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A GLOBAL CONSTRUCTION OF ALGEBRAS OF GENERALIZED FUNCTIONS

Abstract. An original method for the construction of algebras of generalized functions is given, which covers both Colombeau simplified algebras and Rosinger algebras cases, the main used such algebras. Examples are given showing how this method works for the most known algebras in this area.

1. Introduction

There is no general method for the construction of algebras of generalized functions covering all the main used until now. The more frequently used algebras of generalized functions are those of Colombeau [1, 2] and Rosinger [9, 10] in that order. Colombeau algebras exist in two versions: The simplified type algebra and the full type algebra (see eg. [8, 5, 12]). Presently we are concerned with the simplified one. In [6] an analysis of the structure of Colombeau simplified algebras has been elaborated in [7] leading to the concept of the so-called $(\mathcal{C},\mathcal{E},\mathcal{P})-$algebras which generates a large range of algebras of generalized functions. The more known of them are those of Colombeau and Egorov [4]. Unfortunately Rosinger algebras of generalized functions are not covered by that framework. In fact from the view point of their construction, Colombeau and Rosinger algebras appear to be quite different. In the present work is given an new method for the construction of algebras of generalized functions that covers all the above mentioned algebras. To achieve this goal the new concept of $(\mathcal{M},\mathcal{N},V_{\mathcal{P}})-$algebra is introduced. This concept is based on the idea that algebras of generalized functions may be represented as a factor of *moderate* elements by the corresponding *null* ones as it is formulated in Colombeau's construction. It is shown that this method covers $(\mathcal{C},\mathcal{E},\mathcal{P})$ and the Rosinger type algebras. Nevertheless explicit constructions for Colombeau and Egorov algebras are also given. In the above notations $V_{\mathcal{P}}$ denotes an associative and commutative algebra V with $\mathcal{P}$ a family of families of maps defined in V and valued in a given ordered set F, $\mathcal{M}$ and $\mathcal{N}$ are sets of $F-$valued maps defined in a filtered set E.

2. Bounds spaces

2.1. Definitions

Let E denote a set equipped with a filter $\mathcal{F}$. Consider an ordered set $(F,\leq)$ where $\leq$ is not necessarily total, equipped with a multiplication, an addition and the multiplication by a nonnegative number. We suppose that the same rules as the ones of an associative and commutative algebra are fulfilled when the field is replaced by $\mathbb{R}_+^* = (0,\infty)$. In particular if $a \in F$ and $\lambda,\mu \in \mathbb{R}_+^*$, then $\lambda(\mu a) = (\lambda\mu)a$. Furthermore we assume the

compatibility properties with respect to the order relation are satisfied:
For all $(a,b,x,y) \in F^4$ and $(\lambda,\mu) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, one has:

$$\begin{array}{lcl} (a \le b, \lambda \le \mu) & \Rightarrow & (\lambda a \le \mu b) \\ (a \le b, x \le y) & \Rightarrow & (a + x \le b + y) \\ (a \le b, x \le y) & \Rightarrow & (ax \le by) \end{array}$$

As examples E may be one of the sets $(0,1], (0,\infty), \mathbb{N}$ equipped with usual filters converging to 0 or ∞. In the sequel we consider essentially the case $F = \mathbb{R}_+$. Nevertheless we may consider others cases: Let F denote the set of polynomials with positive real coefficients. If $P = \sum_0^\infty a_n X^n$ and $Q = \sum_0^\infty b_n X^n$ are two elements of F, let $P \le Q$ if for all $n \in \mathbb{N}$, one has $a_n \le b_n$. One can verify that all the above necessary conditions are fulfilled.
We denote by $\mathcal{A}(E,F)$ the set of all maps from E to F. We define a binary relation $\prec$ between two elements f and g of $\mathcal{A}(E,F)$ by:

$$f \prec g \Leftrightarrow \exists X \in \mathcal{F}, \forall x \in X, f(x) \le g(x).$$

Let $\mathcal{M}$ denote a subset of $\mathcal{A}(E,F)$ such that :

$$\forall (\varphi_1, \varphi_2) \in \mathcal{M}^2, \exists \varphi \in \mathcal{M}, \varphi_1 + \varphi_2 \prec \varphi. \tag{1}$$

$$\forall (\varphi_1, \varphi_2) \in \mathcal{M}^2, \exists \varphi \in \mathcal{M}, \varphi_1 \varphi_2 \prec \varphi. \tag{2}$$

$$\exists \nu \in F, \exists \varphi_0 \in \mathcal{M}, \nu \prec \varphi_0 \, (\nu \,\text{is considered as a constant map}). \tag{3}$$

We denote by $\mathcal{N}$ a subset of $\mathcal{A}(E,F)$ such that :

$$\forall \varphi \in \mathcal{M}, \forall \psi \in \mathcal{N}, \exists \psi_1 \in \mathcal{N}, \varphi \psi_1 \prec \psi. \tag{4}$$

We call $(\mathcal{M},\mathcal{N})$ a *couple of bounds spaces*, that will be justified in section 3.3

2.2. Examples

In all the following examples we take $F = \mathbb{R}_+$. For the four first ones we take $E = (0,1]$ and a basis of $\mathcal{F}$ is the set of all open intervals $(0,\eta)$ with $\eta \in E$.

