Rend. Sem. Mat. Univ. Politec. Torino
Vol. 70, 3 (2012), 297 – 309

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KÄHLER–WEYL MANIFOLDS OF DIMENSION 4

Abstract. We determine the space of algebraic pseudo-Hermitian Kähler–Weyl curvature tensors and the space of para-Hermitian Kähler–Weyl curvature tensors in dimension 4 and show that every algebraic possibility is geometrically realizable. We establish the Gray identity for pseudo-Hermitian Weyl manifolds and for para-Hermitian Weyl manifolds in arbitrary dimension.
MSC: 53B05, 15A72, 53A15, 53C07, 53C25

1. Introduction

Let (M,g) be a pseudo-Riemannian manifold of dimension $m = 2m \geq 4$ with vanishing first de Rham cohomology group: $H^1(M;\mathbb{R}) = 0$. Let ∇ be a torsion-free connection on the tangent bundle TM of M. The triple (M,g,∇) is said to be a *Weyl structure* if $\nabla g = -2\phi \otimes g$ for some smooth 1-form ϕ on M. Let ∇^g be the Levi-Civita connection of g and let $\phi^\star$ be the associated dual vector field. One has [11]:

$$\nabla_x y := \nabla^g_x y + \phi(x)y + \phi(y)x - g(x,y)\phi^\star\,. \tag{1.1}$$

These geometries were first introduced by Weyl [28] and remain an active area of investigation today – see, for example, the discussion in [8, 18, 19, 27]. Weyl structures are intimately linked with conformal geometry. If $\tilde{g} = e^{2f}g$ is a conformally equivalent metric, then $(M,\tilde{g},\nabla)$ is again a Weyl structure where $\tilde{\phi} = \phi - df$. A Weyl structure is said to be *trivial* if $\phi = df$ for some smooth function f or, equivalently, if $\nabla = \nabla^{\tilde{g}}$ where $\nabla^{\tilde{g}}$ is the Levi-Civita connection of the conformally equivalent metric $\tilde{g} = e^{2f}g$. Since we have assumed that $H^1(M;\mathbb{R}) = 0$, the Weyl structure is trivial if and only if $d\phi = 0$.

We say that J_- is an *almost complex structure* on the tangent bundle TM if J_- is an automorphism of TM so that $J_-^2 = -\mathrm{id}$. We say that J_+ is an *almost para-complex structure* on TM is J_+ is an automorphism of TM with $J_+^2 = \mathrm{id}$ and $\mathrm{Tr}(J_+) = 0$; this latter condition is automatic for an almost complex structure but must be imposed here.

Let J_- (resp. J_+) be an almost complex (resp. para-complex) structure on TM. It is convenient to use a common notation $J_\pm$ even though we shall never be considering both structures simultaneously. One says that $J_\pm$ is *integrable* if there exists a cover of M by coordinate charts $(x^1,...,x^m,y^1,...,y^m)$ so that

$$J_\pm : \partial_{x_i} \to \partial_{y_i} \quad \text{and} \quad J_\pm : \partial_{y_i} \to \pm\partial_{x_i}\,.$$

We say that a torsion free connection ∇ is *Kähler* if $\nabla J_\pm = 0$; the existence of such a connection then implies $J_\pm$ is integrable.

Let (M,g) be a pseudo-Riemannian manifold of signature (p,q). An integrable complex structure J_- on M is said to define *a pseudo-Hermitian structure* and the triple

(M,g,J_-) is said to be a *pseudo-Hermitian* manifold if $J_-^*g = g$. Similarly if J_+ is an integrable para-complex structure on M, then J_+ is said to define a *para-Hermitian structure* and the triple (M,g,J_+) is said to be a *para-Hermitian* manifold if $J_+^*g = -g$. Again, to have a common notation, we will say that $(M,g,J_\pm)$ is a *para/pseudo-Hermitian manifold*. If the Levi-Civita connection ∇^g is *Kähler*, then $(M,g,J_\pm)$ is said to be *Kähler*.

We wish to study the interaction of these two structures. One says that a quadruple $(M,g,J_\pm,\nabla)$ is a *Kähler–Weyl* structure if $(M,g,J_\pm)$ is a para/pseudo-Hermitian manifold, if (M,g,∇) is a Weyl structure, and if $\nabla J_\pm = 0$. The following is well known – see, for example, the discussion in [20] in the Riemannian setting (which uses results of [25, 26]) and the generalization given in [10] to the more general context:

THEOREM 1. *Let $m \geq 6$. If $(M,g,J_\pm,\nabla)$ is a Kähler–Weyl structure, then the associated Weyl structure is trivial, i.e. there exists a conformally equivalent metric $\tilde{g} = e^{2f}g$ so that $(M,\tilde{g},J_\pm)$ is Kähler and so that $\nabla = \nabla^{\tilde{g}}$.*

Examples in [6, 21] show that Theorem 1 fails if $m = 4$ and motivate our present investigation. Let $\Omega_\pm$ be the Kähler form:

$$\Omega_\pm(x,y) := g(x,J_\pm y)\,.$$

Let d be the exterior derivative and let δ be the dual operator, the interior coderivative. The *Lee form* is given, modulo a suitable normalizing constant, by $J_\pm^*\delta\Omega_\pm$ and plays a crucial role. The following result was established [16] in the Riemannian setting; the proof extends without change to this more general context:

THEOREM 2. *Every para/pseudo-Hermitian manifold of dimension 4 admits a unique Kähler-Weyl structure where $\phi = \pm\frac{1}{2}J_\pm^*\delta\Omega_\pm$.*

