

Ghislain R. Franssens

THE CONVOLUTION AND MULTIPLICATION OF ONE-DIMENSIONAL ASSOCIATED HOMOGENEOUS DISTRIBUTIONS

Abstract. The set of Associated Homogeneous Distributions (AHDs) with support in R is an important subset of the tempered distributions, since it contains the majority of the (one-dimensional) distributions typically encountered in physics applications. In previous work of the author, a convolution and multiplication product for AHDs on R was defined and fully investigated. The aim of this paper is to give an easily readable introduction to these new distributional algebras.

The constructed algebras are internal to Schwartz' theory of distributions and, when one restricts to AHDs, provide a simple alternative for any of the larger generalized function algebras, now used in non-linear models. Our approach belongs to the same class as certain methods of renormalization used in quantum field theory and which are known in the distributional literature as multi-valued methods. Products of AHDs on R , based on our definition, are generally multi-valued only at critical degrees of homogeneity. Unlike other definitions proposed earlier in this class of methods, the multi-valuedness of our products is canonical in the sense that it involves at most one arbitrary constant.

As a simple example of the convolution and multiplication products that are obtained by our method, the product tables of HDs of integer degree are presented.

Keywords: Generalized function, Associated homogeneous distribution, Convolution, Multiplication.

1. Introduction

Homogeneous Distributions (HDs) are the distributional analogue of homogeneous functions, such as $|x|^z : R \rightarrow \mathbb{C}$, which is homogeneous with complex degree z . Associated to homogeneous functions are power-log functions, which arise when taking the derivative with respect to the degree of homogeneity z . The set of Associated Homogeneous Distributions (AHDs) with support in the line R , and which we denote by $\mathcal{H}'(R)$, is the distributional analogue of the set of power-log functions with domain in R . For compactness of notation, we will drop (R) in $\mathcal{H}'(R)$ and other sets.

The set \mathcal{H}' is an interesting and important subset of the distributions of slow growth (or tempered distributions), \mathcal{S}' , [22], [25]. For instance, \mathcal{H}' is, just as \mathcal{S}' , closed under Fourier transformation. In addition, \mathcal{H}' contains the majority of the (one-dimensional) distributions one typically encounters in physics applications, such as the delta distribution δ , the eta distribution $\eta \triangleq \frac{1}{\pi}x^{-1}$ (a normalized Cauchy's principal value $\text{Pv} \frac{1}{x}$), the Heaviside step distributions 1_{\pm} , several so called pseudo-functions generated by taking Hadamard's finite part of certain divergent integrals, associated Riesz kernels, generalized Heisenberg distributions, all their generalized derivatives and primitives, and many familiar others, [16].

One-dimensional AHDs were first considered in [14]. The various (and sometimes inconsistent) definitions under which they have been introduced in the literature were closely examined in [23]. In a series of papers, [11]–[12], the present author undertook a detailed study of the set \mathcal{H}' , resulting in the construction of a convolution product and an isomorphic multiplication product. The multiplication product for AHDs on R provides a non-trivial example of how multiplication can be defined for a useful subset of distributions, containing a derivation and the delta distribution, and how this construction is influenced by Schwartz' "impossibility theorem", [21]. Our definition results in a convolution product and a multiplication product that is non-commutative and non-associative. However, these products deviate from commutativity and associativity only at critical degrees of homogeneity and in a minimal and interesting way, as explained below. The resulting product structures $(\mathcal{H}', *)$ and (\mathcal{H}', \cdot) are derivation magmas with identity.

Our approach towards defining distributional products for AHDs on R belongs to what in the distribution literature is called the class of multi-valued products, [18]. It remains entirely within the scope of Schwartz' distribution theory and thus provides a simpler alternative for the multiplication of AHDs on R than the larger generalized function algebras, such as e.g., [3]. It is related, but not identical to and in a sense more natural than, certain methods of renormalization in quantum field theory, such as in [15], [1], [24], [17].

In the following sections, we start with giving the definition for AHDs on R and list some of their immediate properties. Then, we introduce a few distributional concepts that were found useful in the treatment of AHDs on R . Thereafter, we proceed to show how a closed convolution product on \mathcal{H}' can be defined, first for non-critical products and then for critical products. Based on the generalized convolution theorem, the definition of a multiplication product on \mathcal{H}' is then stated. We discuss the non-commutativity of critical products and the non-associativity of critical triple products and exhibit the elegant nature of this lack of commutativity and associativity.

In order to guide the reader who wishes to consult or verify the underlying calculations we provide the following road map to the previous papers:

(i) Notation, definitions, properties and an extensive list of properties of basis AHDs on R are derived and collected in [11].

(ii) Structure theorems for AHDs on R , being complex holomorphic in their degree of homogeneity in some $\Omega \subseteq \mathbb{C}$, are proved in [7].

(iii) The construction of the convolution product for AHDs on R , in case the resulting degree of homogeneity is not a natural number, called a non-critical product, is developed in [5]. Here is also proved the associativity of non-critical triples under convolution.

