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SOME NEW CRITERIA FOR STARLIKENESS AND CONVEXITY OF ANALYTIC FUNCTIONS

Abstract. We, here, study a differential inequality involving a multiplier transformation and obtain certain new criteria for starlikeness and convexity of p-valent and univalent analytic functions.

1. Introduction

A function f is said to be analytic at a point z in a domain $\mathbb{D}$ if it is differentiable not only at z but also in some neighborhood of the point z. A function f is said to be analytic in a domain $\mathbb{D}$ if it is analytic at each point of $\mathbb{D}$. Let $\mathcal{A}$ be the class of all functions f which are analytic in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions that $f(0) = f'(0) - 1 = 0$. Thus, $f \in \mathcal{A}$ has the Taylor series expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, p \in \mathbb{N} = \{1, 2, 3, \cdots\},$$

analytic and multivalent in the open unit disk $\mathbb{E}$. Note that $\mathcal{A}_1 = \mathcal{A}$. For $f \in \mathcal{A}_p$, define the multiplier transformation $I_p(n,\lambda)$ as follows:

$$I_p(n,\lambda)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k, \ (\lambda \geq 0, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

The special case $I_1(n,0)$ of above defined operator is the well-known Sălăgean [4] derivative operator D^n, defined for $f \in \mathcal{A}$ as under:

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k.$$

A function $f \in \mathcal{A}_p$ is said to be p-valent starlike of order α $(0 \leq \alpha < p)$ in $\mathbb{E}$, if it satisfies the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \ z \in \mathbb{E}.$$

Let $\mathcal{S}_p^*(\alpha)$ denote the class of all p-valent starlike functions of order α $(0 \le \alpha < p)$. A function $f \in \mathcal{A}_p$ is said to be p-valent convex of order α $(0 \le \alpha < p)$ in $\mathbb{E}$, if it satisfies the inequality

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha,\ z \in \mathbb{E}.$$

We denote by $\mathcal{K}_p(\alpha)$, the class of all functions $f \in \mathcal{A}_p$ that are p-valent convex of order α $(0 \le \alpha < p)$ in $\mathbb{E}$. Note that $\mathcal{S}^*(\alpha) = \mathcal{S}_1^*(\alpha)$ and $\mathcal{K}(\alpha) = \mathcal{K}_1(\alpha)$ are the usual classes of univalent starlike functions (w.r.t. the origin) of order α $(0 \le \alpha < 1)$ and univalent convex functions of order α $(0 \le \alpha < 1)$.

For two analytic functions f and g in the unit disk $\mathbb{E}$, we say that f is subordinate to g in $\mathbb{E}$ and write as $f \prec g$ if there exists a Schwarz function w analytic in $\mathbb{E}$ with $w(0) = 0$ and $|w(z)| < 1,\ z \in \mathbb{E}$ such that $f(z) = g(w(z)),\ z \in \mathbb{E}$. In case the function g is univalent, the above subordination is equivalent to: $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

Obradovič [2], introduced and studied the class $\mathcal{N}(\alpha),\ 0 < \alpha < 1$ of functions $f \in \mathcal{A}$ satisfying the following inequality

$$\Re\left\{f'(z)\left(\frac{z}{f(z)}\right)^{1+\alpha}\right\} > 0,\ z \in \mathbb{E}.$$

He called it as the class of non-Bazilevič functions.

In 2005, Wang et al. [5] introduced the generalized class $\mathcal{N}(\lambda, \alpha, A, B)$ of non-Bazilevič functions which is analytically defined as under:

$$\mathcal{N}(\lambda,\alpha,A,B) = \left\{f \in \mathcal{A} : (1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha} - \lambda\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha} \prec \frac{1+Az}{1+Bz},\right\}$$

where $0 < \alpha < 1,\ \lambda \in \mathbb{C},\ -1 \le B \le 1,\ A \ne B,\ A \in \mathbb{R}$.

Wang et al. [5] studied the class $\mathcal{N}(\lambda, \alpha, A, B)$ and made some estimates on $\left(\frac{z}{f(z)}\right)^{\alpha}$.

Using the concept of differential subordination, Shanmugam et al. [3] studied the differential operator $(1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha} - \lambda\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha}$ and obtained the best dominant for $\left(\frac{z}{f(z)}\right)^{\alpha}$.

Differential inequalities play an important role in the theory of analytic functions. A number of criteria for starlikeness and convexity of analytic have been developed in terms of differential inequalities. It has always been a matter of interest for the researchers either to find a new criterion for starlikeness and convexity of analytic functions or to generalize or improve certain known ones. Keeping this in mind, we, here, study a differential inequality involving the multiplier transformation $I_p(n, \lambda)$. The main objective of this paper is to make estimates on $\frac{zf'(z)}{f(z)}$ and $1+\frac{zf''(z)}{f(z)}$ in terms of certain differential inequalities and consequently obtain certain new criteria for starlikeness and convexity of functions $f \in \mathcal{A}_p$.