The results of Theorem 1 and of Theorem 2 are closely related to curvature decompositions. Let R be the curvature tensor, let $\mathcal{R}$ be the curvature operator, and let ρ be the Ricci tensor of a Weyl structure (M,g,∇). They are defined by:

$$\begin{aligned}
&\mathcal{R}(x,y) := \nabla_x\nabla_y - \nabla_y\nabla_x - \nabla_{[x,y]},\\
&R(x,y,z,w) := g(\mathcal{R}(x,y)z,w),\\
&\rho(x,y) := \operatorname{Tr}\{z \to \mathcal{R}(z,x)y\}\,.
\end{aligned}$$

Let $\rho_a(x,y) := \frac{1}{2}\{\rho(x,y)-\rho(y,x)\}$ be the alternating part of the Ricci tensor. The following facts are well known (see, for example, [7, 11, 21, 22]):

$$\begin{aligned}
&R(x,y,z,w) = -R(y,x,z,w),\\
&R(x,y,z,w) + R(y,z,x,w) + R(z,x,y,w) = 0,\\
&R(x,y,z,w) + R(x,y,w,z) = -\tfrac{4}{m}\rho_a(x,y)g(z,w)\,.
\end{aligned} \tag{1.2}$$

We also have the relation:

$$d\phi = -\tfrac{2}{m}\rho_a\,. \tag{1.3}$$

If $\nabla = \nabla^g$ is the Levi-Civita connection, then we have the additional symmetry:

$$R(x,y,z,w) + R(x,y,w,z) = 0\,. \tag{1.4}$$

The Weyl structure is trivial if and only if Equation (1.4) is satisfied [11]. If ∇ is Kähler, then $\mathcal{R}(x,y)J_\pm = J_\pm\mathcal{R}(x,y)$ for all x,y or, equivalently:

$$R(x,y,J_\pm z,J_\pm w) = \mp R(x,y,z,w)\,. \tag{1.5}$$

We now pass to the algebraic context. Let $(V,\langle\cdot,\cdot\rangle)$ be an inner product space. The space of *Weyl curvature tensors* $\mathfrak{W}\subset\otimes^4V^*$ is defined by imposing the symmetry of Equation (1.2). The space of *Riemann curvature tensors* $\mathfrak{R}\subset\mathfrak{W}$ is obtained by requiring in addition the symmetry of Equation (1.4). Let $J_\pm$ be a para/pseudo-Hermitian structure on $(V,\langle\cdot,\cdot\rangle)$. We define the space of *Kähler tensors* $\mathfrak{K}_\pm$ by imposing Equation (1.5). The space of *Kähler–Weyl tensors* $\mathfrak{K}_{\pm,\mathfrak{W}} := \mathfrak{K}_\pm\cap\mathfrak{W}$ is obtained by imposing the symmetries of Equation (1.2) and of Equation (1.5) and the space of *Kähler–Riemann tensors* $\mathfrak{K}_{\pm,\mathfrak{R}} := \mathfrak{K}_\pm\cap\mathfrak{R}$ is obtained by imposing in addition the symmetry of Equation (1.4). The structure groups are given by:

$$\begin{aligned}
&\mathcal{O} := \{T\in\mathrm{GL} : T^*\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle\},\\
&\mathcal{U}_\pm := \{T\in\mathcal{O} : TJ_\pm = J_\pm T\},\\
&\mathcal{U}_\pm^\star := \{T\in\mathcal{O} : TJ_\pm = J_\pm T \text{ or } TJ_\pm = -J_\pm T\}\,.
\end{aligned}$$

It is convenient to work with the $\mathbb{Z}_2$ extensions $\mathcal{U}_\pm^\star$ which permits us to interchange the roles of $J_\pm$ and $-J_\pm$. Let χ be the $\mathbb{Z}_2$ valued character of $\mathcal{U}_\pm^\star$ so that if $T\in\mathcal{U}_\pm^\star$, then

$$J_\pm T = \chi(T)TJ_\pm\,.$$

One then has that $T^*\Omega_\pm = \chi(T)\Omega_\pm$. Let

$$\Lambda^2_{0,J_\pm} = \{\Phi\in\Lambda^2(V^*) : \Phi\perp\Omega_\pm\}\,.$$

THEOREM 3. *Let $(V,\langle\cdot,\cdot\rangle,J_\pm)$ be a para/pseudo-Hermitian vector space.*

1. *If $m\geq 6$, then $\mathfrak{K}_{\pm,\mathfrak{W}} = \mathfrak{K}_{\pm,\mathfrak{R}}$.*
2. *If $m = 4$, then $\mathfrak{K}_{\pm,\mathfrak{W}} = \mathfrak{K}_{\pm,\mathfrak{R}}\oplus L_{0,J_\pm}$ where $L_{0,J_\pm}\approx\Lambda^2_{0,J_\pm}$ as a $\mathcal{U}_\pm^\star$ module.*

This is one of the facts about 4-dimensional geometry that distinguishes it from the higher dimensional setting; the module $L_{0,J_\pm}$ provides additional curvature possibilities if $m = 4$.

Let $(V,\langle\cdot,\cdot\rangle,J_\pm)$ be a para/pseudo-Hermitian vector space and let $A\in\mathfrak{K}_{\pm,\mathfrak{W}}$. We say that A is *geometrically realizable* if there exists a point $P\in M$, a Kähler–Weyl structure $(M,g,J_\pm,\nabla)$, and an isomorphism $\phi : T_PM\to V$ so that $\phi^*\langle\cdot,\cdot\rangle = g_P$, $\phi^*J_\pm = J_{\pm,P}$, and $J^*A = R_P$.