(iv) The completion of the construction of the convolution product for AHDs on R , in case the resulting degree of homogeneity is a natural number, called a critical product, is given in terms of a functional extension process in [6]. Combining the results from [5] and [6] then shows that $(\mathcal{H}', *)$ is closed.

(v) The general convolution product formula for AHDs on R is derived in [8].

Here is also proved the particular form of the non-associativity of critical triple convolution products.

(vi) The general multiplication product formula for AHDs on R is derived in [10]. Here is also proved the particular form of the non-associativity of critical triple multiplication products.

(vii) The structures $(\mathcal{H}', *)$ and (\mathcal{H}', \cdot) contain various interesting abstract algebraic substructures. These are investigated in [12]. The most important substructure of $(\mathcal{H}', *)$ is a particular subgroup, which contains AHDs that, when used as kernels of convolution operators, allow to define integration operators of complex degree over the whole line R .

We use the notation and definitions introduced in [11].

2. Associated Homogeneous Distributions on R

2.1. Definition

A distribution $f_0^z \in \mathcal{D}'$ is called a (positively) homogeneous distribution of degree of homogeneity $z \in \mathbb{C}$ iff it satisfies for any $r > 0$,

$$(1) \quad \langle f_0^z, \varphi(x/r) \rangle = r^{z+1} \langle f_0^z, \varphi(x) \rangle, \forall \varphi \in \mathcal{D},$$

or, using [11, eq. (63)], $(f_0^z)_{(rx)} = r^z (f_0^z)_{(x)}$. A homogeneous distribution is also called an associated homogeneous distribution of order $m = 0$.

A distribution $f_m^z \in \mathcal{D}'$ is called an associated (positively) homogeneous distribution of degree of homogeneity $z \in \mathbb{C}$ and order of association $m \in \mathbb{Z}_+$, iff there exists a sequence of associated homogeneous distributions f_{m-l}^z of degree of homogeneity z and associated order $m-l$, $\forall l \in \mathbb{Z}_{[1,m]}$, not depending on r and with $f_0^z \neq 0$, satisfying,

$$(2) \quad \langle f_m^z, \varphi(x/r) \rangle = r^{z+1} \left\langle f_m^z + \sum_{l=1}^m \frac{(\ln r)^l}{l!} f_{m-l}^z, \varphi(x) \right\rangle, \forall \varphi \in \mathcal{D}.$$

Differentiate (2) l times with respect to r , put $r = 1$, use the definitions [11, eqs. (29) and (32)] of the generalized derivatives, and [11, eq. (38)]. This yields the system, $\forall l \in \mathbb{Z}_{[1,m+1]}$,

$$(3) \quad X_z f_{m-(l-1)}^z = f_{m-l}^z,$$

wherein we set $f_{-1}^z = 0$ and

$$(4) \quad X_z \triangleq \mathbf{X} \cdot \mathbf{D} - z \text{Id},$$

wherein $\mathbf{X} \cdot \mathbf{D}$ is the generalized Euler operator and Id the identity operator. The system (3) can be used as an equivalent for definition (2) and generalizes Euler's theorem on homogeneous functions to AHDs on R .

2.2. General properties

The following properties of AHDs on R easily follow from (2) or (3).

(i) AHDs of the same order m , but of different degrees $\{z_1, \dots, z_k\}$, are linearly independent.

(ii) AHDs of the same degree z , but of different orders $\{m_1, \dots, m_k\}$, are linearly independent. Any such linear combination is again an AHD of degree z and of order $m \leq \max\{m_1, \dots, m_k\}$.

(iii) Let f_m^z be an AHD of order m which is complex analytic in its degree $z \in \Omega \subset \mathbb{C}$. If f_m^z has an analytic extension $(f_m^z)_{a.e}$ to a region $\Omega_1 \supset \Omega$, then $(f_m^z)_{a.e}$ is a unique AHD of degree z and order m due to the uniqueness of the process of analytic continuation, [14, p. 150].

(iv) AHDs are distributions of slow growth: $\mathcal{H}' \subset \mathcal{S}'$. A proof for homogeneous distributions can be found in [4, pp. 154–155]. By using in addition [7, Theorem 1], the fact that $D_z f_m^z$ is associated of order $m+1$ and of the same degree z , linearity and induction, it follows that any AHD is a distribution of slow growth.

(v.1) If $f_p^a \cdot g_q^b$ exists (as a distribution) and neither f_p^a nor g_q^b is a zero divisor, then this multiplication product is associated of order $m = p+q$ and of degree $a+b$. Under these conditions, an injective multiplication operator with a homogeneous kernel of degree 0 is a map from $\mathcal{H}'_m \rightarrow \mathcal{H}'_m$. In particular, the parity reversal transformation (i.e., the multiplication operator $S \triangleq -i \operatorname{sgn}.$, see [11, eq. (86)]) then preserves the degree of homogeneity and order of association.

