Rend. Sem. Mat. Univ. Politec. Torino
Vol. 70, 2 (2012), 139 – 149

A. Kamiński and S. Sorek

REMARKS ON PROOFS OF DIAGONAL THEOREM AND ITS APPLICATIONS IN THE THEORY OF DISTRIBUTIONS

Abstract. We recall and discuss main ideas of an elementary proof of the equivalence of the functional and sequential theories of distributions presented in [5], offering certain clarifications of its details. In particular, we present a modified proof of the theorem on the equivalence of the strong and weak boundedness in sequential Köthe spaces and discuss some of its nuances to avoid any doubts concerning its completeness (see Remark 2). We also present an alternative proof of Diagonal Theorem (see Remark 1) playing the main role in showing the latter equivalence.

1. Introduction

That the functional theory of distributions, initiated by S. L. Sobolev in [16] and created as a whole by L. Schwartz in his famous book [15], and the more elementary sequential approach, offered in [14] by J. Mikusiński and R. Sikorski (and extended by them in collaboration with P. Antosik in the book [5]), are equivalent is a consequence of the fact that the linear topologies of spaces considered in the functional theory are sequential. An elementary proof of the equivalence of the two approaches, without considering various topologies but the corresponding types of convergence of sequences of distributions, is given in [5]. The whole proof is very elegant and its beauty justifies one's wish to clarify all details and to resolve any potential readers' doubts concerning completeness of the reasoning.

The main idea of the proof consists in the use of Hermite expansions in the space of tempered distributions and the replacement of the two types of convergence in this space by the corresponding convergences of matrices of Hermite coefficients of the considered tempered distributions. This reduces the problem to the equivalence of strong and weak boundedness and, consequently, of strong and weak convergence in sequential Köthe spaces. In section 4, we present an essential modification of the original proof of the latter equivalence given in [5] (see also [11] and [12]) which, in our opinion, contains a subtle gap discussed in Remark 2 (at the end of section 4) and filled in due to our modification.

The chief tool in the proof of the equivalence of strong and weak boundedness in Köthe spaces is a version of Diagonal Theorem which has turned out to be very useful in proving numerous theorems in measure theory and functional analysis (see [7], [18] and some references thereof) as well as in the theory of generalized functions (see [10] for a recent application). In section 3, we recall this beautiful theorem in the form given in [1] (see also [5, pp. 217-219]), presenting its alternative proof and discussing some of its details (see Remark 1 in section 3).

The first author during the conference on generalized functions GF 2011 in Martinique presented several ideas connected with the equivalence of the two approaches to the theory of distributions. In this note we confine ourselves to an exact formulation of the principal result (Theorem 1 in section 2) and discussions concerning the proofs of the two theorems used in [5] as the chief tools in the proof of the equivalence of the approaches: Diagonal Theorem and the equivalence of the two types of boundedness in Köthe spaces.

2. Theories of distributions and Köthe spaces

Distributions, studied by L. Schwartz in [15] in terms of functional analysis, are defined in the sequential approach (see [14] and [5]) as equivalence classes, Mikusiński-Sikorski distributions, of so-called fundamental sequences of (continuous or smooth) functions on an open set $\Omega \subseteq \mathbb{R}^d$ (see [5, pp. 6–10, 63–65]). Consider the space $\mathcal{M}(\Omega)$ of all such classes with the sequentially strong $\xrightarrow{ss}$ and sequentially weak $\xrightarrow{sw}$ convergences (see [5, pp. 86, 232]), the space $\mathcal{T}$ of tempered distributions in the sense of [5, p. 165] and the inner product $(f,\varphi)_\Omega$ of $f \in \mathcal{M}(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, defined in [5, p. 174]. The equivalence of the functional and sequential theories of distributions discussed in [5, pp. 180–235] (cf. [17]) can be formulated as follows:

THEOREM 1. *Let $\Omega \subseteq \mathbb{R}^d$ be an open set. For every Schwartz distribution $T \in \mathcal{D}'(\Omega)$ there is a unique $f = f_T \in \mathcal{M}(\Omega)$ given by $(f,\varphi)_\Omega := < T,\varphi >$ for $\varphi \in \mathcal{D}(\Omega)$. Moreover, if $T_n \in \mathcal{D}'(\Omega)$ for $n \in \mathbb{N}_0$, then*

$$T_n \to T_0 \ \text{ in } \mathcal{D}'(\Omega) \quad \Leftrightarrow \quad f_{T_n} \xrightarrow{ss} f_{T_0} \ \text{ in } \Omega \quad \Leftrightarrow \quad f_{T_n} \xrightarrow{sw} f_{T_0} \ \text{ in } \Omega.$$

Conversely, if $f \in \mathcal{M}(\Omega)$, then there is a unique Schwartz distribution $T = T_f \in \mathcal{D}'(\Omega)$, given by $< T,\varphi >:= (f,\varphi)_\Omega$. If $f_n \in \mathcal{M}(\Omega)$ $(n \in \mathbb{N}_0)$, then

$$f_n \xrightarrow{ss} f_0 \ \text{ in } \Omega \quad \Leftrightarrow \quad f_n \xrightarrow{sw} f_0 \ \text{ in } \Omega \quad \Leftrightarrow \quad T_{f_n} \to T_{f_0} \ \text{ in } \mathcal{D}'(\Omega).$$

The mappings $\mathcal{M}(\Omega) \ni f \mapsto T_f \in \mathcal{D}'(\Omega)$ and $\mathcal{D}'(\Omega) \ni T \mapsto f_T \in \mathcal{M}(\Omega)$ are isomorphisms.

