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ON WEAKLY-NOETHERIAN RINGS

Abstract. In this paper, we introduce a weak version of Noetherianity that we call weakly-Noetherian property and we study the transfer of weakly-Noetherian property to the trivial ring extensions, to the direct product of rings, and to the amalgamated duplication of a ring along an ideal. We also exhibit several examples of rings which are weakly-Noetherian and are not Noetherian.

1. Introduction

Throughout this paper, all rings are commutative with identity element, and all modules are unital. Recall that a ring R is Noetherian if all ideals of R are finitely generated. So we are lead to ask the following question: Is R Noetherian if all finitely generated ideals of R are Noetherian R-modules? In view of this we introduce the concept of "weakly-Noetherian ring". A ring R is called weakly-Noetherian if all finitely generated ideals of R are Noetherian R-modules. Equivalently, a ring is weakly-Noetherian if for any pair of ideals I and J such that $I \subseteq J$ and J is a finitely generated proper ideal, then I is finitely generated. A Noetherian ring is naturally a weakly-Noetherian ring. Observe that the definition of weakly Noetherian ring by Hinohara in [7] is different from the one given in this paper.

Our aim in this paper is to prove that weakly-Noetherian rings are not Noetherian, in general.

Let A be a ring, E be an $A-$module and $R := A \propto E$ be the set of pairs (a,e) with pairwise addition and multiplication given by $(a,e)(a',e') = (aa', ae' + a'e)$. R is called the trivial ring extension of A by E. An ideal of R of the form $I \propto IE$, where I is an ideal of A, is finitely generated if and only if I is finitely generated ([5], page 141). Trivial ring extensions have been studied extensively; the basic properties of the trivial ring extensions are summarized in Glaz's book [5] and Huckaba's book [8]. These extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. See for instance [1, 5, 8, 9].

Let R be a ring and I be a proper ideal of R. The amalgamated duplication of a ring R along an ideal I is a subring of $R \times R$, defined by:

$$R \bowtie I = \{(r, r+i) \mid r \in R, i \in I\}.$$

This extension has been studied, in the general case, and from the different point of view of pullbacks, by D'Anna and Fontana in [4, 3], they have considered the case of the amalgamated duplication of a ring, is a non necessarily Noetherian setting, along a multiplicative-canonical ideal in the sense Heinzer-Huckaba-Papick [6]. In [2], D'Anna has studied some properties of $R \bowtie I$, in order to construct reduced Gorenstein rings associated to Cohen-Macaulay rings.

In this paper, we investigate the possible transfer of weakly-Noetherian property to various trivial extension constructions. Also, we study the direct product of rings with the weakly-Noetherian property. Finally, we examine the transfer of weakly-Noetherian property to the amalgamated duplication of ring along an ideal. Using these results, we construct several classes of examples of non-Noetherian weakly-Noetherian rings.

2. Main Results

We begin this paper by giving an example of a weakly-Noetherian ring which is not a Noetherian ring.

Example 1. Let K be a field, $E := K^{\infty}$ be a K-vector space of infinite rank and let $R := K \propto E$ be the trivial ring extension of K by E. Then:
(1) R is a local weakly-Noetherian ring.
(2) R is not a Noetherian ring.

Proof. (1) Remark that all proper ideal of R has the form $0 \propto E'$, where E' is a K-vector subspace of E (since (a,e) is invertible in R if and only if a is invertible in K (by [8, Theorem 25.1]), that is $a \neq 0$).
Let $I \subseteq J$ be two ideals of R such that J is a finitely generated proper ideal. We claim that I is finitely generated. Indeed, let $J = 0 \propto E'$ be a finitely generated proper ideal of R, where E' is a finitely generated K-vector subspace of E. Then, $I = 0 \propto E$", where E" is a K-vector subspace of E'. Hence, E" is a finitely generated K-vector space and so $I := 0 \propto E$" is a finitely generated ideal of R. Therefore, R is a weakly-Noetherian ring.
(2) R is not Noetherian since its maximal ideal $M = 0 \propto E$ is not finitely generated (since E is a K-vector space of infinite rank). □

Now, we give a sufficient condition to have the equivalence between weakly-Noetherian and Noetherian properties.

Theorem 1. *Let R be a ring. Then:*
(1) *If R is a Noetherian ring, then it is a weakly-Noetherian ring.*
(2) *Assume that R contains a regular element (i.e., neither a unit nor a zerodivisor). Then, R is a Noetherian ring if and only if R is a weakly-Noetherian ring.*
(3) *Assume that* (R,M) *is a local ring, where M is its maximal ideal. Then, R is a*

Noetherian ring if and only if R is a weakly-Noetherian ring and M is a finitely generated ideal.

