1-x^3)(1-x^4)} \right\}_{2g} - \left\{ \frac{x}{(1-x)(1-x^2)(1-x^3)} \right\}_g =$$

$$= \left\{ \frac{1}{(1-x^2)(1-x^3)(1-x^4)} - \frac{x^2}{(1-x^2)(1-x^4)(1-x^6)} \right\}_{2g} =$$

$$= \left\{ \frac{1+x^3-x^2}{(1-x^2)(1-x^4)(1-x^6)} \right\}_{2g} =$$

Since in the denominator only even powers appear, we can remove x^3 from the numerator and we get

$$= \left\{ \frac{1}{(1-x^4)(1-x^6)} \right\}_{2g} = \left\{ \frac{1}{(1-x^2)(1-x^3)} \right\}_g,$$

as we wanted.

For the last assertion, consider the subring
$$\mathbb{C}[I,J] \subset \oplus_m S^m(S^4(\mathbb{C}^2))^{SL(2)}.$$

In order to prove the equality, it is enough to show that the two rings have the same Hilbert series. We have proved that the Hilbert series of the invariant ring

$\oplus_m S^m(S^4(\mathbb{C}^2))^{SL(2)}$ is $\frac{1}{(1-x^2)(1-x^3)}$, which is the Hilbert series of a ring with two algebraically independent generators of degree 2 and 3. The invariants I and J have respectively degree 2 and 3. So it is enough to prove that I, J are algebraically independent.

Assume we have a relation in degree g between I, J, that is a relation $\sum_{2k+3l=g} c_{kl} I^k J^l = 0$ which hold identically for any $a_0, \ldots, a_4$. Since $I(a_0, 0, 0, a_3, a_4) = a_0 a_4, J(a_0, 0, 0, a_3, a_4) = -a_0 a_3^2$ we get

$$\sum_{2k+3l=g} c_{kl}(-1)^l a_0^{k+l} a_3^{2l} a_4^k = 0.$$

All k, as well as all l, are distinct, because every k determines uniquely l and conversely. It follows get $c_{kl} = 0 \; \forall k, l$, as we wanted.

$\square$

Note that in degree 6 there are two independent invariants, I^3 and J^2 given respectively by

1	1	1	1	2	2	2	2	3	3	3	3
4	4	4	4	5	5	5	5	6	6	6	6

and

1	1	1	1	2	2	4	4	4	4	5	5
2	2	3	3	3	3	5	5	6	6	6	6

.

It is not trivial to show directly that all the semistandard 2×12 tableau functions give, under the Plücker relations, a linear combination of these two.

Indeed the Hilbert series we computed in §4.1 is

$$F_4(z, w) = \frac{1 + z^3 w^6}{(1 - z^2)(1 - z^3)(1 - zw^4)(1 - z^2 w^4)}$$

The factors at the denominator correspond respectively at I, J, f, H.

The syzygy represents Q^2 as a combination of Jf^3, If^2H, H^3.

I^3/J^2 is an absolute invariant, $\Delta = I^3 - 27J^2$ is the discriminant.

The table of covariants is the following, where in the first column we read I, J, in the column labeled with 4 we find respectively f, H and in the last column we find $Q = (f, H)_1$.

		0	1	2	3	4	5	6
degree	1					1		
	2	1				1		
	3	1						1

order

The vanishing of Q as covariant (meaning that its seven cubic generators all vanish, express the fact that the quartic is a square). So they give the equations of a

classically well known surface, which is the projection in $\mathbb{P}^4$ of the Veronese surface from a general point in $\mathbb{P}^5$.

For an extension to binary forms of any degree see [1].

$Q = 0$ represents the three pairs of double points for the three involutions which leave f invariant.

4.5. $SL(2)$ as symplectic group. Symplectic construction of invariants for binary quartics

The content of this subsection was suggested by Francesco Gherardelli several years ago. I report here with the hope that somebody could be interested and take this idea further.

The starting point is that $SL(2)$ can be seen as the symplectic group, preserving $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Many invariants for binary forms of even degree can be computed in a symplectic setting. In the example of binary quartics we have $a_0 x^4 + 4a_1 x^3 y + 6a_2 x^2 y^2 +$

$$4a_3 xy^3 + a_4 y^4 = \begin{bmatrix} x^2 & 2xy & y^2 \end{bmatrix} \begin{bmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{bmatrix} \begin{bmatrix} x^2 \\ 2xy \\ y^2 \end{bmatrix}$$

Define for $g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = g \begin{bmatrix} x' \\ y' \end{bmatrix}$ and note that, setting

$$f(g) = S^2 g = \begin{bmatrix} \alpha^2 & \alpha\beta & \beta^2 \\ 2\alpha\gamma & \alpha\delta + \beta\gamma & 2\beta\delta \\ \gamma^2 & \gamma\delta & \delta^2 \end{bmatrix}, \text{ we get}$$

$$\begin{bmatrix} x^2 \\ 2xy \\ y^2 \end{bmatrix} = f(g) \begin{bmatrix} x'^2 \\ 2x'y' \\ y'^2 \end{bmatrix}.$$

Hence $f(g)^t \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{2} & 0 \\ 1 & 0 & 0 \end{bmatrix} f(g) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{2} & 0 \\ 1 & 0 & 0 \end{bmatrix}$

and we get

$$\det \begin{bmatrix} a_0 & a_1 & a_2 + t \\ a_1 & a_2 - \frac{t}{2} & a_3 \\ a_2 + t & a_3 & a_4 \end{bmatrix} = \frac{t^3}{2} + t \cdot I(a_i) + J(a_i).$$

The beauty of this formula is that the two invariants I and J, coming respectively from (4) and (5), are defined at once. What happens for higher degree?

REMARK 7. The reader can find something similar at the end of Procesi's book [47]. In [47], $S^d\mathbb{C}^2$ is considered inside $S^{d-2}\mathbb{C}^2 \otimes S^{d-2}\mathbb{C}^2 \simeq End(S^{d-2}\mathbb{C}^2)$, in the case $d = 4k$. The coefficients of this characteristic polynomial are conjecturally the generators of the invariant ring for $S^d\mathbb{C}^2$.

4.6. The cubic invariant for plane quartics

This is the easiest example of invariant of ternary forms defined by the symbolic representation of §3.9. Let (x_0, x_1, x_2) be coordinates on a 3-dimensional complex space V and (y_0, y_1, y_2) be coordinates on $V^\vee$. Let

$$f(x_0, x_1, x_2) = \sum_{i+j+k=4} \frac{4!}{i!\,j!\,k!} f_{ijk} x_0^i x_1^j x_2^k \in S^4 V$$

be the equation of a plane quartic curve on $\mathbb{P}(V)$. By Corollary 2, all invariants of f have degree which is multiple of 3. The invariant of smallest degree has degree 3 and it is defined by the tableau

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 \\ \hline \end{array}.$$

We denote $A = F_T$. The trilinear form $A(f, g, h)$, for $f, g, h \in S^4 V$ satisfies $A\left(a^4, b^4, c^4\right) = (a \wedge b \wedge c)^4$, or, expanding the linear forms

$$A\left((a_0 x_0 + a_1 x_1 + a_2 x_2)^4, (b_0 x_0 + b_1 x_1 + b_2 x_2)^4, (c_0 x_0 + c_1 x_1 + c_2 x_2)^4\right) = \begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{vmatrix}^4.$$

The explicit expression of the cubic invariant A can be found at art. 293 of Salmon's book [50], it can be checked with the M2 script of 3.9 and it is the sum of the following 23 summands (we denote $A(f)$ for $A(f, f, f)$)

$$
\begin{aligned}
A(f) = {} & f_{400} f_{040} f_{004} + 3(f_{220}^2 f_{004} + f_{202}^2 f_{040} + f_{400} f_{022}^2) + 12(f_{202} f_{121}^2 + f_{220} f_{112}^2 + f_{022} f_{211}^2) + \\
& + 6 f_{220} f_{202} f_{022} - 4(f_{301} f_{103} f_{040} + f_{400} f_{031} f_{013} + f_{310} f_{130} f_{004}) + \\
& + 4(f_{310} f_{103} f_{031} + f_{301} f_{130} f_{013}) - 12(f_{202} f_{130} f_{112} + f_{220} f_{121} f_{103} + f_{211} f_{202} f_{031} + \\
& + f_{301} f_{121} f_{022} + + f_{310} f_{112} f_{022} + f_{220} f_{211} f_{013} + f_{211} f_{121} f_{112}) + \\
& + 12(f_{310} f_{121} f_{013} + f_{211} f_{130} f_{103} + f_{301} f_{112} f_{031}).
\end{aligned}
$$

(23)

This expression for the cubic invariant can be found also by applying the differential operators D_1, D_2 defined in §3.3 to the space of isobaric monomials of degree 3 and total weight $[4, 4, 4]$. Indeed there are 23 such monomials and the only linear combination of these monomials which is killed by the differential operators is a scalar multiple of the cubic invariant.

The differential operators are analogous to (12) and are

$$\sum_{i_0 + \ldots + i_2 = 4} i_1 f_{i_0 + 1, i_1 - 1, i_2} \frac{\partial I}{\partial f_{i_0 \ldots i_2}} = 0,$$

$$\sum_{i_0 + \ldots + i_2 = 4} i_2 f_{i_0, i_1 + 1, i_2 - 1} \frac{\partial I}{\partial f_{i_0 \ldots i_2}} = 0.$$

It is enough to impose $19 + 19 = 38$ conditions to the 23-dimensional space. Compare with [53] example 4.5.3 where similar computations were performed for the ternary cubic.

Note that given $f, g \in S^4 V$, the equation $A(f, g, *) = 0$ defines an element in the dual space $S^4 V^\vee$, possibly vanishing.

PROPOSITION 19. *(i)* $A(f, g, l^4) = 0$ *if and only if the restrictions* $f_{|l}$, $g_{|l}$ *to the line* $l = 0$ *are apolar.*

(ii) Let $A(f, g, *) = H$. *We have* $A(f, g, l^4) = 0$ *if and only if* $H(l) = 0$.

