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LINEAR DIFFERENTIAL EQUATIONS IN THE ALGEBRA OF ALMOST PERIODIC GENERALIZED FUNCTIONS

Abstract. The paper introduces and studies an algebra of almost periodic generalized functions generalizing trigonometric polynomials, classical almost periodic functions as well as almost periodic Schwartz distributions. Then we study a linear system of ordinary differential equations in this algebra of almost periodic generalized functions.

1. Introduction

Trigonometric polynomials are the basic examples of the uniformly almost periodic functions introduced and studied by H. Bohr, see [1]. There exist three equivalent definitions of uniformly almost periodic functions, the first definition of H. Bohr, S. Bochner's definition and the definition based on the approximation property. Bochner's definition is more suitable for extension to distributions. L. Schwartz in [6] introduced the basic elements of almost periodic distributions.

The algebra of generalized functions of Colombeau [3] give an answer to the problem of multiplication of distributions, this algebra contains the space of Schwartz distributions, and is currently the subject of many scientific works, see [5].

We introduce and study an algebra of almost periodic generalized functions generalizing trigonometric polynomials, classical almost periodic functions as well as almost periodic Schwartz distributions. Then we study a linear system of ordinary differential equations in this algebra of almost periodic generalized functions, namely we give results on the existence of generalized solutions of the linear system of ordinary differential equations

$$\dot{u}(t) = Au(t) + f(t),$$

where f is an almost periodic generalized function and A is a matrix of classical complex numbers.

2. Almost periodic functions and distributions

We consider functions and distributions defined on the whole one dimensional space $\mathbb{R}$.

Let $\mathcal{C}_b$ be the space of bounded and continuous complex valued functions on $\mathbb{R}$ endowed with the norm $\| \|_\infty$ of uniform convergence on $\mathbb{R}$, $(\mathcal{C}_b, \| \|_\infty)$ is a Banach algebra.

DEFINITION 1. *(S. Bochner) A complex valued function f defined and continuous on $\mathbb{R}$ is called almost periodic, if for any sequence of real numbers $(h_n)_n$ one can* *extract a subsequence* $\left(h_{n_k}\right)_k$ *such that* $\left(f\left(.+h_{n_k}\right)\right)_k$ *converges in* $(\mathcal{C}_b, \|\ \|_\infty)$. *Denote by* $\mathcal{C}_{ap}$ *the space of almost periodic functions.*

To recall Schwartz almost periodic distributions, we need some function spaces, see [6]. Let $p \in [1,+\infty]$, the space $\mathcal{D}_{L^p} := \left\{\varphi \in \mathcal{C}^\infty : \varphi^{(j)} \in L^p, \forall j \in \mathbb{Z}_+\right\}$ endowed with the topology defined by the countable family of norms $|\varphi|_{k,p} := \sum\limits_{j \le k} \left\|\varphi^{(j)}\right\|_{L^p}$, $k \in \mathbb{Z}_+$, is a differential Frechet subalgebra of $\mathcal{C}^\infty$. The topological dual of $\mathcal{D}_{L^1}$, denoted by $\mathcal{D}'_{L^\infty}$, is called the space of bounded distributions.

Let $h \in \mathbb{R}$ and $T \in \mathcal{D}'$, the translate of T by h, denoted by $\tau_h T$, is defined as $<\tau_h T, \varphi> = <T, \tau_{-h}\varphi>, \varphi \in \mathcal{D}$, where $\tau_{-h}\varphi(x) = \varphi(x+h)$.

The definition and characterizations of an almost periodic distribution are summarized in the following results.

THEOREM 1. *For any bounded distribution* $T \in \mathcal{D}'_{L^\infty}$, *the following statements are equivalent :*

i) The set $\{\tau_h T\ , h \in \mathbb{R}\}$ *is relatively compact in* $\mathcal{D}'_{L^\infty}$.

ii) $T * \varphi \in \mathcal{C}_{ap}, \forall\ \varphi \in \mathcal{D}$.

iii) $\exists\ (f_j)_{j \le k} \subset \mathcal{C}_{ap}$, $T = \sum_{j=1}^k f_j^{(j)}$.