1) $\mathcal{M} = \{\varepsilon \mapsto a, a > 1\}$; $\mathcal{N} = \{\varepsilon \mapsto a, 0 < a < 1\}$.

2)$\mathcal{M} = \{\varepsilon \mapsto \varepsilon^p, p \in \mathbb{Z}\}$; $\mathcal{N} = \{\varepsilon \mapsto \varepsilon^p, p \in \mathbb{Z}_+^*\}$ where $\mathbb{Z}_+^*$ denotes the set of positive integers.
That correspond to the simplified Colombeau algebras with entire powers of the parameter.

3) $\mathcal{M} = \{\varphi > 0, \forall a > 1, \varphi(\varepsilon) < a^{1/\varepsilon}\}$; $\mathcal{N} = \{\varepsilon \mapsto a^{1/\varepsilon}, 0 < a < 1\}$.
These sets correspond to algebras of generalized hyperfunctions on the circle [11, 13].

4) $\mathcal{M} = \{a_n : n \in \mathbb{Z}_-\}$; $\mathcal{N} = \{a_n : n \in \mathbb{Z}_+^*\}$ where $(a_n)_{n\in\mathbb{Z}}$ is an asymptotic scale over $E = (0,1]$ endowed with its Fréchet filter. An asymptotic scale consists of functions defined on E and satisfying some asymptotic properties. This concept was introduced for the construction of the so-called asymptotic algebras [3].

5) We take $E = \mathbb{N}$ endowed with is Fréchet filter. Let $\mathcal{M}$ be the set of all nonnegative sequences and $\mathcal{N}$ the subset of $\mathcal{M}$ formed by the finite support sequences. Here we are in the setting of Egorov's algebra [4].

PROPOSITION 1. *Assume that the following condition*

$$(*)\ \forall r \in F, \exists n \in \mathbb{N}^*, r \prec n\nu$$

is fulfilled with ν *as given in (3). Then, we have:*
(i) $\forall r \in F, \exists \varphi \in \mathcal{M}, r \prec \varphi$
(ii) $\forall (r,s) \in F^2, \forall (\varphi_1, \varphi_2) \in \mathcal{M}^2, \exists \varphi \in \mathcal{M}, r\varphi_1 + s\varphi_2 \prec \varphi$
(iii) $\forall r \in F, \forall \psi \in \mathcal{N}, \exists \psi_1 \in \mathcal{N}, r\psi_1 \prec \psi$
(iv) moreover if there exist $a \in F$ *and* $\psi_0 \in \mathcal{N}$ *such that* $\psi_0 \prec a$*, then we have :* $\forall r \in F, \exists \psi \in \mathcal{N}, r\psi \prec a$.

Proof. (i) Let $r \in F$. Because of $(*)$, there is $n \in N^*$ such that $r \prec 2^n\nu$. Using the property (1) of the definition of $\mathcal{M}$, we find by induction $\varphi \in \mathcal{M}$ such that $2^n\varphi_0 \prec \varphi$. From (3), one has $2^n\nu \prec 2^n\varphi_0$. Hence, it follows that $r \prec \varphi$.
(ii) let $(r,s) \in F^2$ and $(\varphi_1, \varphi_2) \in \mathcal{M}^2$. From (i), there is $(\zeta_1, \zeta_2) \in \mathcal{M}^2$ such that $r \prec \zeta_1$ and $s \prec \zeta_2$; it follows that $r\varphi_1 + s\varphi_2 \prec \zeta_1\varphi_1 + \zeta_2\varphi_2$. Using (2), we find $(\xi_1, \xi_2) \in \mathcal{M}^2$ such that $\zeta_1\varphi_1 \prec \xi_1$ and $\zeta_2\varphi_2 \prec \xi_2$. Property (1) gives $\varphi \in \mathcal{M}$ such that $\xi_1 + \xi_2 \prec \varphi$. Hence we have $r\varphi_1 + s\varphi_2 \prec \varphi$.
(iii) let $r \in F$ and $\psi \in \mathcal{N}$. By (i) we have $r \prec \varphi$ for some $\varphi \in \mathcal{M}$. Now, from (4), there exists $\psi_1 \in \mathcal{N}$ such that $\varphi\psi_1 \prec \psi$; hence $r\psi_1 \prec \varphi\psi_1 \prec \psi$.
(iv) let the hypothesis fulfilled. From (iii) there exists $\psi_1 \in \mathcal{N}$ such that $r\psi_1 \prec \psi_0$; hence $r\psi_1 \prec a$. □