Proof. To prove (i) , consider $f = (\sum_{i=0}^2 a_i x_i)^4$, $g = (\sum_{i=0}^2 b_i x_i)^4$, $l = x_2$. Then

$$
A\left(f, g, l^4\right) = \begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ 0 & 0 & 1 \end{vmatrix}^4 = \begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix}^4 = f_{|l} \cdot g_{|l}.
$$

This formula extends by linearity to any f, g.

(ii) follows because $H(l) = H \cdot l^4$. $\qquad\qquad\square$

REMARK 8. Let l_1, l_2 be two lines. $A\left(l_1^4, l_2^4, f\right) = 0$ gives the condition that f passes through the intersection point $l_1 = l_2 = 0$.

Note also from Prop. 19 that $A(f, f, l^4) = 0$ if and only if f cuts l in an equianharmonic 4-tuple. The quartic curve $A(f, f, *)$ in the dual space is called the *equianharmonic envelope* of f. It is sometimes called a "contravariant". The "transfer principle of Clebsch" says that from the symbolic expression $(ab)(cd)\ldots$ for a invariant it follows $(ab*)(cd*)\ldots$ for the (envelope) contravariant.

This gives the classical geometric interpretation of the cubic invariant for plane quartics. The condition $A(f, f, f) = 0$ means that f is apolar with its own equianharmonic envelope (see [12]), note that it gives a solution to Exercise (1) in the last page of [53].

4.7. The Aronhold invariant as a pfaffian

Another classical invariant of ternary forms is the Aronhold invariant for plane cubics, it is defined by the tableau

$$
T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 & 3 \\ \hline 3 & 4 & 4 & 4 \\ \hline \end{array}.
$$

We denote by Ar the corresponding multilinear form G_T and also its symmetrization F_T. We get

$$
Ar(x^3, y^3, z^3, w^3) = (x \wedge y \wedge z)(x \wedge y \wedge w)(x \wedge z \wedge w)(y \wedge z \wedge w).
$$

The expression of the Aronhold invariant *Ar* has 25 monomials and it can be found in [53] Prop. 4.4.7 or in [16] (5.13.1). The Aronhold invariant is a "lucky" case, were the geometric interpretation follows easily from the symbolic notation. It is not a surprise that was one of the first examples leading Aronhold to the symbolic notation. If a plane cubic f is sum of three cubes (namely, it is $SL(3)$-equivalent to the Fermat cubic $f = x_0^3 + x_1^3 + x_2^3$) we have $Ar(f) = 0$. Indeed $Ar(f,f,f,f)$ splits as a sum of $Ar(x_{i_0}^3, x_{i_1}^3, x_{i_2}^3, x_{i_3}^3)$ where $i_k \in \{0,1,2\}$, so that $\{i_0,i_1,i_2,i_3\}$ contains at least a repetition, in such a way that all summands contributing to $Ar(f,f,f,f)$ vanish.

Let W be a vector space of dimension 3. In particular $\Gamma^{2,1}W = \text{ad } W$ is self-dual and it has dimension 8. We refer to [42] Theor. 1.2 for the proof of the following result, see also [33] example 1.2.1.

THEOREM 26. *For any $\phi \in S^3W$, let $A_\phi \colon \Gamma^{2,1}W \to \Gamma^{2,1}W$ be the $SL(V)$-invariant contraction operator. Then A_ϕ is skew-symmetric and the pfaffian $Pf A_\phi$ is the equation of $\sigma_3(\mathbb{P}(W), \Omega(3))$, i.e. it is the Aronhold invariant Ar.*

The corresponding picture is

The $SL(W)$-module $\text{End}_0 W$ consists of the subspace of endomorphisms of W with zero trace. The contraction

$$A_\phi \colon \text{End}_0 W \to \text{End}_0 W$$

in the case $\phi = v^3$ satisfies

$$A_{v^3}(M)(w) = (M(v) \wedge v \wedge w)v$$

where $M \in \text{End } W$, $w \in W$.

REMARK 9. We recall from [16] the definition of the Scorza map. For any plane quartic F and any point $x \in \mathbf{P}(W)$ we consider the polar cubic $P_x(F)$. Then $Ar(P_x(F))$ is a quartic in the variable x which we denote by $S(F)$. The rational map $S \colon \mathbf{P}(S^4W) \dashrightarrow \mathbf{P}(S^4W)$ is called the Scorza map. Our description of the Aronhold invariant shows that $S(F)$ is defined as the degeneracy locus of a skew-symmetric morphism on $\mathbf{P}(W)$

$$\Omega(-2)^8 \xrightarrow{f} \Omega(-1)^8.$$

It is easy to check (see [6]) that $\text{Coker } f = E$ is a rank two vector bundle over $S(F)$ such that $c_1(E) = K_{S(F)}$.

I owe to A. Buckley the claim that from E it is possible to recover the even theta characteristic θ on $S(F)$ defined in [16, (7.7)] (see also next section), by following a construction by C. Pauly [44] . There are exactly eight maximal line subbundles P_i of

E of maximal degree equal to 1 such that $h^0(E \otimes P_i^\vee) > 0$. These eight line bundles are related by the equality (Lemma 4.2 in [44])

$$\otimes_{i=1}^8 P_i = K_{S(F)}^2.$$

The construction in [44] §4.2 gives a net of quadrics in the following way. For the general stable E as ours, there exists a unique stable bundle E' with $c_1(E') = \Omega$ such that $h^0(E \otimes E')$ has the maximal value 4. The extensions

$$0 \longrightarrow P_i \longrightarrow E \longrightarrow K_{S(F)} \otimes P_i^{-1} \longrightarrow 0$$

$$0 \longrightarrow P_i^{-1} \longrightarrow E' \longrightarrow P_i \longrightarrow 0$$

define eight sections in $Hom(E', E) \simeq E \otimes E'$ as the compositions

$$E' \longrightarrow P_i \longrightarrow E$$

which give eight points in $\mathbb{P}H^0(E \otimes E')$. These eight points are the base locus for a net of quadrics, which gives a symmetric representation of the original quartic curve $S(F)$ and then defines a even theta characteristic.

4.8. Clebsch and Lüroth quartics. Theta characteristics

A plane quartic $f \in S^4 V$ is called *Clebsch* if it has an apolar conic, that is if there exists a nonzero $q \in S^2 V^\vee$ such that $q \cdot f = 0$.

One defines, for any $f \in S^4 V$, the catalecticant map $C_f \colon S^2 V^\vee \to S^2 V$ which is the contraction by f. The equation of the Clebsch invariant is easily seen as the determinant of C_f, that is we have([16], example (2.7))

THEOREM 27 (Clebsch). *A plane quartic f is Clebsch if and only if $\det C_f = 0$. The conics which are apolar to f are the elements of $\ker C_f$.*

It follows ([14], Lemma 6.3.22) that the general Clebsch quartic can be expressed as a sum of five 4-th powers, that is

(24) $$f = \sum_{i=0}^4 l_i^4.$$

A general Clebsch quartic f can be expressed as a sum of five 4-th powers in ∞^1 many ways. Precisely the 5 lines l_i belong to a unique smooth conic Q in the dual plane, which is apolar to f and it is found as the generator of $\ker C_f$. Equivalently, the 5 lines l_i are tangent to a unique conic, which is the dual conic of Q.

We recall that a *theta characteristic* on a general plane quartic f is a line bundle θ on f such that θ^2 is the canonical bundle. Hence $\deg \theta = 2$. There are 64 theta characteristic on f. If the curve is general, every bitangent is tangent in two distinct points P_1 and P_2, and the divisor $P_1 + P_2$ defines a theta characteristic θ such that $h^0(\theta) = 1$,

these are called odd theta characteristic and there are 28 of them. The remaining 36 theta characteristic θ are called even and they satisfy $h^0(\theta) = 0$ (for any curve, even theta characteristics have even $h^0(\theta)$) .

The *Scorza map* is the rational map from $\mathbb{P}^{14} = \mathbb{P}(S^4V)$ to itself which associates to a quartic f the quartic $S(f) = \{x \in \mathbb{P}(V)|Ar(P_x(f)) = 0\}$, where $P_x(f)$ is the cubic polar to f at x and Ar is the Aronhold invariant [14,16,51]. It is well known that it is a $36:1$ map. Indeed the curve $S(f)$ is equipped with an even theta characteristic. For a general quartic curve, its 36 inverse images through the Scorza map give all the 36 even theta characteristic.

A *Lüroth quartic* is a plane quartic containing the ten vertices of a complete *pentalateral*, or the limit of such curves.

If l_i for $i = 0,\ldots,4$ are the lines of the pentalateral, we may consider as divisor (of degree 4) over the curve. Then $l_0 + \ldots + l_4$ consists of 10 double points, the meeting points of the 5 lines. Let $P_1 + \ldots + P_{10}$ the corresponding reduced divisor of degree 10. Then $P_1 + \ldots + P_{10} = 2H + \theta$ where H is the hyperplane divisor and θ is a even theta characteristic, which is called the pentalateral theta. The pentalateral theta was called pentagonal theta in [16], and it coincides with [14], Definition 6.3.30 (see the comments thereafter).

The following result is classical [51], for a modern proof see [14,16].

PROPOSITION 20. *Let f be a Clebsch quartic with apolar conic Q, then $S(f)$ is a Lüroth quartic equipped with the pentalateral theta corresponding to Q.*

Proof. Let $f = \sum_{i=1}^5 l_i^4$. Let l_p and l_q be two lines in the pentalateral and let $x_{pq} = l_p \cap l_q$. Then

$$P_{x_{pq}}(f) = \sum_{i=1}^5 P_{x_{pq}}(l_i^4) = \sum_{i=1}^5 4l_i^3 P_{x_{pq}}(l_i).$$

In the above sum at most three summands survive, because the ones with $i = p, q$ are killed. Then $P_{x_{pq}}(f)$ is a Fermat cubic and $Ar(P_{x_{pq}}(f)) = 0$, hence $x_{pq} \in S(f)$. It follows that $S(f)$ is inscribed in the pentalateral and it is Lüroth. $\square$

The number of pentalateral theta on a general Lüroth quartic, called δ, is equal to the degree of the Scorza map when restricted to the hypersurface of Clebsch quartics.