$T \in \mathcal{D}'_{L^\infty}$ *is said almost periodic if it satisfies any (hence every) of the above conditions.*

DEFINITION 2. *The space of almost periodic distributions is denoted by* $\mathcal{B}'_{ap}$.

Let us introduce the space of regular almost periodic functions.

DEFINITION 3. *The space of almost periodic infinitely differentiable functions on* $\mathbb{R}$ *is defined and denoted by* $\mathcal{B}_{ap} = \left\{\varphi \in \mathcal{D}_{L^\infty} : \varphi^{(j)} \in \mathcal{C}_{ap}, \forall j \in \mathbb{Z}_+\right\}$.

Some, easy to prove, properties of $\mathcal{B}_{ap}$ are given in the following proposition.

PROPOSITION 1. *We have*

i) $\mathcal{B}_{ap}$ *is a closed differential subalgebra of* $\mathcal{D}_{L^\infty}$.

ii) if $T \in \mathcal{B}'_{ap}$ *and* $\varphi \in \mathcal{B}_{ap}$, *then* $\varphi T \in \mathcal{B}'_{ap}$.

iii) $\mathcal{B}_{ap} * L^1 \subset \mathcal{B}_{ap}$.

iv) $\mathcal{B}_{ap} = \mathcal{D}_{L^\infty} \cap \mathcal{C}_{ap}$.

As a consequence of (iv), we have the following result.

COROLLARY 1. *If* $v \in \mathcal{D}_{L^\infty}$ *and* $v * \varphi \in \mathcal{C}_{ap}, \forall\ \varphi \in \mathcal{D}$, *then* $v \in \mathcal{B}_{ap}$.

REMARK 1. It is important to mention that $\mathcal{B}_{ap} \subsetneq \mathcal{C}^\infty \cap \mathcal{C}_{ap}$.

3. Almost periodic generalized functions

Let $I =]0,1]$ and

$$\mathcal{M}_{L^\infty} := \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{D}_{L^\infty})^I, \forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+,\ |u_\varepsilon|_{k,\infty} = O\left(\varepsilon^{-m}\right),\ \varepsilon \longrightarrow 0 \right\}$$
$$\mathcal{N}_{L^\infty} := \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{D}_{L^\infty})^I, \forall k \in \mathbb{Z}_+, \forall m \in \mathbb{Z}_+,\ |u_\varepsilon|_{k,\infty} = O\left(\varepsilon^{m}\right),\ \varepsilon \longrightarrow 0 \right\}$$

DEFINITION 4. *The algebra of bounded generalized functions, denoted by $\mathcal{G}_{L^\infty}$, is defined by the quotient $\mathcal{G}_{L^\infty} = \frac{\mathcal{M}_{L^\infty}}{\mathcal{N}_{L^\infty}}$*

Define

$$\text{(1)} \qquad \begin{aligned} \mathcal{M}_{ap} &= \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{B}_{ap})^I, \forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |u_\varepsilon|_{k,\infty} = O\left(\varepsilon^{-m}\right), \varepsilon \longrightarrow 0 \right\} \\ \mathcal{N}_{ap} &= \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{B}_{ap})^I, \forall k \in \mathbb{Z}_+, \forall m \in \mathbb{Z}_+, |u_\varepsilon|_{k,\infty} = O\left(\varepsilon^{m}\right), \varepsilon \longrightarrow 0 \right\} \end{aligned}$$

The properties of $\mathcal{M}_{ap}$ and $\mathcal{N}_{ap}$ are summarized in the following proposition.

PROPOSITION 2. *i) The space $\mathcal{M}_{ap}$ is a subalgebra of $(\mathcal{D}_{L^\infty})^I$.*
ii) The space $\mathcal{N}_{ap}$ is an ideal of $\mathcal{M}_{ap}$.

