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AN IMPULSIVE DIFFERENTIAL EQUATION IN AN INFINITE DIMENSIONAL FOCK SPACE

Abstract. A scale of infinite dimensional Fock spaces, $\Gamma^{pB} = \cup_{s\geq 1}\Gamma^{p,sB}$, is introduced together with some of their fundamental properties. Each space,$\Gamma^{p,sB_m}, s \geq 1$ is a Frechet space described in [18, 23, 19]. An impulsive differential equation involving a generalized Laplacian [20] is then introduced where at the points, $t_i, 1 \leq i < \infty$ belonging to R receives an impulse. Due to the generalization of this paper, the impulse takes its values in the infinite dimensional Fock space, $\Gamma^{pB} = \cup_{s\geq 1}\Gamma^{p,sB}$. The components of the vector in the space, $\Gamma^{p,sB_m}, s \geq 1$ are tempered distributions thus generalizing the classical Fock space having components in $L^p(R^n)$. An explicit algorithm computing the solution to the problem is given together with a uniqueness technique. The existence technique is in the spirit of an operational calculus.

1. Introduction

A system of impulsive differential equations [1, 2, 13] within the framework of a Euclidean space, R^n, can be described by the following system,

$$\dot{z}(t) = A(t)z(t) + f(t), t \neq t_i, \tag{1}$$

where $t \in R, z(t) : R \to R^n, A(t) : R \to R^n \times R^n, f : R \to R^n$ together with the impulsive conditions,

$$\Delta z(t_i) = z(t_i^+) - z(t_i^-) = b_i, i = \pm 1, \pm 2, ... \tag{2}$$

Moreover, $\{b_i\}_{i=\pm 1}^{\infty}$ is a sequence of n-dimensional constants. To illustrate these notions we included the following fundamental example [2]:

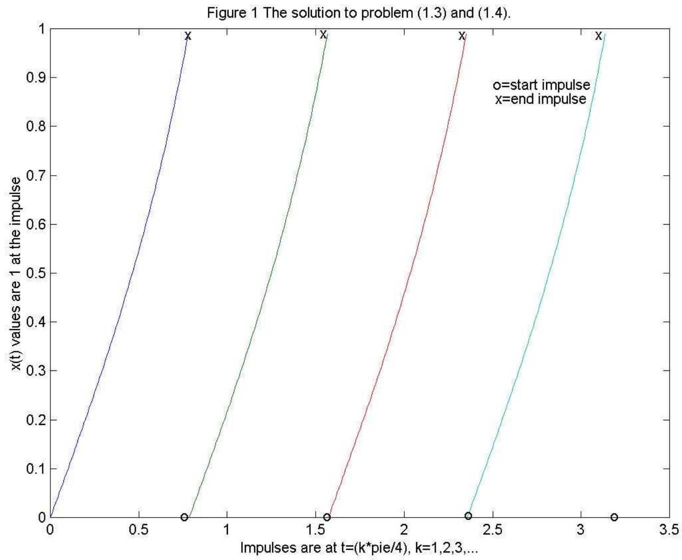
$$\dot{x}(t) = 1 + [x(t)]^2, t \neq \frac{k\pi}{4}, k = 1, 2, ... \tag{3}$$

and

$$\Delta x(t_k) = x(t_x^+) - x(t_x^-) + 1 \ \textit{whenever} \ t_k = \frac{k\pi}{4} \tag{4}$$

together with x(0)=0. Equations (3) and (4) are fundamental examples corresponding to conditions (1) and (2). The solution to (3) and (4) is easily seen to be

$$x(t) = \begin{cases} tan(t), & 0 < t < \frac{1}{4\pi} \\ tan(t - \frac{k\pi}{4}), & t \in (\frac{k\pi}{4}, \frac{(k+1)\pi}{4}) \end{cases} \tag{5}$$



where $x(t)$ is periodic with period = $\frac{\pi}{4}$. The graph of the solution is illustrated in Figure 1.

We note the classical corresponding differential equation (3) without the impulse condition (4) has the solution, $x(t) = tan(t)$ with an interval of existence, $(0, \frac{\pi}{2})$ since $\lim_{t\to\frac{\pi}{2}} x(t) \to \infty$.