THEOREM 4. *Every element of $\mathfrak{K}_{\pm,\mathfrak{W}}$ is geometrically realizable.*

Theorem 4 means that Equation (1.2) and Equation (1.5) generate the universal curvature symmetries of the curvature tensor of a Kähler–Weyl structure; there are no hidden symmetries. The fact that $\mathfrak{K}_{\pm,\mathfrak{W}} \neq \mathfrak{K}_{\pm,\mathfrak{R}}$ in dimension 4 permits us to find Kähler–Weyl structures which do not satisfy the symmetry of Equation (1.4) and which therefore are not trivial. Thus it is the curvature decomposition of Theorem 3 which is at the heart of the difference between the 4-dimensional setting and the higher dimensional setting exemplified by Theorem 1 and by Theorem 2.

The *Gray symmetrizer* is defined by setting:

$$\begin{aligned} \mathcal{G}_\pm(A)(x,y,z,w) &:= A(x,y,z,w) + A(J_\pm x, J_\pm y, J_\pm z, J_\pm w) \\ &\pm A(J_\pm x, J_\pm y, z, w) \pm A(x,y,J_\pm z, J_\pm w) \pm A(J_\pm x, y, J_\pm z, w) \\ &\pm A(x, J_\pm y, z, J_\pm w) \pm A(J_\pm x, y, z, J_\pm w) \pm A(x, J_\pm y, J_\pm z, w)\,. \end{aligned} \tag{1.6}$$

Gray [12] showed that the integrability of the (para)-complex structure gives rise to the additional curvature identity $\mathcal{G}(R^g) = 0$. Although his result was originally stated only in the Hermitian setting, it extends easily to the para/pseudo-Hermitian setting [2, 5]. In fact, this identity remains valid in the context of Weyl geometry:

Theorem 5. *Let $(M,g,J_\pm)$ be a para/pseudo-Hermitian manifold and let ∇ be a Weyl connection. Then $\mathcal{G}(R^\nabla) = 0$.*

Here is a brief outline of this paper. In Section 2, we review some decomposition results that are needed. In Section 3, we establish Theorem 2; we shall not follow the discussion in [16] but rather base our discussion on the decomposition results of [1, 13] given in Theorem 9 as that will be more convenient for our further development. In Section 4, we prove Theorem 3; we restrict to the case $m = 4$ since the case $m \geq 6$ is treated in [10]. We also verify Theorem 4. Since every element of $\mathfrak{K}_{\pm,\mathfrak{R}}$ can be geometrically realized by a para/pseudo-Kähler manifold [3], Theorem 4 follows from Theorem 3 if $m \geq 6$. It therefore suffices to prove Theorem 4 if $m = 4$. In Section 5, we use Theorem 3 to prove Theorem 5.

We remark that an expository version of some of these results is to appear in [9]. The treatment in that paper is quite different - in particular there is no discussion of the Gray symmetrizer (Theorem 5) nor is there a proof of the extension of the results of [16] to the indefinite setting and para-Hermitian settings given in Theorem 2. Finally, the proof of Theorem 4 is very different in the discussion of [9].

2. Decomposition results

In Section 2.1, we recall the fundamental facts of group representation theory that we shall need; we work in the context of $\mathcal{U}_\pm^\star$ modules as many of the relevant results fail for $\mathcal{U}_+$. In Section 2.2, we review the Tricerri-Vanhecke decomposition of $\mathfrak{R}$ as a $\mathcal{U}_\pm^\star$ module. In Section 2.3, we combine the Higa decomposition of $\mathfrak{W}$ with the Tricerri-Vanhecke decomposition to decompose $\mathfrak{W}$ as a $\mathcal{U}_\pm^\star$ module. In Section 2.4, we present the Gray-Hervella decomposition of the space of covariant derivatives of the Kähler form as a $\mathcal{U}_\pm^\star$ module.

2.1. Representation Theory

Let $(V,\langle\cdot,\cdot\rangle,J_\pm)$ be a para/pseudo-Hermitian vector space. Extend $\langle\cdot,\cdot\rangle$ to a non-degenerate symmetric bilinear form on $\otimes^k V^*$ by setting:

$$\langle (v_1\otimes\cdots\otimes v_k),(w_1\otimes\cdots\otimes w_k)\rangle := \prod_{i=1}^{k}\langle v_i,w_i\rangle .$$

Use $\langle\cdot,\cdot\rangle$ to identify $\otimes^k V$ with $\otimes^k V^*$ henceforth. The natural action of $\mathcal{U}_\pm^\star$ on $\otimes^k V^*$ by pullback is an isometry making any $\mathcal{U}_\pm^\star$-invariant subspace of $\otimes^k V^*$ into a $\mathcal{U}_\pm^\star$ module. We refer to [1] for the proof of the following result; this result fails for the group $\mathcal{U}_+$ and for that reason we choose to work with the groups $\mathcal{U}_\pm^\star$.