REMARK 1. **(a)** note that if $\mathcal{M} \cap \mathcal{N} \neq \emptyset$ and $\varphi \in \mathcal{M} \cap \mathcal{N}$ one has $\varphi \prec \varphi$ with $(\varphi, \varphi) \in \mathcal{M} \times \mathcal{N}$. Now, if there is $(\varphi, \psi) \in \mathcal{M} \times \mathcal{N}$ such that $\psi \prec \varphi$, then from (4) there is $\psi_1 \in \mathcal{N}$ such that $\psi\psi_1 \prec \varphi\psi_1$. Hence we have $\psi\psi_1 \prec \psi$. In particular, if $F = \mathbb{R}_+$, we obtain that $\psi_1 \prec 1$. Hence from (i) it follows : $\forall r > 0, \exists \psi \in \mathcal{N}, \psi \prec r$.
(b) Note that the ordered set $(F, \leq)$ of polynomials with positive coefficients does not verify the condition $(*)$ of Proposition 1. Nevertheless if $\mathcal{M}$ has a constant function then (i) becomes a straightforward consequence of (1). It folows that $(ii) - (iv)$ are also valid.

3. Generalized algebras

3.1. Definitions

Here and in the sequel, we suppose that the hypotheses of Proposition 1 are fulfilled. Let V denote a commutative and associative algebra over K where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Consider the set $\mathcal{A}(V,F)$ of all maps from V to F. We suppose that there is a set I with a basis of filter $(I_\gamma)_{\gamma\in\Gamma}$, that is satisfying

$$\forall\gamma\in\Gamma,\ I_\gamma\neq\emptyset \text{ and } \forall(\gamma_1,\gamma_2)\in\Gamma^2,\ \exists\gamma\in\Gamma,\ I_\gamma\subset I_{\gamma_1}\cap I_{\gamma_2} \tag{5}$$

and to each $\gamma\in\Gamma$ are associated a family $\mathcal{P}_\gamma=(p_{\gamma,i})_{i\in I_\gamma}$ of elements of $\mathcal{A}(V,F)$ and a family of maps $(\sigma_\gamma)_{\gamma\in\Gamma}$ from $\mathbb{K}$ to F satisfying the following conditions :

$$\forall(\gamma_1,\gamma_2)\in\Gamma^2, (I_{\gamma_1}\subset I_{\gamma_2})\Rightarrow(\forall i\in I_{\gamma_1}, p_{\gamma_1,i}\le p_{\gamma_2,i}); \tag{6}$$

$$\begin{aligned}&\forall\gamma\in\Gamma,\forall i\in I_\gamma,\forall(\xi,\zeta)\in V^2,\forall(\lambda,\mu)\in\mathbb{K}^2,\\ &p_{\gamma,i}(\lambda\xi+\mu\zeta)\le\sigma_\gamma(\lambda)p_{\gamma,i}(\xi)+\sigma_\gamma(\mu)p_{\gamma,i}(\zeta);\end{aligned} \tag{7}$$

$$\begin{aligned}&\forall\gamma\in\Gamma,\forall i\in I_\gamma,\exists(j,k)\in I_\gamma^2,\exists C\in F,\forall(\xi,\zeta)\in V^2,\\ &p_{\gamma,i}(\xi\zeta)\le Cp_{\gamma,j}(\xi)p_{\gamma,k}(\zeta).\end{aligned} \tag{8}$$

Note that if V has a family $\mathcal{P}=(p_i)_{i\in I}$ of seminorms then (5)-(8) are satisfied with $(I_\gamma)_{\gamma\in\Gamma}$ reduced to I and $\sigma_\gamma=|.|$.

3.2. An example

Set $F=\mathbb{R}_+$ and let I denote the set of all polynomials of degree ≥ 1 with positive coefficients except the $0-$th coefficient which equals zero. For $p,q\in I$, $p\le q$ means that $\deg p\le\deg q$ and $a_{p,m}\le a_{q,m}, 1\le m\le\deg p$ where $a_{p,m}$ (resp. $a_{q,m}$) is the $m-$th coefficient of p (resp. q). We denote by Γ a subset of I with the following property

$$\forall(p,q)\in I^2,\exists\gamma\in\Gamma,\gamma\le\inf\{p,q\} \tag{9}$$

Such a subset Γ exists because if Γ is the set of terms of a sequence in I decreasing to zero, we can take $\gamma\in\Gamma$ such that $a_{\gamma,m}\le\inf\{a_{p,m},a_{q,m}\}, 1\le m\le\inf\{\deg p,\deg q\}$. We set

$$I_\gamma=\{p\in I, p\le\gamma\} \text{ and } \sigma_\gamma(\nu)=2^{\deg\gamma-1}\sup\{|\nu|,|\nu|^{\deg\gamma}\},\ \nu\in\mathbb{K}.$$

It follows straightforwardly from (9) that $(I_\gamma)_{\gamma\in\Gamma}$ satisfies (5).
Now let V be is a $\mathbb{K}-$algebra with a given family of semi-norms $(\rho_i)_{i\in I}$ (indexed by I) such that

$$\forall i\in I,\exists(j,k)\in I^2,\exists D>0,\rho_i(\xi\zeta)\le D\rho_j(\xi)\rho_k(\zeta),\ (\xi,\zeta)\in V^2. \tag{10}$$