This paper generalizes conditions (1) and (2) whereby condition (1) contains a generalized Laplacian [20] and condition (2) becomes $\Delta u(t_i) = I_i(u(t_i))$ where $u(t_i) \in \Gamma^{p,sB}$ for some $s \geq 1$ and $I_i : (\dot{u} \to \Gamma^{p,sB}) \to (\dot{u} \to \Gamma^{p,s'B})$, where $s > s' > 1$.

2. The scale of infinite dimensional fock spaces $\Gamma^{pB} = \cup_{s\geq 1}\Gamma^{p,sB}$

For each $s \geq 1$ the space, $\Gamma^{p,sB}\{(p > 1, B = \{B_i\}_{i=0}^{\infty}, B_i > B_j, j > i)\}$, is called an infinite dimensional Fock space. The p and $B_i, i \geq 0$ are all real numbers. These spaces are topological spaces of real-valued functionals on $S'(R^{3n};R)$, the space of real-valued tempered distributions. The set of functionals belonging to the space $\Gamma^{p,sB}$ are all $C^\infty(S'(R^{3n};R))$. We also require if $\Phi \in \Gamma^{p,sB}$, then

$$\Phi(x) = \sum_{q=0}^{\infty} a_q[x, ...x] = \sum_{q=0}^{\infty} a_q x^q \tag{6}$$

where $a_0 \in R$ and $a_q, q \geq 1$ are q-multilinear symmetric continuous functionals on $S'(R^{3n}) \times ... \times S'(R^{3n}) \to R$. We identify for each $\Phi \in \Gamma^{p,sB}$ the associated state vector,

$$\Phi \Longleftrightarrow \begin{vmatrix} a_0 \\ a_1 \\ . \\ . \\ a_q \\ . \\ . \end{vmatrix} . \tag{7}$$

Each multilinear functional, $a_q, q \geq 1$, has an infinite dimensional domain space. The functionals having representation given in expression (7) are members of our infinite dimensional Fock space. We equip our infinite dimensional Fock space with the following sequence of norms:

$$|||\Phi|||_{sB_m} = \sup_q \frac{||a_q||_m q!^{\frac{1}{p}}}{(sB_m)^q} < \infty \ m = 0, 1, ... \tag{8}$$

where

$$||a_q||_m = \sup_{||x||-m} |a_q x^q| \quad m = 0, 1, ..., \quad x \in S'(R^{3n}) \tag{9}$$

and

$$||x||_{-m} = \sup_{||\phi||_m} |<x, \phi>| \ m = 0, 1, ..., \quad \phi \in S(R^{3n}) \tag{10}$$

and

$$||\phi||_m = \sup_{\substack{\sigma_1+...+\sigma_n \leq m \\ (\tau_1,...\tau_n) \in R^{3n}}} [(1+|\tau_1|^2)...(1+|\tau_n|^2)]^m |\phi^{\sigma_1,...,\sigma_n}(\tau_1,...,\tau_n)| \tag{11}$$

where

$$\phi^{(0,...,\sigma_i,...0)}(\tau_1,...,\tau_n) = \frac{\partial^{\sigma_i}}{\partial \tau_i^{\sigma_i}} \ \phi \quad 1 \leq i \leq n. \tag{12}$$

The functions ϕ, are rapid descent test functions and the functionals, x, are tempered distributions described in Constantinescu [6] and Zemanian [27]. The set of entire functionals belonging to the space, $\Gamma^{p,sB}$, equipped with the natural topology induced by the sequence of norms, (8) is a Frechet space. We then consider $1 \leq s \leq s'$ where clearly $\Gamma^{p,sB} \subset \Gamma^{p,s'B}$. Also the canonical injection, $J_{s's} : \Gamma^{p,sB} \to \Gamma^{p,s'B}$, is continuous.

The multilinear symmetric functional,$a_q, q \geq 1$, will have a square summing property analogous to classical Fock space sum ability in the following sense.

PROPOSITION 1. *The sequence of multilinear symmetric functionals,*$\{a_q\}_{q=1}^{\infty}$, $a_0 \in R$*, described in expression (7) is square summing in each norm,* $\Sigma_{q=0}^{\infty} ||a_q||_m^2 < \infty$, $m = 0, 1, 2, \ldots$.