As shown in [5, pp. 215-235] (cf. [17]), Theorem 1 results from a similar theorem on isomorphism of the spaces $\mathcal{T}$ and $\mathcal{S}'$ following from the equivalence of strong and weak boundedness in Köthe spaces. We give in section 4 a complete proof of this equivalence (see Remark 2). We recall briefly elements of the Köthe theory, in a simplified form but sufficient for our aims (cf. [5], chapter 10).

Let Λ be a fixed countable set. A mapping $A\colon\ \Lambda \to \mathbb{R}$ will be called a *vector* and $a_\lambda := A(\lambda)$, for a given $\lambda \in \Lambda$, its λ-th *coordinate*; we will write $A = [a_\lambda]$. We call a vector A *positive* if its all coordinates are positive. By e_λ denote the vector whose λ-th coordinate is 1 and the remaining ones are 0. Denote the set of all (positive) vectors by $\mathfrak{R}$ (by $\mathfrak{R}_+$). It is clear that $\mathfrak{R}$ is a linear space over $\mathbb{R}$ with the usual coordinatewise operations. For a given $A = [a_\lambda] \in \mathfrak{R}_+$, we define the vector $A^{-1} := [a_\lambda^{-1}] \in \mathfrak{R}_+$. For

$A = [a_\lambda] \in \mathfrak{R}$, we define the vector $|A| := [|a_\lambda|] \in \mathfrak{R}$ and consider the two norms:

$$\|A\|_1 := \sum_{\lambda\in\Lambda} |a_\lambda|; \qquad \|A\|_\infty := \sup_{\lambda\in\Lambda} |a_\lambda|$$

(denoted in a different way in [5, p. 216]: by $\|A\|$ and $|A|$, respectively). If moreover $B = [b_\lambda] \in \mathfrak{R}$, we define $AB := [a_\lambda b_\lambda] \in \mathfrak{R}$ and the scalar product:

$$(A,B) := \sum_{\lambda\in\Lambda} a_\lambda b_\lambda := \lim_{n\to\infty} \sum_{i=1}^{n} a_{\lambda_i} b_{\lambda_i},$$

whenever $\|AB\|_1 = \sum_{\lambda\in\Lambda} |a_\lambda b_\lambda| < \infty$.

DEFINITION 1. *Let $\{V_i\}_{i\in\mathbb{N}}$ be a sequence of positive vectors such that*

$$(v) \qquad \|V_i V_{i+1}^{-1}\|_\infty < \infty \qquad \textit{for} \quad i \in \mathbb{N}.$$

A vector $A \in \mathfrak{R}$ is called rapidly decreasing *if $\|V_i A\|_1 < \infty$ for all $i \in \mathbb{N}$ and* tempered *if $\|V_{i_0}^{-1} A\|_\infty < \infty$ for some $i_0 \in \mathbb{N}$. The sets of all rapidly decreasing and all tempered vectors are denoted by $\mathfrak{S}$ and $\mathfrak{T}$, respectively.*

Under condition (v), for every $\eta > 0$ we may assume that $\|V_i V_{i+j}^{-1}\|_\infty < \eta^j$ for $i, j \in \mathbb{N}$. In fact, if $\|V_i V_{i+1}^{-1}\|_\infty =: \eta_{i+1}$, then denoting $\widetilde{V}_i := \eta^{-i}\eta_1 \cdot \ldots \cdot \eta_i V_i$, we have $\|\widetilde{V}_i \widetilde{V}_{i+j}^{-1}\|_\infty < \eta^j$ for $i, j \in \mathbb{N}$ and the sets $\mathfrak{S}$ and $\mathfrak{T}$ defined by the sequences $\{V_i\}$ and $\{\widetilde{V}_i\}$ are identical. Thus we may and will assume that

$$(1) \qquad \|V_i V_j^{-1}\|_\infty < 2^{i-j}, \qquad i, j \in \mathbb{N},\ i < j.$$

DEFINITION 2. *A set $\mathcal{A} \subseteq \mathfrak{T}$ of tempered vectors is called* 1° strongly bounded *if there are $i_0 \in \mathbb{N}$ and $\alpha > 0$ so that $\|V_{i_0}^{-1} A\|_\infty < \alpha$ for all $A \in \mathcal{A}$;* 2° weakly bounded *if the set $\{(A,S) : A \in \mathcal{A}\}$ is bounded for all $S \in \mathfrak{S}$.*