LEMMA 1. *Let $(V,\langle\cdot,\cdot\rangle,J_\pm)$ be a para/pseudo-Hermitian vector space. Let ξ be a $\mathcal{U}_\pm^\star$ submodule of $\otimes^k V$.*

1. *$\langle\cdot,\cdot\rangle$ is non-degenerate on ξ.*
2. *There is an orthogonal direct sum decomposition $\xi=\eta_1\oplus\cdots\oplus\eta_k$ where the η_i are irreducible $\mathcal{U}_\pm^\star$ modules.*
3. *If ξ_1 and ξ_2 are inequivalent irreducible $\mathcal{U}_\pm^\star$ submodules of ξ, then $\xi_1\perp\xi_2$.*
4. *The multiplicity with which an irreducible representation appears in ξ is independent of the decomposition in* (2).
5. *If ξ_1 appears with multiplicity* 1 *in ξ and if η is any $\mathcal{U}_\pm^\star$ submodule of ξ, then either $\xi_1\subset\eta$ or else $\xi_1\perp\eta$.*
6. *If $0\to\xi_1\to\xi\to\xi_2\to 0$ is a short exact sequence of $\mathcal{U}_\pm^\star$ modules, then we have that $\xi\approx\xi_1\oplus\xi_2$ as a $\mathcal{U}_\pm^\star$ module.*

2.2. The Tricerri-Vanhecke decomposition

Decompose $\otimes^2 V^* = S^2\oplus\Lambda^2$ as the direct sum of the symmetric and of the alternating 2-tensors, respectively. Set

$$\begin{aligned} S^2_{+,J_\pm} &:= \{\theta\in S^2 : J_\pm^*\theta=+\theta\}, &\quad \Lambda^2_{+,J_\pm} &:= \{\theta\in\Lambda^2 : J_\pm^*\theta=+\theta\}\\ S^2_{-,J_\pm} &:= \{\theta\in S^2 : J_\pm^*\theta=-\theta\}, &\quad \Lambda^2_{-,J_\pm} &:= \{\theta\in\Lambda^2 : J_\pm^*\theta=-\theta\}. \end{aligned}$$

We have $\langle\cdot,\cdot\rangle\in S^2_{\mp,J_\pm}$ and $\Omega_\pm\in\Lambda^2_{\mp,J_\pm}$. This permits us to express

$$\begin{aligned} &S^2_{\mp,J_\pm} = \langle\cdot,\cdot\rangle\cdot\mathbb{R}\oplus S^2_{0,\mp,J_\pm} \quad\text{and}\quad \Lambda^2_{\mp,J_\pm} = \Omega_\pm\cdot\mathbb{R}\oplus\Lambda^2_{0,\mp,J_\pm} \quad\text{where}\\ &S^2_{0,\mp,J_\pm} := \{\theta\in S^2_{\mp,J_\pm} : \theta\perp\langle\cdot,\cdot\rangle\} \quad\text{and}\quad \Lambda^2_{0,\mp,J_\pm} := \{\theta\in\Lambda^2_{\mp,J_\pm} : \theta\perp\Omega_\pm\}. \end{aligned}$$

This gives the following orthogonal decomposition of $\otimes^2 V^*$ into irreducible and inequivalent $\mathcal{U}_\pm^\star$ modules:

$$\otimes^2 V^* = S^2_{\pm,J_\pm}\oplus\mathbb{R}\oplus S^2_{0,\mp,J_\pm}\oplus\Lambda^2_{\pm,J_\pm}\oplus\chi\oplus\Lambda^2_{0,\mp,J_\pm}. \tag{2.1}$$

The following decompositions were first established in [17, 23, 24] for almost complex structures in the positive definite case; we refer to [4] for the extension to the higher signature setting and to the almost para-complex case:

THEOREM 6. *Adopt the notation established above. We have an orthogonal direct sum decompositions of non-trivial irreducible $\mathcal{U}_\pm^\star$ modules:*

$$\begin{aligned}\mathfrak{R} &= \left\{\begin{array}{ll}\oplus_{1\le i\le 10} W_{\pm,i} & \text{if } m\ge 8\\ \oplus_{1\le i\le 10, i\ne 6} W_{\pm,i} & \text{if } m=6\\ \oplus_{1\le i\le 10, i\ne 5,6,10} W_{\pm,i} & \text{if } m=4\end{array}\right\},\\ \mathfrak{K}_{\pm,\mathfrak{R}} &= W_{\pm,1}\oplus W_{\pm,2}\oplus W_{\pm,3}\,.\end{aligned}$$

These are inequivalent $\mathcal{U}_\pm^\star$ modules except for the isomorphisms:

$$W_{\pm,1}\approx W_{\pm,4}\approx\mathbb{R}\quad\textit{and}\quad W_{\pm,2}\approx W_{\pm,5}\approx S^2_{0,\mp,J_\pm}\,.$$

We have $W_{\pm,8}\approx S^2_{\pm,J_\pm}$ and $W_{\pm,9}\approx\Lambda^2_{\pm,J_\pm}$ as $\mathcal{U}_\pm^\star$ modules. None of the modules $W_{\pm,i}$ is isomorphic either to χ or to $\Lambda^2_{0,\mp,J_\pm}$ as $\mathcal{U}_\pm^\star$ modules.

The precise nature of the modules $W_{\pm,i}$ for $i=3,6,7,10$ is not relevant and we refer to [24] in the Riemannian setting and to [4] in the general setting for their precise definition.