Proof. We select $\Phi \in \Gamma^{p,sB}$, where $\Phi(x) = \sum_{q=0}^{\infty} a_q x^q$. We consider a norm, $|||\Phi|||_{sB_m} = \sup_q \frac{||a_q||_m q!^{\frac{1}{p}}}{(sB_m)^q} < C_m < \infty$ implying $||a_q||_m < \frac{C_m (sB_m)^q}{q!^{\frac{1}{p}}}$ for every q. From these statements and returning to square summing notion, we obtain

$$\sum_{q=0}^{\infty} ||a_q||_m^2 = \sum_{q=0}^{\infty} \frac{||a_q||_m^2 q!^{\frac{1}{q}} (sB_m)^q}{(sB_m)^q (q!)^{\frac{1}{p}}}$$
$$\leq |||\Phi|||_{sB_m} \sum_{q=0}^{\infty} \frac{||a_q||^m (sB_m)^q}{q!^{\frac{1}{p}}} \leq |||\Phi|||_{sB_m} C_m \sum_{q=0}^{\infty} (\frac{(sB_m)^q}{q^{\frac{1}{p}}})^2 \leq \infty.$$

□

We remark the kernel representation for each of the multilinear symmetric functional, $a_q, q \geq 1$, has the form of a rapid descent test function, $\phi_q, q \geq 1$. This representation then gives the association,

$$\Phi \Longleftrightarrow \left| \begin{array}{c} \phi_0 \\ \phi_1 \\ . \\ . \\ \phi_q \\ . \\ . \end{array} \right| \tag{13}$$

where $\phi_0 = a_0$ and $\phi_q(x_1, \ldots, x_q) = a_q(\delta(t_1 - x_1), \ldots, \delta(t_q - x_q)), q \geq 1$. The $\delta(t_i - x_i), 1 \leq i \leq q$ are translates of the Dirac delta functional. Details regarding the topological properties of this association can be found in reference [24]. We conclude this section by indicating the vector given in expression (13) also enjoys the so-called square summing property.

3. Infinite dimensional laplacian operator

We briefly review cylinder functionals developed by K.O. Friedrichs and H.N. Shapiro [10]. Cylinder functionals have p-variables where each variable takes its value from a set of functions containing the classical piecewise constant functions. The p variable functions can be written as

$$\phi(t) \leftrightarrow \{\phi_1, ..., \phi_p\}, \tag{14}$$

where each $t \in R^3$. A cylinder functional can be written as

$$f_p(\phi(t)) \leftrightarrow f_p(\phi_1(t), ..., \phi_p(t)), \tag{15}$$

where the subscript p denotes the number of components within its representation.

More specifically, when the quantum theory of fields is introduced in Chapter II [10] each t varies in a "cell" contained within R^3 . We consider $\phi_\gamma(t) = 0, 1 \leq \gamma \leq p$, if t does not belong to any n-cell. We then define as in Friedrich and Shapiro [10] $\delta f_p(\phi(t)) = 0$.

If $t \in \gamma^{th}$ cell as in Friedrich and Shapiro [10], we set

$$\frac{\delta}{\delta\phi(t)dt} f_p(\phi(t)) = \frac{1}{\Delta\gamma} \frac{\partial}{\partial\phi_\gamma} f_p(\phi_1(t), ..., \phi_p(t)).$$

The Δ_γ is the "volume" of the γ^{th} cell. Schiff [17] requires the "volume of the cell" to tend to zero. Then Schiff enjoys the presence of the Dirac Delta functional. Friedrich and Shapiro [10] select the following quadratic functional,

$$f_2[\phi] = \int\int b(x', x'')\phi(x')\phi(x'')dx'dx'',$$

and its generalized Laplacian becomes

$$Lf_2[t] = 2\int b(x, x)dx.$$

To avoid computational difficulties we select a $\phi(t, t') \in S(R^2)$, the space of R^2 rapid descent test functions. We then define h as

$$< h, \phi > \equiv 2\int \phi(x, x)dx, \tag{16}$$

proving h to be a tempered distribution. In applications the independent variable, x, are from R^3.