To prove our main result, we shall make use of following lemma due to Hallenbeck and Ruscheweyh [1].

LEMMA 1. *Let G be a convex function in $\mathbb{E}$, with $G(0) = a$ and let γ be a complex number, with $\Re(\gamma) > 0$. If $F(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$, is analytic in $\mathbb{E}$ and $F \prec G$, then*

$$\frac{1}{z^{\gamma}} \int_0^z F(w) w^{\gamma-1}\, dw \prec \frac{1}{n z^{\gamma/n}} \int_0^z G(w) w^{\frac{\gamma}{n}-1}\, dw$$

2. Main Results

THEOREM 1. *Let α, β, δ be real numbers such that $\alpha > 0$, $\beta > 0$, $0 \le \delta < 1$, and let*

$$0 < M \equiv M(\alpha,\beta,\lambda,\delta,p) = \frac{\alpha(1-\delta)[\alpha+\beta(p+\lambda)]}{\alpha[1+\beta(p+\lambda)(1-\delta)]+2\beta(p+\lambda)}, \tag{2.1}$$

If $f \in \mathcal{A}_p$ satisfies the differential inequality

$$\left|\left(\frac{z^p}{I_p(n,\lambda)f(z)}\right)^{\beta}\left[1+\alpha-\alpha\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right]-1\right| < M(\alpha,\beta,\lambda,\delta,p),\ z \in \mathbb{E}, \tag{2.2}$$

then

$$\Re\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right) > \delta,\ z \in \mathbb{E}.$$

Proof. Define

$$\left(\frac{z^p}{I_p(n,\lambda)f(z)}\right)^{\beta} = u(z),\ z \in \mathbb{E}.$$

Differentiate logarithmically, we obtain

$$p - \frac{zI'_p(n,\lambda)f(z)}{I_p(n,\lambda)f(z)} = \frac{zu'(z)}{\beta u(z)} \tag{2.3}$$

In view of the equality

$$zI'_p(n,\lambda)f(z) = (p+\lambda)I_p(n+1,\lambda)f(z) - \lambda I_p(n,\lambda)f(z),$$

(2.3) turns to

$$\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} = 1 - \frac{zu'(z)}{\beta(p+\lambda)u(z)}$$

Therefore, in view of (2.2), we have

$$u(z) + \frac{\alpha}{\beta(p+\lambda)} zu'(z) \prec 1 + Mz. \tag{2.4}$$

Using Lemma 1 $\left(\text{taking } \gamma = \dfrac{\beta(p+\lambda)}{\alpha}\right)$ from (2.4), we have

$$u(z) \prec 1 + \frac{\beta(p+\lambda)Mz}{\alpha+\beta(p+\lambda)},$$

or

$$|u(z)-1| < \frac{\beta(p+\lambda)M}{\alpha+\beta(p+\lambda)} < 1,$$

and therefore, we have

$$|u(z)| > 1 - \frac{\beta(p+\lambda)M}{\alpha+\beta(p+\lambda)} \tag{2.5}$$

Write $\dfrac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} = (1-\delta)w(z)+\delta,\ 0 \le \delta < 1$ and therefore (2.2) reduces to

$$|u(z)\{1+\alpha-\alpha[(1-\delta)w(z)+\delta]\}-1| < M.$$

We need to show that $\Re(w(z)) > 0,\ z \in \mathbb{E}$. If possible, suppose that $\Re(w(z)) \not> 0,\ z \in \mathbb{E}$, then there must exist a point $z_0 \in \mathbb{E}$ such that $w(z_0) = ix, x \in \mathbb{R}$. To prove the required result, it is now sufficient to prove that

$$|u(z_0)\{1+\alpha-\alpha[(1-\delta)ix+\delta]\}-1| \ge M. \tag{2.6}$$

By making use of (2.5), we have

$$\begin{aligned}
&|u(z_0)\{1+\alpha-\alpha[(1-\delta)ix+\delta]\}-1| \\
&\ge |[1+\alpha(1-\delta)-\alpha(1-\delta)ix]\,u(z_0)|-1 \\
&= \sqrt{[1+\alpha(1-\delta)]^2+\alpha^2(1-\delta)^2x^2}\ |u(z_0)|-1 \\
&\ge |1+\alpha(1-\delta)||u(z_0)|-1 \\
&\ge |1+\alpha(1-\delta)|\left(1-\frac{\beta(p+\lambda)M}{\alpha+\beta(p+\lambda)}\right)-1 \ge M. \qquad (2.7)
\end{aligned}$$

Now (2.7) is true in view of (2.1) and therefore, (2.6) holds. Hence $\Re(w(z)) > 0$ i.e.