DEFINITION 3. *Let $S_n \in \mathfrak{S}$ for $n \in \mathbb{N}_0$. We say that S_n converges to S_0 in $\mathfrak{S}$ and write $S_n \xrightarrow{\mathfrak{S}} S_0$, whenever $\|V_i(S_n - S_0)\|_1 \to 0$ for all $i \in \mathbb{N}$.*

3. Diagonal Theorem

To prove Theorem 3 on the equivalence of two types of boundedness in Köthe spaces we need Diagonal Theorem, a formalization of the known sliding-hump technique. It was first shaped by Jan Mikusiński in [13] and then reformulated by Piotr Antosik in [1] (see also [5, p. 217]). The theorem and its later versions are very convenient for applications in measure theory and functional analysis (see e.g. [13], [7], [3], [18]). We will recall and discuss Diagonal Theorem in the version given in [1], for quasi-normed groups, with care over its formulation (see the comments below) and its proof. Notice that this version, due to a nice result proved in [8], implies the theorem formulated for topological groups (cf. [2]). An interesting extension of Diagonal Theorem was given by M. Florencio, P. J. Paúl and J. M. Virués in [9] (see also [6] and [4]). We begin with introducing some definitions and notations.

DEFINITION 4. *By a* quasi-normed group $(X,+,|\cdot|)$ *we mean an Abelian group* $(X,+)$ *endowed with a functional* $|\cdot|$*, a quasi-norm, satisfying the conditions:*

(N_1) $|0| = 0$;
(N_2) $|-x| = |x|, \qquad x \in X$;
(N_3) $|x+y| \leq |x| + |y|, \qquad x,y \in X$.

From conditions $(N_1)-(N_3)$ it follows that $|x| \geq 0$ for all $x \in X$ and

$$\min\{|x+y|,|x-y|\} \geq \big||x|-|y|\big|, \qquad x,y \in X. \tag{2}$$

If we put $d(x,y) := |x-y|$ for a given quasi-norm $|\cdot|$ and all $x,y \in X$, then d is clearly not a metric, in general, but a *quasi-metric* in X, i.e. it satisfies the symmetry and triangle conditions and $d(x,x) = 0$ for $x \in X$; in [1], the second sentence on page 306 is an obvious mistake. If the convergence $x_n \to x$ in X means that $d(x_n,x) = |x_n - x| \to 0$ as $n \to \infty$, then limits of convergent sequences are evidently not unique, in general.

Let $X = (X,+,|\cdot|)$ be a quasi-normed group (not necessarily complete) and let $\{x_n\}$ be a sequence in X. Assuming, for a fixed infinite subset J of $\mathbb{N}$, the condition:

$$\sum_{j\in J} |x_j| < \infty, \tag{3}$$

we introduce the notation:

$$\Big|\sum_{j\in J} x_j\Big| := \lim_{p\to\infty} \Big|\sum_{k=1}^{p} x_{j_k}\Big|, \tag{4}$$

where $\{j_k\}$ is the increasing sequence of all elements of J. No matter that limits in X may be not unique and X not complete, the limit in (4) exists and is unique, so the notation makes sense. This is because the sequence $\{y_p\}$, where $y_p := \big|\sum_{k=1}^{p} x_{j_k}\big|$ for $p \in \mathbb{N}$, is a numerical Cauchy sequence, in view of the inequalities (resulting from (2)):

$$|y_r - y_p| \leq \Big|\sum_{k=p+1}^{r} x_{j_k}\Big| \leq \sum_{k=p+1}^{r} |x_{j_k}|, \qquad p < r$$

and due to condition (3). The same applies to the numerical sequence $\{z_p\}$, where $z_p := \big|\sum_{k=1}^{p} x_{j_{\pi(k)}}\big|$ for an arbitrary bijection $\pi\colon \mathbb{N} \to \mathbb{N}$ and $p \in \mathbb{N}$. Moreover we have

$$\lim_{p\to\infty} \Big|\sum_{k=1}^{p} x_{j_k}\Big| = \lim_{p\to\infty} \Big|\sum_{k=1}^{p} x_{j_{\pi(k)}}\Big|. \tag{5}$$

Identity (5), which can be shown in a standard way (see e.g. [17]), justifies the use of the same symbol as in (4) for the following more general notation:

$$\Big|\sum_{i\in J} x_i\Big| := \lim_{n\to\infty} \Big|\sum_{i\in J_n} x_i\Big|, \tag{6}$$

where $\{J_n\}$ is an arbitrary nondecreasing sequence of finite subsets of J such that $\bigcup_{n=1}^{\infty} J_n = J$. Notation (6) is consistent also in case J is finite.