PROPOSITION 2. *The functional, h, defined in expression (16) is a tempered distribution.*

Proof. The linearity of h is obvious and we prove the continuity by proving boundedness of h. We select any rapid descent test function, $\phi(t, t')\phi$ and compute the following

$$
\begin{aligned}
|<h,\phi(t,t')>| &= 2|\int \phi(t,t)dt| = 2|\int \frac{\phi(t,t)(1+t^2)^4}{(1+t^2)^4}dt| \\
&\le 2 \sup_{\substack{|\sigma|<2 \\ t\in R}} |(1+t^2)^4\phi^{(\sigma)}(t,t)|\pi < 2\pi||\phi(t_1,t_2)||_2,
\end{aligned} \tag{17}
$$

If we select a special $\Phi \in \Gamma^{p,sB}$where its Fock representation (7) is given as

$$
\Phi \Longleftrightarrow \begin{vmatrix} a_0 \\ a_1 \\ . \\ . \\ a_q \\ . \\ . \end{vmatrix} \tag{18}
$$

and apply two generalized differentiations, then

$$
D_h^2\Phi \leftrightarrow \begin{vmatrix} 2a_2[h,h] \\ 0 \\ . \\ . \\ 0 \end{vmatrix}. \tag{19}
$$

Selecting, $h = \delta(t-t')$and implementing an integral operator on (19) results in

$$
\Delta\Phi \leftrightarrow \begin{vmatrix} 2\int a_2[\delta(t-t'),\delta(t-t')]dt \\ 0 \\ . \\ . \\ . \end{vmatrix}. \tag{20}
$$

□

Noting that $a_2[\delta(t-t'),\delta(t-t')] = \phi(t,t)$ is a member of $S(R^2)$ as shown in Schmeelk [2], we have an analogy to expression (3.3). Our Laplacian is mathematically developed in Schmeelk [20] but here we have its Fock representation. If $\Phi \in \Gamma^{p,sB}$ where Φ has its representation given in expression (18), then its generalized Laplacian becomes

$$
\begin{vmatrix} 2\int a_2[\delta(t-t'),\delta(t-t')]dt \\ 2\times 3\int a_3[\delta(t-t'),\delta(t-t'),]dt \\ . \\ . \\ . \\ (q+1)(q+2)\int a_{q+2}[\delta(t-t'),\delta(t-t'),.,.,.]dt \end{vmatrix}. \tag{21}
$$

We observe each a_q is a multilinear symmetric functional so for convenience we insert the pair of translated Dirac delta functionals in the first two arguments. We can now develop an infinite dimensional impulsive differential equation within this setting.

4. Impulsive differential equation solutions in $\Gamma^{pB} = \cup_{s\geq 1}\Gamma^{p,sB}$

Let $u(t) : (\alpha,\beta) \to \Gamma^{pB}, t \neq t_i$ where (α,β) is an interval in R. Moreover for $t \neq t_i$ the vector, $u(t) \in \Gamma^{p,sB}$, and thus must satisfy

$$\sup_{\substack{q \\ t\in(\alpha,\beta)}} \frac{||a_q(t)||_m q!^{\frac{1}{p}}}{(sB_m)^q} < M_m < \infty,\ m = 1,2,... \tag{22}$$

For $t = t_i$we have the impulse component,

$$\Delta u(t_i) = u(t_i^{+0}) - u(t_i^{-0}) = \tilde{u}(t_i),\ i = 1,2,... \tag{23}$$

The vectors, $u(t_i^{+0})$ and $u(t_i^{-0})$ together with the vector, $\tilde{u}(t_i)$, all must satisfy condition (22).