Proof. (1) Straightforward.

(2) If R is a Noetherian ring, then it is a weakly-Noetherian ring by (1). Conversely, assume that R is a weakly-Noetherian ring and let I be a proper ideal of R. We claim that I is finitely generated. Indeed, let a be a regular element of R. Then $aI \subseteq aR$, where aR is a finitely generated principal proper ideal of R, and so aI is a finitely generated ideal of R (since R is a weakly-Noetherian ring). It follows that, I is a finitely generated ideal of R since $aI \cong I$ (since a is a regular element of R). Therefore, R is a Noetherian ring.

(3) By (1), only the sufficiency has to be proved. Assume that R is weakly-Noetherian and let J be a proper ideal of A. Hence, $J \subseteq M$ (since A is a local ring) and so J is finitely generated (since M is a finitely generated proper ideal of A and A is a weakly-Noetherian ring). Therefore, A is a Noetherian ring. □

Next, we examine the transfer of weakly-Noetherian property to trivial ring extensions.

Theorem 2. *Let A be a ring, E be an A-module, and* $R := A \propto E$ *be the trivial ring extension of A by E. Then:*
(1) (*a*) *If R is a weakly-Noetherian ring, then so is A.*
(*b*) *Assume that E is a Noetherian A-module. Then R is a weakly-Noetherian ring if and only if so is A .*
(2) *R is Noetherian if and only if A is Noetherian and E is a finitely generated A-module.*

Before proving the previous Theorem, we need some Lemmas.

Lemma 1. *Let A be a ring and let I be an ideal of A. Then:*
(1) *If A is a weakly-Noetherian ring and I is a finitely generated ideal, then* $\frac{A}{I}$ *is a weakly-Noetherian ring.*
(2) *If* $\frac{A}{I}$ *is a weakly-Noetherian ring and I is a Noetherian A-module, then A is a weakly-Noetherian ring.*

Proof. (1) Assume that A is weakly-Noetherian and I is a finitely generated ideal. Let $\frac{J_1}{I} \subseteq \frac{J_2}{I}$ be two ideals of A/I such that J_2/I is a finitely generated proper ideal of A/I, where $I \subseteq J_1 \subseteq J_2$ are ideals of A. Hence, J_2 is finitely generated since I is finitely generated. Then, J_1 is finitely generated since $J_1 \subseteq J_2$ and A is weakly-Noetherian. Therefore, $\frac{J_1}{I}$ is a finitely generated ideal of A/I and A/I is weakly-Noetherian.

(2) Assume that $\frac{A}{I}$ is weakly-Noetherian and I is a Noetherian A-module. Let $I_1 \subseteq I_2$ be two ideals of A such that I_2 is a finitely generated proper ideal of A. We claim that I_1 is finitely generated. Set $J_1 = I_1 + I$ and $J_2 = I_2 + I$ be two ideals of A which contain I. Hence, $\frac{J_2}{I}$ is a finitely generated ideal since $\frac{J_2}{I} \cong \frac{I_2}{I_2 \cap I}$ and I_2 is finitely generated. Then, $\frac{J_1}{I}$ is finitely generated since $\frac{J_1}{I} \subseteq \frac{J_2}{I}$ and $\frac{A}{I}$ is weakly-Noetherian. But $I_1 \cap I$ is finitely generated since $I_1 \cap I \subseteq I$ and I is a Noetherian A-module. Therefore, I_1 is a finitely generated ideal since $\frac{J_1}{I} \cong \frac{I_1}{I_1 \cap I}$). Hence, A is a weakly-Noetherian ring. □

Lemma 2. *[10, Theorem 8, p. 5]*
Let R be a ring, then R is a Noetherian ring if and only if every prime ideal in R is finitely generated.