THEOREM 2 (cf. [1] and [5, pp. 217–219]). *Let* $x_{i,j} \in X$ $(i, j \in \mathbb{N})$

$$(H) \qquad \lim_{j\to\infty} |x_{i,j}| = 0 \qquad \textit{for} \;\; i \in \mathbb{N},$$

where $(X, +, |\cdot|)$ *is a quasi-normed group. Then there are an infinite set* $I \subseteq \mathbb{N}$ *and a set (finite or infinite)* $J \subseteq I$ *such that*

$$(A1) \quad \sum_{j\in J} |x_{i,j}| < \infty \quad (i \in I) \qquad \textit{and} \qquad (A2) \quad \Big|\sum_{j\in J} x_{i,j}\Big| \geq \frac{|x_{i,i}|}{2} \quad (i \in I).$$

Proof. I. We may assume that, for any finite $\Delta \subset \mathbb{N}$, the implication holds:

$$(*) \qquad \Big|\sum_{j\in\Delta} x_{i,j}\Big| \geq \frac{|x_{i,i}|}{2} \text{ for } i \in \Delta \;\Rightarrow\; \exists_{i_0>\Delta} \; \Big|\sum_{j\in\Delta} x_{i,j}\Big| < \frac{|x_{i,i}|}{2} \text{ for } i \geq i_0.$$

For, if $(*)$ does not hold for some finite Δ, then there is an increasing sequence (depending on Δ) of indices i_n such that $\left|\sum_{j\in\Delta} x_{i,j}\right| \geq \frac{1}{2}|x_{i,i}|$ for $i \in \Delta \cup I_0$, where $I_0 := \{i_n : n \in \mathbb{N}\}$, i.e. assertions $(A1)$ and $(A2)$ hold for $J := \Delta$ and $I := \Delta \cup I_0$. From now we will thus assume that implication $(*)$ is true.

II. Clearly, the left hand side of the implication is satisfied for $\Delta := \{1\}$. Hence, by $(*)$, there is an index $i_1 > 1$ such that

$$(7) \qquad |x_{i,i}| > 2\Big|\sum_{j\in\Delta} x_{i,j}\Big| \geq 0 \qquad \text{for} \quad i \geq i_1.$$

We will inductively construct an increasing sequence of indices $i_n \in \mathbb{N}$ with i_1 as above (and so of the sets $\Delta_n := \{i_1, \ldots, i_n\}$) and a sequence of numbers $\varepsilon_n \in (0, 1/2]$ with $\varepsilon_1 := 1/2$ satisfying the two conditions:

$$(8) \qquad \Big|\sum_{j\in\Delta_{r-1}} x_{i_r,j}\Big| = \left(\frac{1}{2} - \varepsilon_r\right)|x_{i_r,i_r}| \qquad \text{for} \quad r \in \mathbb{N}\setminus\{1\},$$

$$(9) \qquad \sum_{q=p+1}^{r} |x_{i_p,i_q}| < \varepsilon_p |x_{i_p,i_p}| \qquad \text{for} \quad p \in \mathbb{N}, \quad p < r \in \mathbb{N}\setminus\{1\}$$

and the third one:

$$(10) \qquad \Big|\sum_{j\in\Delta_r} x_{i,j}\Big| > \frac{1}{2}|x_{i,i}| \qquad \text{for} \quad i \in \Delta_r, \quad r \in \mathbb{N},$$

which is the antecedent of implication $(*)$ for $\Delta = \Delta_r$.

III. By (7), inequality (10) for $r = 1$ is obvious. Hence, by the consequent of implication $(*)$ for $\Delta = \Delta_1$, there is a natural index $i_1' > i_1$ such that

$$(11) \qquad |x_{i,i_1}| < \frac{1}{2}|x_{i,i}| \qquad \text{for} \quad i \geq i_1'.$$

In view of (H) and (7), we can find a natural $i_2 \geq i_1' > i_1$ such that

$$|x_{i_1,i_2}| < \frac{1}{2}|x_{i_1,i_1}|, \tag{12}$$

i.e. condition (9) is fulfilled for $p = 1, r = 2$. Defining

$$\varepsilon_2 := \left(\frac{1}{2}|x_{i_2,i_2}| - |x_{i_2,i_1}|\right)|x_{i_2,i_2}|^{-1}, \tag{13}$$

we see that (8) holds for $r = 2$ and $\varepsilon_2 \in (0,1/2]$, by (11) and (13). Moreover, we have $|x_{i_1,i_1} + x_{i_1,i_2}| > \frac{1}{2}|x_{i_1,i_1}|$ and $|x_{i_2,i_1} + x_{i_2,i_2}| > \frac{1}{2}|x_{i_2,i_2}|$, due to (12) and (11), respectively, which means that (10) holds for $r = 2$.