We define $p_{\gamma,i}$ by

$$p_{\gamma,i}=\sup_{q\in I_\gamma}q\circ\rho_i.$$

For nonnegative numbers α,β,r,s we have

$$(\alpha r+\beta s)^k\le 2^{k-1}[(\alpha r)^k+(\beta s)^k],\ k\in\mathbb{N}.$$

Then if $q \in \Gamma$ with $\deg q = n$ it follows that

$$q(\alpha r + \beta s) \leq 2^{n-1}[\sup\{\alpha, \alpha^n\}q(r) + \sup\{\beta, \beta^n\}q(s)].$$

which means

$$q(\alpha r + \beta s) \leq \sigma_q(\alpha)q(r) + \sigma_q(\beta)q(s). \tag{11}$$

Set $p_i = q \circ \rho_i$. Let $x, y \in V$ and $\lambda, \mu \in \mathbb{K}$. Then we have

$$p_i(\lambda x + \mu y) = q[\rho_i(\lambda x + \mu y)] \leq q(|\lambda|\rho_i(x) + |\mu|\rho_i(y))$$

Invoking (11) yields

$$p_i(\lambda x + \mu y) \leq \sigma_q(\lambda)p_i(x) + \sigma_q(\mu)p_i(y). \tag{12}$$

Furthermore if $q = a_n X^n + ... + a_1 X$ and $C = \sup\{D^m a_m^{-1}, a_m \neq 0, 1 \leq m \leq n\}$ we have

$$a_m D^m r^m s^m \leq C a_m r^m a_m s^m,\ 1 \leq m \leq n.$$

It follows that $q(\alpha rs) \leq Cq(r)q(s)$ and then

$$q(\rho_i(xy)) \leq q(D\rho_j(x)\rho_k(y)) \leq Cq(\rho_j(x))q(\rho_k(y))$$

which means that

$$p_i(xy) \leq Cp_j(x)p_k(y). \tag{13}$$

Hence (6), (7), (8) are satisfied.

3.3. Generalized algebra of type $(\mathcal{M}, \mathcal{N}, V_{\mathcal{P}})$

We keep notations of section 3.1 and we denote by $\mathcal{X}(V)$ the set of all families $(u_\varepsilon)_\varepsilon$ with $\varepsilon \in E$ and $u_\varepsilon \in V$. If $u = (u_\varepsilon)_\varepsilon \in \mathcal{X}(V)$ and $i \in I_\gamma$, we set

$$\tilde{p}_{\gamma,i}[u](\varepsilon) = p_{\gamma,i}(u_\varepsilon).$$

Hence $\tilde{p}_{\gamma,i}[u]$ is a well defined map from E to F. We define :

$$\mathcal{X}_{\mathcal{M}}(V) = \{u = (u_\varepsilon)_\varepsilon \in \mathcal{X}(V), \exists\gamma \in \Gamma, \forall i \in I_\gamma, \exists\varphi \in \mathcal{M}, \tilde{p}_{\gamma,i}[u] \prec \varphi\}$$

$$\mathcal{X}_{\mathcal{N}}(V) = \{u = (u_\varepsilon)_\varepsilon \in \mathcal{X}(V), \exists\gamma \in \Gamma, \forall i \in I_\gamma, \forall\psi \in \mathcal{N}, \tilde{p}_{\gamma,i}[u] \prec \psi\}$$

THEOREM 1. *Suppose that the following condition holds:*

$$(**)\ \forall\psi \in \mathcal{N}, \exists\varphi \in \mathcal{M}, \psi \prec \varphi.$$

Then $\mathcal{X}_{\mathcal{M}}(V)$ is a subalgebra of $\mathcal{X}(V)$ and $\mathcal{X}_{\mathcal{N}}(V)$ is an ideal of $\mathcal{X}_{\mathcal{M}}(V)$.

Proof. From the hypothesis we have straightforwardly $\mathcal{X}_{\mathcal{N}}(V) \subset \mathcal{X}_{\mathcal{M}}(V)$. Let $u, v \in \mathcal{X}_{\mathcal{M}}(V)$ and $\lambda, \mu \in \mathbb{K}$. From the definition of $\mathcal{X}_{\mathcal{M}}(V)$ there exist $(\gamma_1, \gamma_2) \in \Gamma^2$ and $(\varphi_1, \varphi_2) \in \mathcal{M}^2$ such that

$$\tilde{p}_{\gamma_1, j}[u] \prec \varphi_1 \text{ and } \tilde{p}_{\gamma_2, k}[v] \prec \varphi_2,\ (j,k) \in I_{\gamma_1} \times I_{\gamma_2}.$$