Proof. *i*) It follows from the fact that $\mathcal{B}_{ap}$ is an differential algebra.
ii) Let $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{ap}$ and $(v_\varepsilon)_\varepsilon \in \mathcal{M}_{ap}$, we have

$$\forall k \in \mathbb{Z}_+, \exists m' \in \mathbb{Z}_+, \exists c_1 > 0, \exists \varepsilon_0 \in I, \forall \varepsilon < \varepsilon_0, |v_\varepsilon|_{k,\infty} < c_1 \varepsilon^{-m'}.$$

Take $m \in \mathbb{Z}_+$, then for $m'' = m + m'$, $\exists c_2 > 0$ such that $|u_\varepsilon|_{k,\infty} < c_2 \varepsilon^{m''}$. Since the family of the norms $|u_\varepsilon|_{k,\infty}$ is compatible with the algebraic structure of $\mathcal{D}_{L^\infty}$, then $\forall k \in \mathbb{Z}_+$, $\exists c_k > 0$ such that

$$|u_\varepsilon v_\varepsilon|_{k,\infty} \leq c_k\ |u_\varepsilon|_{k,\infty}\ |v_\varepsilon|_{k,\infty},$$

consequently

$$|u_\varepsilon v_\varepsilon|_{k,\infty} < c_k c_2 \varepsilon^{m''} c_1 \varepsilon^{-m'} \leq C \varepsilon^m, \text{ where } C = c_1 c_2 c_k.$$

Hence $(u_\varepsilon v_\varepsilon)_\varepsilon \in \mathcal{N}_{ap}$. □

The following definition introduces the algebra of almost periodic generalized functions.

DEFINITION 5. *The algebra of almost periodic generalized functions is the quotient algebra*

$$\mathcal{G}_{ap} = \frac{\mathcal{M}_{ap}}{\mathcal{N}_{ap}}$$

We have a characterization of elements of $\mathcal{G}_{ap}$ similar to the result (ii) of theorem 1 for almost periodic distributions.

THEOREM 2. *Let $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{L^\infty}$, the following assertions are equivalent :*
$i)$ u is almost periodic.
$ii)$ $u_\varepsilon * \varphi \in \mathcal{B}_{ap}$, $\forall \varepsilon \in I, \forall \varphi \in \mathcal{D}$.

Proof. $i) \Longrightarrow ii)$ If $u \in \mathcal{G}_{ap}$, so for every $\varepsilon \in I$ we have $u_\varepsilon \in \mathcal{B}_{ap}$, then $u_\varepsilon * \varphi \in \mathcal{B}_{ap}, \forall \varepsilon \in I, \forall \varphi \in \mathcal{D}$.

$ii) \Longrightarrow i)$ Let $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{L^\infty}$ and $u_\varepsilon * \varphi \in \mathcal{B}_{ap}, \forall \varepsilon \in I, \forall \varphi \in \mathcal{D}$, therefore $u \in \mathcal{B}_{ap}$ follows from Theorem 2.2 (ii); it suffices to show that

$$\forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |u_\varepsilon|_{k,\infty} = O\left(\varepsilon^{-m}\right), \varepsilon \longrightarrow 0,$$

which follows from the fact that $u \in \mathcal{G}_{L^\infty}$. □

REMARK 2. The characterization (ii) does not depend on representatives.

DEFINITION 6. *Denote by Σ the set of functions $\rho \in \mathcal{S}$ satisfying $\int \rho(x)\, dx = 1$ and $\int x^k \rho(x)\, dx = 0$, $\forall\, k = 1, 2, \ldots..$ Set $\rho_\varepsilon(.) := \frac{1}{\varepsilon}\rho\left(\frac{\cdot}{\varepsilon}\right)$, $\varepsilon > 0$.*

PROPOSITION 3. *Let $\rho \in \Sigma$, the map*

$$\begin{array}{rccc} i_{ap}: & \mathcal{B}'_{ap} & \longrightarrow & \mathcal{G}_{ap} \\ & u & \longrightarrow & (u * \rho_\varepsilon)_\varepsilon + \mathcal{N}_{ap}, \end{array}$$

is a linear embedding which commutes with derivatives.