We now introduce the transformation,

$$I_i : ((\alpha,\beta) \to \Gamma^{p,sB}) \to ((\alpha,\beta) \to \Gamma^{p,s'B}) \tag{24}$$

where $I_i(u(t_i)) \equiv \Delta(u(t_i)),\ i = 1,2,....$ The functional, $u(t) \in \Gamma^{p,sB}, t \neq t_i$ is said to be *strongly continuous* if and only if whenever, $\varepsilon > 0 \exists m \in N_+$ (positive reals) then there $\exists \delta_{\varepsilon,t',m} > 0$ such that whenever

$$|t-t'| < \delta_{\varepsilon,t',m} \Rightarrow |||u(t) - u(t')|||_{sB_m} < \varepsilon. \tag{25}$$

Strongly differentiable has a somewhat similar requirement whereby

$$||| \frac{u(t') - u(t)}{t'-t} - v(t')|||_{sB_m} < \varepsilon \tag{26}$$

whenever $0 < |t-t'|, \delta_{\varepsilon,t',m}$ and $v(t') \in \Gamma^{p,sB}$. The impulsive differential equation in $\Gamma^{p,sB}$ is now defined as

$$\frac{\partial}{\partial t} u(t)(x) = -P(\Delta^2)u(t,x), t \neq t_i \tag{27}$$

where $u(t) \in ((\alpha,\beta) \to \Gamma^{p,sB}),\ x \in S'(R^{3n})$ and $P(\Delta^2)$ is an infinite dimensional Laplacian polynomial operator of order n where $P(\Delta^2) = d_n(\Delta^2) + d_{n-1}(\Delta^2) + ... + d_0$, where $d_i \in (R), (0 \leq i \leq n)$. The impulsive conditions are given by

$$\begin{aligned} &u(t_i^{+0}) = u(t_i) + \Delta(u(t_i)) = u(t_i) + I_i(u(t_i)), \\ &\tfrac{\partial}{\partial t} u(t_i^{+0}) = \tfrac{\partial}{\partial t} u(t_i) + \Delta \tfrac{\partial}{\partial t} u(t_i) + I_i(\tfrac{\partial}{\partial t} u(t_i)), \\ &\tfrac{\partial^{l-1}}{\partial t^{l-1}} u(t_i^{+0}) = \tfrac{\partial^{l-1}}{\partial t^{l-1}} u(t_i) + \Delta(\tfrac{\partial^{l-1}}{\partial t^{l-1}} u(t_i)) = \tfrac{\partial^{l-1}}{\partial t^{l-1}} u(t_i) + I_i(\tfrac{\partial^{l-1}}{\partial t^{l-1}} u(t_i)) \end{aligned} \tag{28}$$

We also require for $t = t_0^{+0} \in R$ the conditions,

$$
(29) \qquad \begin{array}{l} u(t)(\bullet)|_{t=t_0^{+0}} \in \Gamma^{p,sB} \\ . \\ . \\ \frac{\partial^{l-1}}{\partial t^{l-1}} u(t)(\bullet)|_{t=t_0^{+0}} \in \Gamma^{p,sB} \end{array}
$$

are also satisfied. We now define the following sets in R.

$$
(30) \qquad \{G_k = t \in R : t_{k-1} < t < t_k, k \in N\}
$$

$$
(31) \qquad \{D_k = t \in R : t_{k-1} < t \leq t_k, k \in N\}
$$

$$
(32) \qquad \{F_k = t \in R : t_{k-1} \leq t < t_k, k \in N\}
$$

We then view equation (27) in operational form,

$$
(33) \qquad [\frac{\partial}{\partial t} + P(\Delta^2)]u(t)(x) = 0
$$

and introduce the standard integral operator,

$$
(34) \qquad (Q_{t'}^t)(u(\tau)(x)) = \int_{t'}^t u(\tau)(x) d\tau
$$

for $t, t' \in G_k$. Applying the operator (34) to equation (33) will formally give us

$$
(35) \qquad \lfloor I - \lfloor Q_{t'}^t P_1(\Delta^2) + (Q_{t'}^t)^2 P_2(\Delta^2) + ... + (Q_{t'}^t)^i P_i(\Delta^2) \rfloor \rfloor
$$

We now develop the mathematical formulation for the inverse operator given in (34) and its generalization given in (35).

5. Inverse operators

We introduce the usual inverse to $(\frac{d}{dt})^\lambda$, λ a positive integer, on the space, $C^0((\alpha,\beta);\Gamma^{p,sB})$. The operator, $(Q_{t_0}^t)^\lambda$, is defined as $(Q_{t_0}^t)^\lambda u(\tau)(x) = \int_{t_0}^t \frac{(t-\tau)^{\lambda-1}}{(\lambda-1)!} u(\tau)(x) d\tau$ where $[t_0, t] \subset (\alpha,\beta) \subset R$.