2.3. The Higa decomposition

We refer to [14, 15] for the proof of:

THEOREM 7. *There is an orthogonal direct sum decomposition $\mathfrak{W}=\mathfrak{R}\oplus L$ where $L\approx\Lambda^2$ as an O module.*

Decomposing $\Lambda^2=\Omega_\pm\cdot\mathbb{R}\oplus\Lambda^2_{0,\mp,J_\pm}\oplus\Lambda^2_{\pm,J_\pm}$ as a $\mathcal{U}_\pm^\star$ module and then applying Theorem 6 and Theorem 7 yields:

THEOREM 8. *Let $(V,\langle\cdot,\cdot\rangle,J_\pm)$ be a para/pseudo-Hermitian vector space. We have an orthogonal direct sum decomposition of $\mathcal{U}_\pm^\star$ modules:*

$$\mathfrak{W}=\left\{\begin{array}{ll}\oplus_{1\le i\le 13} W_{\pm,i} & \text{if } m\ge 8\\ \oplus_{1\le i\le 13, i\ne 6} W_{\pm,i} & \text{if } m=6\\ \oplus_{1\le i\le 13, i\ne 5,6,10} W_{\pm,i} & \text{if } m=4\end{array}\right\}.$$

We have $W_{\pm,11}\approx\chi$, $W_{\pm,12}\approx\Lambda^2_{0,\mp,J_\pm}$, and $W_{\pm,13}\approx\Lambda^2_{\pm,J_\pm}$ as $\mathcal{U}_\pm^\star$ modules. These are non-trivial inequivalent $\mathcal{U}_\pm^\star$ modules except for the isomorphisms:

$$W_{\pm,1}\approx W_{\pm,4}\approx\mathbb{R},\quad W_{\pm,2}\approx W_{\pm,5}\approx S^2_{0,\mp,J_\pm},\quad W_{\pm,9}\approx W_{\pm,13}\approx\Lambda^2_{\pm,J_\pm}\,.$$

2.4. The Gray-Hervella decomposition

We follow [1, 13]. We assume $J_\pm$ is integrable. The covariant derivative $\nabla^g\Omega_\pm$ has the symmetries:

$$\begin{aligned}(\nabla^g\Omega_\pm)(x,y;z) &= -(\nabla^g\Omega_\pm)(y,x;z) = \pm(\nabla^g\Omega_\pm)(J_\pm x, J_\pm y; z)\\ &= \mp(\nabla^g\Omega_\pm)(x, J_\pm y; J_\pm z)\,.\end{aligned} \tag{2.2}$$

Let $(V,\langle\cdot,\cdot\rangle,J_\pm)$ be a para/pseudo-Hermitian vector space. Let $\varepsilon_{ij} := \langle e_i, e_j\rangle$ where $\{e_i\}$ is a basis for V. Let $\phi \in V^*$. Let $H \in \otimes^3 V^*$. Let $U_\pm$ be the space of tensors satisfying Equation (2.2). Set

$$\begin{aligned}&\sigma_\pm(\phi)(x,y;z) := \phi(J_\pm x)\langle y,z\rangle - \phi(J_\pm y)\langle x,z\rangle + \phi(x)\langle J_\pm y,z\rangle - \phi(y)\langle J_\pm x,z\rangle,\\ &(\tau_1 H)(x) := \varepsilon^{ij}H(x,e_i;e_j)\,.\end{aligned}$$

The map τ_1 appears in elliptic operator theory. Let δ be coderivative – δ is the formal adjoint of the exterior derivative d. If Φ is a smooth 2-form, then

$$\delta\Phi = \tau_1\nabla^g\Phi\,. \tag{2.3}$$

One has (see, for example, the discussion in [1]) that:

$$\tau_1\sigma_\pm = (m-2)J_\pm^\star\,. \tag{2.4}$$

Thus $\mathrm{Range}(\sigma_\pm) \perp \ker(\tau_1)$ and these are $\mathcal{U}_\pm^\star$ modules. We therefore set:

$$U_{\pm,3} := U_\pm \cap \ker(\tau_1) \quad \text{and} \quad U_{\pm,4} := \mathrm{Range}(\sigma_\pm)\,.$$

The following result follows from a more general result of [13] in the Hermitian setting; we refer to [1] for the extension to the pseudo-Hermitian and the para-Hermitian settings:

THEOREM 9. *Let $(V,\langle\cdot,\cdot\rangle,J_\pm)$ be a para/pseudo-Hermitian vector space. We have a direct sum orthogonal decomposition of $U_\pm$ into non-trivial irreducible and inequivalent $\mathcal{U}_\pm^\star$ modules in the form:*

$$U_\pm = \left\{\begin{array}{ll} U_{\pm,3}\oplus U_{\pm,4} & \text{if } m\ge 6\\ U_{\pm,4} & \text{if } m = 4\end{array}\right\}.$$

3. The proof of Theorem 2

We adopt the notation of Theorem 9. We begin by establishing the following result which is of interest in its own right.

THEOREM 10. *Let $(M,g,J_\pm)$ be a para/pseudo-Hermitian manifold.*

1. The following assertions are equivalent:

(a) $\nabla^g\Omega_\pm \in U_{\pm,4}$ *for all points of* M.

(b) There exists ∇ *so* $(M, g, J_\pm, \nabla)$ *is a Kähler–Weyl structure.*

2. *If* $(M, g, J_\pm, \nabla)$ *is a Kähler–Weyl structure, then* $\phi = \pm\frac{1}{m-2}J_\pm^*\delta\Omega_\pm$.

Remark 1. By Assertion (2) and by Equation (1.1), the connection in Assertion (1b) is uniquely determined by $(M, g, J_\pm)$.