Proof. Let $u \in \mathcal{B}'_{ap}$, by a characterization of almost periodic distributions we have $u = \sum\limits_{\beta \le m} f_\beta^{(\beta)}$, where $f_\beta \in \mathcal{C}_{ap}$, so $\forall \alpha \in \mathbb{Z}$,

$$\left|\left(u^{(\alpha)} * \rho_\varepsilon\right)(x)\right| \le \sum_{\beta \le m} \frac{1}{\varepsilon^{\alpha+\beta}} \int_{\mathbb{R}} \left| f_\beta (x - \varepsilon y)\, \rho^{(\alpha+\beta)}(y)\right| dy,$$

consequently, there exists $c > 0$ such that

$$\sup_{x \in \mathbb{R}} \left|\left(u^{(\alpha)} * \rho_\varepsilon\right)(x)\right| \le \sum_{\beta \le m} \frac{1}{\varepsilon^{\alpha+\beta}} \left\| f_\beta \right\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \left|\rho^{(\alpha+\beta)}(y)\right| dy \le \frac{c}{\varepsilon^{\alpha+m}},$$

i.e.

$$|u * \rho_\varepsilon|_{m',\infty} = \sum_{\alpha \le m'} \sup_{x \in \mathbb{R}} \left|\left(u^{(\alpha)} * \rho_\varepsilon\right)(x)\right| \le \frac{c'}{\varepsilon^{m+m'}}, c' = \sum_{\alpha \le m'} \frac{c}{\varepsilon^{\alpha}},$$

this shows that $(u * \rho_\varepsilon)_\varepsilon \in \mathcal{M}_{ap}$. Let $(u * \rho_\varepsilon)_\varepsilon \in \mathcal{N}_{ap}$, then $\lim\limits_{\varepsilon \longrightarrow 0} u * \rho_\varepsilon = 0$ in $\mathcal{D}'_{L^\infty}$, but $\lim\limits_{\varepsilon \longrightarrow 0} u * \rho_\varepsilon = u$ in $\mathcal{D}'_{L^\infty}$, this shows that i_{ap} is an embedding. Finally we note that i_{ap}

is linear, this results from the fact that the convolution is linear and that $i_{ap}\left(w^{(j)}\right) = \left(w^{(j)} * \rho_\varepsilon\right)_\varepsilon = (w * \rho_\varepsilon)_\varepsilon^{(j)} = (i_{ap}(w))^{(j)}$. □

The space $\mathcal{B}_{ap}$ is embedded into $\mathcal{G}_{ap}$ canonically, i.e.

$$\begin{array}{rccc} \sigma_{ap}: & \mathcal{B}_{ap} & \longrightarrow & \mathcal{G}_{ap} \\ & f & \longrightarrow & [(f)_\varepsilon] = (f)_\varepsilon + \mathcal{N}_{ap} \end{array}$$

There are two ways to embed $f \in \mathcal{B}_{ap}$ into $\mathcal{G}_{ap}$. Actually, we have the same result.

PROPOSITION 4. *The following diagram*

$$\begin{tikzcd} \mathcal{B}_{ap} \arrow[r] \arrow[dr, "\sigma_{ap}"] & \mathcal{B}'_{ap} \arrow[d, "i_{ap}"] \\ & \mathcal{G}_{ap} \end{tikzcd}$$

is commutative.