(v.2) If $f_p^{a-1} * g_q^{b-1}$ exists (as a distribution) and neither f_p^{a-1} nor g_q^{b-1} is a zero divisor, then this convolution product is associated of order $m = p+q$ and of degree $a+b-1$. Under these conditions, an injective convolution operator with a homogeneous kernel of degree -1 is a map from $\mathcal{H}'_m \rightarrow \mathcal{H}'_m$. In particular, the Hilbert transformation (i.e., the convolution operator $H \triangleq \eta*$, see [11, eq. (172)]) then preserves the degree of homogeneity and order of association.

(vi) The Fourier transformation \mathcal{F} maps any AHD to an AHD, such that:

- (a) the order of association m is preserved,
- (b) the degree of homogeneity z is mapped to $-(z+1)$,
- (c) the parity of the distribution is preserved.

A more detailed overview of the properties of AHDs on R and many specific properties of several important basis AHDs on R can be found in [11].

3. Method

3.1. Preliminaries

DEFINITION 1. A partial distribution is a linear and sequentially continuous functional that is only defined on a proper subset $\mathcal{D}_r \subset \mathcal{D}$. Similarly, a partial tempered distribution is a linear and sequentially continuous functional that is only defined on a proper subset $\mathcal{S}_r \subset \mathcal{S}$.

DEFINITION 2. An extension f_ε from \mathcal{D}_r to \mathcal{D} of a partial distribution f is a distribution $f_\varepsilon \in \mathcal{D}'$, defined $\forall \varphi \in \mathcal{D}$, such that $\langle f_\varepsilon, \psi \rangle = \langle f, \psi \rangle$, $\forall \psi \in \mathcal{D}_r \subset \mathcal{D}$. Similarly, an extension f_ε from \mathcal{S}_r to \mathcal{S} of the partial tempered distribution f is a tempered distribution $f_\varepsilon \in \mathcal{S}'$, defined $\forall \varphi \in \mathcal{S}$, such that $\langle f_\varepsilon, \psi \rangle = \langle f, \psi \rangle$, $\forall \psi \in \mathcal{S}_r \subset \mathcal{S}$.

\mathcal{D} is a sequentially complete, locally convex, Hausdorff topological linear space, [20, p. 152], [2, pp. 427–431]. Schwartz' space \mathcal{S} is a Fréchet space, [20, p. 184], [13, Appendix], and also a Fréchet space is locally convex [20, p. 9]. Since $\mathcal{D}(\mathcal{S})$ is locally convex the continuous extension version of the Hahn-Banach theorem applies, [19, p. 56]. This theorem ensures that an extension f_ε of the partial distribution f , only defined on $\mathcal{D}_r \subset \mathcal{D}$ ($\mathcal{S}_r \subset \mathcal{S}$), exists as a sequentially continuous linear functional on $\mathcal{D}(\mathcal{S})$ and that $\langle f_\varepsilon, \psi \rangle = \langle f, \psi \rangle$, $\forall \psi \in \mathcal{D}_r$ (\mathcal{S}_r), [20, p. 61]. If \mathcal{D}_r (\mathcal{S}_r) is dense in \mathcal{D} (\mathcal{S}), then the extension f_ε is unique. Otherwise, an extension f_ε may or may not be unique, [20, p. 56], [2, p. 424]. Let \mathcal{D}'_r (\mathcal{S}'_r) denote the continuous dual of \mathcal{D}_r (\mathcal{S}_r). The subset of \mathcal{D}'_r (\mathcal{S}'_r) which maps \mathcal{D}_r (\mathcal{S}_r) to zero is called the annihilator of \mathcal{D}_r (\mathcal{S}_r) and denoted by $\mathcal{D}'_r{}^\perp$ ($\mathcal{S}'_r{}^\perp$). Any two extensions $f_{\varepsilon,1}$ and $f_{\varepsilon,2}$ from \mathcal{D}_r (\mathcal{S}_r) to \mathcal{D} (\mathcal{S}) differ by a generalized function $g \in \mathcal{D}'_r{}^\perp$ ($\mathcal{S}'_r{}^\perp$).

The classical process of the regularization of divergent integrals, as given in [14] and which goes back to Hadamard, is here placed in the more general context of a functional extension process, justified by the Hahn-Banach theorem. We will refer to this method as “extension of partial distributions”. The advantage of regarding regularization as an extension process is that it clearly exhibits the unavoidable non-uniqueness of regularization.

The Hahn-Banach theorem does not say how an extension is to be constructed. A natural extension process however is the following, inspired by the regularization of divergent integrals, [14, p. 10 and p. 45]. This procedure consists in the construction of a projection operator $T : \mathcal{D} \rightarrow \mathcal{D}_r$ and to replace an integral, convergent $\forall \psi \in \mathcal{D}_r$ but divergent for test functions $\varphi \in \mathcal{D} \setminus \mathcal{D}_r$,

$$(5) \quad \int_{-\infty}^{+\infty} f(x) \psi(x) dx,$$

by the integral

$$(6) \quad \int_{-\infty}^{+\infty} f(x) (T\varphi)(x) dx,$$

which now by construction is convergent $\forall \varphi \in \mathcal{D}$. The integral (6) is then said to be a regularization of the integral (5). Equivalently, we could say that (6) defines a distribution f_ε which is an extension of the partial distribution f defined by (5). The non-uniqueness of this particular regularization method stems from the non-uniqueness of the projection operator T . Particular choices for the projection operator T give also rise to: (i) analytic continuations of AHDs on \mathcal{R} , see [11, eq. (100)], and (ii) the special extensions, denoted with subscript 0, which coincide with M. Riesz' finite analytic part. For details, see [11].