Suppose that, given a natural $s > 2$, we have selected positive integers $i_1 < \ldots < i_{s-1}$ and numbers $\varepsilon_1, \ldots, \varepsilon_{s-1} \in (0,1/2]$ so that (10) holds for all natural $r \leq s-1$, (8) holds for all $r \in \mathbb{N}$ such that $1 < r \leq s-1$ and (9) holds for all $p, r \in \mathbb{N}$ such that $p < r \leq s-1$. By (10) for $r = s-1$, the antecedent of implication $(*)$ and so its consequent hold for $\Delta = \Delta_{s-1}$. Hence, by (H), (7) and (9), assumed for $r = s-1$ and $p < s-1$, there is a common index $i_s > i_{s-1}$ such that $\sum_{q=p+1}^{s} |x_{i_p,i_q}| < \varepsilon_p |x_{i_p,i_p}|$ for $1 \leq p < s$ and

$$\Big|\sum_{j\in\Delta_{s-1}} x_{i,j}\Big| < \frac{1}{2}|x_{i,i}| \qquad \text{for} \quad i \geq i_s. \tag{14}$$

Hence (9) holds for $p < r = s$. Define

$$\varepsilon_s := \Big(\frac{1}{2}|x_{i_s,i_s}| - \Big|\sum_{j\in\Delta_{s-1}} x_{i_s,j}\Big|\Big)|x_{i_s,i_s}|^{-1}. \tag{15}$$

By (14) and (15), $0 < \varepsilon_s \leq 1/2$ and (8) for $r = s$ follows from (15). Also

$$\Big|\sum_{j\in\Delta_s} x_{i_s,j}\Big| \geq |x_{i_s,i_s}| - \Big|\sum_{j\in\Delta_{s-1}} x_{i_s,j}\Big| > \frac{1}{2}|x_{i_s,i_s}|,$$

in view of (14), and

$$\Big|\sum_{j\in\Delta_s} x_{i_p,j}\Big| \geq |x_{i_p,i_p}| - \Big|\sum_{j\in\Delta_{p-1}} x_{i_p,j}\Big| - \sum_{q=p+1}^{s} |x_{i_p,i_q}| > \frac{1}{2}|x_{i_p,i_p}|, \tag{16}$$

whenever $1 \leq p \leq s-1$ (with the sum over Δ_{p-1} vanishing for $p = 1$), by (8) for $r = p < s$ and (9) for $p < r = s$, i.e. condition (10) holds for $r = s$. By induction, conditions (10), (8) and (9) hold for all $p, r \in \mathbb{N}$ as indicated.

IV. We define $I = J := \{i_n : n \in \mathbb{N}\}$, where $\{i_n\}$ is the sequence just constructed. We conclude from (9) that $\sum_{q=p+1}^{\infty} |x_{i_p,i_q}| \leq \varepsilon_p |x_{i_p,i_p}| < \infty$ for $p \in \mathbb{N}$, i.e. $(A1)$ holds. By (8) and (9), we have, for all $p, n \in \mathbb{N}$, $p < n$,

$$\Big|\sum_{q=1}^{n} x_{i_p,i_q}\Big| \geq |x_{i_p,i_p}| - \Big|\sum_{j\in\Delta_{p-1}} x_{i_p,j}\Big| - \sum_{q=p+1}^{n} |x_{i_p,i_q}| > \frac{1}{2}|x_{i_p,i_p}|,$$

where the sum over Δ_{p-1} is 0 in case $p = 1$. Hence, as $n \to \infty$,

$$\Big|\sum_{j\in J} x_{i_p,j}\Big| = \lim_{n\to\infty}\Big|\sum_{q=1}^{n} x_{i_p,i_q}\Big| \geq \frac{1}{2}|x_{i_p,i_p}|, \qquad p \in \mathbb{N},$$

i.e. also assertion $(A2)$ of the theorem holds. □

REMARK 1. After reducing the above proof to the case where implication $(*)$ is true, we used $(*)$ to construct inductively sequences $\{i_n\}$ and $\{\varepsilon_n\}$ satisfying conditions (8), (9) and (10). Of these three conditions only (8) and (9) were used in the final part IV of the proof to show assertions (A1) and (A2) for $I = J = \{i_1, i_2, \ldots\}$. Condition (10) was needed merely to guarantee that the antecedent of implication $(*)$ is satisfied (and so $(*)$ can be properly used) for every $n \in \mathbb{N}$ in the construction. As a matter of fact, condition (10) can be deduced, for arbitrary $r \in \mathbb{N}\setminus\{1\}$, directly from (8) and (9), due to inequality (16) (with s replaced by r), which is a consequence of (N_3) and just conditions (8) and (9). This is another way to conclude (10) than it is shown above.

The above comment means that the original proof of Diagonal Theorem given in [1] (see also [5], section 10.5) does not contain a gap, though nothing is mentioned there that the set $\Delta = \Delta_p = \{i_1, \ldots, i_p\}$ satisfies the antecedent of implication $(*)$ for every $p \in \mathbb{N}\setminus\{1\}$.