This section requires the use of multinomial coefficients, specialized Euclidean n-space points, specialized factorials and Euclidean n space summands. To enhance the readability of this section and in particular theorem 1, we introduce the following compact notation:

$|i| = i_1 + ... + i$,
$(\frac{|i|}{i}) = (i_1 + ... + i_i)!(i_1!i_2! \times ... \times i_i!)^{-1}$,
$n \bullet i = n_1 i_1 + ... + n_i i_i$,
$2n \bullet i = 2n_1 i_1 + ... + 2n_i i_i$,
$l \bullet i = 1i_i + 2i_2 + ... + li_i$,
$n \bullet i \uparrow = n_1 i_1! n_2 i_2 ... n_i i_i!$,
$l \bullet i \uparrow = 1i_1!2i_2...li_i!$,
$\sum_{|i|=0}^{\infty} = \sum_{|i_1|=0}^{\infty} \sum_{|i_2|=0}^{\infty} \cdots \sum_{|i_l|=0}^{\infty}$
$\sum_{|i|=0}^{n} = \sum_{|i_1|=0}^{n} \sum_{|i_2|=0}^{n} \cdots \sum_{|i_l|=0}^{n}$

THEOREM 1. *Given: (i) a continuous $u(t) \in C^0((\alpha,\beta);\Gamma^{p,sB}), (\alpha,\beta \subset R)$; (ii) a real valued scalar function, $B(\xi) = \sum_{|i|=0}^{\infty} b_i(\xi)^i$, with real coefficients, b_i, and nonzero radius of convergence of p; (iii) a polynomial transformation of the form, $P(Q_{t_0}^t \Delta^2) = \sum_{\lambda=1}^{i} (Q_{t_0}^t)^{\lambda} P_{\lambda}(\Delta^2)$, where P_λ, $(1 \le \lambda \le l)$are real valued polynomials of degree, n_λ, $(1 \le \lambda \le l)$ and where we formally replace the independent real variable with the infinite dimensional Laplacian operator of section 3, ie. $P_\lambda(\Delta^2) = \sum_{i=0}^{n_\lambda} d_i^{\lambda}(\Delta^2)^i$, $d_i^{\lambda} \in R$, $0 \le i \le n_\lambda, 1 \le \lambda \le l$; (iv) that p satisfies $1 < p < \frac{1}{1-\mu_\Delta}$ if $\mu_\Delta \ne 1$ and $p < \infty$ if $\mu_\Delta = 1$ where we define the order of $P(Q_{t_0}^t \Delta^2)$ to be $\mu_\Delta = min_{1 \le \lambda \le i}(\frac{\lambda}{2n_\lambda})$.*

Then the series $B(Q_{t_0}^t \Delta^2) = \sum_{i=0}^{\infty} b_i(P_\lambda(Q_{t_0}^t \Delta^2))^i$, obtained by substituting $P(Q_{t_0}^t \Delta^2)$ for ξ into $B(\xi)$ and formally expanding into a series of $(Q_{t_0}^t \Delta^2)$ monomials is applied to $u(t)$. This process gives another $v(t) \in \Gamma^{p,s'B}$ continuous for $t \in [t_0,t] \subset (\alpha,\beta)$ where $s' \ge (l+1)s$.

Proof. Use the multinomial expansion and formally obtain

$$
\begin{aligned}
B(P(Q_{t_0}^t \Delta^2)) &= \sum_{i=0}^{\infty} b_i(P(Q_{t_0}^t \Delta^2))^i = \sum_{i=0}^{\infty} b_i \sum_{|i|}(\tfrac{|b_i|}{i})(Q_{t_0}^t)^{l \bullet i} P_1^{i_1}...P_l^{i_l} \\
&= \sum_{|i|=0}^{\infty} b_{|i|}(\tfrac{|i|}{i})(Q_{t_0}^t)^{i \bullet i} P_1^{i_l}...P_l^{i_l} \\
&= \sum_{|i|=0}^{\infty} b_{|i|}(\tfrac{|i|}{i}(Q_{t_0}^t)^{l \bullet i} l[d_{n\bullet i}(\Delta^2)^{n \bullet i} + d_{n \bullet i-1}(\Delta^2)^{n \bullet i-1} + ... + d_0].
\end{aligned} \tag{36}
$$