DEFINITION 3. *The equivalence class of extensions $[f_e]$ from S_r to S of a partial distribution $f \in \mathcal{H}'$ is the set of all extensions $f_e \in \mathcal{H}'$ from S_r to S of f , together with the equivalence relation given in (7) or in (8).*

In the theory of AHDs on R , regularization is only required at integer degrees of homogeneity. In addition, it turns out that partial distributions in \mathcal{H}' are defined on only (i) $S_{\{k\}}$ (the subspace of S whose members have zero k -th order moment) or (ii) $S_{\{-l\}}$ (the subspace of S whose members have zero $|-l+1|$ -th order derivative at the origin), with $k \in \mathbb{N}$ and $l \in \mathbb{Z}_+$. Therefore, only the following two equivalence classes, $\sim_{\mathbb{N}}$ and $\sim_{\mathbb{Z}_-}$, of extensions in \mathcal{H}' are needed,

$$(7) \quad [f^k] \triangleq \left\{ f^k + cx^k \in \mathcal{H}' : f^k + cx^k \sim_{\mathbb{N}} f^k, \forall c \in \mathbb{C} \right\}, \forall k \in \mathbb{N},$$

$$(8) \quad [f^{-l}] \triangleq \left\{ f^{-l} + c\delta^{(l-1)} \in \mathcal{H}' : f^{-l} + c\delta^{(l-1)} \sim_{\mathbb{Z}_-} f^{-l}, \forall c \in \mathbb{C} \right\}, \forall l \in \mathbb{Z}_+.$$

Finally, we will need the following.

DEFINITION 4. *The convolution product of any two AHDs on R of degrees $a-1$ and $b-1$ is called a critical (convolution) product, iff the resulting degree $a+b-1 \triangleq k \in \mathbb{N}$.*

The multiplication product of any two AHDs on R of degrees a and b is called a critical (multiplication) product, iff the resulting degree $a+b \triangleq -l \in \mathbb{Z}_-$.

3.2. Convolution

Let \mathcal{D}'_R denote the distributions based on R with support bounded on the left and \mathcal{D}'_L denote the distributions based on R with support bounded on the right, [22, vol II, p. 28-30]. One of our structure theorems (the normalized half-lines representation) states that any AHD on R is the sum of an AHD in \mathcal{D}'_L and an AHD in \mathcal{D}'_R , [7, Theorem 1]. Our definition of the convolution product of AHDs on R then comprises three cases.

Any degree

Case 1. The factors in the convolution product have one-sided support, bounded at the same side. In this case we use the standard definition, involving the direct product, e.g., [25, p. 123, eq. (2) and Theorem 5.4-1].

Non-critical degree

Case 2. The factors in the convolution product have one-sided support, bounded at different sides. In this case, the convolution $f * g$, with $f \in \mathcal{D}'_L$ and $g \in \mathcal{D}'_R$, can not straightforwardly be defined in terms of a direct product, because $\text{supp}(f * g) \cap \text{supp}(\varphi \in \mathcal{D}(R^2))$ is generally non-compact. In [5] however it is shown that if f^a and g^b are AHDs on R with degrees of homogeneity $a-1$ and $b-1$, whose support is

bounded on different sides, one can still construct their convolution, provided the resulting degree of homogeneity $a + b - 1$ is not a natural number. This is a consequence of the existence of the mixed-sided convolution product $D_a^m \Phi_-^a * D_b^n \Phi_+^b$, $\forall m, n \in \mathbb{N}$ and $\forall a, b \in \mathbb{C}$ such that $a + b - 1 \notin \mathbb{N}$. Herein are the normalized half-line basis AHDs $D_a^m \Phi_-^a \in \mathcal{D}'_L$ and $D_b^n \Phi_+^b \in \mathcal{D}'_R$ defined as $\Phi_{\pm}^z \triangleq x_{\pm}^{z-1} / \Gamma(z)$, [14], [11].

Let $T \triangleq \{(a, b) \in \mathbb{C}^2 : 0 < \operatorname{Re}(a), 0 < \operatorname{Re}(b) \text{ and } \operatorname{Re}(a + b) < 1\}$. In T , $D_a^m \Phi_-^a$ and $D_b^n \Phi_+^b$ are regular distributions. A direct calculation, using the standard convolution integral, shows that $D_a^m \Phi_-^a * D_b^n \Phi_+^b$ exists in T and is also a regular distribution. Let $R \triangleq \{(a, b) \in \mathbb{C}^2 : a + b - 1 \notin \mathbb{N}\}$. The distribution $D_a^m \Phi_-^a * D_b^n \Phi_+^b$ is subsequently defined in R by analytic continuation. Its explicit form was derived in [5].