The example below shows the situation where the set J cannot be taken to be equal I in the assertion of Theorem 2 (cf. [13], [1] and [5, p. 217]).

EXAMPLE 1. Consider the infinite matrix defined as follows: $x_{i,i} = 2, x_{i,j} = -1$ if $j < i$ and $x_{i,j} = 0$ if $j > i$ (for $i, j \in \mathbb{N}$). Obviously, assumption (H) is satisfied and the assertion holds for an arbitrary infinite $I \subset \mathbb{N}$ and a respective finite $J \subset I$. In fact, if $I := \{i_n \in \mathbb{N} \colon n \in \mathbb{N}\}$, where $i_n \uparrow \infty$, then $(A1)$ and $(A2)$ hold for $J := \{i_1\}$. On the other hand, for each infinite $I \subset \mathbb{N}$ and $J := I$, written in the form $I = J := \{i_n \in \mathbb{N} \colon n \in \mathbb{N}\}$ with $i_n \uparrow \infty$, there exists an $i \in I$, namely $i := i_3$, such that $\big|\sum_{j\in J} x_{i,j}\big| = 0$, so $(A2)$ is not satisfied for the indicated i.

4. Boundedness of sets in Köthe spaces

THEOREM 3. *Every countable set $\mathcal{A}$ of tempered vectors is strongly bounded if and only if it is weakly bounded.*

Proof. If the set $\mathcal{A} = \{A_n : n \in \mathbb{N}\} \subseteq \mathfrak{T}$ is strongly bounded, then $\|V_j^{-1}A_n\|_\infty \leq \alpha$ for some $j \in \mathbb{N}$, $\alpha > 0$ and for all $n \in \mathbb{N}$. If $S \in \mathfrak{S}$, then $\|V_iS\|_1 < \beta < \infty$ for all $i \in \mathbb{N}$. Hence $|(A_n, S)| \leq \alpha\beta$ for all $n \in \mathbb{N}$, i.e. the set $\mathcal{A}$ is weakly bounded. Let $V_i := [v_{i,j}]$ and $A_n := [a_{n,j}]$ $(i, j, n \in \mathbb{N})$. Suppose that $\mathcal{A}$ is weakly but not strongly bounded, i.e. (a) the sequences $\{\|V_i^{-1}A_n\|_\infty\}$ are unbounded for all $i \in \mathbb{N}$, and (b) for any $i, r \in \mathbb{N}$ there are constants $\alpha_{i,r} \geq 1$ such that

$$|(A_n, V_i^{-1}e_r)| = |v_{i,r}^{-1}a_{n,r}| = \|V_i^{-1}e_rA_n\|_\infty \leq \alpha_{i,r} \tag{17}$$

for all $n \in \mathbb{N}$, since $V_i^{-1} e_r \in \mathfrak{S}$. Using (a) and (b), we will construct inductively increasing sequences $\{n_i\}$ and $\{r_i\}$ of positive integers such that the following inequalities, given below in two equivalent forms, are true:

$$(18) \qquad \|V_i^{-1} e_{r_i} A_{n_i}\|_\infty > \|V_i^{-1} A_{n_i}\|_\infty - 1 > \alpha_i, \qquad i \in \mathbb{N}$$

and

$$(19) \qquad |v_{i,r_i}^{-1} a_{n_i,r_i}| > \sup_{j\in\mathbb{N}} |v_{i,j}^{-1} a_{n_i,j}| - 1 > \alpha_i, \qquad i \in \mathbb{N},$$

where α_i are defined, via constants $\alpha_{i,r}$ satisfying (17), as follows:

$$(20) \qquad \alpha_1 := \alpha_{1,1}; \qquad \alpha_i := \max\{\alpha_{i,r} :\ r \le r_{i-1}\} + \alpha_{i-1}, \qquad i > 1.$$

Clearly, since $\alpha_{i,r} \ge 1$ for all $i, r \in \mathbb{N}$, the definition in (20) guarantees that $\alpha_n \uparrow \infty$. Find, by (a), an $n_1 \in \mathbb{N}$ to fulfill the second inequality and then, by the definition of $\|\cdot\|_\infty$, an $r_1 \in \mathbb{N}$ to satisfy, for $i = 1$, the first inequality in (18) or, equivalently, in (19). Assume that indices $n_1 < \ldots < n_p$ and $r_1 < \ldots < r_p$ satisfying (18) for $i = 1, \ldots, p$ are chosen. Apply (a) for $i = p+1$ to find an index $n_{p+1} > n_p$ such that $\|V_{p+1}^{-1} A_{n_{p+1}}\|_\infty > \alpha_{p+1} + 1$. By the definition of $\|\cdot\|_\infty$, there is an $r_{p+1} \in \mathbb{N}$ such that the first inequality in (18) holds for $i = p+1$, i.e. $\|V_{p+1}^{-1} e_{r_{p+1}} A_{n_{p+1}}\|_\infty > \alpha_{p+1}$. But, by (17) and (20), we have