$$\Re\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right) > \delta,\ 0 \le \delta < 1,\ z \in \mathbb{E}.$$

□

Remark 1. From Theorem 1, it follows, if $\alpha,\ \beta,\ \delta$ are real numbers such that $\alpha > 0,\ \beta > 0,\ 0 \le \delta < 1$ and if $f \in \mathcal{A}_p$ satisfies

$$\left|\left(\frac{z^p}{I_p(n,\lambda)f(z)}\right)^{\beta}\left[\frac{1}{\alpha}+1-\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right]-\frac{1}{\alpha}\right| < \frac{(1-\delta)[\alpha+\beta(p+\lambda)]}{\alpha[1+\beta(p+\lambda)(1-\delta)]+2\beta(p+\lambda)},$$

then

$$\Re\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right) > \delta,\ z \in \mathbb{E}.$$

Letting $\alpha \to \infty$ in above remark, we get the following result.

THEOREM 2. *Let* β, δ *be real numbers such that* $\beta > 0$, $0 \le \delta < 1$ *and let* $f \in \mathcal{A}_p$ *satisfy*

$$\left| \left(\frac{z^p}{I_p(n,\lambda)f(z)} \right)^{\beta} \left(1 - \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \right) \right| < \frac{1-\delta}{1+\beta(p+\lambda)(1-\delta)},$$

then

$$\Re\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \right) > \delta,\ z \in \mathbb{E}.$$

For $p = 1$ and $\lambda = 0$ in Theorem 1, we get the following result involving Sălăgean operator.

THEOREM 3. *If* α, β, δ *are real numbers such that* $\alpha > 0$, $\beta > 0$, $0 \le \delta < 1$ *and if* $f \in \mathcal{A}$ *satisfies the differential inequality*

$$\left| \left(\frac{z}{D^n f(z)} \right)^{\beta} \left[1 + \alpha - \alpha \frac{D^{n+1} f(z)}{D^n f(z)} \right] - 1 \right| < \frac{\alpha(\alpha+\beta)(1-\delta)}{\alpha[1+\beta(1-\delta)]+2\beta},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{D^{n+1} f(z)}{D^n f(z)} \right) > \delta,\ z \in \mathbb{E}.$$

Select $p = 1$ and $\lambda = 0$ in Theorem 2, we obtain:

THEOREM 4. *If* β, δ *are real numbers such that* $\beta > 0$, $0 \le \delta < 1$ *and* $f \in \mathcal{A}$ *satisfies*

$$\left| \left(\frac{z}{D^n f(z)} \right)^{\beta} \left(1 - \frac{D^{n+1} f(z)}{D^n f(z)} \right) \right| < \frac{1-\delta}{1+\beta(1-\delta)},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{D^{n+1} f(z)}{D^n f(z)} \right) > \delta,\ z \in \mathbb{E}.$$

3. Criteria for Starlikeness and Convexity

Setting $\lambda = n = 0$ in Theorem 1, we obtain the following result.

COROLLARY 1. *Let* α, β, δ *be real numbers such that* $\alpha > 0$, $\beta > 0$, $0 \le \delta < 1$ *and let* $f \in \mathcal{A}_p$ *satisfy the differential inequality*

$$\left| (1+\alpha) \left(\frac{z^p}{f(z)} \right)^{\beta} - \alpha \frac{z f'(z)}{p f(z)} \left(\frac{z^p}{f(z)} \right)^{\beta} - 1 \right| < \frac{\alpha(\alpha+p\beta)(1-\delta)}{\alpha[1+p\beta(1-\delta)]+2p\beta},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > p\delta = \gamma,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{S}_p^*(\gamma),\ 0 \leq \gamma < p.$

Writing $\beta = 1$ in above corollary, we obtain:

COROLLARY 2. *Suppose that* $\alpha,\ \delta$ *are real numbers such that* $\alpha > 0,\ 0 \leq \delta < 1$ *and suppose that* $f \in \mathcal{A}_p$ *satisfies*

$$\left|(1+\alpha)\frac{z^p}{f(z)} - \alpha\frac{z^{p+1}f'(z)}{p(f(z))^2} - 1\right| < \frac{\alpha(\alpha+p)(1-\delta)}{\alpha[1+p(1-\delta)]+2p},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > p\delta = \gamma,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{S}_p^*(\gamma),\ 0 \leq \gamma < p.$

Setting $n = 1$ and $\lambda = 0$ in Theorem 1, we obtain the following result.