Setting $K_\lambda = max(1, |d_n^\lambda| + ... + |d_0^\lambda|)$, $1 \le \lambda l$, it is clear that the d_j coefficients in (5.3) are majorized with $|d_j| \le K_1^{i_1} K_2^{i_2} ... K_i^{i_i} \equiv K^{|i|}$. The hypothesis, $u(t) \in \Gamma^{p,sB}$, implies $\sup_{\substack{t \in (\alpha,\beta) \\ q}} ||a_q(t)||_m q!^{\frac{1}{p}}(sB_m)^{-q} < M_m < \infty$.

This leads us to $||a_{q+j}(t)||_m \le M_m(q+j)!^{\frac{1}{p}}(sB_m)^{-q-j}$ for all t $in(\alpha,\beta)$, $j = 1,2,...$. Selecting $s' \ge (l+1)s$, we consider the following partial sums:

$$
B_n(P(Q_{t_0}^t \Delta^2)) = \sum_{|i|=0}^{n} b_{|i|}(\frac{|i|}{i})(Q_{t_0}^t)^{l \bullet i} \times [d_{n \bullet i}(\Delta^2)^{n \bullet i} + d_{n \bullet i-1}(\Delta^2)^{n \bullet i-1} + ... + d_0]. \tag{37}
$$

This then gives us

(38)

$$|||B_n(P(Q_{t_0}^t\Delta^2))u(t)|||_{s'B} \leq \sup_q \sum_{|i|}^n b_{|i|}(\tfrac{|i|}{i})(Q_{t_0}^t)^{l\bullet i}\bullet$$
$$[|d_{n\bullet i}|(q+2n\bullet i)!q!^{-1}||\int_{R^n}..\int_{R^n} a_{q+2n\bullet i}[\bullet,\delta_{\tau_1},\delta_{\tau_2},...,\delta_{\tau_{n-1}}]d\tau_1...d_{2n-1}||_m+...$$
$$...+d_{v_1}]q!(s'B_m)^{-q}$$

Introducing $(1+\tau_1^2)^2...(1+\tau_{n\bullet l}^2)^2$ into the numerator and denominator of the integrand in series (38) majorizes (38) with

$$M_m\sup_q\sum_{|i|=0}^n b_i(\frac{|i|}{i})\frac{|t-t_0|^{l\bullet i}}{(l\bullet i)!}\times[K^{|i|}(q+2n\bullet i)!^{-1}\frac{(sB_m)^{q+2n\bullet i}}{(q+2n\bullet i)!^{\frac{1}{p}}}+...+k^{|i|}]\bullet q!^{\frac{1}{p}}(s'B_m)^{-q}.$$

Note the following inequalities:

(39) $2n\bullet i<2\bullet 2^{n\bullet i},(\frac{|i|}{i})<l^{|i|},|b_{|i|}|<k\rho^{-|i|},(q+2n\bullet i)!\leq q!(2n\bullet i)\uparrow(l+1)^{q+2n\bullet i},$

and

(40) $$((l\bullet i)!)^{-1}\leq((l\bullet i)\uparrow)^{-1}.$$

Using inequalities (39) and (40) in the majorized form of expression (38), we obtain

(41)

$$2K'M_m\sup_q\sum_{|i|=0}^n\rho^{-|i|}\bullet l^{|i|}\times\tfrac{(t-t_0)^{l\bullet i}}{(l\bullet i)\uparrow}\times$$
$$2^{n\bullet i}K^{|i|}(2n\bullet i)\uparrow\bullet(l\bullet i)^{q+2n\bullet i}\times\tfrac{(sB_m)^{q+2n\bullet i}}{(q+2n\bullet i)!^{\frac{1}{p}}}\times\tfrac{q!^{\frac{1}{p}}}{(s'b_m)^q}\leq$$
$$2K'M_m\sup_q\tfrac{(s(l+1))^q}{(s')^q}\sum_{|i|=0}^n(\tfrac{K_1l}{e})^{i_l}...(\tfrac{K_ll}{e})^{i_l}\bullet|t-t_0|^{l\bullet i}(2\bullet(l+1)^2(sB_m^2)^{n_1i}...$$
$$\bullet 2\bullet(l+1)^2(sB_m^2)^{n_li_l})\bullet\tfrac{((2n\bullet i)\uparrow)^{1-\frac{1}{p}}}{l\bullet i\uparrow}$$

.