Explicitly, if f is Clebsch with equation

$$l_0^4 + \ldots + l_4^4,$$

then $S(f)$ has equation

$$\sum_{i=0}^4 k_i \prod_{j \neq i} l_j,$$

where $k_i = \prod_{p<q<r, i \notin \{p,q,r\}} |l_p l_q l_r|$ (see [14], Lemma 6.3.26) so that $l_0,\ldots,l_4$ is a pentalateral inscribed in $S(f)$. Note that the conic where the five lines which are the summands of f are tangent, is the same conic where the pentalateral inscribed in $S(f)$ is tangent.

REMARK 10. The degree of Lüroth invariant is 54. This has been proved by Morley in 1919 [38], see [43] for a review of his nice proof. To have the flavour of the complexity, think that the space of monomials of degree 54 in the 15 variables a_{ijk} with $i+j+k=4$ has dimension $\binom{68}{14} \simeq 10^{14}$, while the space of isobaric ones has dimension $62,422,531,333 \simeq 10^{11}$.

The dimension of the space of invariants was computed first by Shioda [52], it is 1165. Ohno computes this space as dimension 1380 with 215 relations. This is reviewed by Basson, Lercier, Ritzenthaler, Sijsling in [2], where the Lüroth invariant has been described as linear combination of these monomials. This computation allows to detect if a given plane quartic is Lüroth. Moreover it disproves a guess by Morley at the end of [38] about the explicit form of the Lüroth invariant. Still the existence of a determinantal formula or other simple descriptions for the Lüroth invariant is sought.

Recently, a determinantal description for the undulation invariant of degree 60 has been found [45]. It vanishes on quartic curves that have an undulation point, that is a line meeting the quartic in a single point with multiplicity 4. A beautiful classical source about plane quartics is Ciani monograph [12].

PROPOSITION 21. *Let $Y_{10} = \sigma_9(v_6(\mathbb{P}^2))$ be the determinantal hypersurface in $\mathbb{P}S^6\mathbb{C}^3$ of sextics having a apolar cubic. The dual variety $Y_{10}^\vee$ is the variety of double cubics.*

Proof. By Remark 1, the dual variety corresponds to the sextics which are singular in 9 points, hence they are double cubics. □

PROPOSITION 22. *The 3-secant variety $\sigma_3(Y_{10}^\vee)$ is the theta divisor, that is the locus of sextic curves which admit an effective even theta-characteristic. Its degree is 83200*

Proof. The sextics in the variety of 3-secant to Y_{10} can be written as $A^2 + B^2 + C^2$ where A, B, C are cubics. Since all plane conics are projectively equivalent, they can be written as $AC - B^2 = 0$, that is as a 2×2 symmetric determinant with cubic entries. Write

$$\Omega(-3)^2 \xrightarrow{M} \Omega^2$$

with M symmetric. The cokernel is a effective theta-characteristic and conversely every effective theta-characteristic arises in this way (see [6] and remark 4 in [9]). The computation of the degree is a nontrivial result proved in [9]. □

Question What is the degree of the theta divisor for plane curves of degree d, that is the locus of plane curves of degree d which admit an effective even theta-characteristic ?

5. Invariants of points. Cremona equations for the cubic surface and invariants of six points

5.1. The two Fundamental Theorems for invariants of points

Given $p_1, \ldots p_d \in \mathbb{P}V = \mathbb{P}^n$, we can write their coordinates in a $(n+1) \times d$ matrix, writing the coordinates of p_i in the i-th column. The ring of polynomials over these coordinates $\mathbb{C}[V \otimes \mathbb{C}^d] = \oplus_m S^m(V \otimes \mathbb{C}^d)$ has a natural multigraduation

$$\oplus_{m_1, \ldots, m_d} \left(S^{m_1}V \otimes \ldots \otimes S^{m_d}V \right),$$

where the coordinates of p_i appear with total degree m_i. The group $SL(n+1)$ acts on $\mathbb{P}V$, then it acts on the multigraded ring. Classically, these rings have been studied in the "democratic" case when all n_i are equal. So the invariant ring to be studied was $\oplus_m \underbrace{S^m V \otimes \ldots \otimes S^m V}_{d}^{SL(n+1)}$. In these cases, there is the additional action of Σ_d on the points and then on the invariant ring. The $SL(n+1) \times \Sigma_d$-invariants were called "rational", while the ones invariant just for the smaller subgroup $SL(n+1) \times Alt(d)$ were called "irrational".

After GIT has been developed, it has been understood that it is convenient to fix a weight (polarization) $h = (h_1, \ldots, h_d)$, so getting

$$\mathbb{C}[V \otimes \mathbb{C}^d]_{(h)} = \oplus_p S^{ph_1}V \otimes \ldots \otimes S^{ph_d}V.$$

The invariant subring $\mathbb{C}[V \otimes \mathbb{C}^d]_{(h)}^{SL(n+1)}$ is called the invariant ring of d ordered points on $\mathbb{P}V$ with respect to the weight h. When the weight h is not specified, it is understood that it is $h = 1^d$.

The invariant subring $\mathbb{C}[V \otimes \mathbb{C}^d]^{SL(V) \times \Sigma_d}$ is called the invariant ring of d unordered points on $\mathbb{P}V$.

In the case $n = 1$, the invariant ring of d unordered points on the line coincides with the invariant ring of binary forms of degree d. This is clear associating to any binary form its scheme of roots.

For the convenience of the reader, we repeat with slight changes the construction of Definition 4.

DEFINITION 6 (From tableau to invariants of points).
Let $h_1 + \ldots + h_d = (n+1)g$. For any tableau T over a Young diagram of size $(n+1) \times g$, filled with numbers from 1 appearing h_1 times, until d appearing h_d times, we denote by G_T the multilinear function $G_T \colon S^{h_1}V^\vee \times \ldots \times S^{h_d}V^\vee \to \mathbb{C}$ defined by (compare with Definition 4)

$$G_T(x_1^{h_1}, \ldots, x_d^{h_d}) = \prod_{j=1}^{d} \left(x_{t(1,j)} \wedge \ldots \wedge x_{t(n+1,j)} \right).$$

G_T is well defined by Theorem 1 and it is SL(V)-invariant by Proposition 16.

The geometric meaning of the vanishing of G_T, where

$$T = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array},$$

is that the corresponding points $x_1, \ldots x_4 \in V^\vee$ are dependent.

THEOREM 28 (1FT for ordered points).

The invariant ring $\mathbb{C}[V \otimes \mathbb{C}^d]^{SL(n+1)}_{(h)}$ of d ordered points on $\mathbb{P}V$ with respect to h is generated by functions G_T like in Definition 6 for tableau T having weight multiple of h.

Proof. The decomposition (see Theorem 11) $S^p(V \otimes \mathbb{C}^d) = \oplus_\lambda S^\lambda V \otimes S^\lambda \mathbb{C}^m$, where the sum is extended to all Young diagrams λ with $|\lambda| = p$, shows that $S^p(V \otimes \mathbb{C}^d)^{SL(n+1)} = 0$ for p which is not multiple of $(n+1)$ and $S^p(V \otimes \mathbb{C}^d)^{SL(n+1)} = S^\mu \mathbb{C}^d$ if $p = (n+1)g$ and μ is the Young diagram with $(n+1)$ rows and g columns.

By Theorem 9, $S^\mu \mathbb{C}^d$ has a basis consisting of semistandard Young tableau T, where the numbers $1, \ldots, d$ appear.

This basis has a natural multigraduation, depending on partitions $p = m_1 + \ldots + m_d$, where in T the number 1 appears m_1 times, 2 appears m_2 times, until d appears m_d times. Moreover, this basis fits with the other decomposition

$$S^p(V \otimes \mathbb{C}^d) = \oplus (S^{m_1} V \otimes \ldots \otimes S^{m_d} V),$$

where the sum is extended to all the partitions with d summands $p = m_1 + \ldots m_d$ which induce

$$S^p(V \otimes \mathbb{C}^d)^{SL(n+1)} = \oplus (S^{m_1} V \otimes \ldots \otimes S^{m_d} V)^{SL(n+1)}.$$

In other words, the summand $(S^{m_1} V \otimes \ldots \otimes S^{m_d} V)^{SL(n+1)}$ has a natural basis consisting of semistandard Young tableau, consisting in the Young diagram $(n+1) \times \frac{p}{n+1}$ filled with 1 appearing m_1 times, 2 appearing m_2 times, until d appearing m_d times.

These semistandard tableau T correspond to the multilinear function G_T of Definition 6. $\qquad \square$

COROLLARY 6 (1FT for unordered points). *All invariants of d unordered points are polynomials in the tableau functions G_T, where in T the numbers from 1 to d appear equally, which moreover are symmetric under permutation of points.*

The invariant ring of d unordered points on $\mathbb{P}V$ is isomorphic to

$$\oplus_m \left[S^d(S^m V) \right]^{SL(n+1)}.$$

REMARK 11. It is interesting to compare the invariant ring of d unordered points with the invariant ring for forms in $S^d V$, which is $\oplus_m S^m(S^d V)^{SL(n+1)}$. Note the swapping between m and d. Note also that on P^1 the swapping makes no difference, by Hermite reciprocity (Corollary 4).

Proof of Theorem 21, 1FT for forms Consider, in the proof of Theorem 28, the case $m_1 = \ldots = m_d = m$, so that $p = md$.

We get that $(\underbrace{S^m V \otimes \ldots \otimes S^m V}_{d})^{SL(n+1)}$ has a natural basis consisting of G_T, where

T is a semistandard Young tableau, filling the diagram $(n+1) \times \frac{md}{n+1}$ with 1 appearing m times, 2 appearing m times, until d appearing m times.

Considering the subspace of Σ_d-invariants, we get the space of symmetric multi-linear functions F_T, like in Definition 5, which indeed is $S^d(S^m V)^{SL(n+1)}$. By swapping m with d, we get exactly the construction performed in §3.9.   □

Note that by 1FT, $S^{mh_1} V \otimes \ldots \otimes S^{mh_d} V \neq 0$ if and only if $m(h_1 + \ldots + h_d)$ is a multiple of $(n+1)$. The invariants of minimal degree are those with

$$m = \frac{lcm(h_1 + \ldots + h_d, n+1)}{h_1 + \ldots + h_d}.$$

If $g(n+1) = md$ we have the weight $(\underbrace{1, \ldots, 1}_{d})$ and the graded ring $\oplus_m S^m V \otimes$

$\ldots \otimes S^m V$.