Proof. Let $f \in \mathcal{B}_{ap}$, we prove that $(f * \rho_\varepsilon - f)_\varepsilon \in \mathcal{N}_{ap}$. By Taylor's formula and the fact that $\rho \in \Sigma$, we obtain

$$\|f * \rho_\varepsilon - f\|_{L^\infty} \leq \varepsilon^m \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} \left| \frac{(-y)^m}{m!} f^{(m)}(x - \theta(x)\varepsilon y)\, \rho(y)\, dy \right|,$$

then $\exists C_m > 0$, such that

$$\|f * \rho_\varepsilon - f\|_{L^\infty} \leq \varepsilon^m C_m \left\| f^{(m)} \right\|_{L^\infty} \|y^m \rho\|_{L^1}.$$

The same result is obtained for all derivatives of f. Hence $(f * \rho_\varepsilon - f)_\varepsilon \in \mathcal{N}_{ap}$. □

The Colombeau algebra of tempered generalized functions on $\mathbb{C}$ is denoted $\mathcal{G}_{\mathcal{T}}(\mathbb{C})$, for more details on $\mathcal{G}_{\mathcal{T}}(\mathbb{C})$ see [3].

PROPOSITION 5. *Let $u \in \mathcal{G}_{ap}$ and $F \in \mathcal{G}_{\mathcal{T}}(\mathbb{C})$, then $F \circ u = [(F \circ u_\varepsilon)_\varepsilon]$ is a well defined element of $\mathcal{G}_{ap}$.*

Proof. It follows from the classical case of composition in the context of Colombeau algebra, we have $F \circ u_\varepsilon \in \mathcal{B}_{ap}$ in view of the classical results of composition and convolution. □

We recall a characterization of integrable distributions.

DEFINITION 7. *A distribution $v \in \mathcal{D}'$ is said an integrable distribution, denoted $v \in \mathcal{D}'_{L^1}$, if and only if $v = \sum\limits_{i \leq l} f_i^{(i)}$, where $f_i \in L^1$.*

PROPOSITION 6. *If $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{ap}$ and $v \in \mathcal{D}'_{L^1}$, then the convolution $u * v$ defined by $(u * v)(x) = \left(\int\limits_{\mathbb{R}} u_\varepsilon (x - y) v(y) dy \right)_\varepsilon + \mathcal{N}_{ap}$ is a well defined almost periodic generalized function.*

Proof. Let $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{ap}$ be a representative of u, then $\forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, \exists C > 0, \exists \varepsilon_0 \in I, \forall \varepsilon \leq \varepsilon_0, |u_\varepsilon|_{k,\infty} < C\varepsilon^{-m}$, since $v \in \mathcal{D}'_{L^1}$, then $v = \sum\limits_{i \leq l} f_i^{(i)}$, where $f_i \in L^1$. For each $\varepsilon \in I$, $u_\varepsilon * v$ is an almost periodic infinitely differentiable function. By Young inequality there exists $C > 0$ such that $\left\| (u_\varepsilon * v)^{(j)} \right\|_{L^\infty} \leq C \sum \|f_i\|_{L^1} \left\| u_\varepsilon^{(i+j)} \right\|_{L^\infty}$, consequently $|u_\varepsilon * v|_{k,\infty} = O(\varepsilon^{-m}), \varepsilon \longrightarrow 0$, this shows that $(u_\varepsilon * v)_\varepsilon \in \mathcal{M}_{ap}$. Suppose that $(w_\varepsilon)_\varepsilon \in \mathcal{M}_{ap}$ is another representative of u, then there exists $C > 0$ such that

$$\begin{aligned}
\|(u_\varepsilon * v - w_\varepsilon * v)\|_{L^\infty} &\leq \sum_{i \leq l} \sup_{\mathbb{R}} \int_{\mathbb{R}} \left| (u_\varepsilon - w_\varepsilon)^{(i)} (x - y) \right| |f_i(y)| dy \\
&\leq C \sum_{i \leq l} \|f_i\|_{L^1} \left\| (u_\varepsilon - w_\varepsilon)^{(i)} \right\|_{L^\infty},
\end{aligned}$$

as $(u_\varepsilon - w_\varepsilon)_\varepsilon \in \mathcal{N}_{ap}$, so $\forall m \in \mathbb{Z}_+$, $|(u_\varepsilon * v - w_\varepsilon * v)(x)| = O(\varepsilon^m), \varepsilon \longrightarrow 0$. We obtain the same result for $(u_\varepsilon * v - w_\varepsilon * v)_\varepsilon^{(j)}$. Hence $(u_\varepsilon * v - w_\varepsilon * v)_\varepsilon \in \mathcal{N}_{ap}$. □

If $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{ap}$, taking the integral of each element u_ε on a compact, we obtain an element of $\mathbb{C}^I$.