Proof. We compute directly that:

$$\begin{aligned}(\nabla\Omega_\pm)(x,y;z) &= zg(x,J_\pm y) - g(\nabla_z x, J_\pm y) - g(x, J_\pm\nabla_z y)\\ &= zg(x,J_\pm y) - g(\nabla_z x, J_\pm y) - g(x, \nabla_z J_\pm y) + g(x,(\nabla_z J_\pm)y)\\ &= (\nabla_z g)(x, J_\pm y) + g(x,(\nabla_z J_\pm)y)\\ &= -2\phi(z)g(x,J_\pm y) + g(x,(\nabla_z J_\pm)y)\,.\end{aligned}$$

We use Equation (1.1) and the definition of $\sigma_\pm$ to compute that:

$$\begin{aligned}(\nabla\Omega_\pm)(x,y;z) &= zg(x,J_\pm y) - g(\nabla^g_z x, J_\pm y) - g(x, J_\pm\nabla^g_z y)\\ &\quad -\phi(z)g(x,J_\pm y) - \phi(x)g(z,J_\pm y) + g(x,z)g(\phi^*, J_\pm y)\\ &\quad -\phi(z)g(x,J_\pm y) - \phi(y)g(x,J_\pm z) + g(y,z)g(x,J_\pm\phi^*)\\ &= (\nabla^g\Omega)(x,y;z) - 2\phi(z)g(x,J_\pm y) - \sigma_\pm(\phi)(x,y;z)\,.\end{aligned}$$

This leads to the relation:

$$(\nabla^g\Omega_\pm)(x,y;z) = (\sigma_\pm\phi)(x,y;z) + g(x,(\nabla_z J_\pm)y)\,. \tag{3.1}$$

Suppose that there exists a torsion free connection ∇ so that $\nabla g = -2\phi\otimes\phi$ and so that $\nabla J_\pm = 0$. By Equation (3.1),

$$\nabla^g\Omega_\pm \in \mathrm{Range}(\sigma_\pm) = U_{\pm,4}\,.$$

Consequently, Assertion (1b) implies Assertion (1a). Conversely, suppose that there exists a 1-form ϕ so $\nabla^g\Omega_\pm = \sigma_\pm(\phi)$. By Equation (2.4),

$$\phi = \pm\frac{1}{m-2}J_\pm^*\tau_1\nabla^g\Omega_\pm\,.$$

Consequently, ϕ is smooth. Motivated by Equation (1.1), we define a connection ∇ by setting:

$$\nabla_x y := \nabla^g_x y + \phi(x)y + \phi(y)x - g(x,y)\phi^\star\,.$$

Since $\nabla_x y - \nabla_y x = \nabla^g_x y - \nabla^g_y x = [x,y]$, ∇ is torsion free. Furthermore,

$$\begin{aligned}(\nabla_x g)(y,z) &= xg(y,z) - g(\nabla_x y, z) - g(y, \nabla_x z)\\ &= xg(y,z) - g(\nabla^g_x y, z) - g(y, \nabla^g_x z)\\ &- \phi(x)g(y,z) - \phi(y)g(x,z) + g(x,y)\phi(z)\\ &- \phi(x)g(z,y) - \phi(z)g(x,y) + g(x,z)\phi(y)\\ &= -2\phi(x)g(y,z)\,.\end{aligned}$$

This shows $\nabla g = -2\phi\otimes g$ so (M,g,∇) is a Weyl structure. We apply Equation (3.1) to conclude $\nabla J_\pm = 0$ and thus $(M,g,J_\pm,\nabla)$ is a Kähler–Weyl structure. This shows that Assertion (1a) implies Assertion (1b) and completes the proof of Assertion (1).

If $(M,g,J_\pm,\nabla)$ is a Kähler–Weyl structure, then $\nabla^g\Omega_\pm = \sigma_\pm\phi$ by Equation (3.1). We use Equation (2.3) and Equation (2.4) to compute:

$$\begin{aligned} &\tau_1(\sigma_\pm(\phi))(x) = (m-2)(J_\pm^*\phi)(x)\\ = \quad &(\tau_1\nabla^g\Omega_\pm)(x) = (\varepsilon^{ij}\nabla^g\Omega_\pm)(x,e_i;e_j) = (\delta\Omega_\pm)(x)\,. \end{aligned}$$

This shows that $(m-2)J_\pm^\star\phi = \delta\Omega_\pm$. Since $J_\pm^\star J_\pm^\star = \pm\,\mathrm{id}$, Assertion (2) follows. □

Let $m=4$. By Theorem 9, $\nabla^g\Omega_\pm = \sigma_\pm(\phi)$ for some ϕ. By Theorem 10, $(M,g,J_\pm,\nabla)$ is a Kähler–Weyl structure where $\phi = \pm\frac{1}{2}J_\pm^*\delta\Omega_\pm$. This completes the proof of Theorem 2. □

4. The proof of Theorem 3 and of Theorem 4

We begin with a simple example. Let (x^1,x^2,x^3,x^4) be the usual coordinates on $\mathbb{R}^4$. Define the canonical (para)-complex structure $J_\pm$ on $\mathbb{R}^4$ by setting:

$$J_\pm(\partial_{x_1}) = \partial_{x_2},\quad J_\pm(\partial_{x_2}) = \pm\partial_{x_1},\quad J_\pm(\partial_{x_3}) = \partial_{x_4},\quad J_\pm(\partial_{x_4}) = \pm\partial_{x_3}\,. \tag{4.1}$$

If g is a para/pseudo-Hermitian metric on $\mathbb{R}^4$, set

$$g(\partial_{x_i},\partial_{x_j};\partial_{x_k}) = \partial_{x_k}g(\partial_{x_i},\partial_{x_j})\,.$$