THEOREM 29 (2FT for points). *In the invariant ring of d (ordered or un-ordered) points, all the relations between the tableau functions are generated by the Plücker relations $\sum_{s=0}^{k+1}(-1)^s G_{T_s} = 0$, exactly like in Theorem 22, with the tableau T_s having the correct weight.*

A strong improvement of 2FT for ordered points on $\mathbb{P}^1$ is relatively recent, and it will be treated in Theorem 33.

The theory is better explained by examples.

PROPOSITION 23.
(i) The conic through $P_0, \ldots, P_4$ has equation

$$[014][234][02x][13x] - [024][134][01x][23x] = 0.$$

(ii) 6 points in the plane $\mathbb{P}^2$ lie on a conic if and only if

(25) $d_2 := [014][234][025][135] - [024][134][015][235] = 0.$

(it is an irrational invariant, indeed it is Alt(6)-invariant but not Σ_6-invariant).

Proof. The singular conics between $P_0, \ldots, P_3$ are $[01x][23x]$, $[02x][13x]$,
$[03x][12x]$. We ask that there exist A and B such that $A[01x][23x] + B[02x][13x]$ vanishes for $x = P_4$. Hence we have $A[014][234] + B[024][134] = 0$ which is satisfied for $A = -[024][134]$ and $B = [014][234]$.   □

REMARK 12. The expression in (ii) of Proposition 23 is the symbolic expression for the 6×6 determinant having in the i-th row the coefficients $x_0^2, \ldots x_2^2$ computed at P_i.

Note again that the skew-symmetry is not at all evident from the symbolic expression. It can be showed by using the Plücker relations, in this case there are 35 quadratic relations.

5.2. The graphical algebra for the invariants of d points on the line. Kempe's Lemma

In 1894 A. Kempe [29] introduced a graphical representation of invariants.

Let fix a weight $(h_1, \ldots, h_d)$ and consider the algebra

$$\mathbb{C}[\mathbb{C}^2 \otimes \mathbb{C}^d]_{(h)} = \oplus_m S^{mh_1}\mathbb{C}^2 \otimes \ldots \otimes S^{mh_d}\mathbb{C}^2.$$

Any d points on a line $I\!\!P^1$ can be represented as d vertex of a regular polygon, numbered clockwise from 1 to d.

The bracket function (ij) between i and j is represented as an arrow from i to j. A tableau function of weight mh is represented as a graph with valence mh_i at the vertex i.

For example $(12)(34)(56)$ represents an invariant with respect to the weight 1^6 and it corresponds to the graph
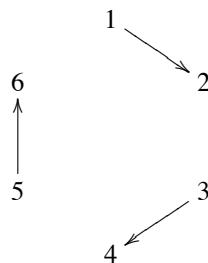
Inverting one arrow corresponds to a sign change. The relation $(ij)(kl) - (ik)(jl) + (il)(jk) = 0$ is represented graphically by

(26)

The graphical algebra is the same algebra generated by tableau functions. Every linear combination of tableau functions transfers to a linear combination of corresponding graphs and conversely. The product of two graphs corresponds to the union of the corresponding arrows, like in

This graphical algebra has been carefully studied in a series of recent papers by Howard, Millson, Snowden and Vakil. In order to review their approach and their main results, we recall the following basic result from graph theory.

THEOREM 30 (Hall Marriage Theorem). *Consider a graph G with* 2*m vertices, m being positive and m being negative. A perfect matching is a collection of m edges, each one joining one positive vertex with one negative vertex, in such a way that every vertex belongs to one edge. The necessary and sufficient condition that G contains a perfect matching, is that for every subset Y of positive vertices, the cardinality of the set of negative vertices which are connected to at least one member of Y is bigger or equal than the cardinality of Y.*

THEOREM 31 (Kempe's Lemma, 1FT for ordered points on $\mathbb{P}^1$). *All invariants of ordered points on* $\mathbb{P}^1$ *are generated by tableau functions of minimal degree*

$$\frac{lcm(h_1+\ldots+h_d,(n+1))}{h_1+\ldots+h_d}.$$

With respect to the weight $(1,1,\ldots,1)$*, the generating invariants have degree* 1 *when d is even and degree* 2 *when d is odd.*

Proof. We follow [25], who gave a shorter proof than Kempe original one. We assume for simplicity that d is even and $h = (1,\ldots,1)$, and refer to [25] for the general case. Consider the graphical description of an invariant of weight $(m,\ldots,m)$. Divide the vertices into two subsets of equal cardinality, called positive and negative.

So the edges have three possible types: positive (both vertices positive), negative (both vertices negative) and neutral (two opposite vertices).

Since every monomial is homogeneous, every vertex has valence m. It follows that the number of positive edges is equal to the number of negative edges. Applying the relation (26) to a pair given by a positive and a negative edge, we get all neutral edges. Continuing in this way, we get a combination of graphs, each one with all neutral edges. Then the assumption of Hall Marriage Theorem is satisfied, because from every subset Y of p positive vertices start pm edges. Since the valence of each vertex is m, Y must connect to at least p negative vertices. So a perfect matching exists. Note that a perfect matching corresponds to a generator of minimal degree 1. By factoring this generator we can conclude by induction on m.   □

From Theorem 9 and from the proof of Theorem 28, a basis of tableau functions G_T of minimal degree is given by semistandard tableau. In the case of points of $\mathbb{P}^1$, an alternative description is possible. A graph is said to be noncrossing if no two edges cross in an interior point.

THEOREM 32 (Kempe). *A basis of the tableau functions of minimal degree is given by noncrossing complete matching of minimal degree.*

Proof. We apply the relation (26) to a pair of noncrossing edges. We get a combination of graphs, in each of them the total euclidean length of the edges is strictly smaller. This

process must terminate, because there is a finite number of complete matching, hence a finite number of total euclidean length. When the process terminates, we have a combination of noncrossing complete matching, otherwise the process could be repeated. This shows that the noncrossing complete matching span. For the proof of independence of noncrossing graphs with the same weight h, assume we have a nonzero relation involving a minimal number n of vertices. Not all the graphs appearing contain the edge $(n-1)n$, otherwise we could remove it, obtaining a smaller relation. Now identify the vertices $(n-1)$ and n, so that the graphs containing the edge $(n-1)n$ go to zero. This gives a bijection between the graphs on n vertices with weight h, not containing the edge $(n-1)n$, and the graphs on $(n-1)$ vertices with weight $(h_1, \ldots, h_{n-2}, h_{n-1} + h_n)$. We get a nonzero relation on $(n-1)$ vertices, contradicting the minimality. □

REMARK 13. The above proof gives a graphical version of the straightening algorithm, as pointed out in [25] Prop. 2.5.

For example, the space of invariants of 6 points on $\mathbb{P}^1$ with weight 1^6 is generated by the noncrossing complete matching as follows. We draw all the arrows from even to odd.

(27)
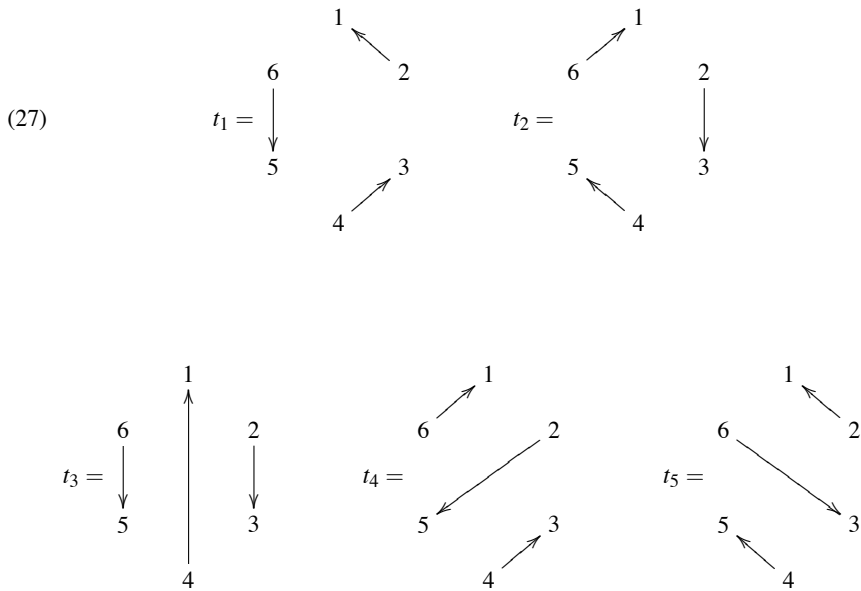
THEOREM 33 (2FT for ordered points on $\mathbb{P}^1$, Howard, Millson, Snowden and Vakil [27]). *In the invariant ring for d ordered points on $\mathbb{P}^1$, with any weight $w \neq 1^6$, the relation among the generators of minimal degree are generated by quadric relations.*

In the case $w = 1^6$, we will see in Theorem 37 that there is a unique cubic relation among the generators of minimal degree of the invariant ring of six ordered points on $\mathbb{P}^1$ (this relation gives the Segre cubic primal, see Remark 16).

REMARK 14. Theorem 39 will show that, in the case of d unordered points, the relations are more complicated, certainly not generated by quadric relations.

5.3. Molien formula and elementary examples

The following Theorem by Molien shows that the Hilbert series of the invariant subring is the average of the inverse of the characteristic polynomial.

PROPOSITION 24. *Let G be a finite group acting on U. The induced action on the symmetric algebra S^*U has Hilbert series*

$$(28) \qquad \sum_{i=0}^{+\infty} \dim \left(S^i U \right)^G q^i = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - qg)}$$

where g acts on U.

Proof. [53] Theor. 2.2.1 $\square$

We analyze now some invariant rings for d points on $\mathbb{P}^1$, for small d. The simplest cases is the following.

THEOREM 34. *The Hilbert series of the invariant ring of three ordered points on $\mathbb{P}^1$ is*

$$\frac{1}{1 - t^2}.$$

The ring is generated by the only noncrossing matching of valence 2 which is
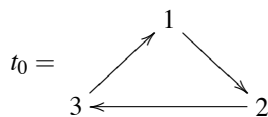
Proof. Immediate from Theorem 32. $\square$

We get the second promised proof of the Corollary 3.