COROLLARY 3. *Let* $\alpha,\ \beta,\ \delta$ *be real numbers such that* $\alpha > 0,\ \beta > 0,\ 0 \leq \delta < 1$ *and let* $f \in \mathcal{A}_p$ *satisfy the differential inequality*

$$\left|(1+\alpha)\left(\frac{pz^{p-1}}{f'(z)}\right)^{\beta} - \frac{\alpha}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\left(\frac{pz^{p-1}}{f'(z)}\right)^{\beta} - 1\right| < \frac{\alpha(\alpha+p\beta)(1-\delta)}{\alpha[1+p\beta(1-\delta)]+2p\beta},$$

then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > p\delta = \gamma,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{K}_p(\gamma),\ 0 \leq \gamma < p.$

Writing $\beta = 1$ in above corollary, we obtain:

COROLLARY 4. *If* $\alpha,\ \delta$ *are real numbers such that* $\alpha > 0,\ 0 \leq \delta < 1$ *and if* $f \in \mathcal{A}_p$ *satisfies*

$$\left|(1+\alpha)\frac{pz^{p-1}}{f'(z)} - \alpha\frac{z^{p-1}}{f'(z)}\left(1+\frac{zf''(z)}{f'(z)}\right) - 1\right| < \frac{\alpha(\alpha+p)(1-\delta)}{\alpha[1+p(1-\delta)]+2p},\ z \in \mathbb{E},$$

then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > p\delta = \gamma,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{K}_p(\gamma),\ 0 \leq \gamma < p.$

Writing $\lambda = n = 0$ in Theorem 2, we get:

Corollary 5. *If β, δ are real numbers such that $\beta > 0$, $0 \le \delta < 1$ and if $f \in \mathcal{A}_p$ satisfies*

$$\left|\left(\frac{z^p}{f(z)}\right)^{\beta}\left(1-\frac{zf'(z)}{pf(z)}\right)\right| < \frac{1-\delta}{1+p\beta(1-\delta)},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > p\delta = \gamma,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{S}_p^*(\gamma),\ 0 \le \gamma < p$.

Setting $\lambda = 0$ and $n = 1$ in Theorem 2, we obtain:

Corollary 6. *Assume that β, δ be real numbers such that $\beta > 0$, $0 \le \delta < 1$ and assume that $f \in \mathcal{A}_p$ satisfies*

$$\left|\left(\frac{pz^{p-1}}{f'(z)}\right)^{\beta}\left[1-\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\right]\right| < \frac{1-\delta}{1+p\beta(1-\delta)},\ z \in \mathbb{E},$$

then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > p\delta = \gamma,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{K}_p(\gamma),\ 0 \le \gamma < p$.

Taking $p = 1$ in Corollary 1, we get:

Corollary 7. *If α, β, δ are real numbers such that $\alpha > 0$, $\beta > 0$, $0 \le \delta < 1$ and if $f \in \mathcal{A}$ satisfies*

$$\left|(1+\alpha)\left(\frac{z}{f(z)}\right)^{\beta} - \alpha f'(z)\left(\frac{z}{f(z)}\right)^{1+\beta} - 1\right| < \frac{\alpha(\alpha+\beta)1-\delta)}{\alpha[1+\beta(1-\delta)]+2\beta},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \delta,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{S}^*(\delta)$.

Setting $p = 1$ in Corollary 3, we get:

Corollary 8. *If α, β, δ are real numbers such that $\alpha > 0$, $\beta > 0$, $0 \le \delta < 1$ and if $f \in \mathcal{A}$ satisfies*

$$\left|\left(\frac{1}{f'(z)}\right)^{\beta}\left[1+\alpha-\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right] - 1\right| < \frac{\alpha(\alpha+\beta)(1-\delta)}{\alpha[1+\beta(1-\delta)]+2\beta},\ z \in \mathbb{E},$$

then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \delta,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{K}(\delta)$.

Put $\lambda = p = 1$ and $n = 0$ in Theorem 1, we get:

Corollary 9. *Suppose that* α, β, δ *are real numbers such that* $\alpha > 0$, $0 \le \delta < 1$, $\beta > 0$ *and suppose that* $f \in \mathcal{A}$ *satisfies*

$$\left|\left(1+\frac{\alpha}{2}\right)\left(\frac{z}{f(z)}\right)^{\beta} - \frac{\alpha}{2}f'(z)\left(\frac{z}{f(z)}\right)^{1+\beta} - 1\right| < \frac{\alpha(\alpha+2\beta)(1-\delta)}{\alpha[1+2\beta(1-\delta)]+4\beta},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 2\delta - 1,\ z \in \mathbb{E}.$$

Put $\lambda = p = 1$ and $n = 0$ in Theorem 2, we obtain the following result.

Corollary 10. *If* $f \in \mathcal{A}$ *satisfies*

$$\left|\left(\frac{z}{f(z)}\right)^{\beta}\left(1 - \frac{zf'(z)}{f(z)}\right)\right| < \frac{2(1-\delta)}{1+2\beta(1-\delta)},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 2\delta - 1,\ z \in \mathbb{E},$$

where β, δ *are real numbers such that* $\beta > 0$, $0 \le \delta < 1$.

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