Critical degree

Case 3. Let f^{a-1} and g^{b-1} be AHDs on R of degree $a - 1$ and $b - 1$, respectively, and $a + b - 1 = k \in \mathbb{N}$. The standard technique of the preceding subsection, when applied to a critical convolution product of AHDs, does not generate a distribution. It was shown in [6] however that:

(i) The convolution product $f^{a-1} * g^{b-1}$ is now in general a partial distribution $(f^{a-1} * g^{b-1})_{p,d}$, defined only on $\mathcal{S}_{\{k\}}$.

(ii) It is natural to consider as a particular extension of $(f^{a-1} * g^{b-1})_{p,d}$ from $\mathcal{S}_{\{k\}}$ to \mathcal{S} , the analytic finite part. However, this finite part, being a limit in \mathbb{C}^2 , was also found to be non-unique in general. Fortunately, a detailed investigation revealed that this non-uniqueness also vanishes on $\mathcal{S}_{\{k\}}$. This can be read off from the explicit expression of the analytic finite part, $(f^{a-1} * g^{b-1})_0$, given in [6, eq. (27)].

Now, any critical convolution product of AHDs on R , of a pair f^{a-1} and g^{b-1} with $a + b - 1 = k \in \mathbb{N}$ and which results in the partial distribution $(f^{a-1} * g^{b-1})_{p,d}$, is defined as any extension of $(f^{a-1} * g^{b-1})_{p,d}$ from $\mathcal{S}_{\{k\}}$ to \mathcal{S} . For instance, $x^k * x^l \triangleq (x^k * x^l)_e = 0 + c x^{k+l+1}$, with $c \in \mathbb{C}$ arbitrary, $\forall k, l \in \mathbb{N}$.

The general formula for the convolution product of two arbitrary AHDs on R is given in [8, Theorem 6].

3.3. Multiplication

Let f^a and g^b be AHDs on R of degree a and b , respectively. Then, $(\mathcal{F}_{\mathcal{H}'}^{-1} f^a)$ and $(\mathcal{F}_{\mathcal{H}'}^{-1} g^b)$ are also AHDs on R . Owing to the results from [5] and [6], $(\mathcal{F}_{\mathcal{H}'}^{-1} f^a) * (\mathcal{F}_{\mathcal{H}'}^{-1} g^b)$ exists, so we define

$$(9) \quad f^a . g^b \triangleq \mathcal{F}_{\mathcal{H}'} \left((\mathcal{F}_{\mathcal{H}'}^{-1} f^a) * (\mathcal{F}_{\mathcal{H}'}^{-1} g^b) \right),$$

and $f^a . g^b$ is again an AHD on R of degree $a + b$.

The multiplication product $f^a . g^b$ in the critical case, i.e., when $a + b = -l \in \mathbb{Z}_-$, is a partial distribution only defined on $\mathcal{S}_{\{-l\}}$. Any critical multiplication product

of AHDs on R , f^a and f^b , which results in a partial distribution $f^a \cdot f^b$, is then defined as any extension from $\mathcal{S}_{\{-l\}}$ to \mathcal{S} of the partial distribution $f^a \cdot f^b$. For instance, $\delta^{(k-1)} \cdot \delta^{(l-1)} \triangleq \left(\delta^{(k-1)} \cdot \delta^{(l-1)} \right)_e = 0 + c \delta^{(k+l-1)}$, with $c \in \mathbb{C}$ arbitrary, $\forall k, l \in \mathbb{Z}_+$.

The general formula for the multiplication product of two arbitrary AHDs on R is given in [10, Theorem 6].

Having a multiplication product for AHDs on R , now also allows us to judge the appropriateness of the suggestive notation commonly used for these distributions. For instance, with $l, m \in \mathbb{Z}_+$, the distributions $x_{\pm}^{-l} \ln^m |x|$ are often tacitly interpreted as the distributional multiplication products $x_{\pm}^{-l} \cdot \ln^m |x|$, which is correct in this case. On the other hand, the notation $(x \pm i0)^{-l} \ln^m (x \pm i0)$, used in e.g., [14, pp. 96-98] for the distributions $D_z^m (x \pm i0)^z$ at $z = -l$, is prone to be read as the distributional multiplication products $(x \pm i0)^{-l} \cdot \ln^m (x \pm i0)$, but which is incorrect (subject to our definition of multiplication), see [11, eq. (229)].

3.4. Non-commutativity

Convolution

The non-commutativity of the convolution product is as follows.

(i) Non-critical convolution products are always commutative, [5].

(ii) Critical convolution products are generally non-commutative as a result of their definition as any extension of a partial distribution.