$$\|V_{p+1}^{-1} e_r A_{n_{p+1}}\|_\infty \le \alpha_{p+1,r} \le \alpha_{p+1} \qquad \text{for all } r \le r_p,$$

i.e. the index r_{p+1} just found cannot be among indices $r \le r_p$. Consequently, it must be $r_{p+1} > r_p$. Thus the inductive construction of increasing sequences $\{n_i\}$ and $\{r_i\}$ satisfying (18) and (19) is completed. Put $x_{i,j} := (A_{n_i}, V_j^{-1} e_{r_j}) \in \mathbb{R}$ for $i, j \in \mathbb{N}$. Since A_{n_i} are tempered, there are $p_i \in \mathbb{N}$ and $\beta_i > 0$ so that $\|V_{p_i}^{-1} A_{n_i}\|_\infty \le \beta_i < \infty$ for all $i \in \mathbb{N}$. Hence

$$|x_{i,j}| \le \|V_{p_i}^{-1} A_{n_i}\|_\infty \cdot \|V_{p_i} V_j^{-1}\|_\infty \le \beta_i \cdot 2^{p_i - j}, \qquad i, j \in \mathbb{N},$$

due to (1), and so $\lim_{j\to\infty} |x_{i,j}| = 0$ for every $i \in \mathbb{N}$. It follows from Diagonal Theorem that there exist an infinite set $I \subseteq \mathbb{N}$ and its subset J (finite or infinite) such that $\sum_{j\in J} |(A_{n_i}, V_j^{-1} e_{r_j})| < \infty$ and we have, for all $i \in I$,

$$(21) \qquad \frac{1}{2}|(A_{n_i}, V_i^{-1} e_{r_i})| \le \Big|\sum_{j\in J} (A_{n_i}, V_j^{-1} e_{r_j})\Big| = \lim_{n\to\infty} \Big|\sum_{j\in J_n} (A_{n_i}, V_j^{-1} e_{r_j})\Big|,$$

where J_n are finite sets forming a nondecreasing sequence such that $\bigcup_{n=1}^\infty J_n = J$ (i.e. $J_n = J$ for sufficiently large n in case J is finite). We are now in a position to define the vector R whose r_j-th coordinate coincides with the r_j-th coordinate of V_j^{-1} for all $j \in J$ and the remaining coordinates are equal to 0 (we have $0 \le R \le V_j^{-1}$), i.e.

$$(22) \qquad R := \sum_{j\in J} e_{r_j} V_j^{-1} = \lim_{n\to\infty} R_n,$$

where $R_n := \sum_{j\in J_n} e_{r_j} V_j^{-1}$ and the limit is meant coordinatewise (convergence in a stronger sense is shown below). Since the constructed sequence $\{r_j\}$ of indices is strictly increasing, the above definition of R is correct also in case J is infinite (see Remark 2). Only the latter case is sufficient to be considered in the proof that $R \in \mathfrak{S}$ and that (22) holds also in the sense of the convergence in $\mathfrak{S}$. Fix $i \in \mathbb{N}$ and denote $J_{p,q} := \{j \in J : \ p < j \leq q\}$, whenever $p, q \in \mathbb{N}$ and $i < p < q$. For any triple of such indices, we have

$$\Big\|V_i \sum_{j \in J_{p,q}} e_{r_j} V_j^{-1}\Big\|_1 \leq \sum_{j \in J_{p,q}} \|V_i V_j^{-1}\|_\infty \leq \sum_{j=p+1}^{q} 2^{i-j},$$

which implies that $\|V_i R\|_1 < \infty$ and $\lim_{n\to\infty} \|V_i(R - R_n)\|_1 = 0$ for all $i \in \mathbb{N}$, i.e. $R \in \mathfrak{S}$ and $R_n \xrightarrow{\mathfrak{S}} R$. Hence, by (21), we have

$$|(A_{n_i}, R)| = \lim_{n\to\infty} |(A_{n_i}, R_n)| = \Big|\sum_{j \in J} (A_{n_i}, V_j^{-1} e_{r_j})\Big| \geq \frac{1}{2} |(A_{n_i}, V_i^{-1} e_{r_i})|,$$

which means, by (18), that $|(A_{n_i}, R)| \to \infty$ and this contradicts the assumption that the set $\mathcal{A}$ is weakly bounded. $\square$

REMARK 2. In the original proof of the assertion of Theorem 3, presented in [5], section 10.6 (see also earlier proofs in [11, pp. 73-76], and in [12]), the construction of the sequences $\{n_i\}$ and $\{r_i\}$ does not guarantee that $\{r_i\}$, contrary to $\{n_i\}$, is increasing. This fact seems to be marked in the cited texts: in [5], p. 220, lines 13 and 16, and even more clearly in [11], p. 74, lines 2-3, and in [12], p. 18, line 19. This means that á priori the sequence $\{r_i\}$, constructed in the original proof, may be bounded. In fact, to get the first of the two inequalities in each of the equivalent formulas (18) and (19) (corresponding to the inequality in formula (3) in [5], section 10.6), one applies just the definition of the norm $\|\cdot\|_\infty$ to choose the r_i-th coordinate of the vector $|V_i^{-1} A_{n_i}|$ sufficiently close to the supremum of all its coordinates. Evidently, one cannot conclude, without an additional reasoning, that the indices r_i increase as i grow to infinity.