According to (5) we may choose $\gamma \in \Gamma$ such that $I_\gamma \subset I_{\gamma_1} \cap I_{\gamma_2}$. Hence for all $i \in I_\gamma$, we have $\tilde{p}_{\gamma,i}[u] \prec \varphi_1$ and $\tilde{p}_{\gamma,i}[v] \prec \varphi_2$. It follows that

$$\tilde{p}_{\gamma,i}[\lambda u + \mu v] \le \sigma_\gamma(\lambda)\tilde{p}_{\gamma,i}[u] + \sigma_\gamma(\mu)\tilde{p}_{\gamma,i}.$$

Hence we have

$$\tilde{p}_{\gamma,i}[\lambda u + \mu v] \le \sigma_\gamma(\lambda)\varphi_1 + \sigma_\gamma(\mu)\varphi_2.$$

Using [(ii), Proposition 1] gives $\varphi \in \mathcal{M}$ such that $\tilde{p}_{\gamma,i}[\lambda u + \mu v] \prec \varphi$. Hence $\lambda u + \mu v \in \mathcal{X}_{\mathcal{M}}(V)$.
From (6), there exist $C \in F$ and $(j,k)\ I_\gamma^2$ such that

$$\tilde{p}_{\gamma,i}[uv] \le C\tilde{p}_{\gamma,j}[u]\tilde{p}_{\gamma,k}[v] \prec C\varphi_1\varphi_2.$$

Using condition (2) of the definition of $\mathcal{M}$ and [(i), Proposition 1] gives $\varphi \in \mathcal{M}$ such that $\tilde{p}_{\gamma,i}[uv] \prec \varphi$, showing that $uv \in \mathcal{X}_{\mathcal{M}}(V)$. Hence $\mathcal{X}_{\mathcal{M}}(V)$ is a subalgebra of $\mathcal{X}(V)$.
Let $u, v \in \mathcal{X}_{\mathcal{N}}(V)$ and $\lambda, \mu \in \mathbb{K}$. From the definition of $\mathcal{X}_{\mathcal{N}}(V)$, there are $(\gamma_1, \gamma_2) \in \Gamma^2$ such that for all $(j,k) \in I_{\gamma_1} \times I_{\gamma_2}$ and all $\psi_1, \psi_2 \in \mathcal{N}$, we have

$$\tilde{p}_{\gamma_1, j}[u] \prec \psi_1 \text{ and } \tilde{p}_{\gamma_1, k}[v] \prec \psi_2.$$

As above take $I_\gamma \subset I_{\gamma_1} \cap I_{\gamma_2}$ and $\psi \in \mathcal{N}$. From [(iii), Proposition 1] there are $\psi_1, \psi_2 \in \mathcal{N}$ such that

$$2\sigma_\gamma(\lambda)\psi_1 \prec \psi \text{ and } 2\sigma_\gamma(\mu)\psi_2 \prec \psi.$$

Since $I_\gamma \subset I_{\gamma_1} \cap I_{\gamma_2}$, then for all $i \in I_\gamma$, we have $\tilde{p}_{\gamma,i}[u] \prec \psi_1$ and $\tilde{p}_{\gamma,i}[v] \prec \psi_2$. It follows that

$$2\tilde{p}_{\gamma,i}[\lambda u + \mu v] \prec 2(\sigma_\gamma(\lambda)\psi_1 + \sigma_\gamma(\mu)\psi_2) \prec 2\psi.$$

Hence, multiplying by $1/2$ gives $\tilde{p}_{\gamma,i}[\lambda u + \mu v] \prec \psi$, that is $\lambda u + \mu v \in \mathcal{X}_{\mathcal{N}}(V)$.
Let $u \in \mathcal{X}_{\mathcal{N}}(V)$ and $v \in \mathcal{X}_{\mathcal{M}}(V)$. There is $\gamma_1 \in \Gamma$ such that for all $\psi_1 \in \mathcal{N}$, we have $\tilde{p}_{\gamma_1, j}[u] \prec \psi_1$ for all $j \in I_{\gamma_1}$. In the same way, there are $\gamma_2 \in \Gamma$ and $\varphi_1 \in \mathcal{M}$ such that if $k \in I_{\gamma_2}$, then $\tilde{p}_{\gamma_2, k}[v] \prec \varphi_1$. Now, let $\gamma \in \Gamma$ such that $I_\gamma \subset I_{\gamma_1} \cap I_{\gamma_2}$ and $\psi \in \mathcal{N}$. Let $i \in I_\gamma$. There exist $C \in F$ and $(j,k) \in I_\gamma^2$ such that

$$\tilde{p}_{\gamma,i}[uv] \prec C\tilde{p}_{\gamma,j}[u]\tilde{p}_{\gamma,k}[v].$$

Let $\psi \in \mathcal{N}$ and choose $\psi_1 \in \mathcal{N}$ such that $C\varphi_1\psi_1 \prec \psi$. The above inequality shows that $\tilde{p}_{\gamma,i}[uv] \prec \psi$ that is $uv \in \mathcal{X}_{\mathcal{N}}(V)$. Hence $\mathcal{X}_{\mathcal{N}}(V)$ is an ideal of $\mathcal{X}_{\mathcal{M}}(V)$. $\square$