Proof of Theorem 2. (1) (a) Assume that R is a weakly-Noetherian ring. Let $I_1 \subseteq I_2$ be two ideals of A such that I_2 is a finitely generated proper ideal of A. Our aim is to show that I_1 is a finitely generated ideal of A. Hence, $I_2 \propto I_2E$ is a finitely generated ideal of R and so $I_1 \propto I_1E$ is a finitely generated ideal of R (since $I_1 \propto I_1E \subseteq I_2 \propto I_2E$ and R is a weakly-Noetherian ring). Therefore, I_1 is a finitely generated ideal of A and so A is a weakly-Noetherian ring.
(b) If R is weakly-Noetherian, then so is A by (1). Conversely, assume that A is weakly-Noetherian and E is an A-module Noetherian. Then $0 \propto E$ is an R-module Noetherian and so R is weakly-Noetherian by Lemma 1 (2) (since $\frac{R}{0 \propto E} \cong A$), as desired.
(2) Assume that R is Noetherian and let I be an ideal of A. Then, $J = I \propto E$ is a finitely generated ideal of R since R is Noetherian. Set $J := \sum_{i=1}^{n} R(a_i, e_i)$ where $a_i \in I$ and $e_i \in E$ for all i. Then, $I = \sum_{i=1}^{n} Aa_i$ is a finitely generated ideal of A and so A is a Noetherian ring.
Now, we show that E is a finitely generated A-module. The ideal $J := 0 \propto E$ of R is finitely generated. So, there exists $(0, e_i) \in J$ such that $J = \sum_{i=1}^{n} R(0, e_i) = 0 \propto \sum_{i=1}^{n} Ae_i$. Then, $E = \sum_{i=1}^{n} Ae_i$ is a finitely generated A-module, as desired.
Conversely, assume that A is a Noetherian ring and E is a finitely generated A-module. We wish to show that R is a Noetherian ring. For that, let us consider a prime ideal J of R and prove that J is finitely generated. By ([8],Theorem 25.1.3), there exists a prime ideal I of A such that $J := I \propto E$. Let $I = \sum_{i=1}^{n} Aa_i$ for some $a_i \in I$ since A is a Noetherian ring and let $E := \sum_{i=1}^{n} Ae_i$ for some $e_i \in E$ as it is a finitely generated A-module. Therefore, it is clear that $J = \sum_{i=1}^{n} R(a_i, 0) + \sum_{i=1}^{n} R(0, e_i)$ and this completes the proof of Theorem 2. □

The following Corollary is an immediate consequence of Theorem 2.

Corollary 1. *Let D be a domain, $K := qf(D)$, E be a K-vector space, and $R := D \propto E$ be the trivial ring extension of D by E. Then:*
(1) *R is a weakly-Noetherian ring if and only if D is a field.*
(2) *R is Noetherian if and only if D is a field and E is a K-vector space with finite rank.*

Proof. Let R be a weakly-Noetherian ring. We claim that D is a field. Deny. Let d be

a regular element of D. Then $(d,0)$ is a regular element of R and so R is Noetherian by Theorem 1 (2) since it is weakly-Noetherian, a contradiction with [6, Theorem 2.8 (1)]. Hence, D is a field, as desired.
(2) Straightforward. □

PROPOSITION 1. *Let (A,M) be a local ring where M is its maximal ideal, E be a finitely generated A-module with $ME = 0$, and $R := A \propto E$ be the trivial ring extension of A by E. Then R is a weakly-Noetherian ring if and only if A is a weakly-Noetherian ring.*

Before proving Proposition 1, we need the following Lemma.

LEMMA 3. *Let (A,M) be a local ring where M is its maximal ideal, and E be a finitely generated A-module with $ME = 0$. Then, E is an A-module Noetherian.*

Proof. Let F be an A-submodule of E. We claim that F is a finitely generated A-module. Indeed, F is an (A/M)-vector subspace of E since $MF \subseteq ME = 0$. Hence, F is a finitely generated (A/M)-vector space as E and so F is a finitely generated A-module. Therefore, E is an A-module Noetherian, as desired. □

Proof of Proposition 1. If R is weakly-Noetherian, then so is A by Theorem 1 (1). Conversely, if A is weakly-Noetherian, then so is R by Theorem 2 (1) (b) and Lemma 3. □

Proposition 1 enriches the literature with new examples of non-Noetherian weakly-Noetherian rings, as shown below.

EXAMPLE 2. Let K be a field and $R = (K \propto K^{\infty}) \propto (\frac{K \propto K^{\infty}}{0 \propto K^{\infty}})$. Then:
(1) R is a weakly-Noetherian ring by Proposition 1 since $K \propto K^{\infty}$ is weakly-Noetherian.
(2) R is not Noetherian by Theorem 2 (2) since $K \propto K^{\infty}$ is not Noetherian.

We know that a Noetherian ring is weakly-Noetherian and coherent ring too. The following examples show that there is no relationship between weakly-Noetherian and coherent properties.

EXAMPLE 3. Let K be a field, $X_1, X_2, ...$ be an indeterminates over K, and $R := K[[X_1, X_2, X_3 ... X_n, ...]]$ be the ring of power series in $X_1, X_2, ...$ over K. Then:
(1) R is coherent.
(2) R is not weakly-Noetherian.