We then have [1]:

$$\begin{aligned} (\nabla^g\Omega_\pm)(\partial_{x_i},\partial_{x_j};\partial_{x_k}) = \tfrac{1}{2}\{&g(\partial_{x_i},\partial_{x_k};J_\pm\partial_{x_j}) - g(\partial_{x_j},\partial_{x_k};J_\pm\partial_{x_i})\\ &+g(J_\pm\partial_{x_i},\partial_{x_k};\partial_{x_j}) - g(J_\pm\partial_{x_j},\partial_{x_k};\partial_{x_i})\}\,. \end{aligned} \tag{4.2}$$

We consider a flat background metric

$$g_0 := \varepsilon_{11}(dx^1\otimes dx^1 \mp dx^2\otimes dx^2) + \varepsilon_{22}(dx^3\otimes dx^3 \mp dx^4\otimes dx^4)\,. \tag{4.3}$$

We take $\varepsilon_{11}=\varepsilon_{22}=1$ to define a Hermitian metric, $\varepsilon_{11}=1$ and $\varepsilon_{22}=-1$ to define a pseudo-Hermitian metric of signature $(2,2)$, and $\varepsilon_{11}=\varepsilon_{22}=-1$ to define a pseudo-Hermitian metric of signature $(4,0)$. We take $\varepsilon_{11}=\varepsilon_{22}=1$ (and change the sign on ∂_{x_2} and ∂_{x_4}) to define a para-Hermitian metric.

LEMMA 2. *Let $f = f(x_1,x_3)$ be a smooth function on $\mathbb{R}^4$. Perturb the metric of* Equation (4.3) *to define:*

$$g_f := \varepsilon_{11}e^{2f}(dx^1\otimes dx^1 \mp dx^2\otimes dx^2) + \varepsilon_{22}(dx^3\otimes dx^3 \mp dx^4\otimes dx^4)\,.$$

This is a para/pseudo-Hermitian metric on $\mathbb{R}^4$. Apply Theorem 10 *to choose ∇ so $(M,g_f,J_\pm,\nabla)$ is a Kähler–Weyl structure. Then $\rho_a = \pm4\partial_{x_1}\partial_{x_3}f\,dx^1\wedge dx^3$.*

Proof. We apply Equation (4.2) to see

$$(\nabla^{g_f}\Omega_\pm)(\partial_{x_1},\partial_{x_3};\partial_{x_k}) = \left\{\begin{array}{rll} \mp\varepsilon_{11}e^{2f}\partial_{x_3}f & \text{if} & k=2 \\ 0 & \text{if} & k\neq 2 \end{array}\right\}.$$

We apply Equation (2.2) to see that the non-zero components of $\nabla^{g_f}\Omega_\pm$ are given, up to the $\mathbb{Z}_2$ symmetry in the first components, by:

$$(\nabla^{g_f}\Omega_\pm)(\partial_{x_1},\partial_{x_3};\partial_{x_2}) = \mp\varepsilon_{11}e^{2f}\partial_{x_3}f, \quad (\nabla^{g_f}\Omega_\pm)(\partial_{x_1},\partial_{x_4};\partial_{x_1}) = \pm\varepsilon_{11}e^{2f}\partial_{x_3}f,$$
$$(\nabla^{g_f}\Omega_\pm)(\partial_{x_2},\partial_{x_4};\partial_{x_2}) = -\varepsilon_{11}e^{2f}\partial_{x_3}f, \quad (\nabla^{g_f}\Omega_\pm)(\partial_{x_2},\partial_{x_3};\partial_{x_1}) = \pm\varepsilon_{11}e^{2f}\partial_{x_3}f.$$

We contract indices and apply Theorem 10 to see:

$$\phi = \pm\tfrac{1}{2}J_\pm^*\delta\Omega_\pm = \pm\tfrac{1}{2}J_\pm^*\tau_1(\nabla^{g_f}\Omega_\pm) = \pm J_\pm^*\{\mp\partial_{x_3}f\cdot dx^4\} = \mp\partial_{x_3}f\cdot dx^3\,.$$

Since $f = f(x_1,x_3)$, the desired conclusion now follows from Equation (1.3). □

4.1. The proof of Theorem 3

Let $m=4$. We apply Lemma 1, Theorem 6, and Theorem 8. Let ξ be an irreducible $\mathcal{U}_\pm^\star$ submodule of $\mathfrak{K}_{\pm,\mathfrak{W}}$. If ξ is not isomorphic to a submodule of Λ^2, then ξ must be a submodule of $\mathfrak{R}$ and hence

$$\xi\subset\mathfrak{R}\cap\mathfrak{K}_\pm = W_{\pm,1}\oplus W_{\pm,2}\oplus W_{\pm,3}\,.$$

Since the modules $W_{\pm,i}$ are inequivalent and irreducible for $i=1,2,3$, we have $\xi = W_{\pm,i}$ for $i=1,2,3$.