DEFINITION 1. *Assume that [$(*)$, Proposition 1] and [$(**)$, Theorem 1] are satisfied. Then, the generalized algebra of type $(\mathcal{M}, \mathcal{N}, V_{\mathcal{P}})$ is the factor algebra:*

$$\mathcal{G}_{\mathcal{M},\mathcal{N},\mathcal{P}}(V) = \mathcal{X}_{\mathcal{M}}(V)/\mathcal{X}_{\mathcal{N}}(V).$$

4. Usual algebras as $(\mathcal{M},\mathcal{N},V_{\mathcal{P}})-$algebras

In this section Ω denotes an open set of $\mathbb{R}^n$ and $\mathcal{E}(\Omega)$ the space of smooth functions in Ω. We denote by $(K_l)_l$ an increasing sequence of compact sets exhausting Ω with $K_0 \neq \emptyset$.

4.1. Simplified Colombeau algebra

We consider on $\mathcal{E}(\Omega)$ the sequence of seminorms

$$\mu_l(f) = \sup\{|\partial^\alpha f(x)|, \alpha \in \mathbb{N}^n, |\alpha| \leq l, x \in K_l\}, l \in \mathbb{N}.$$

We note here and for the sequel that $\mu_l(fg) \leq 2^l \mu_l(f)\mu_l(g)$, $f,g \in \mathcal{E}(\Omega)$. The set $\mathcal{E}_M(\Omega)$ of moderate nets consists of nets $(f_\varepsilon)_\varepsilon \in \mathcal{E}(\Omega)^{(0,1]}$ with the properties

$$\forall l \in \mathbb{N}, \exists r \in \mathbb{R}, \exists \eta > 0, \mu_l(f_\varepsilon) \leq \varepsilon^r, 0 < \varepsilon < \eta$$

and the set $\mathcal{N}(\Omega)$ of null nets consists of nets $(f_\varepsilon)_\varepsilon \in \mathcal{E}(\Omega)^{(0,1]}$ with the properties

$$\forall l \in \mathbb{N}, \forall q \in \mathbb{R}, \exists \eta > 0, \mu_l(f_\varepsilon) \leq \varepsilon^q, 0 < \varepsilon < \eta.$$

These spaces are both algebras and $\mathcal{N}(\Omega)$ is an ideal of $\mathcal{E}_M(\Omega)$. The simplified Colombeau algebra $\mathcal{G}^s(\Omega)$ is defined as the factor

$$\mathcal{G}^s(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega).$$

Let $E = (0,1]$ endowed with its Fréchet filter, $F = \mathbb{R}_+$ with its usual order $\leq$, set

$$\mathcal{M} = \{\varepsilon \mapsto \varepsilon^p, p \in \mathbb{R}\} \text{ and } \mathcal{N} = \{\varepsilon \mapsto \varepsilon^p, p \in \mathbb{R}_+^*\}.$$

Since $\mathcal{N} \subset \mathcal{M}$ it is easily seen that (1)-(4) and condition $(*)$ of Proposition 1 are satisfied, we may take $\nu = 1$ in (3). Condition $(**)$ is also trivially satisfied. Taking $V = \mathcal{E}(\Omega), I = \mathbb{N}$ with the basis of filter $\{\mathbb{N}\}$ and $\mathcal{P} = (\mu_l)_{l\in\mathbb{N}}$ we find

$$\mathcal{X}_{\mathcal{M}}(V) = \mathcal{E}_M(\Omega) \text{ and } \mathcal{X}_{\mathcal{N}}(V) = \mathcal{N}(\Omega)$$

showing that $\mathcal{G}^s(\Omega) = \mathcal{G}_{\mathcal{M},\mathcal{N},\mathcal{P}}(V)$.

4.2. Egorov algebra

Denote by $\mathcal{N}_E(\Omega)$ the subset of $\mathcal{E}(\Omega)^{\mathbb{N}}$ whose elements $(u_n)_n$ satisfy the properties

$$\forall K \,\text{compact set} \subset \Omega,\ \exists n_0 \in \mathbb{N},\ u_n(x) = 0,\ x \in K,\ n \geq n_0. \tag{14}$$

It is seen that $\mathcal{N}_E(\Omega)$ is an ideal of $\mathcal{E}(\Omega)^{\mathbb{N}}$. The Egorov algebra $\mathcal{G}_E(\Omega)$ is defined as the factor

$$\mathcal{G}_E(\Omega) = \mathcal{E}(\Omega)^{\mathbb{N}}/\mathcal{N}_E(\Omega).$$