Let $f_m^{a-1}, g_n^{b-1} \in \mathcal{H}'$ and $a + b - 1 = k \in \mathbb{N}$. Then, $f_m^{a-1} * g_n^{b-1}$ and $g_n^{b-1} * f_m^{a-1}$ exist as the partial distributions $(f_m^{a-1} * g_n^{b-1})_{p.d.}$ and $(g_n^{b-1} * f_m^{a-1})_{p.d.}$, respectively, and $(f_m^{a-1} * g_n^{b-1})_{p.d.} = (g_n^{b-1} * f_m^{a-1})_{p.d.}$. It is natural to define the distributions

$$\begin{aligned} f_m^{a-1} * g_n^{b-1} &\triangleq \text{any extension } \left(f_m^{a-1} * g_n^{b-1} \right)_e \text{ of } \left(f_m^{a-1} * g_n^{b-1} \right)_{p.d.}, \\ g_n^{b-1} * f_m^{a-1} &\triangleq \text{any extension } \left(g_n^{b-1} * f_m^{a-1} \right)_e \text{ of } \left(g_n^{b-1} * f_m^{a-1} \right)_{p.d.}. \end{aligned}$$

This makes that the distributions $f_m^{a-1} * g_n^{b-1}$ and $g_n^{b-1} * f_m^{a-1}$ are generally different, since the arbitrary constants, in both extensions, do not necessarily have to be equal. Hence, in general $f_m^{a-1} * g_n^{b-1} - g_n^{b-1} * f_m^{a-1} = cx^k$ with $c \in \mathbb{C}$ arbitrary.

Thus, if $a + b - 1 \in \mathbb{N}$, we do not need to pay attention to the order in which the product $f_m^{a-1} * g_n^{b-1}$ is evaluated, since we can always compensate for the effect that a change of order induces, by a change of extension of the final result. In other words, if $a + b - 1 \in \mathbb{N}$, calculating $f_m^{a-1} * g_n^{b-1}$ in different orders merely results in different extensions of the critical product.

This can equivalently be stated as: the convolution product on $\mathcal{H}' / \sim_{\mathbb{N}}$ is commutative.

Multiplication

The non-commutativity of the multiplication product is isomorphic to the non-commutativity of the convolution product.

(i) Non-critical multiplication products are always commutative.

(ii) Critical multiplication products are generally non-commutative as a result of their definition as any extension of a partial distribution.

Let $f_m^a, g_n^b \in \mathcal{H}'$ and $a + b = -l \in \mathbb{Z}_-$. Then, $f_m^a \cdot g_n^b - g_n^b \cdot f_m^a = c\delta^{(l-1)}$ with $c \in \mathbb{C}$ arbitrary, or equivalently: the multiplication product on $\mathcal{H}' / \sim_{\mathbb{Z}_-}$ is commutative.

3.5. Non-associativity

Convolution

The non-associativity of triple convolution products is as follows.

(i) Non-critical triple convolution products are always associative, [5, Theorem 9].

(ii) Critical triple convolution products are generally non-associative.

From [8, Theorem 3] combined with linearity and since critical convolution products are defined as any extension, it follows that the critical triple products $f_{m_1}^{a-1} * (f_{m_2}^{b-1} * f_{m_3}^{c-1})$, $f_{m_2}^{b-1} * (f_{m_1}^{a-1} * f_{m_3}^{c-1})$ and $(f_{m_1}^{a-1} * f_{m_2}^{b-1}) * f_{m_3}^{c-1}$, $\forall f_{m_1}^{a-1}, f_{m_2}^{b-1}, f_{m_3}^{c-1} \in \mathcal{H}' : a + b + c - 1 = k \in \mathbb{N}$, differ by a distribution of the form rx^k , with $r \in \mathbb{C}$ arbitrary. Moreover, if an intermediate product is critical while the final product is not, say $a + b - 1 \in \mathbb{N}$ while $a + b + c - 1 \notin \mathbb{N}$, then the intermediate non-uniqueness of the extension process does not propagate into the final product, due to [8, eq. (3)].

Thus, if $a + b + c - 1 \in \mathbb{N}$, we do not need to pay attention to the order in which the product $f_{m_1}^{a-1} * f_{m_2}^{b-1} * f_{m_3}^{c-1}$ is evaluated, since we can always compensate for the effect that a change of order induces, by a change of extension of the final result. In other words, if $a + b + c - 1 \in \mathbb{N}$, calculating $f_{m_1}^{a-1} * f_{m_2}^{b-1} * f_{m_3}^{c-1}$ in different orders merely results in different extensions of the critical triple product. This can equivalently be stated as: the convolution product on $\mathcal{H}' / \sim_{\mathbb{N}}$ is associative.

Multiplication

The (non-)associativity of triple multiplication products immediately follows from the (non-)associativity of triple convolution products, since the multiplication product is homomorphic to the convolution product under Fourier transformation. We thus have the following.

(i) Non-critical triple multiplication products are associative.

(ii) Critical triple multiplication products are generally non-associative.