COROLLARY 7. *The Hilbert series of the invariant ring of three unordered points on $\mathbb{P}^1$ is*

$$\frac{1}{1 - t^4}.$$

The ring is generated by the discriminant $\Delta = t_0^2$.

Proof. The function t_0 is $Alt(3)$-invariant (for even permutations), while t_0^2 is Σ_3-invariant. $\qquad\square$

THEOREM 35. *The Hilbert series of the invariant ring of four ordered points on $\mathbb{P}^1$ is*

$$\frac{1}{(1-t)^2}.$$

The ring is generated by the two noncrossing matchings which are

$$j_0 = \begin{array}{c} 1 \longleftarrow 2 \\[10pt] 4 \longrightarrow 3 \end{array} \qquad\qquad j_1 = \begin{array}{cc} 1 & 2 \\ \uparrow & \downarrow \\ 4 & 3 \end{array}$$

Proof. j_0 and j_1 generate the invariant ring by Theorem 32. Moreover j_0 and j_1 are algebraically independent. This can be shown directly or by observing that the geometric quotient is one dimensional. $\qquad\square$

In the point (i) of the following Theorem we get a second proof of Theorem 25.

THEOREM 36.

(i) The Hilbert series of the invariant ring of four unordered points on $\mathbb{P}^1$ (binary quartics) is

$$\frac{1}{(1-t^2)(1-t^3)}.$$

The ring is generated by I, J defined in (4), (5).

(ii) The Hilbert series of the $Alt(4)$-invariant ring of four unordered points on $\mathbb{P}^1$ (binary quartics) is

$$\frac{1+t^3}{(1-t^2)(1-t^3)} = \frac{1-t^6}{(1-t^2)(1-t^3)^2}.$$

The ring is generated by $I, J, \sqrt{D}$ where $\sqrt{D}$ has degree 3 and it is the product of differences of the roots of the quartic. The only relation is $(\sqrt{D})^2 = I^3 - 27J^2$.

Proof. The result is elementary, but we give the details of the representation theoretic approach as warming up for the more interesting case of six points (Theorem 39). The two generators j_0, j_1 of Theorem 35 span the unique irreducible representation W of dimension 2 of Σ_4. So we have to compute the Hilbert series of $\oplus_p S^p(W)^{\Sigma_4}$. The proof is a straightforward computation by using Molien's formula (28), by summing over the five conjugacy classes in Σ_4.

The result is

$$\frac{1}{24}\left[\frac{1}{(1-t)^2} + \frac{6}{1-t^2} + \frac{8}{1+t+t^2} + \frac{6}{1-t^2} + \frac{3}{(1-t)^2}\right] =$$
$$= \frac{1}{(1-t^2)(1-t^3)}.$$

Let I, J be the second and third elementary symmetric function of $j_0, -j_1, j_1 - j_0$. I and J correspond, up to scalar factor, to (4), (5) and they are the generators of the invariant ring corresponding to the two factors in the denominator of the Hilbert series. This proves (i). (ii) can be proved by restricting the sum to the even conjugacy classes, which are the ones with numerator 1, 8, and 3. $\square$

REMARK 15. The cross ratio $\frac{(x_3 - x_1)(x_4 - x_2)}{(x_3 - x_2)(x_4 - x_1)}$ of four points with affine coordinates x_i for $i = 1, \ldots, 4$ has the expression $\frac{j_1 - j_0}{j_1}$. It parametrizes the moduli space of 4 ordered points on $\mathbb{P}^1$, which is isomorphic to $\mathbb{P}^1$ itself.

5.4. Digression about the symmetric group Σ_6 and its representations

We list a representative for each of the 11 conjugacy classes of Σ_6.

		even	number of elements
C1	(1)	*	1
C2	(12)		15
C3	(12)(34)	*	45
C4	(12)(34)(56)		15
C5	(123)	*	40
C6	(123)(45)		120
C7	(123)(456)	*	40
C8	(1234)		90
C9	(1234)(56)	*	90
C10	(12345)	*	144
C11	(123456)		120

The natural action of Σ_d on $\mathbf{C}^d$ splits into the trivial representation U (dimension one) and the standard representation V (dimension $d - 1$). The subspace $V \subset \mathbf{C}^d$ is given by $\sum e_i = 0$.

The exterior representation $U' : \Sigma_d \to \mathbf{C}^*$ which sends $p \in \Sigma_d$ to its sign $\varepsilon(p)$ has again dimension one. For any representation W, it is customary to denote $W' = W \otimes U'$. W' corresponds to the transpose Young diagram of W. Each representation of Σ_n is self-dual.

We list the irreducible representations in the case $d = 6$ and we put in evidence

the transpose diagrams.

name	shape	dimension	name	shape	dimension
$X1 = U$		1	$X7 = \wedge^3 V$		10
$X2 = V$		5	$X8 = X5'$		5
$X3$		9	$X9 = X3'$		9
$X4 = \wedge^2 V$		10	$X10 = \wedge^4 V$		5
$X5$		5	$X11 = U'$		1
$X6 = X6'$		16			

One peculiarity of the above list is that there are *four* irreducible representations of dimension $5 = d - 1$. This happens only in the case $d = 6$. S_6 is the only symmetric group which admits an automorphism which is not inner, which can be defined indeed by means of $X5$ (or dually by $X8$).

We list the character table of Σ_6, from the appendix of [20]. Note that the first column gives the dimension.

	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11
$X1$	1	1	1	1	1	1	1	1	1	1	1
$X2$	5	3	1	−1	2	0	−1	1	−1	0	−1
$X3$	9	3	1	3	0	0	0	−1	1	−1	0
$X4$	10	2	−2	−2	1	−1	1	0	0	0	1
$X5$	5	1	1	−3	−1	1	2	−1	−1	0	0
$X6$	16	0	0	0	−2	0	−2	0	0	1	0
$X7$	10	−2	−2	2	1	1	1	0	0	0	−1
$X8$	5	−1	1	3	−1	−1	2	1	−1	0	0
$X9$	9	−3	1	−3	0	0	0	1	1	−1	0
$X10$	5	−3	1	1	2	0	−1	−1	−1	0	1
$X11$	1	−1	1	−1	1	−1	1	−1	1	1	−1

Other useful formulas are ([19] ex. 2.2)

$$\chi_{S^2W}(g) = \frac{1}{2}\left[\chi_W(g)^2 + \chi_W(g^2)\right],$$

(29)
$$\chi_{S^3W}(g) = \frac{1}{6}\chi_W(g)^3 + \frac{1}{2}\chi_W(g)\chi_W(g^2) + \frac{1}{3}\chi_W(g^3).$$

We have the following table explaining how the powers of elements divide among the conjugacy classes.

(30)

g	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11
g^2	C1	C1	C1	C1	C5	C5	C7	C3	C3	C10	C7
g^3	C1	C2	C3	C4	C1	C2	C1	C8	C9	C10	C4
g^4	C1	C1	C1	C1	C5	C5	C7	C1	C1	C10	C7
g^5	C1	C2	C3	C4	C5	C6	C7	C8	C9	C1	C11

5.5. The invariant ring of six points on the line

THEOREM 37. *The Hilbert series of the invariant ring of six ordered points on* $\mathbb{P}^1$ *is*

$$\frac{1-t^3}{(1-t)^5}.$$

The ring is generated by the five noncrossing matching $t_1,\ldots,t_5$ *as in* (27). *We have the unique relation in degree* 3

(31)
$$t_1t_2(-t_1 - t_2 + t_3 + t_4 + t_5) - t_3t_4t_5 = 0.$$

Proof. The five noncrossing matching $t_1, \ldots, t_5$ listed in (27) generate by Theorems 31 and 32. They are easily identified as a basis for the representation $X5$ of Σ_6. By dimensional reasons (the quotient has dimension three), only one relation is expected.

The fact that there are no relations in degree two can be proved by counting the number of semistandard tableau 2×6 of weight 2^6, which are indeed 15. In order to understand the relation, we add a sixth tableau which is a linear combination of the first noncrossing $t_1, \ldots, t_5$ as

(32)
$$t_0 =$$
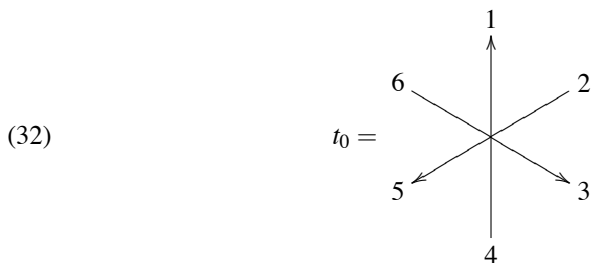
After some computation with the straightening algorithm, as in the proof of Theorem 32, we get $t_0 = -t_1 - t_2 + t_3 + t_4 + t_5$.

Looking at the graphs there is the obvious relation $t_0 t_1 t_2 - t_3 t_4 t_5 = 0$ that we show in the following picture.
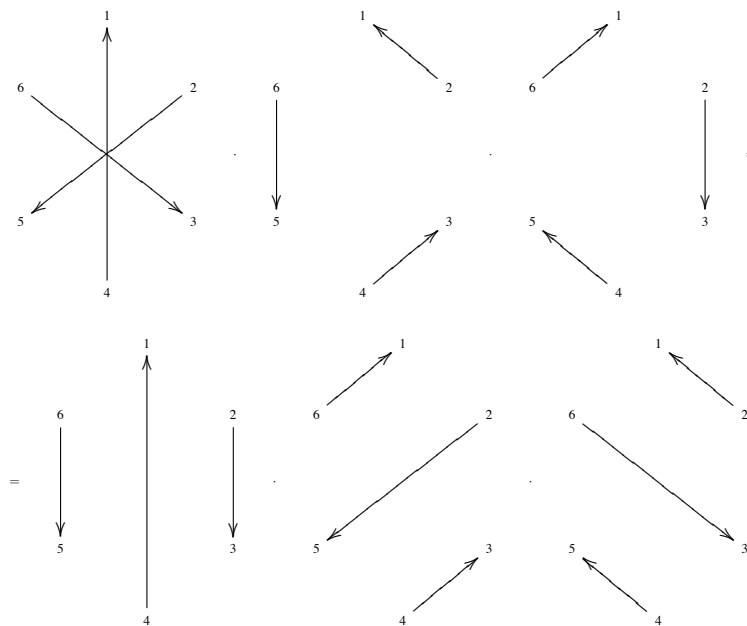
This concludes the proof.