Proof. (1) R is a coherent domain by [[5], Corollary 2.3.4, p.48].
(2) By Theorem 1 (2) since R is a non-Noetherian domain. □

EXAMPLE 4. Let K be a field and $R := K \propto K^{\infty}$. Then:

(1) R is weakly-Noetherian by Example 2.1.
(2) R is not coherent by [11, Theorem 3.4].

In the polynomial ring, we have:

Proposition 2. *Let R be a ring and X be an indeterminate over R. Then $R[X]$ is weakly-Noetherian if and only if R is Noetherian.*

Proof. Assume that $R[X]$ is weakly-Noetherian. Then, $R[X]$ is Noetherian since X is a regular element of $R[X]$ and so R is Noetherian. The converse is clear. □

Next, we study the transfer of the weakly-Noetherain propery to direct products.

Proposition 3. *Let $(R_i)_{i=1,..,n}$ be a family of ring. Then, $\prod_{i=1}^{n} R_i$ is weakly-Noetherian if and only if R_i is Noetherian for each $i = 1,\dots,n$ (i.e., if and only if $\prod_{i=1}^{n} R_i$ is Noetherian).*

Proof. By induction on n, it suffices to prove the assertion for $n = 2$.
If R_1 and R_2 are Noetherian, then $R_1 \times R_2$ is Noetherian and so weakly-Noetherian.
Conversely, assume that $R_1 \times R_2$ is weakly-Noetherian. We claim that R_1 is Noetherian (the same proof holds for R_2). Indeed, let I be a proper ideal of R_1. Then, $I \times 0 \subseteq R_1 \times 0$ are two proper ideals of $R_1 \times R_2$. Hence, $I \times 0$ is a finitely generated ideal of $R_1 \times R_2$ since $R_1 \times R_2$ is weakly-Noetherian and $R_1 \times 0$ is a finitely generated ideal of $R_1 \times R_2$. Therefore, I is a finitely generated ideal of R_1 and this completes the proof of Theorem 2.13.

□

Let R be a ring and I be a proper ideal of R. The amalgamated duplication of a ring R along an ideal I is a subring of $R \times R$, defined by $R \bowtie I := \{(r, r+i)/r \in R, i \in I\}$. It is easy to see that, if Π_i $(i = 1,2)$ are the projection of $R \times R$ on R, then $\Pi_i(R \bowtie I) = R$. Hence, if $O_i = ker(\Pi_i|_{R \bowtie I})$, then $(R \bowtie I)/O_i \cong R$. Moreover, $O_1 = \{(0,i)/i \in I\}$, $O_2 = \{(i,0)/i \in I\}$ and $O_1 \cap O_2 = (0)$.
As consequence of the previous fact we have the following result.

Theorem 3. *Let R be a ring, I be a proper ideal of R, and $R \bowtie I$ the amalgamated duplication of a ring R along I. Then:*
(1) *If $R \bowtie I$ is a weakly-Noetherian ring and I is a finitely generated ideal of R, then R is a weakly-Noetherian ring.*
(2) *If R is a weakly-Noetherian ring and I is a Noetherian R-module, then $R \bowtie I$ is weakly-Noetherian.*
(3) *Assume that R contains a regular element. Then R is weakly-Noetherian if and only if $R \bowtie I$ is Noetherian.*

Proof. (1) Assume that $R \bowtie I$ is weakly-Noetherian. Then R is weakly-Noetherian by Lemma 2.4(1) since O_1 (or since O_2) is finitely generated ideal of $R \bowtie I$ (since I is finitely generated ideal of R) and $(R \bowtie I)/O_i \cong R$ for $i = 1,2$.
(2) Suppose R is weakly-Noetherian. Then $R \bowtie I$ is weakly-Noetherian by Lemma 1 (2) since O_1 (or since O_2) is Noetherian $(R \bowtie I)$-modules (because I is a Noetherian R-module) and $(R \bowtie I)/O_i \cong R$ for $i = 1,2$.
(3) Assume that R is weakly-Noetherian. Then, R is Noetherian since it contains a regular element and so $R \bowtie I$ is Noetherian. The converse is clear. □

Theorem 3 enriches the literature with new examples of non-Noetherian weakly-Noetherian rings, as shown below.

Example 5. Let K be a field, $R = K \propto K^{\infty}$, and let $I := 0 \propto K^{\infty}$. Then:
(1) $R \bowtie I$ is a weakly-Noetherian ring by Theorem 3 (2) since $K \propto K^{\infty}$ is weakly-Noetherian and I is a Noetherian R-module.
(2) $R \bowtie I$ is not Noetherian since R is not Noetherian.

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