In this case the critical triple products $f_{m_1}^a \cdot (f_{m_2}^b \cdot f_{m_3}^c)$, $f_{m_2}^b \cdot (f_{m_1}^a \cdot f_{m_3}^c)$ and $(f_{m_1}^a \cdot f_{m_2}^b) \cdot f_{m_3}^c$, $\forall f_{m_1}^a, f_{m_2}^b, f_{m_3}^c \in \mathcal{H}' : a + b + c = -l \in \mathbb{Z}_-$, differ by a distribution of the form $r\delta^{(l-1)}$, with $r \in \mathbb{C}$ arbitrary, or equivalently: the multiplication product of triples

| | | | | |
|-------------------------------------------------|-----------------------------|------------------------------|------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------|
| * | $\delta^{(l)}$ | $\eta^{(l)}$ | $\frac{1}{\pi} \frac{x^l}{l!}$ | $\frac{1}{2} \frac{x^l \operatorname{sgn}}{l!}$ |
| $\delta^{(k)}$ | $\delta^{(k+l)}$ (C.1.1) | $\eta^{(k+l)}$ (C.1.2) | $1_{k \leq l} \frac{1}{\pi} \frac{x^{l-k}}{(l-k)!}$ (C.1.3) | $1_{l < k} \delta^{(k-l-1)} + 1_{k \leq l} \frac{1}{2} \frac{x^{l-k} \operatorname{sgn}}{(l-k)!}$ (C.1.4) |
| $\eta^{(k)}$ | | $-\delta^{(k+l)}$ (C.2.2) | $1_{k \leq l} c \frac{1}{\pi} \frac{x^{l-k}}{(l-k)!}$ (C.2.3) | $1_{l < k} \eta^{(k-l-1)} + 1_{k \leq l} \frac{1}{\pi} \frac{x^{l-k}}{(l-k)!} (\ln x + c)$ (C.2.4) |
| $\frac{1}{\pi} \frac{x^k}{k!}$ | | | $c \frac{1}{\pi} \frac{x^{k+l+1}}{(k+l+1)!}$ (C.3.3) | $c \frac{1}{\pi} \frac{x^{k+l+1}}{(k+l+1)!}$ (C.3.4) |
| $\frac{1}{2} \frac{x^k \operatorname{sgn}}{k!}$ | | | | $\frac{1}{2} \frac{x^{k+l+1} \operatorname{sgn}}{(k+l+1)!} + c \frac{1}{\pi} \frac{x^{k+l+1}}{(k+l+1)!}$ (C.4.4) |

Table 4.1: Some particular convolution products of homogeneous distributions.

| | | | |
|----------------------------------------|-------------------|--------------------------------------------------|-------------------------------------------------------------------|
| . | x^l | $\frac{1}{2} x^l \operatorname{sgn}$ | $(-1)^l \frac{\delta^{(l)}}{l!}$ |
| x^k | x^{k+l} (M.1.1) | $\frac{1}{2} x^{k+l} \operatorname{sgn}$ (M.1.2) | $1_{k \leq l} (-1)^{l-k} \frac{\delta^{(l-k)}}{(l-k)!}$ (M.1.3) |
| $\frac{1}{2} x^k \operatorname{sgn}$ | | $\frac{1}{4} x^{k+l}$ (M.2.2) | $1_{k \leq l} c (-1)^{l-k} \frac{\delta^{(l-k)}}{(l-k)!}$ (M.2.3) |
| $(-1)^k \frac{\delta^{(k)}}{k!}$ | | | $c (-1)^{k+l+1} \frac{\delta^{(k+l+1)}}{(k+l+1)!}$ (M.3.3) |
| $(-1)^{k+1} \pi \frac{\eta^{(k)}}{k!}$ | | | |

Table 4.2: Some special multiplication products of homogeneous distributions (part A).

on $\mathcal{H}' / \sim_{\mathbb{Z}_-}$ is associative.

4. Selected results

4.1. Convolution products

A direct calculation, using the general results obtained in [5] and [6], produces the convolution products of HDs of integer degree of homogeneity, as summarized in the Table 4.1, $\forall k, l \in \mathbb{N}$ and wherein $c \in \mathbb{C}$ is an arbitrary constant.

This table is symmetric, modulo the choice for the constant c . More general and new convolution products were obtained in [8].

4.2. Multiplication products

Starting from Table 4.1 and using results [10, eqs. (1)–(5)] leads to the multiplication products of HDs on R of integer degree of homogeneity, as summarized in the Tables 4.2, 4.3, $\forall k, l \in \mathbb{N}$ and wherein $c \in \mathbb{C}$ is an arbitrary constant.