□

REMARK 16. The cubic 3-fold (31) is called the Segre cubic primal. It has 10 singular points and it contains 15 planes (see [15] example 11.6). It can be expressed as the sum of six cubes, and not eight like the general cubic 3-fold (see [42,48]). An explicit expression as a sum of cubes will be obtained in (34).

REMARK 17. In [15] example 11.6 it is reported an alternative combinatorial proof of Theorem 37, by counting the number of semistandard tableau $2 \times 3m$ of weight m^6.

Since the relation (31) is $Alt(6)$-invariant but not Σ_6-invariant, we need a preliminary step to get the Σ_6-invariants.

By following Coble [10] §3, we define the Joubert invariants (see [14] for an intrinsic way to compute these invariants).

$$
\begin{aligned}
(33) \quad
& A = (25)(13)(46) + (51)(42)(36) + (14)(35)(26) + (43)(21)(56) + (32)(54)(16) \\
& B = (53)(12)(46) + (14)(23)(56) + (25)(34)(16) + (31)(45)(26) + (42)(51)(36) \\
& C = (53)(41)(26) + (34)(25)(16) + (42)(13)(56) + (21)(54)(36) + (15)(32)(46) \\
& D = (45)(31)(26) + (53)(24)(16) + (41)(25)(36) + (32)(15)(46) + (21)(43)(56) \\
& E = (31)(24)(56) + (12)(53)(46) + (25)(41)(36) + (54)(32)(16) + (43)(15)(26) \\
& F = (42)(35)(16) + (23)(14)(56) + (31)(52)(46) + (15)(43)(26) + (54)(21)(36)
\end{aligned}
$$

In terms of redundant $t_0, \ldots, t_5$ we have nicer expressions

$$
\begin{cases}
A/2 & = & -t_0 & +t_1 & +t_2 & & & \\
B/2 & = & & & & -t_3 & -t_4 & +t_5 \\
C/2 & = & & & & +t_3 & -t_4 & -t_5 \\
D/2 & = & +t_0 & -t_1 & +t_2 & & & \\
E/2 & = & +t_0 & +t_1 & -t_2 & & & \\
F/2 & = & & & & -t_3 & +t_4 & -t_5
\end{cases}
$$

Note that

$$
A + B + C + D + E + F = 0.
$$

so that any five among them give a basis of the space of invariant tableau functions for six ordered points.

A direct inspection shows that an even permutation of the points effects an even permutation of the functions and that an odd permutation of the points effects an odd permutation of the functions accompanied by a change of sign. For example after the permutation (12) we get

$$
A \to -D \qquad B \to -E \qquad C \to -F,
$$

after the permutation (13) we get

$$
A \to -F \qquad B \to -D \qquad C \to -E.
$$

Let $b_{15} = (A-B)(A-C)\ldots(E-F)$ be the product of differences. The previous permutation rules about $A,\ldots,F$ show that b_{15} is Σ_6-invariant.

We denote by a_i the i-th elementary symmetric function of $A,\ldots,F$. It follows that a_i are $Alt(6)$-invariant and Σ_6-invariant for even i.

Consider $\prod_{i<j}(x_j - x_i) = \Delta$ which is a $Alt(6)$-invariant, and it is the square root of the discriminant of the polynomial having the points as roots.

Every $Alt(6)$-invariant can be written as $a + b\Delta$ where a,b are Σ_6-invariants, see [53] Prop. 1.1.3. It follows that Δ is equal to a_5 (up to scalar multiples) and by degree reasons we get that $a_1 = a_3 = 0$. The last equation can be written in equivalent way as

$$(34) \qquad\qquad A^3 + B^3 + C^3 + D^3 + E^3 + F^3 = 0,$$

which is the promised expression of the Segre cubic primal as a sum of six cubes.

THEOREM 38 (Coble). *The Hilbert series of the $Alt(6)$-invariant ring of six (unordered) points on $\mathbb{P}^1$ is*

$$\frac{1+t^{15}}{(1-t^2)(1-t^4)(1-t^5)(1-t^6)}.$$

The ring is generated by $a_2, a_4, a_6, \Delta, b_{15}$ with the relation in degree 30 expressing b_{15}^2 as a polynomial in the other generators.

Proof. $X5$ and $X8$ restrict to the same representation of $Alt(6)$, that we call again $X5$. We compute first the Hilbert series of $\oplus_p S^p(X5)^{Alt(6)}$. The proof is a straightforward computation by using Molien formula (28). The characteristic polynomial of any $g \in Alt(6)$ can be computed with the following trick. The trace of the action of g^i is computed by the character table and by the table (30). It gives the Newton sums $\sum_{j=1}^{5} \lambda_j^i$, where λ_j are the eigenvalues of the 5×5 matrix representing the action of g on $X5$. Then, by the Newton identities, we can compute the i-th elementary symmetric functions.

We sum in Molien formula over the six even conjugacy classes in Σ_6, as in the proof of Theorem 36.

The result is

$$\frac{1}{360}\left[\frac{1}{(t-1)^5} + \frac{45}{(t-1)^3(t+1)^2} + \frac{40}{(t-1)(t^2+t+1)^2} + \frac{40}{(t-1)^3(t^2+t+1)} + \right.$$
$$\left. + \frac{90}{(t-1)(t+1)^2(t^2+1)} + \frac{144}{(t-1)(t^4+t^3+t^2+t+1)}\right] =$$

$$(35) \qquad\qquad = \frac{1+t^{15}}{(1-t^2)(1-t^3)(1-t^4)(1-t^5)(1-t^6)}.$$

By considering the relation (31), the factor $(1-t^3)$ cancels from the denominator in the Hilbert series. The remaining factors in the denominator correspond to a_2, a_4, Δ, a_6. This suggests that there is another generator of degree 15, which is identified with b_{15}. By dimensional reasons, we expect a single relation. The square b_{15}^2 is Σ_6-invariant and it corresponds to the discriminant of the polynomial having $A, \ldots, F$ as roots, so it can be expressed as a polynomial in a_i. $\qquad\square$

THEOREM 39 (Coble). *The Hilbert series of the Σ_6-invariant ring of six (unordered) points on $I\!\!P^1$ (binary sextic) is*

$$\frac{1+t^{15}}{(1-t^2)(1-t^4)(1-t^6)(1-t^{10})}.$$

The ring is generated by $a_2, a_4, a_6, \Delta^2, b_{15}$ with the relation in degree 30 expressing b_{15}^2 as a polynomial in $a_2, \ldots, a_6$.

Proof. The Hilbert series is obtained as in the proof of Theorem 38, by adding the contribution of the remaining five conjugacy classes. Among the generators of the $Alt(6)$-invariant ring, a_2, a_4, a_6, b_{15} are already Σ_6-invariant. Δ^2 is another independent invariant. The ring generated by these invariant has the claimed Hilbert series. $\qquad\square$

The Hilbert function of d ordered points on $I\!\!P^1$ has been found by Howe.

THEOREM 40. *The dimension of the space of invariants for d ordered points on $I\!\!P^1$ and multidegree k^d with the weight $\begin{cases} 1^d \text{ if } d \text{ is even} \\ 2^d \text{ if } d \text{ is odd} \end{cases}$ is*

$$\sum_{j=0}^{(d-1)/2} (-1)^j \binom{d}{j} \binom{k(d/2-j)+d-2-j}{d-2}.$$

In this formula it is understood that a binomial coefficient $\binom{a}{b}$ is zero if $a < b$. In the case d even, this formula gives the degree k part of the invariant ring with respect to 1^d.

In the case d odd, the formula is meaningful for k even, and gives the degree $k/2$ part of the invariant ring with respect to 2^d.

Proof. ([28] 5.4.2.3)). $\qquad\square$

As an application, from this formula, in [18] Theorem 1.6, it has been computed the Hilbert series of the invariant ring of 8 ordered points on $I\!\!P^1$, which is

$$\frac{1+8t+22t^2+8t^3+t^4}{(1-t)^6} = \frac{1-14t^2+175t^4-512t^5+700t^6-512t^7+175t^8-14t^{10}+t^{12}}{(1-t)^{14}}.$$

The second formulation is reported because the coefficients are the Betti number of the resolution of the GIT quotient $I\!\!P(\mathbb{C}^2 \otimes \mathbb{C}^8)_{1^8}/SL(2)$, which is the moduli space of 8 ordered points on a line (see [18] Lemma 1.4, [26] Prop. 7.2).

Howe's formula in Theorem 40 has been generalized to arbitrary weights in [23].

5.6. The invariant ring of six points on the plane. Cremona hexahedral equation for the cubic surface

Let $a_1, \ldots a_6$ be six points on the projective plane. The aim of this section is to report in representation theoretic language the results by Coble [10] about six points on the plane, and to apply them to the construction of Cremona hexahedral equations.

We denote by (ijk) the 3×3 determinant of three points labeled with i, j, k. In particular (ijx) is the equation in the coordinate x of the line through a_i and a_j. This fits particularly well with the graphical description of the previous section, indeed the line (ijx) can be seen just by prolongation of the arrow between i and j.

Any relation between invariants (ij) on $\mathbb{P}^1$ transfers to an analogous relation among covariants (ijx) on $\mathbb{P}^2$. This is "Clebsch transfer principle".

In general, an invariant is a function $F(a_1, \ldots, a_6)$ which is symmetric (classically called rational) or skew-symmetric (classically called irrational) in the points, of degree q in the coordinates of each point, such that for every $g \in SL(3)$ $F(ga_1, \ldots, ga_6) = (det\, g)^{(q/3)} F(a_1, \ldots, a_6)$.

The covariants are polynomials in the invariants (ijk) and in the (ijx), which again are symmetric or skew-symmetric in the six points.