However, the definition of the vector R in (22) (in case the set J is infinite) is incorrect if, for a certain $r \in \mathbb{N}$, the equation $r_i = r$ holds for infinitely many indices $i \in \mathbb{N}$, i.e. if the sequence $\{r_i\}$ contains constant subsequences. Fortunately, this bad possibility was eliminated in the above proof by imposing on the inductively constructed sequences $\{n_i\}$ and $\{r_i\}$ of positive integers the additional requirement in the form of the second inequality in (18) (or, equivalently, in (19)) with the constants α_i defined by formula (20), forcing the strict increase of $\{r_i\}$. Consequently, the requirement (satisfied due to conditions (a) and (b)) guaranteed the strict increase of both sequences $\{n_i\}$ and $\{r_i\}$ and the correctness of the definition of the vector R.

Acknowledgments. The first author wishes to express his gratitude to the organizers of the GF 2011 conference in Fort de France for invitation and financial support during the conference. We would like to thank Professors Piotr Antosik, Józef Burzyk and Zbigniew Lipecki for valuable discussions and thank the anonymous reviewer for the suggestions which allowed us to improve the final form of this article.

References

[1] Antosik, P. On the mikusiński diagonal theorem. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 19 (1971), 305–310.

[2] Antosik, P. A generalization of the diagonal theorem. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 20 (1972), 373–377.

[3] Antosik, P. A lemma on matrices and its applications. *Contemporary Math.*, 52 (1986), 89–95.

[4] Antosik, P., Kamiński, A., and Sorek, S. On the proof of the florencio-paúl-virués diagonal theorem. preprint.

[5] Antosik, P., Mikusiński, J., and Sikorski, R. *Theory of Distributions. The Sequential Approach*. Elsevier-PWN, Amsterdam-Warszawa, 1973. the Russian improved and extended edition: Mir, Moscow 1976.

[6] Antosik, P., and Saeki, S. A lemma on set functions and its applications. *Dissertationes Math*, 340 (1995), 13–21.

[7] Antosik, P., and Swartz, C. Matrix methods in analysis. In *Lecture Notes in Math*, no. 1113. Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1985.

[8] Burzyk, J., and Mikusiński, P. On normability of semigroups. *Bull. Acad. Polon. Sci. Sér. Sci. Math.*, 28 (1980), 33–35.

[9] Florencio, M., Paúl, P. J., and Virués, J. M. On the mikusiński-antosik diagonal theorem. *Bull. Pol. Acad. Sci. Math.*, 40 (1992), 189–195.

[10] Kamiński, A., and Mincheva-Kamińska, S. On the equivalence of the mikusiński-shiraishi-itano products in $\mathcal{S}'$ for various classes of delta-sequences. *Integral Transforms Spec. Func.*, 20 (2009), 207–214.

[11] Mikusiński, J. Lectures on the constructive theory of distributions. *Univ. of Florida, Dept. of Math.* (1969).

[12] Mikusiński, J. On spaces of sequences. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 17 (1969), 17–20.

[13] Mikusiński, J. A theorem on vector matrices and its applications in measure theory and functional analysis. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 18 (1970), 193–196.

[14] Mikusiński, J., and Sikorski, R. The elementary theory of distributions. *Dissertationes Math. 1, 2*, 12, 25 (1957, 1961).

[15] Schwartz, L. *Théorie des distributions*, vol. 1, 2. Hermann, Paris, 1950–51.

[16] Sobolev, S. L. Méthode nouvelle à résoudre le problème de cauchy pour les équations hyperboliques normales. *Recuell Math. (Math. Sbornik)*, 1 (1936), 39–71.

[17] Sorek, S. Some irregular operations on distributions and their applications. Ph.D. Thesis (supervisor: A. Kamiński), in Polish.

[18] Swartz, C. *Infinite Matrices and the Gliding Hump*. World Scientific, Singapure, 1996.

AMS Subject Classification: Primary 46F05, 15A45; Secondary 40C05, 46A45

Andrzej KAMIŃSKI, Sławomir SOREK,
Institute of Mathematics, University of Rzeszów
Rejtana 16A, 35-310 Rzeszów, POLAND
e-mail: akaminsk@univ.rzeszow.pl, ssorek@univ.rzeszow.pl

Lavoro pervenuto in redazione il 28.09.2012.