Employing inequality,

(42) $$\frac{n^n}{e^n}\leq n!\leq n^n$$

in expression (41), we obtain

(43)

$$2K'M_m\sup_q\tfrac{(s(l+1))^q}{(s')^q}\sum_{|i|=0}^n(\tfrac{K_1le}{p})^{i_1}...(\tfrac{K_lle}{p})^{i_l}.\ |t-t_0|^{l\bullet i}(2\bullet(l+1)^2(sB_m)^2(2n_1)^{2n_1i_1}...$$
$$2\bullet(l+1)^2(sB_m)^2(2n_i)^{2n_li_l})\bullet\tfrac{(i_1^{2n_1i_1}...i_l^{2n_li_l})^{1-\frac{1}{p}}}{i_1^{-i_1}...i_l^{-i_1}}.$$

Since hypothesis (iv) in the order, μ_Δ imples $1-\mu_\Delta+\varepsilon=\frac{1}{p}$ for $\varepsilon>0$, we have $n_\lambda(1-\frac{1}{p})\leq 1-n_\lambda$. Implementing this observation in expression (43) we have

(44)

$$2K'M_m \sup_q \tfrac{(s(l+1))^q}{(s')^q} \textstyle\sum_{|i|=0}^{n} (\tfrac{K_1 le}{p})^{i_1} \ldots (\tfrac{K_l le}{p})^{i_l} \; |t-t_0|^{l \bullet i} (2 \bullet (l+1)^2 (sB_m)^2 (2n_1)^{2n_1 i_1} \ldots . 2 \bullet$$

$$(l+1)^2 (sB_m)^2 (2n_l)^{2n_l i_l}) \bullet \tfrac{1}{[i_1^{-2n_1 i_1} \ldots i^{2n_1 i_l}]}.$$

We let $n \to \infty$ in expression (44). Consequently each series converges by the root test. If we select $s' \geq (l+1)s$, is bounded. As a result when $\varepsilon > 0, n \in N_+$, there exists an $N \in N_+$, such that $|||B_n(P(Q_{t_0}^t \Delta^2)) - B_m(P(Q_{t_0}^t \Delta^2))|||_{s'B_m} < \varepsilon$ for all $\varepsilon > 0, n, m > N$. $\square$

Therefore we have a Cauchy sequence in a Frechet space, $\Gamma^{p,s'B}$. It is clear that any rearrangement of the series converges to the same limit, $v(t) \in \Gamma^{p,s'B}$. The continuity of $v(t)$ follows for our infinite dimensional Laplacian operator in much the same manner as it did for the operator D_h in reference [13].

6. Impulsive differential equation solutions in Γ^{PB}

We generalize the impulsive differential equation, (1) , to be

$$\frac{\partial^i u(t)(x)}{\partial t^i} = \frac{\partial^{i-1}}{\partial t^{i-1}} P_1(\Delta^2) u(t)(x) + \ldots + P_i(\Delta^2) u(t)(x) \tag{45}$$

The Δ^2 in expression (45) is the infinite dimensional Laplacian operator developed in section III.

The $P_\lambda(\Delta^2), 1 \leq \lambda \leq l$ are polynomials having the independent variable formally replaced with the infinite dimensional Laplacian operator. We equip the solution of equation (45) to satisfy the initial impulsive state conditions, (29) and (30) where both the initial and impulsive conditions are in $\Gamma^{p,sB}$. A direct computation reveals the solution to be $v(t) = u(t_0^+)(x) + [I - Q_{t_0}^t P_1(\Delta^2) + \ldots + Q_{t_0}^t P_l(\Delta^2)] u(t)(x) + \sum_{t_{t_0} < t_k < t} I_k(u(t_k))$ for $t \in J^+$ as in reference [13]. Similarly for $t \in J^-$.

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