For example $(12x)(34x)(56x)$ represents a cubic splitting in three lines. There are 15 such cubics (as symbols), which span the 4-dimensional space of all cubics through the six points.

The six generators $t_0, \ldots, t_5$ of (27) and (32) give six cubics through the six lines. The relation $t_0 + t_1 + t_2 = t_3 + t_4 + t_5$ transfers to an analogous relation between cubics.

It is more convenient to consider the six invariants $A \ldots F$ of (33) which induce the following list of cubics

$$
\begin{aligned}
a &= (25x)(13x)(46x) + (51x)(42x)(36x) + (14x)(35x)(26x) + (43x)(21x)(56x) + (32x)(54x)(16x) \\
b &= (53x)(12x)(46x) + (14x)(23x)(56x) + (25x)(34x)(16x) + (31x)(45x)(26x) + (42x)(51x)(36x) \\
c &= (53x)(41x)(26x) + (34x)(25x)(16x) + (42x)(13x)(56x) + (21x)(54x)(36x) + (15x)(32x)(46x) \\
d &= (45x)(31x)(26x) + (53x)(24x)(16x) + (41x)(25x)(36x) + (32x)(15x)(46x) + (21x)(43x)(56x) \\
e &= (31x)(24x)(56x) + (12x)(53x)(46x) + (25x)(41x)(36x) + (54x)(32x)(16x) + (43x)(15x)(26x) \\
f &= (42x)(35x)(16x) + (23x)(14x)(56x) + (31x)(52x)(46x) + (15x)(43x)(26x) + (54x)(21x)(36x)
\end{aligned}
$$

For a representation-theoretic way to get these expressions, see [14] 9.4.4.

They satisfy the relation

$$a + b + c + d + e + f = 0$$

and indeed the six cubics span $X5$.

Again an even permutation of the points effects an even permutation of the cubics and an odd permutation of the points effects an odd permutation of the cubics accompanied by a change of sign. For example after the permutation (12) we get

$$a \rightarrow -d \qquad b \rightarrow -e \qquad c \rightarrow -f.$$

By the same argument given for invariants of points, the relation (34) transfers

to the relation $a^3 + b^3 + c^3 + d^3 + e^3 + f^3 = 0$ (in equivalent way the third elementary symmetric function of $a, \dots, f$ vanishes).

Since the space of cubics through 6 points is four dimensional, there is a further relation

$$\bar{a}a + \bar{b}b + \bar{c}c + \bar{d}d + \bar{e}e + \bar{f}f = 0,$$

which is uniquely determined considering the additional condition $\bar{a} + \bar{b} + \bar{c} + \bar{d} + \bar{e} + \bar{f} = 0$.

In order to find the additional relation between $a, \dots, f$, Coble considers a second interesting formulation. Define (ij, kl, mn) to be the function which represents that the three lines cij, ckl, cmn are concurrent in a point. It is a 3×3 determinant whose rows are $i \wedge j$, $k \wedge l$, $m \wedge n$. This function can be expressed in terms of (ijk) by the Lagrange identity

$$(v \wedge w) \cdot (m \wedge n) = (v \cdot m)(w \cdot n) - (v \cdot n)(w \cdot m).$$

With $v = i \wedge j$, $w = k \wedge l$, we get

$$(ij, kl, mn) = (ijm)(kln) - (ijn)(klm) = (ijl)(kmn) - (ijk)(lmn) =$$

$$= (ikl)(jmn) - (jkl)(imn),$$

where the last two identities are obtained by a permutation of the rows of the mixed product. They can be seen also as a consequence of the Plücker relations.

Coble considers ([10]page 170) the following expressions, obtained by formally replacing in (33) $(ij)(kl)(mn)$ with (ij, kl, mn).

$$\bar{a} = (25, 13, 46) + (51, 42, 36) + (14, 35, 26) + (43, 21, 56) + (32, 54, 16)$$
$$\bar{b} = (53, 12, 46) + (14, 23, 56) + (25, 34, 16) + (31, 45, 26) + (42, 51, 36)$$
$$\bar{c} = (53, 41, 26) + (34, 25, 16) + (42, 13, 56) + (21, 54, 36) + (15, 32, 46)$$
$$\bar{d} = (45, 31, 26) + (53, 24, 16) + (41, 25, 36) + (32, 15, 46) + (21, 43, 56)$$
$$\bar{e} = (31, 24, 56) + (12, 53, 46) + (25, 41, 36) + (54, 32, 16) + (43, 15, 26)$$
$$\bar{f} = (42, 35, 16) + (23, 14, 56) + (31, 52, 46) + (15, 43, 26) + (54, 21, 36)$$

which satisfy

(36) $\bar{a} + \bar{b} + \bar{c} + \bar{d} + \bar{e} + \bar{f} = 0.$

Now any permutation of the points effects a permutation of the cubics, without any change of sign. For example after the permutation (12) we get

$$\bar{a} \to \bar{d}, \qquad \bar{b} \to \bar{e}, \qquad \bar{c} \to \bar{f}.$$

Indeed $\bar{a}, \dots, \bar{f}$ span the representation $X8$.

PROPOSITION 25.

$$\bar{a}a + \bar{b}b + \bar{c}c + \bar{d}d + \bar{e}e + \bar{f}f = 0.$$

Proof. Consider the point at the intersection of $(12x)$ and $(34x)$.

By a direct computation, see for example [14] Theor. 9.4.13, the cubic at the left-hand side of our statement vanishes. By the invariance, the cubic vanishes at all the 45 intersection points of the lines (ijx) and (klx), hence it has to contain the 15 lines and must vanish. □

Let a_i for $i = 2, \ldots, 6$ be the i-th elementary symmetric function in $\bar{a}, \ldots, \bar{f}$ (remind that by (36) we have $a_1 = 0$). Remind the expression d_2 in (25), which is an irrational invariant expressing that the six points lie on a conic. Let Δ be the product of the differences of $\bar{a}, \ldots, \bar{f}$. It is an irrational invariant of degree 15. Note that its square is the discriminant of the polynomial with roots $\bar{a}, \ldots, \bar{f}$ and it can be expressed as a polynomial in $a_2, \ldots, a_6$.

THEOREM 41. *The Hilbert series of the invariant ring of six ordered points on* $\mathbb{P}^2$ *is*

$$\frac{1+t^2}{(1-t)^5} = \frac{1-t^4}{(1-t)^5(1-t^2)}.$$

The ring is generated by $\bar{a}, \ldots, \bar{e}$ *and by* d_2. *We have the relation in degree 4* $d_2^2 = a_2^2 - 4a_4$.

Proof. The Hilbert series reduces to a computation of semistandard tableau, that can be achieved by counting the integral points in a certain polytope, see [15] §11.2. The invariants of degree 1 are generated by $\bar{a}, \ldots, \bar{f}$ with the relation (36). In alternative, also the tableau functions $(123)(456)$, $(124)(356)$, $(125)(346)$, $(134)(256)$, $(135)(246)$ can be taken as generators. Note that d_2 cannot be obtained as a polynomial in $\bar{a}, \ldots, \bar{f}$ because otherwise it should be a multiple of Δ. Indeed all $Alt(6)$-invariants can be written as $\Delta \cdot p$ where p is a symmetric polynomial in $\bar{a}, \ldots, \bar{f}$ (see also Theorem 42). □

REMARK 18. This description shows that Kempe's Lemma (Theorem 31) fails for points on $\mathbb{P}^2$, indeed d_2 is a generator which has degree 2, while the minimal degree is 1.

THEOREM 42 (Coble). *The Hilbert series of the* $Alt(6)$-*invariant ring of six (unordered) points on* $\mathbb{P}^2$ *is*

$$\frac{1+t^{15}}{(1-t^2)^2(1-t^3)(1-t^5)(1-t^6)}.$$

The ring is generated by $a_2, d_2, a_3, a_5, a_6, \Delta$ *with the relation in degree 30 expressing* Δ^2 *as a polynomial in the other generators.*

Proof. $X8$ is an irreducible representation of $Alt(6)$. We already computed in the proof of Theorem 38 the Hilbert series of $\oplus_p S^p(X8)^{Alt(6)}$ by using Molien formula.

The result is

$$(37) \qquad \frac{1+t^{15}}{(1-t^2)(1-t^3)(1-t^4)(1-t^5)(1-t^6)}.$$

The factors in the denominator correspond to $a_2, \ldots, a_6$. This suggests that there is another generator of degree 15, which is identified with Δ. Now from Theorem 41 we have to add d_2 to the generators, which is already $Alt(6)$-invariant. We get that a_4 can be deleted by the generators. $\square$

THEOREM 43 (Coble). *The Hilbert series of the Σ_6-invariant ring of six (un-ordered) points on $I\!\!P^2$ is*

$$\frac{1 + t^{17}}{(1 - t^2)(1 - t^3)(1 - t^4)(1 - t^5)(1 - t^6)}.$$

The ring is generated by $a_2, a_3, a_4, a_5, a_6, d_2\Delta$ with the relation in degree 34 expressing $(d_2\Delta)^2$ as a polynomial in $a_2, \ldots, a_6$.

Proof. Among the generators of the $Alt(6)$-invariant ring, a_2, a_3, a_5, a_6 are already Σ_6-invariant. The relation $d_2^2 = a_2^2 - 4a_4$ allows to add a_4 at the generators at the place of d_2. Also $d_2\Delta$ is a Σ_6-invariant of degree 17. $\square$

REMARK 19. By adding the contribution of the remaining five odd conjugacy classes to (37) we get

$$\frac{1}{(1 - t^2)(1 - t^3)(1 - t^4)(1 - t^5)(1 - t^6)}.$$

REMARK 20. The Morley covariant

$$\bar{a}^2 a + \bar{b}^2 b + \bar{c}^2 c + \bar{d}^2 d + \bar{e}^2 e + \bar{f}^2 f$$

is a cubic quite important in the study of Lüroth quartics. Its property is that given six points on $I\!\!P^2$, then a seventh point belongs to the Morley cubic if and only if the ramification quartic obtained by the $2\colon 1$ map given linear system of cubics trough the seven points is a Lüroth quartic (see §4.8) and moreover the seven points give a Aronhold system of bitangents on the quartic which is associated to the pentalateral theta [38] or [43] Coroll. 4.2., Theor. 6.1 and Remark 10.7.