We show that $\mathcal{G}_E(\Omega)$ is a $(\mathcal{M},\mathcal{N},V_{\mathcal{P}})-$type algebra. For let I denote the set of compacts subsets of Ω. Let $V=\mathcal{E}(\Omega)$ and $(K_l)_l$ be as in the previous section. If $l\in\mathbb{N}$ and $u\in V$ we set $p_l(u)=\sup\{|u(x)|,x\in K_l\}$ and $\mathcal{P}=(p_l)_l$. We take $E=\mathbb{N}$, it follows that $\mathcal{X}(V)=\mathcal{E}(\Omega)^{\mathbb{N}}$. Let $\mathcal{M}$ be the set of all sequences of nonnegative real numbers and $\mathcal{N}$ the subset of $\mathcal{M}$ consisting of all sequences which finite support. If $(u_n)_n\in\mathcal{X}(V)$ and $l\in\mathbb{N}$ we have $(p_l(u_n))_n\in\mathcal{M}$, it follows that $\mathcal{X}_{\mathcal{M}}(V)=\mathcal{X}(V)=\mathcal{E}(\Omega)^{\mathbb{N}}$. It is easily seen that (1)-(8) with $(*)$ and $(**)$ are satisfied. From the definition of $\mathcal{N}$ it follows that $(u_n)_n\in\mathcal{N}$ if and only if for each $l\in\mathbb{N}$, $(u_n|K_l)_n$ have a finite support. Hence the corresponding $(\mathcal{M},\mathcal{N},V_{\mathcal{P}})-$algebra is Egorov's algebra.

4.3. $(\mathcal{C},\mathcal{E},\mathcal{P})-$algebras

The $(\mathcal{C},\mathcal{E},\mathcal{P})$ type algebras are constructed as follows. If $r\in\mathbb{K}^{(0,1]}$, we set $r=(r_\varepsilon)_\varepsilon$ and $|r|=(|r_\varepsilon|)_\varepsilon$. Moreover if $r,t\in\mathbb{R}^{(0,1]}$, $t\leq r$ means $t_\varepsilon\leq r_\varepsilon,\varepsilon\in(0,1]$. If $T\subset\mathbb{K}^{(0,1]}$ one defines

$$T^+=\{t\in T,t_\varepsilon\geq 0,\varepsilon\in(0,1]\}\text{ and }|T|=\{|t|,t\in T\}$$

Consider the following two conditions for $T\subset\mathbb{K}^{(0,1]}$:

$$\forall t\in T^+,\forall r\in\mathbb{K}^{(0,1]},|r|\leq t\Rightarrow r\in T \qquad (S)$$

and

$$T^+=|T|. \qquad (MS)$$

Here (S) stands for *solidity* and (MS) for *modulus stability*. Note that if (S) is satisfied then (MS) can be replaced by the condition $|S|\subset S$.

Now let A denote a subring of $\mathbb{K}^{(0,1]}$ and I_A an ideal of A satisfying both (S) and (MS). Let $\mathcal{E}$ be a topological $\mathbb{K}-$algebra endowed with a family $\mathcal{P}=(p_l)_{l\in L}$ of seminorms such that

$$\forall l\in L,\exists(j,k)\in L^2,\exists C\in\mathbb{R}_+^*,p_l(fg)\leq Cp_j(f)p_k(g),f,g\in\mathcal{E}.$$

Then one defines

$$\begin{aligned}&\mathcal{H}_{(A,\mathcal{E}_{\mathcal{P}})}=\left\{(u_\varepsilon)_\varepsilon\in\mathcal{E}^{(0,1]},\forall l\in L,(p_l(u_\varepsilon))_\varepsilon\in A^+\right\};\\&\mathcal{J}_{(I_A,\mathcal{E}_{\mathcal{P}})}=\left\{(u_\varepsilon)_\varepsilon\in\mathcal{E}^{(0,1]},\forall l\in L,(p_l(u_\varepsilon))_\varepsilon\in I_A^+\right\};\\&\mathcal{C}=A/I_A.\end{aligned}$$

It is shown that both spaces $\mathcal{H}_{(A,\mathcal{E}_{\mathcal{P}})}$ and $\mathcal{J}_{(I_A,\mathcal{E}_{\mathcal{P}})}$ are algebras, the latter being an ideal of the former. The associated $(\mathcal{C},\mathcal{E},\mathcal{P})-$algebra is the algebra $\mathcal{A}$ defined by

$$\mathcal{A}=\mathcal{H}_{(A,\mathcal{E}_{\mathcal{P}})}/\mathcal{J}_{(I_A,\mathcal{E}_{\mathcal{P}})}.$$

It is easily seen that $\mathcal{H}_{(A,\mathbb{K}_{|.|})}=A$ and $\mathcal{J}_{(A,\mathbb{K}_{|.|})}=I_A$, then the corresponding quotient gives the ring $\mathcal{C}$.