DEFINITION 8. *Let $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{ap}$ and $x_0 \in \mathbb{R}$, define the primitive of u by*

$$U(x) = \left(\int_{x_0}^{x} u_\varepsilon(t) dt \right)_\varepsilon + \mathcal{N}[\mathbb{C}].$$

We give a generalized version of the classical Bohl-Bohr theorem.

PROPOSITION 7. *The primitive of an almost periodic generalized function is almost periodic if and only if it is a bounded generalized function.*

Proof. We will obtain this result as a corollary of our study of linear ordinary differential equations in the algebra of almost periodic generalized functions. This study is done in the next section. □

4. A system of ordinary differential equations in $\mathcal{G}_{ap}$

Consider the linear system of ordinary differential equations

$$(2) \qquad \dot{u} = Au + f,$$

where $f = [(f_\varepsilon)_\varepsilon] = \left[(f_{1,\varepsilon}, ..., f_{n,\varepsilon})_\varepsilon\right]$ with components almost periodic generalized functions and $A = (a_{ij})_{1 \le i,j \le n} \in \mathcal{M}_n(\mathbb{C})$ is a square matrix of complex numbers. The unknown function $u = \left[(u_{1,\varepsilon}, ..., u_{n,\varepsilon})_\varepsilon\right]$.

REMARK 3. Note that a generalized function $u = [(u_\varepsilon)_\varepsilon] = \left[(u_{1,\varepsilon}, ..., u_{n,\varepsilon})_\varepsilon\right]$ is called bounded (resp. almost periodic) if all components $u_{i,\varepsilon}, 1 \le i \le n$, are bounded (resp. almost periodic).

DEFINITION 9. *A generalized function $u \in (\mathcal{G}_{ap})^n$ is called solution of the differential equation* (2) *if it satisfies*

$$\left(\dot{u}_\varepsilon\right)_\varepsilon - (Au_\varepsilon)_\varepsilon - (f_\varepsilon)_\varepsilon \in \mathcal{N}_{ap},$$

where $(u_\varepsilon)_\varepsilon$ and $(f_\varepsilon)_\varepsilon$ are respectively representatives of u and f.

We will need the following classical lemma.

LEMMA 1. *If $A = (a_{ij})_{1 \le i,j \le n} \in \mathcal{M}_n(\mathbb{C})$, then there exists $C = (c_{ij})_{1 \le i,j \le n} \in \mathcal{M}_n(\mathbb{C})$ such that*

(i) $\det C \neq 0$

(ii) *$C^{-1}AC$ is a triangular matrix, i.e.*

$$C^{-1}AC = \begin{pmatrix} \lambda_1 & b_{12} & ... & b_{1n} \\ 0 & \lambda_2 & ... & b_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & ... & 0 & \lambda_n \end{pmatrix},$$

where $\lambda_i, 1 \le i \le n$, are the eigenvalues of A, and $b_{ij}, 1 \le i < j \le n$, are complex numbers.

The first result is a generalization of Bohr-Neugebauer theorem.

THEOREM 3. *The solution u of* (2) *is an almost periodic generalized function if and only if it is a bounded generalized function.*