We therefore suppose that ξ is isomorphic to a submodule of Λ^2. If $\psi\in\Lambda^2$, set:

$$\begin{aligned}\Xi(\psi)(x,y,z,w) := {}& 2\psi(x,y)\langle z,w\rangle + \psi(x,z)\langle y,w\rangle - \psi(y,z)\langle x,w\rangle \\ & - \psi(x,w)\langle y,z\rangle + \psi(y,w)\langle x,z\rangle\,.\end{aligned}$$

We then have [14, 15, 24] that the module L of Theorem 6 is the image of Ξ. Suppose that $\xi\approx\chi$. Then ξ appears with multiplicity 1 and thus

$$\xi = W_{\pm,11} = \Xi(\Omega_\pm)\cdot\mathbb{R}\,.$$

Let $J_\pm$ be the (para)-complex structure on $\mathbb{R}^4$ given in Equation (4.1) and let g be the metric of Equation (4.3). We show $\Xi(\Omega_\pm)$ is not a Kähler tensor and thus $\xi\not\approx\chi$ by computing:

$$\begin{aligned}\Xi(\Omega_\pm)(e_1,e_4,e_3,e_1) &= -g(e_4,J_\pm e_3)g(e_1,e_1) = -g_{11}g_{44},\\ \mp\Xi(\Omega_\pm)(e_1,e_4,J_\pm e_3,J_\pm e_1) &= \pm g(e_1,J_\pm J_\pm e_1)g(e_4,J_\pm e_3) = g_{11}g_{44}\,.\end{aligned}$$

Let $f = \pm\frac{1}{4}x_1x_3$. By Lemma 2, $\rho_a(f) = dx^1\wedge dx^3$; this is perpendicular to $\Omega_\pm$. Clearly $\rho_a(f)$ has non-trivial components in both $\Lambda^2_{0,\mp,J_\pm}$ and $\Lambda^2_{\pm,J_\pm}$. By Lemma 1, this means that both of the modules $\Lambda^2_{\pm,J_\pm}$ and $\Lambda^2_{0,\mp,J_\pm}$ appear with multiplicity at least 1 in $\mathfrak{K}_{\pm,\mathfrak{W}}$. By Theorem 8, $\Lambda^2_{0,\mp,J_\pm}$ appears with multiplicity 1 in $\mathfrak{W}$. Thus $\Lambda^2_{0,\mp,J_\pm}$ appears with multiplicity 1 in $\mathfrak{K}_{\pm,\mathfrak{W}}$. Since $\Lambda^2_{\pm,J_\pm}$ appears with multiplicity 2 in $\mathfrak{W}$ and since $W_{\pm,9}\approx\Lambda^2_{\pm,J_\pm}\subset\mathfrak{R}$ does not appear in $\mathfrak{K}_{\mathfrak{R}}$, we conclude that $\Lambda^2_{\pm,J_\pm}$ appears with multiplicity 1 in $\mathfrak{K}_{\pm,\mathfrak{W}}$. Theorem 3 now follows. □

4.2. The proof of Theorem 4

Let $m = 4$. Consider the space $\mathcal{S}$ of all germs of para/pseudo-Hermitian metrics g on $\mathbb{R}$ with the canonical (para)-complex structure given in Equation (4.1) so that $g(0) = g_0$ is the inner product of Equation (4.3).

$$g(0) = \varepsilon_{11}(dx^1 \otimes dx^1 \mp dx^2 \otimes dx^2) + \varepsilon_{22}(dx^3 \otimes dx^3 \mp dx^4 \otimes dx^4)$$

and so that $dg(0) = 0$. We let ∇ be the associated Kähler–Weyl connection and let $R = R(0)$. Let $\tilde{\mathfrak{K}}_{\pm,\mathfrak{W}}$ be the range of this map; this is $\mathcal{U}_{\pm}^{\star}$ module. Results of [3] in the Kähler setting show every element of $\mathfrak{K}_{\pm,\mathfrak{R}}$ can be geometrically realized by such a Kähler metric; set $\nabla = \nabla^g$ to take the trivial Weyl structure. Thus $W_{\pm,i} \subset \tilde{\mathfrak{K}}_{\pm,\mathfrak{W}}$ for $i = 1,2,3$. Lemma 2 shows $\tilde{\mathfrak{K}}_{\pm,\mathfrak{W}}$ contains submodules isomorphic to $\Lambda^2_{\pm,J_\pm}$ and to $\Lambda^2_{0,\mp,J_\pm}$. We may now apply Theorem 3 to conclude $\tilde{\mathfrak{K}}_{\pm,\mathfrak{W}} = \mathfrak{K}_{\pm,\mathfrak{W}}$ and to complete the proof. □

5. The proof of Theorem 5

Let $\mathcal{G}_\pm$ be the Gray symmetrizer defined in Equation (1.6). Then $\frac{1}{8}\mathcal{G}_\pm$ is orthogonal projection on the $\mathcal{U}_{\pm}^{\star}$ module $W_{\pm,7}$ appearing in Theorem 8 [4, 24]. Let $(M, g, J_\pm)$ be a para/pseudo-Hermitian manifold and let ∇ be a torsion free connection such that $\nabla g = -2\phi \otimes g$. Choose $f \in C^\infty(M)$ so that $df(P) = \phi(P)$. If we replace g by the conformally equivalent metric $\tilde{g} = e^{2f} g$, then we replace ϕ by $\tilde{\phi} = \phi - df$. Thus without loss of generality, we may assume that $\phi(P) = 0$. The map $\phi \to R^\nabla(P) - R^g(P)$ is then a linear map in the second derivatives of ϕ and can be regarded as defining a map $\Theta : \otimes^2 T_P^* M \to \mathfrak{W}_P$. Since $W_{\pm,7}$ is not isomorphic to any $\mathcal{U}_{\pm}^{\star}$ submodule of $\otimes^2 T_P^* M$, we may apply Lemma 1 to see that $\mathcal{G} \circ \Theta = 0$ and thus $\mathcal{G}_\pm(R^\nabla) = \mathcal{G}_\pm(R^g)$. Since $J_\pm$ is integrable, $\mathcal{G}_\pm(R^g) = 0$ [4, 12]. □

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AMS Subject Classification: 53B05, 15A72, 53A15, 53C07, 53C25

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Lavoro pervenuto in redazione il 04.07.2012.