We now show that every $(\mathcal{C},\mathcal{E},\mathcal{P})-$algebra is a $(\mathcal{M},\mathcal{N},E_{\mathcal{P}})-$algebra. For let $E=(0,1]$ with its Fréchet filter, $F=\mathbb{R}_+$ equipped with the order $\leq$ and $I=L$ with $\{L\}$ as basis of filter. If $a\in|A|$ we define $f_a:E\to F$ by $f_a(\varepsilon)=|a_\varepsilon|$ and we set

$$\mathcal{M}=\{f_a,a\in|A|\}\text{ and }\mathcal{N}=\{f_a,a\in|I_A|\}.$$

We note that $\mathcal{N} \subset \mathcal{M}$, then $(**)$ is satisfied. We get (3) with $\nu = 1$ and $\varphi_0 = f_{\mathbf{1}}$ where $\mathbf{1}$ is the unit of $\mathbb{K}^{(0,1]}$. Then $(*)$ is satisfied with any integer $n \geq r+1$. From (S) and (MS) it is seen that (1), (2) are verified and finally we have (4) with $\psi_1 = 0$.

From (S) and (MS), if $r \in \mathbb{R}_+^{(0,1]}$, then $r \in |A|$ (resp. $r \in |I_A|$) if and only if there exists $a \in |A|$ (resp. $a \in |I_A|$) such that $r \leq a$. It follows that

$$X_{\mathcal{M}}(\mathcal{E}) = \mathcal{H}_{(A,\mathcal{E}_{\mathcal{P}})} \text{ and } X_{\mathcal{N}}(\mathcal{E}) = \mathcal{J}_{(I_A,\mathcal{E}_{\mathcal{P}})}$$

proving that $\mathcal{A} = \mathcal{H}_{(A,\mathcal{E}_{\mathcal{P}})}/\mathcal{J}_{(I_A,\mathcal{E}_{\mathcal{P}})} = \mathcal{G}_{\mathcal{M},\mathcal{N},\mathcal{P}}(\mathcal{E})$.

4.4. Rosinger algebra

Rosinger's nowhere dense ideal $I_{nd}(\Omega)$ is the set of all $u = (u_\varepsilon)_\varepsilon \in \mathcal{E}(\Omega)^{(0,\infty)}$ such that there is a nowhere dense closed set $\gamma \subset \Omega$ such that for all $x \in \Omega \setminus \gamma$ there exist $\eta > 0$ and an open neighborhood W_x of x in $\Omega \setminus \gamma$ such that $u_\varepsilon(y) = 0$ for all $y \in W_x$ and $\varepsilon < \eta$. Rosinger's algebra is

$$\mathcal{R}(\Omega) = \mathcal{E}(\Omega)^{(0,\infty)}/I_{nd}(\Omega).$$

Let $u \in I_{nd}(\Omega)$ and choose γ as above defined. Let ω denote a relatively compact open set in $\Omega \setminus \gamma$. For each $x \in \omega$ there are $\eta_x > 0, W_x$ open set of $\Omega \setminus \gamma$ such that $x \in W_x$ and $u_\varepsilon|_{V_x} \equiv 0$, $0 < \varepsilon < \eta_x$. Using classical compactness arguments gives $\eta > 0$ such that $u_\varepsilon|_\omega \equiv 0$ for $0 < \varepsilon < \eta$. It follows that $u = (u_\varepsilon)_\varepsilon \in I_{nd}(\Omega)$ if and only if there is a nowhere dense closed set $\gamma \in \Omega$ such that for all relatively compact open set of $\Omega \setminus \gamma$, there is $\eta > 0$ such that $u_\varepsilon|_\omega \equiv 0$ for $0 < \varepsilon < \eta$.

Let $E = (0,\infty)$ endowed with its Fréchet filter converging to 0 and let $V = \mathcal{E}(\Omega)$. We define Γ as the set of all nowhere dense closed sets in Ω and I the set of all relatively compact open sets in Ω. For each $\gamma \in \Gamma$, we denote by I_γ the set of $\omega \in I$ such that $\omega \cap \gamma = \emptyset$ and we set $\sigma_\gamma = |.|$. Note that $I_\gamma \neq \emptyset$ and if $(\gamma_1, \gamma_2) \in \Gamma^2$, then $\gamma = \gamma_1 \cup \gamma_2 \in \Gamma$ and $I_\gamma = I_{\gamma_1} \cap I_{\gamma_2}$. It follows that $(I_\gamma)_{\gamma \in \Gamma}$ satisfies (5). Let $\zeta \in V$. If $\gamma \in \Gamma$ and $\omega \in I_\gamma$ we set $p_{\gamma,\omega} = \sup\{|\zeta(x)|, x \in \omega\}$. Let $F = \mathbb{R}_+$ and $\mathcal{N} = \{0\}$. It is easily seen that $u = (u_\varepsilon)_\varepsilon \in I_{nd}(\Omega)$ if and only if there is $\gamma \in \Gamma$ such that for any $\omega \in I_\gamma$ one has $\tilde{p}_{\gamma,\omega}[u] \prec 0$. That is $I_{nd}(\Omega) = X_{\mathcal{N}}(V)$. If $\mathcal{M}$ is the set of all positive maps from E to $\mathbb{R}_+$, we obtain $\mathcal{R}(\Omega) = \mathcal{G}_{\mathcal{M},\mathcal{N},\mathcal{P}}(V)$.

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