Proof. Since $\mathcal{G}_{ap} \subset \mathcal{G}_{L^\infty}$, it remains to prove that if the solution u of (2) is a bounded generalized function, i.e. $u \in \mathcal{G}_{L^\infty}$, then it is almost periodic, i.e. $u \in \mathcal{G}_{ap}$. Let $C = (c_{ij})_{1 \le i,j \le n} \in \mathcal{M}_n(\mathbb{C})$. By taking $u = Cv$, the system of differential equations

$$(3) \qquad \dot{u}(t) = Au(t) + f(t)$$

is equivalent in $\mathcal{G}(\mathbb{R})^n$ to the following linear system of differential equations

$$(4)\qquad \begin{cases} \dot{v}_{1,\varepsilon}(t) &= \lambda_1 v_{1,\varepsilon}(t) + b_{12}v_{2,\varepsilon}(t) + \ldots + b_{1n}v_{n,\varepsilon}(t) + g_{1,\varepsilon}(t) & (1) \\ \dot{v}_{2,\varepsilon}(t) &= \lambda_2 v_{2,\varepsilon}(t) + \ldots + b_{2n}v_{n,\varepsilon}(t) + g_{2,\varepsilon}(t) & (2) \\ \vdots & & \vdots \\ \dot{v}_{n,\varepsilon}(t) &= \lambda_n v_{n,\varepsilon}(t) + g_{n,\varepsilon}(t) & (n) \end{cases},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A and the generalized function g is given by $Cg = f$. Thus, it is sufficient to study the scalar linear ordinary differential equation

$$(5)\qquad \dot{v}(t) = \lambda v(t) + g(t),$$

where $g = [(g_\varepsilon)_\varepsilon] \in \mathcal{G}_{ap}$ and $\lambda \in \mathbb{C}$. Any solution of this equation is given by the class of the generalized function

$$(v_\varepsilon(t))_\varepsilon = \left(e^{\lambda t}(C_\varepsilon + \int_0^t e^{-\lambda s} g_\varepsilon(s)\, ds) \right)_\varepsilon,$$

where $(C_\varepsilon)_\varepsilon \in \widetilde{\mathbb{C}}$ is an arbitrary generalized complex constant. From the assumption that the solution u de (2) is a bounded generalized function, i.e. $u \in \mathcal{G}_{L^\infty}$, we distinguish the following cases : (1) $\mathrm{Re}\,\lambda > 0$, (2) $\mathrm{Re}\,\lambda < 0$, (2) $\mathrm{Re}\,\lambda = 0$.

(1) If $\mathrm{Re}\,\lambda > 0$, then we obtain that

$$v_\varepsilon(t) = -\int_t^{+\infty} e^{\lambda(t-s)} g_\varepsilon(s)\, ds.$$

Moreover, we have

$$\begin{aligned} \sup_{t\in\mathbb{R}} |v_\varepsilon(t+\tau) - v_\varepsilon(t)| &= \sup_{t\in\mathbb{R}} \left| e^{\lambda(t+\tau)} \int_{t+\tau}^{+\infty} e^{-\lambda s} g_\varepsilon(s)\, ds - e^{\lambda t} \int_t^{+\infty} e^{-\lambda s} g_\varepsilon(s)\, ds \right| \\ &= \frac{1}{\mathrm{Re}\,\lambda} \sup_{t\in\mathbb{R}} |g_\varepsilon(t+\tau) - g_\varepsilon(t)|, \end{aligned}$$

which shows the almost periodicity of v_ε. Similarly, we prove that $\forall j \in \mathbb{Z}_+, v_\varepsilon^{(j)}$ is almost periodic.

(2) The case $\mathrm{Re}\,\lambda < 0$ is analogous to the case $\mathrm{Re}\,\lambda > 0$.

(3) If $\mathrm{Re}\,\lambda = 0$, then $e^{\lambda t} = e^{i\theta t}$, and the generalized solution of (5) is of the form

$$v_\varepsilon(t) = e^{i\theta t} \left(C_\varepsilon + \int_0^t e^{-i\theta s} g_\varepsilon(s)\, ds \right), \varepsilon \in I,$$

where $C_\varepsilon \in \mathcal{E}_M[\mathbb{C}]$ is an arbitrary generalized complex constant. The almost periodicity of $(v_\varepsilon)_\varepsilon$ results from the almost periodicity of $(\int_0^t e^{-i\theta s} g_\varepsilon(s)\, ds)_\varepsilon$, which is a bounded primitive of an almost periodic generalized function, see [2]. □

REMARK 4. The previous theorem does not give the existence of solutions.