Again, this table is symmetric, modulo the choice for the constant c . More general and new multiplication products were obtained in [10].

| | |
|--------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| . | $(-1)^l \pi \frac{\eta^{(l)}}{l!}$ |
| x^k | $1_{l < k} x^{k-l-1} + 1_{k \leq l} (-1)^{l-k} \pi \frac{\eta^{(l-k)}}{(l-k)!}$ (M.1.4) |
| $\frac{1}{2} x^k \operatorname{sgn}$ | $1_{k \leq l} (-1)^{l-k} \pi \frac{(\eta^{(l-k)} \operatorname{sgn})_0 + c \delta^{(l-k)}}{(l-k)!} + 1_{l < k} \frac{1}{2} x^{k-l-1} \operatorname{sgn}$ (M.2.4) |
| $(-1)^k \frac{\delta^{(k)}}{k!}$ | $c (-1)^{k+l+1} \frac{\delta^{(k+l+1)}}{(k+l+1)!}$ (M.3.4) |
| $(-1)^k \pi \frac{\eta^{(k)}}{k!}$ | $(-1)^{k+l+1} \pi \frac{\eta^{(k+l+1)}}{(k+l+1)!} + c (-1)^{k+l+1} \frac{\delta^{(k+l+1)}}{(k+l+1)!}$ (M.4.4) |

Table 4.3: Some special multiplication products of homogeneous distributions (part B).

References

- [1] BOGOLIUBOV, N. N., LOGUNOV, A. A., AND TODOROV, I. T. *Introduction to Axiomatic Quantum Field Theory*. Benjamin, Reading, 1975.
- [2] CHOQUET-BRUHAT, Y., DEWITT-MORETTE, C., AND DILLARD-BLEICK, M. *Analysis, Manifolds and Physics*, 2 ed. Elsevier, 1982.
- [3] COLOMBEAU, J. New generalized functions and multiplication of distributions. In *Math. Studies*, no. 84. North-Holland, Amsterdam, 1984.
- [4] DONAHUE, W. F. Distributions and fourier transforms.
- [5] FRANSSSENS, G. R. The convolution of associated homogeneous distributions on r . *Appl. Anal.*, 88 (2009), 309–331.
- [6] FRANSSSENS, G. R. The convolution of associated homogeneous distributions on r – part ii. *Appl. Anal.* 88 (2009), 333–356.
- [7] FRANSSSENS, G. R. Structure theorems for associated homogeneous distributions based on the line. *Math. Methods Appl. Sci.*, 32 (2009), 986–1010.
- [8] FRANSSSENS, G. R. Convolution product formula for associated homogeneous distributions on r . *Math. Methods Appl. Sci.*, 34 (2011), 703–727.
- [9] FRANSSSENS, G. R. Erratum: Convolution product formula for associated homogeneous distributions on r . *Math. Methods Appl. Sci.*, 34 (2011), 620.
- [10] FRANSSSENS, G. R. Multiplication product formula for associated homogeneous distributions on r . *Math. Methods Appl. Sci.*, 34 (2011), 1460–1471.
- [11] FRANSSSENS, G. R. One-dimensional associated homogeneous distributions. *Bull. of Math. Anal. and Appl.*, 3 (2011), 1–60.
- [12] FRANSSSENS, G. R. *Substructures in algebras of associated homogeneous distributions on R* , vol. 19. 2012.
- [13] FRIEDLANDER, G., AND JOSHI, M. *Introduction to The Theory of Distributions*, 2 ed. Cambridge Univ. Press, 1998.
- [14] GEL'FAND, I., AND SHILOV, G. E. *Generalized Functions*, vol. 1. Academic Press, 1964.
- [15] GÜTTINGER, W. Products of improper operators and the renormalization problem in quantum field theory. *Prog. Theor. Phys.*, 134 (1955), 612–626.
- [16] KANWAL, R. P. *Generalized Functions, Theory and Technique*, 2 ed. Birkhäuser, 1998.
- [17] LEUTWYLER, H. Anomalies. *Helv. Phys. Acta*, 59 (1986), 201–219.

- [18] OBERGUGGENBERGER, M. Multiplication of distributions and applications to partial differential equations. In *Pitman Research Notes in Mathematics Series*, vol. 259. Longman, 1992.
- [19] RUDIN, W. *Principles of Mathematical Analysis*. McGraw-Hill, 1976.
- [20] RUDIN, W. *Functional Analysis*. McGraw-Hill, 1991.
- [21] SCHWARTZ, L. Sur l'impossibilité de la multiplication des distributions. *C. R. Math. Acad. Sci. Paris*, 239 (1954), 847–848.
- [22] SCHWARTZ, L. *Théorie des Distributions*, vol. 1, 2. Hermann, 1957.
- [23] SHELKOVICH, V. M. Associated and quasi-associated homogeneous distributions (generalized functions). *J. Math. Anal. Appl.*, 338 (2008), 48–70.
- [24] TAYLOR, J. G. The renormalization constants in perturbation theory. *Nuovo Cimento*, 17 (1960), 695–702.
- [25] ZEMANIAN, A. H. *Distribution Theory and Transform Analysis*. Dover, 1965.

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Ghislain FRANSENS
Belgian Institute for Space Aeronomy
Ringlaan 3, B-1180 Brussels, BELGIUM
e-mail: ghislain.franssens@aeronomy.be

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