REMARK 21. The quadratic symmetric invariant a_2 of six points is the 2nd elementary symmetric function in $\bar{a}, \ldots, \bar{f}$. Note that given 5 points on $I\!\!P^2$, the sixth point defines a covariant conic curve, not passing through the points. What is the geometric interpretation of this conic ?

The previous construction can be resumed in the following steps. Start from 6 points in the plane.

Then the linear system of cubics through the six points embed the plane blown up at these six points in the cubic surface with equation

$$\begin{cases} a + b + c + d + e + f = 0 \\ \bar{a}a + \bar{b}b + \bar{c}c + \bar{d}d + \bar{e}e + \bar{f}f = 0 \\ a^3 + b^3 + c^3 + d^3 + e^3 + f^3 = 0 \end{cases}$$

These are known as *Cremona hexahedral equation* [11, 14]. They show that the general cubic surface is a hyperplane section of the Segre cubic primal defined in Remark 16.

The following line

$$a + b = c + d = e + f = 0$$

belongs to the cubic surface. By applying permutations, we get 15 such lines.

The remaining 12 lines correspond to the proper transform of the six lines and of the six conics through five among the six points. They form a so called *double-six* on the cubic surface.

References

[1] A. ABDESSALAM, J. CHIPALKATTI, *On Hilbert covariants*, Canad. J. Math. 66 (2014), no. 1, 3–30.

[2] R. BASSON, R. LERCIER, C. RITZENTHALER, J. SIJSLING, *An explicit expression of the Lüroth invariant*, ISSAC 2013 — Proceedings of the 38th International Symposium on Symbolic and Algebraic Computation, 31–36, ACM, New York, 2013.

[3] M. BRION, *Invariants et covariants des groupes algébriques réductifs* Notes de l'école d'été "Théorie des invariants" (Monastir, 1996), available on M. Brion web page.

[4] M. BRION, *Sur certains modules gradués associés aux produits symétriques*, Algèbre non commutative, groupes quantiques et invariants (Reims, 1995), 157–183, Sémin. Congr., 2, Soc. Math. France, Paris, 1997.

[5] L. BÁEZ-DUARTE, *Hardy-Ramanujan's asymptotic formula for partitions and the central limit theorem*, Adv. Math. 125 (1997), no. 1, 114–120.

[6] A. BEAUVILLE , *Determinantal hypersurfaces*, Michigan Math. J. 48 (2000), 39-64 .

[7] L. BEDRATYUK , *Analogue of Sylvester-Cayley formula for invariants of ternary form*, Ukrainian Mathematical Journal, 2010, 62 (11), 1810–1821.

[8] L. BEDRATYUK , *Analogue of Sylvester-Cayley formula for invariants of n-ary form*, Linear and Multilinear Algebra, 59, (11), 2011, 1189–1199 .

[9] G. BLEKHERMANN, J.HAUENSTEIN, J.C. OTTEM, K. RANESTAD, B. STURMFELS, *Algebraic boundaries of Hilbert's SOS cones* , Comp. Math. 148 (2012) 1717–1735.

[10] A. COBLE, *Point sets and allied Cremona groups*, Trans. AMS, part I 16 (1915), part II 17 (1916).

[11] L. CREMONA, *Ueber die Polar-Hexaeder bei den Flächen dritter Ordnung*, Math. Ann. 13 (1877).

[12] E. CIANI, *Le curve piane di quarto ordine*, Giornale di Matematiche, 48 (1910), 259–304.

[13] H. DERKSEN, G. KEMPER, *Computational Invariant Theory*, Encyclopaedia of Mathematical Sciences, 130, Springer, Berlin, 2002.

[14] I. DOLGACHEV, *Classical Algebraic Geometry, a modern view* , Cambridge Univ. Press, 2012.

[15] I. DOLGACHEV, *Lectures on invariant theory*, London Math. Soc. Lecture Note Series, 296. Cambridge Univ. Press, Cambridge, 2003.

[16] I. DOLGACHEV, V. KANEV, *Polar covariants of plane cubics and quartics*, Advances in Math. 98 (1993), 216-301.

[17] F. ENRIQUES AND O. CHISINI, *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche*, Zanichelli, Bologna 1915.

[18] E. FREITAG,R. SALVATI MANNI, *The modular variety of hyperelliptic curves of genus three*, Trans. Amer. Math. Soc. 363 (2011), 281–312.

[19] W. FULTON AND J. HARRIS, *Representation theory*, Springer GTM 129.

[20] G. JAMES AND A. KERBER, *The representation theory of the symmetric group*, Encyclopedia of Mathematics and its Applications, Addison Wesley 1981.

[21] J. GRACE AND A. YOUNG, *The algebra of invariants*, Cambridge, 1903.

[22] D. GRAYSON AND M. STILLMAN, MACAULAY 2: *a software system for research in algebraic geometry*. Available at `http://www.math.uiuc.edu/Macaulay2`.

[23] M. HERING AND B. HOWARD, *The ring of evenly weighted points on the line*, Math. Z. 277 (2014), no. 3-4, 691–708.

[24] D. HILBERT, *The theory of algebraic invariants*, Cambridge 1993.

[25] B. HOWARD, J. MILLSON, A. SNOWDEN, R. VAKIL, *The equations for the moduli space of n points on the line*, Duke Math. J. 146 (2009), no. 2, 175–226.

[26] B. HOWARD, J. MILLSON, A. SNOWDEN, R. VAKIL, *The geometry of eight points in projective space: Representation theory, Lie theory, dualities*, Proc. Lond. Math. Soc. (3) 105 (2012), no. 6, 1215–1244.

[27] B. HOWARD, J. MILLSON, A. SNOWDEN, R. VAKIL, *The ideal of relations for the ring of invariants of n points on the line*, J. Eur. Math. Soc. 14 (2012), no. 1, 1–60.

[28] R. HOWE, *The classical groups and invariants of binary forms*, in The Mathematical Heritage of Herman Weyl, 133–166, Proc. Sympos. Pure Math. 48, Amer. Math. Soc., Providence, 1988.

[29] A. KEMPE, *On regular difference terms*, Proc. London Math. Soc. 25 (1894), 343–350.

[30] H. KRAFT, C. PROCESI, *Classical invariant theory, a primer*, available on H. Kraft web page.

[31] J. KUNG, G. ROTA, *The invariant theory of binary forms*, Bull. Amer. Math. Soc. 10 (1984), no. 1, 27–85.

[32] J.M. LANDSBERG, *Tensors: Geometry and Applications*, Graduate Studies in Mathematics, 128, AMS, Providence, 2012.

[33] J.M. LANDSBERG AND G. OTTAVIANI, *Equations for secant varieties of Veronese and other varieties*, Ann. Mat. Pura Appl. (4) 192 (2013), no. 4, 569–606.

[34] J. LE POTIER AND A. TIKHOMIROV, *Sur le morphisme de Barth*, Ann. Sci. École Norm. Sup. (4) **34** (2001), no. 4, 573–629.

[35] J. LÜROTH, *Einige Eigenshaften einer gewissen Gathung von Curves vierten Ordnung*, Math. Annalen 1 (1868), 38–53.

[36] L. MANIVEL, *Symmetric functions, Schubert polynomials and degeneracy loci*, SMF/AMS Texts and Monographs, vol. 6, 2001.

[37] L. MANIVEL, *Prehomogeneous spaces and projective geometry*, note for School on Invariant Theory and Projective Geometry (Trento, 2012), Rend. Sem. Mat. Univ. Politec. Torino, 71 (2013), no.1, 35–118.

[38] F. MORLEY, *On the Lüroth Quartic Curve*, American J. of Math. 41 (1919), 279–282.

[39] S. MUKAI, *An introduction to invariants and moduli*, Cambridge Studies in Advanced Mathematics, 81. Cambridge University Press, Cambridge, 2003.

[40] J. MÜLLER, M. NEUNHÖFFER, *Some computations regarding Foulkes' conjecture*, Experiment. Math. 14 (2005), no. 3, 277–283.

[41] P. OLVER, *Classical invariant theory*, London Mathematical Society Student Texts, 44, Cambridge University Press, Cambridge, 1999.

[42] G. OTTAVIANI, *An invariant regarding Waring's problem for cubic polynomials*, Nagoya Math. J., 193 (2009), 95–110 .

[43] G. OTTAVIANI AND E. SERNESI, *On the hypersurface of Lüroth quartics*, Michigan Math. J. 59 (2010), 365–394.

[44] C. PAULY, *Self-duality of Coble's quartic hypersurface and applications*, Michigan Math. J. 50 (2002), no. 3, 551–574.

[45] A. POPOLITOV, SH. SHAKIROV, *On Undulation Invariants of Plane Curves*, arXiv:1208.5775 .

[46] V.L.POPOV, E.B. VINBERG, *Invariant Theory* in Encyclopaedia of Math. Sciences, Algebraic geometry IV, 55, Springer, Berlin, 1994.

[47] C. PROCESI, *Lie groups, an approach through invariants and representations*, Universitext, Springer, New York, 2007.

[48] K. RANESTAD AND F.O. SCHREYER, *Varieties of sums of powers*, Journal für die reine und angew. Math., 525, (2000), 147–181 .

[49] G.C. ROTA, *Two turning points in invariant theory*, The Mathematical Intelligencer, Volume 21(1), (1999), 20–27.

[50] G. SALMON, *A treatise on higher plane curves*, Hodges and Smith, Dublin, 1852 (reprinted from the 3d edition by Chelsea Publ. Co., New York, 1960).

[51] G. SCORZA, *Un nuovo teorema sopra le quartiche piane generali*, Math. Ann. 52, (1899), 457-461.

[52] T. SHIODA, *On the graded ring of invariants of binary octavics*, American J. of Math., 89(4), (1967), 1022–1046.

[53] B. STURMFELS, *Algorithms in Invariant Theory*, Springer, 1993, Wien.

[54] F. VACCARINO, *Generalized symmetric functions and invariants of matrices*, Math. Z. 260 (2008), no. 3, 509–526.

[55] J. WEYMAN, *Cohomology of vector bundles and syzygies*, Cambridge University Press, 2003.

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