We have the following result.

THEOREM 4. *If the matrix A has eigenvalues λ satisfying* $\operatorname{Re}\lambda \neq 0$, *then there exists a solution $u = \left[(u_{1,\varepsilon}, ..., u_{n,\varepsilon})_\varepsilon\right]$ bounded generalized function (and then almost periodic) of* (2).

Proof. From the proof of theorem (3), it suffices to show that the system (4), where the eigenvalues λ satisfy $\operatorname{Re}\lambda \neq 0$ has a bounded generalized solution. The system (4) shows that by defining $v_{n,\varepsilon}$ as

$$v_{n,\varepsilon}(t) = -\int_t^{+\infty} e^{\lambda_n(t-s)} g_{n,\varepsilon}(s)\,ds, \text{ if } \operatorname{Re}\lambda_n > 0 \tag{6}$$

or

$$v_{n,\varepsilon}(t) = \int_{-\infty}^{t} e^{\lambda_n(t-s)} g_{n,\varepsilon}(s)\,ds, \text{ if } \operatorname{Re}\lambda_n < 0 \tag{7}$$

and then replacing the solution $v_{n,\varepsilon}$ in the equation $(n-1)$ of (4), we obtain for $v_{n-1,\varepsilon}$ an equation of the same type as (5). Since $\operatorname{Re}\lambda_{n-1} \neq 0$, we obtain the same possible cases $(6), (7)$ for $v_{n-1,\varepsilon}$, and so on. Consequently, it remains to show that the generalized solution of the system (4) thus constructed $(v_{1,\varepsilon}, ..., v_{n,\varepsilon})_\varepsilon$ is a bounded generalized function. Indeed, let $\mu > 0$ be such that

$$|\operatorname{Re}\lambda_i| \geq \mu, \; 1 \leq i \leq n. \tag{8}$$

From $(6), (7)$, we have $\forall\varepsilon \in I$,

$$\|v_{n,\varepsilon}(t)\|_{L^\infty} \leq \frac{\|g_{n,\varepsilon}\|_{L^\infty}}{\mu} \tag{9}$$

From the equation

$$\dot{v}_{n-1,\varepsilon}(t) = \lambda_{n-1} v_{n-1,\varepsilon}(t) + b_{n-1n} v_{n,\varepsilon}(t) + g_{n-1,\varepsilon}(t),$$

and by giving $v_{n-1,\varepsilon}$ one of the formulas $(6), (7)$, we obtain $\forall\varepsilon \in I$,

$$\|v_{n-1,\varepsilon}\|_{L^\infty} \leq \frac{\left(|b_{n-1n}| \frac{\|g_{n,\varepsilon}\|_{L^\infty}}{\mu} + \|g_{n-1,\varepsilon}\|_{L^\infty}\right)}{\mu} \tag{10}$$

$$\leq \left(\frac{|b_{n-1n}|}{\mu^2} + \frac{1}{\mu}\right) \max\left(\|g_{n,\varepsilon}\|_{L^\infty}, \|g_{n-1,\varepsilon}\|_{L^\infty}\right), \tag{11}$$

an so on, for $i = n-2, n-3, ..., 1$, we obtain $\exists C > 0, \forall\varepsilon \in I$,

$$\max_{i \leq n} \|v_{i,\varepsilon}\|_{L^\infty} \leq C \max_{i \leq n}\left(\|g_{j,\varepsilon}\|_{L^\infty}\right). \tag{12}$$

So it clear haw to obtain the following estimates

(13) $$\exists C = C(A) > 0, \forall k \in \mathbb{Z}_+, \forall \varepsilon \in I,\ \left|(v_{1,\varepsilon}, \ldots, v_{n,\varepsilon})\right|_{k,\infty} \leq C\left|(g_{1,\varepsilon}, \ldots, g_{n,\varepsilon})\right|_{k,\infty}$$

which give the result. □

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